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ORIGINAL ARTICLE

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On the approximate schemes for the evaluation of the acoustic radiation by a thin elastic layer

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Abstract A plane strain problem for forced time-harmonic vibrations of an elastic layer lying on an acoustic half-space is considered. The validity of the approximate formulation involving the classical Kirchhoff theory for plate bending as well as its shortened forms is investigated. The developed asymptotic framework demonstrates that the aforementioned theory is not able to predict the effect of the plate stiffness on the acoustic radiation. A consistent low-frequency approximation relying on plate transverse compression instead of plate bending is derived.

Keywords Fluid-loaded elastic layer · Acoustic radiation · Asymptotic · Kirchhoff plate

1 Introduction

The asymptotic derivation of the low-dimensional theories for thin elastic bodies was the subject of numerous publications, e.g., see [1–13] and references therein. Most of them originate from the Neumann-type boundary-value problems in 3D elasticity corresponding to traction-free faces or stresses prescribed along the faces. Another limitation is concerned with the one-parametric nature of the underlying procedures. In particular, for forced vibrations it is usually assumed that the relation between typical wave length and frequency characterising loading is the same as for free vibrations, see [1]. These limitations restrict the range of scenarios related to the justification and refinement of ad hoc engineering models, including the problems in fluid- and soil-structure interaction.

Only a few recent papers consider interaction of plates and shells with the environment imposing full contact conditions along the interface instead of treating acoustic or elastic pressure as the terms in 2D approximate equations of motion standing for transverse loads. Among them we mention [14–16], analysing a thin elastic layer resting on a Winkler foundation, and [17–19] dealing with fluid loaded structures. The cited contributions reveal a number of inconsistencies in previous developments, in particular adapting the asymptotic scaling, suitable for elastic structures with traction-free faces instead of that governing coupled problems in fluid and soil structure interaction. At the same time, all of the cited papers develop a one-parametric approach including analysis of a free fluid-borne bending wave, see also [20], which does not radiate into the fluid. Therefore, the evaluation of acoustic radiation by an elastic layer assumes a special insight.

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Generally, forced vibrations of a fluid-loaded layer imply two-parametric treatment, similar to that for a layer not contacting fluid, see [1,21] with independent wave number and frequency, defined by the prescribed loading. In this paper, however, we restrict ourselves to a single typical setup supporting radiation. For the sake of simplicity, we consider a plane strain problem for an elastic layer resting on an acoustic half-space subject to a normal force applied to its upper surface.

We start with testing of three approximate formulations governing the fluid–structure interactions, including a traditional scheme relying on the Kirchhoff plate theory along with its two shortened forms. The first of them ignores plate stiffness taking into consideration plate's inertia only, while the other one fully neglects the presence of the plate, i.e. the force is readily transmitted to the acoustic half-space.

It is worth noting that Kirchhoff theory has been applied to the modelling of a wide variety of the important problems in fluid–structure interaction for a long time. It is hardly possible to oversee all the contributions on the subject within the scope of a single technical paper. Among numerous developments in this area, we just mention the books [22,23] and the 1988 Rayleigh medal lecture [24], as well as the journal publications [25–32] and references therein.

The results obtained from the above mentioned approximations are compared with the exact solution of the full plane strain problem. The calculations are presented for a force taken in the form of a low-frequency travelling wave, for which the wave number is proportional to the vibration frequency. Such regime is different to that for free vibrations manifesting a fluid-borne bending wave propagation, e.g., see [18,20]. In each of the trials, three-term asymptotic expansions in a small wave number are given for the acoustic potential at the interface. The approximations ignoring the plate effect or taking into account its inertia only catch one or two terms in the expansions of the exact solution, respectively. In this case, the model involving a Kirchhoff plate does not add much reproducing only two terms in the three-term expansion. Thus, it is not useful for investigating the effect of the elastic stiffness on radiation. This observation motivates us to revisit the asymptotic model governing forced vibrations of a thin fluid-loaded layer. For this purpose, we return back to the original plane strain equations setting a new scaling oriented to plate transverse compression rather than to plate bending, e.g., see [1] and references therein. As a result, a consistent refined framework is established, including a 1D equation for plate motion and a contact condition along the interface. The latter catches all three terms in the long-wave expansion of the exact solution, enabling to determine the effect of the extensional stiffness of the layer.

The paper is organised as follows. The statement of the problem, governing equations and adapted assumptions are presented in Sect. 2. Section 3 studies popular approximate models and provides a comparative numerical analysis. Thereafter, a consistent asymptotic model is established in Sect. 4. Finally, concluding remarks are given in Sect. 5.

2 Statement of the problem

Consider forced vibrations of an isotropic linearly elastic layer of thickness $2h$, resting on an acoustic half-space. Assume that the normal stress F is applied to the upper surface of the layer. For the sake of simplicity restrict ourselves to the plane strain problem, adapting the Cartesian coordinates (x_1, x_2), see Fig. 1 below.

Throughout the paper, the following notations are utilised: E is Young's modulus, ν is the Poisson's ratio, λ and μ are Lamé elastic constants, ρ is solid density, ρ_0 is the density of the acoustic media, c_1 and c_2 are the longitudinal and transverse wave speeds in solid, respectively, and c_0 is the acoustic wave speed.

The equations of motion in linear elasticity take the form

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} - \rho \frac{\partial^2 v_1}{\partial t^2} = 0, \\ &\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} - \rho \frac{\partial^2 v_2}{\partial t^2} = 0, \end{aligned} \tag{1}$$

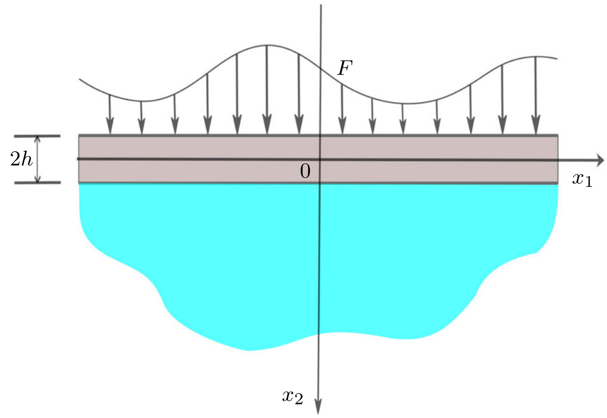

Fig. 1 Forced vibrations of an elastic layer resting on an acoustic half-space

where σ_{ij} $(i, j = 1, 2)$ are stresses, v_n $(n = 1, 2)$ are displacements and t denotes time. The stresses and displacements can be related through the formulae, e.g., see [1]

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1-\nu^2}\frac{\partial v_1}{\partial x_1} + \frac{\nu}{1-\nu}\sigma_{22},\\
\sigma_{33} &= \frac{E\nu}{1-\nu^2}\frac{\partial v_1}{\partial x_1} + \frac{\nu}{1-\nu}\sigma_{22},\\
\frac{\partial v_2}{\partial x_2} &= \frac{1}{E}\Big(\sigma_{22} - \nu(\sigma_{11}+\sigma_{33})\Big),\\
\frac{\partial v_1}{\partial x_2} &= -\frac{\partial v_2}{\partial x_1} + \frac{2(1+\nu)}{E}\sigma_{21}.
\end{aligned} \tag{2}$$

In addition, we present the wave equation for the acoustic potential $\varphi(x_1, x_2, t)$

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = 0. \tag{3}$$

The boundary condition along the faces of the layer are given by

$$\begin{aligned}
&\sigma_{21} = 0, \quad \sigma_{22} = F, && \text{at } x_2 = -h,\\
&\sigma_{21} = 0, \quad \sigma_{22} = \rho_0\frac{\partial^2\varphi}{\partial t^2}, \quad v_2 = \frac{\partial\varphi}{\partial x_2}, && \text{at } x_2 = h,
\end{aligned} \tag{4}$$

where $F = F(x_1, t)$. We also impose the radiation condition at infinity, which is specified below for time-harmonic setup.

The low-frequency vibrations of a fluid-loaded thin layer are often modeled by the approximate Kirchhoff theory for plate bending. In this case, the approximate equation of motion becomes

$$\frac{Eh^3}{3(1-\nu^2)}\frac{\partial^4 w}{\partial x_1^4} + \rho h\frac{\partial^2 w}{\partial t^2} - \frac{1}{2}\rho_0\frac{\partial^2\varphi}{\partial t^2}\bigg|_{x_2=h} = -\frac{1}{2}F, \tag{5}$$

see for example [33], where $w(x_1, t) \approx v_2(x_1, h, t)$. This equation is considered together with (3) and the impenetrability condition

$$w = \frac{\partial\varphi}{\partial x_2}\bigg|_{x_2=h}, \tag{6}$$

which is the counterpart of the last formula in (4).

Below we will also study the approximation which neglects the plate stiffness, i.e. instead of (5) we take

$$\rho h \frac{\partial^2 w}{\partial t^2} - \frac{1}{2}\rho_0 \frac{\partial^2 \varphi}{\partial t^2}\bigg|_{x_2=h} = -\frac{1}{2}F. \tag{7}$$

Finally, the load may be applied to the surface of the acoustic half-space, resulting in the boundary condition

$$\rho_0 \frac{\partial^2 \varphi}{\partial t^2}\bigg|_{x_2=h} = F, \tag{8}$$

imposed for the wave Eq. (3).

The initial goal of the paper is to test the approximate solutions following from the Kirchhoff plate-based model and its simplifications by comparing them with the exact solution of the original plane strain problem. For the sake of simplicity, we take the prescribed load in the form of a traveling wave $F = F_0 e^{i(kx_1-\omega t)}$, where k is the wave number, ω is the angular frequency, and F_0 is a constant. In what follows the exponential factor is always omitted. We also assume that

$$kh = K \ll 1, \quad \frac{\omega h}{c_2} = \Omega \ll 1, \tag{9}$$

and

$$\Omega > \frac{K}{\delta}, \tag{10}$$

with $\delta = c_2/c_0$. The last inequality stands for the radiation into the acoustic media, see the consideration in the next section.

3 Testing of approximate models

First, derive the exact solution of the problem (1)–(4). It is well known, e.g., see [34], that the equations of dynamic elasticity can be rewritten in terms of Lamé potentials ϕ and ψ as

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \zeta^2} - \alpha^2 \phi = 0, \\ \frac{\partial^2 \psi}{\partial \zeta^2} - \beta^2 \psi = 0, \end{aligned} \tag{11}$$

where $\zeta = x_2/h$ and

$$\alpha = \sqrt{K^2 - \varkappa^2 \Omega^2}, \qquad \beta = \sqrt{K^2 - \Omega^2}, \tag{12}$$

with $\varkappa = c_2/c_1 = \sqrt{\frac{1-2\nu}{2-2\nu}}$.

We also have for the elastic quantities in the contact conditions (4)

$$\begin{aligned} \sigma_{21} &= \frac{\mu}{h^2}\left(\frac{\partial^2 \psi}{\partial \zeta^2} + 2iK\frac{\partial \phi}{\partial \zeta} + K^2 \psi\right), \\ \sigma_{22} &= \frac{\mu}{h^2}\left(\varkappa^{-2}\frac{\partial^2 \phi}{\partial \zeta^2} - 2iK\frac{\partial \psi}{\partial \zeta} + (2-\varkappa^{-2})K^2 \phi\right), \\ v_2 &= \frac{1}{h}\left(\frac{\partial \phi}{\partial \zeta} - iK\psi\right). \end{aligned} \tag{13}$$

In addition, the Eq. (3) becomes

$$\frac{\partial^2 \varphi}{\partial \zeta^2} + \gamma^2 \varphi = 0, \tag{14}$$

with

$$\gamma = \sqrt{\delta^2\Omega^2 - K^2}, \tag{15}$$

where, due to inequality (10), the expression under the square root is positive. The solution of the last equation satisfying the radiation condition is

$$\varphi(\zeta) = \Phi e^{i\gamma(\zeta-1)}, \qquad \zeta \geqslant 1, \tag{16}$$

where Φ is the sought for constant and the exponential factor, corresponding to a traveling wave, is omitted as above.

Now, solving the boundary value problem for ordinary differential Eqs. (11) subject to conditions (4), taken with (13) and (16), we readily arrive at, e.g., see [1,33]

$$\Phi = \frac{F_0 h^2}{\mu}\frac{M}{N}, \tag{17}$$

where

$$\begin{aligned} M &= \left(-\alpha\beta K^2 \sinh(2\alpha) + \sinh(2\beta)\left(K^2 - \frac{\Omega^2}{2}\right)^2\right)\Omega^2\alpha, \\ N &= \left(r\Omega^4\alpha \sinh(2\alpha)\cosh(2\beta) - 8i\gamma\left(K^2 - \frac{\Omega^2}{2}\right)^2 (\cosh(2\alpha)\cosh(2\beta) - 1)\right)\alpha\beta K^2 \\ &\quad + \left(4i\gamma\left(K^8 - 2K^6\Omega^2 + \left(\alpha^2\beta^2 + \frac{3\Omega^4}{2}\right)K^4 \right.\right. \\ &\quad \left.\left. - \frac{K^2\Omega^6}{2} + \frac{\Omega^8}{16}\right)\sinh(2\alpha) - r\Omega^4\alpha\left(K^2 - \frac{\Omega^2}{2}\right)^2\cosh(2\alpha)\right)\sinh(2\beta), \end{aligned} \tag{18}$$

with

$$r = \frac{\rho_0}{\rho}. \tag{19}$$

In addition, for the sake of simplicity, we restrict ourselves to the scenario for which $\Omega \sim K$, setting that

$$\Omega_0 = \frac{\Omega}{K} \sim 1. \tag{20}$$

Here we also require $\Omega_0 > 1/\delta$, implying radiation of vibration energy into the acoustic half-space, see (14) and (15). In fact, general analysis of forced vibrations has to start from a two-parametric procedure, when Ω and K are independent of each other, in contrast to free vibrations. For the latter, the connection $\Omega \sim K^{5/2}$, corresponding to a non-radiating fluid-borne bending wave, holds, see [18,35]. The discussion below demonstrates that the chosen one-parametric setup (20) is useful for evaluating validity of the studied approximate models, avoiding a full two-parametric treatment.

Consider a Kirchhoff plate under acoustic loading. Denoting $\Phi = \Phi_1$ in Eq. (16), we derive from (5), (6) and (16)

$$\Phi_1 = -\frac{F_0 h^2}{\mu}\frac{3(1-\nu)}{K^2\left(3\Omega_0^2 r(1-\nu) + 2i(2K^2 - 3\Omega_0^2(1-\nu))\sqrt{\Omega_0^2\delta^2 - 1}K\right)}. \tag{21}$$

Similarly, we have for other approximations (7) and (8)

$$\Phi_2 = -\frac{F_0 h^2}{\mu}\frac{1}{\Omega_0^2 K^2\left(r - 2i\sqrt{\Omega_0^2\delta^2 - 1}K\right)}. \tag{22}$$

and

$$\Phi_3 = -\frac{F_0 h^2}{\mu}\frac{1}{r\Omega_0^2 K^2}, \tag{23}$$

respectively. Deriving last formulae, we adapted the substitutions $\Phi = \Phi_n$, $n = 2, 3$ in formula (16). It may be shown that formula (23), as expected, does not involve the elastic parameters.

Next, obtain three-term asymptotic expansions at $K \ll 1$ of the exact solution (17) and the approximate formulae (21) and (22). We have

$$\Phi \approx -\frac{F_0 h^2}{\mu}\frac{1}{r\Omega_0^2 K^2}\left(1 + \frac{2iK\sqrt{\Omega_0^2\delta^2 - 1}}{r} + \frac{2K^2\left(4 + (2\delta^2(1-\nu) + \frac{1}{2}r^2(2\nu - 1))\Omega_0^4 + ((r^2-2)(1-\nu) - 4\delta^2)\Omega_0^2\right)}{(2 + \Omega_0^2(\nu - 1))r^2}\right), \tag{24}$$

$$\Phi_n \approx -\frac{F_0 h^2}{\mu}\frac{1}{r\Omega_0^2 K^2}\left(1 + \frac{2iK\sqrt{\Omega_0^2\delta^2 - 1}}{r} - \frac{4K^2(\Omega_0^2\delta^2 - 1)}{r^2}\right), \quad n = 1, 2. \tag{25}$$

Inspection of the derived formulae results in important observations. First, we conclude that the presence of the elastic layer does not affect the leading term in the asymptotic expansions (24)–(25), which is identical to the expression given by (23). In this case, the simplest approximate setup (3) and (8), corresponding to a boundary value problem in acoustics, is appropriate. We can also see that the second term in the asymptotic expansion of the exact solution (24) follows from another approximate formulation, see the expansion (25) and governing Eqs. (3), (6), and (7), taking into account the transverse inertia of the layer only. At the same time, the effect of bending stiffness of the layer does not add much into the accuracy of the sought for asymptotic expansion, see (24). Thus, the implementation of the full Kirchhoff plate-based model, governed by relations (3), (5), and (6) in the considered problem, does not seem to be rational.

Numerical results are presented for the pairs aluminium-water and magnesium-water. In this case, $\nu = 0.33$, $\delta = 2.073$, $r = 0.369$ and $\nu = 0.29$, $\delta = 2.112$, $r = 0.575$, respectively. Here and below, we set $\Omega_0 = 3$. In Figs. 2, 3, we present one-, two-, and three-term expansions given by (24) versus the exact solution for the fluid potential (17), in terms of (20). The modulus of all sought-for complex-valued quantities, multiplied by AK^2, where $A = \frac{\mu r \Omega_0^2}{F_0 h^2}$ is plotted. As might be expected, the accuracy of the derived approximations gets worse for the materials with a larger density due to the presence of the factor K/r in the asymptotic expansions.

The normalised absolute error of the approximate solutions given by (21) and (22) is plotted in Fig. 4. In contrast to the previous figures, the normalisation factor is A. It is remarkable that all the curves take non-zero values at the long-wave limit $K \to 0$. It means, in particular, that the Kirchhoff plate-based model does not catch the third order term in the long-wave asymptotic expansion of the exact solution. At the same time, the coincidence of the curves corresponding to both studied models confirms the theoretical finding above stating that the plate stiffness does not make a significant contribution into the radiated energy.

The spotted issue with the Kirchhoff theory for plate bending has a fundamental reason behind. It originates from the aforementioned two-parametric nature of forced vibrations, for which the relation between the wave number K and angular frequency Ω is generally not the same as that for free vibrations, following from appropriate dispersion equations. A full two-parametric analysis of 3D Neumann-type boundary value problem in linear elasticity for a thin elastic layer in vacuum subject to arbitrary low-frequency loading prescribed along its faces, see [1,21], indicates that the Kirchhoff theory is not robust for certain parametric setups outside the domain corresponding to the free vibration scaling ($\Omega \sim K^2$). The cited contributions are apparently the only examples of a general two-parametric insight, involving treatment of various possible scenarios. For the sake of simplicity, the asymptotic derivations of the lower-dimensional models for thin structures usually adapt the relation between K and Ω characteristic for free vibrations, e.g., see [1] and references therein. Thus, forced vibrations under the imposed condition (20) have to be revisited using the original plane strain equations, presented in Sect. 1, in order to establish a consistent approximate model instead of that involving the Kirchhoff plate theory.

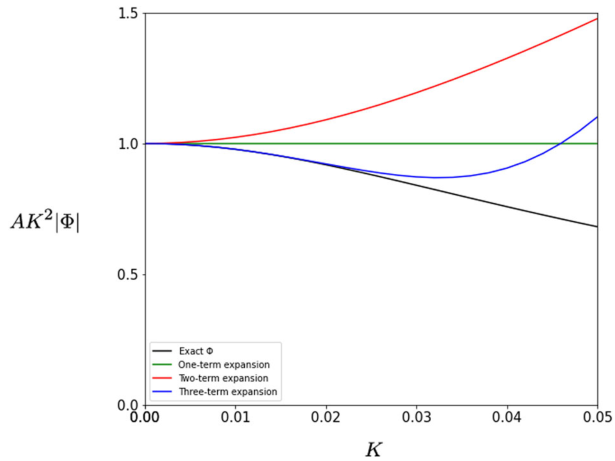

Fig. 2 Comparison of the exact potential Φ (17) (solid black line), with one-term (solid green line), two-term (solid red line) and three-term (solid blue line) expansions from (24), for the magnesium-water pair

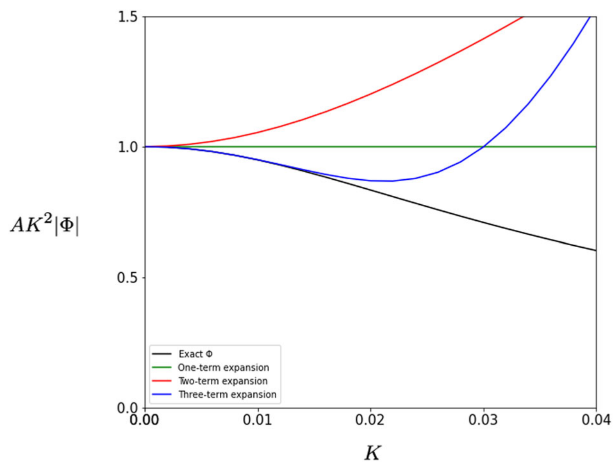

Fig. 3 Comparison of the exact potential Φ (17) (solid black line), with one-term (solid green line), two-term (solid red line) and three-term (solid blue line) expansions from (24), for the aluminium-water pair

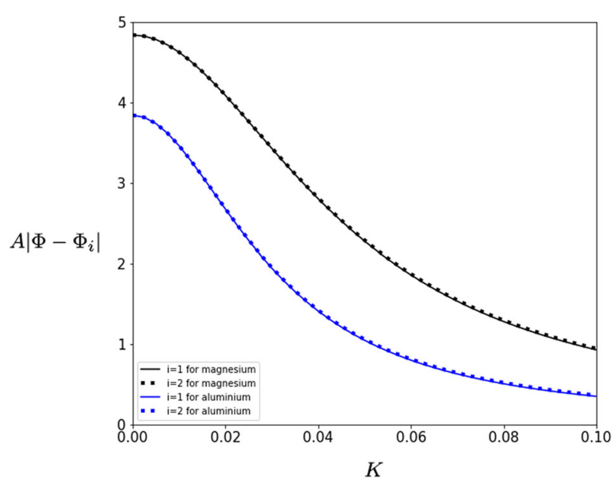

Fig. 4 Comparison showing the absolute error of the approximate solutions (21) and (22), denoted by Φ_1 and Φ_2, respectively, versus the exact solution (17) for the potential Φ, calculated for magnesium-water and aluminium-water pairs

4 Asymptotically consistent model

We begin by scaling the independent variables specified in Sect. 2

$$t = T\tau, \qquad x_1 = L\xi, \qquad x_2 = \begin{cases} h\zeta, & \text{if } |x_2| < h, \\ L\zeta_f, & \text{otherwise,} \end{cases} \tag{26}$$

where typical time T and length scale L are related by

$$T = \frac{L}{c_2}, \tag{27}$$

arising from the assumption that $\Omega \sim K$, as specified above.

Now, introduce the dimensionless quantities by

$$\begin{aligned} \sigma_{11} &= \mu\sigma_{11}^*, & \sigma_{22} &= \mu\sigma_{22}^*, \\ \sigma_{21} &= \epsilon^2\mu\sigma_{21}^*, & \sigma_{33} &= \mu\sigma_{33}^*, \\ v_1 &= Lv_1^*, & v_2 &= Lv_2^*, \\ F &= \mu F^*, & \varphi &= L^2\varphi^*, \end{aligned} \tag{28}$$

where $\epsilon = h/L \ll 1$ is a small geometric parameter and all the starred quantities are assumed to be of the same asymptotic order.

Therefore, Eqs. (1)–(2), taking into account relations (26) and (28), can be written in dimensionless form as

$$\begin{aligned} \epsilon\frac{\partial\sigma_{21}^*}{\partial\zeta} &= -\frac{\partial\sigma_{11}^*}{\partial\xi} + \frac{\partial^2 v_1^*}{\partial\tau^2}, \\ \frac{\partial\sigma_{22}^*}{\partial\zeta} &= -\epsilon^3\frac{\partial\sigma_{21}^*}{\partial\xi} + \epsilon\frac{\partial^2 v_2^*}{\partial\tau^2}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \sigma_{11}^* &= \frac{2}{1-\nu}\frac{\partial v_1^*}{\partial\xi} + \frac{\nu}{1-\nu}\sigma_{22}^*, \\ \sigma_{33}^* &= \frac{2\nu}{1-\nu}\frac{\partial v_1^*}{\partial\xi} + \frac{\nu}{1-\nu}\sigma_{22}^*, \\ \frac{\partial v_2^*}{\partial\zeta} &= \epsilon\frac{1}{2(1+\nu)}\sigma_{22}^* - \epsilon\frac{\nu}{2(1+\nu)}(\sigma_{11}^* + \sigma_{33}^*), \\ \frac{\partial v_1^*}{\partial\zeta} &= -\epsilon\frac{\partial v_2^*}{\partial\xi} + \epsilon^3\sigma_{21}^*. \end{aligned} \tag{30}$$

Likewise, the dimensionless form of relations (3) and (4) can be obtained using (26)–(28), which gives the following

$$\frac{\partial^2\varphi^*}{\partial\xi^2} + \frac{\partial^2\varphi^*}{\partial\zeta_f^2} - \delta^2\frac{\partial^2\varphi^*}{\partial\tau^2} = 0, \tag{31}$$

and

$$\begin{aligned} &\sigma_{21}^*\Big|_{\zeta=1} = 0, \quad \sigma_{22}^*\Big|_{\zeta=1} = r\frac{\partial^2\varphi^*}{\partial\tau^2}\Big|_{\zeta_f=\epsilon}, \quad \frac{\partial\varphi^*}{\partial\zeta_f}\Big|_{\zeta_f=\epsilon} = v_2^*\Big|_{\zeta=1}, \quad \text{at } \zeta = 1, \\ &\sigma_{21}^*\Big|_{\zeta=-1} = 0, \qquad \sigma_{22}^*\Big|_{\zeta=-1} = F^*, \qquad\qquad \text{at } \zeta = -1. \end{aligned} \tag{32}$$

Next, expand the starred stress, displacement and potential components in an asymptotic series as

$$
\begin{aligned}
v_i^* &= v_i^{(0)} + \epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \cdots \\
\sigma_{nn}^* &= \sigma_{nn}^{(0)} + \epsilon \sigma_{nn}^{(1)} + \epsilon^2 \sigma_{nn}^{(2)} + \cdots \\
\sigma_{21}^* &= \sigma_{21}^{(0)} + \epsilon \sigma_{21}^{(1)} + \epsilon^2 \sigma_{21}^{(2)} + \cdots \\
\varphi^* &= \varphi^{(0)} + \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + \cdots
\end{aligned} \tag{33}
$$

resulting in shortened forms of the original 2D plane strain equations. Here $i = 1, 2$ and $n = 1, 2, 3$.

Our attention first is on the leading-order approximation corresponding to the terms with the suffix (0) in the asymptotic series (33). Integrating $(29)_2$ and $(30)_3$-$(30)_4$ along the thickness variable ζ, we obtain, respectively,

$$
\sigma_{22}^{(0)} = S_{22}^{(0)}(\xi, \tau), \quad v_2^{(0)} = V_2^{(0)}(\xi, \tau) \quad \text{and} \quad v_1^{(0)} = V_1^{(0)}(\xi, \tau), \tag{34}
$$

where $S_{22}^{(0)}$, $V_1^{(0)}$ and $V_2^{(0)}$ are arbitrary functions. Next, substitute these into Eqs. $(30)_1$-$(30)_2$ to get

$$
\begin{aligned}
\sigma_{11}^{(0)} &= \frac{2}{1-\nu}\frac{\partial V_1^{(0)}}{\partial \xi} + \frac{\nu}{1-\nu} S_{22}^{(0)}, \\
\sigma_{33}^{(0)} &= \frac{2\nu}{1-\nu}\frac{\partial V_1^{(0)}}{\partial \xi} + \frac{\nu}{1-\nu} S_{22}^{(0)}.
\end{aligned} \tag{35}
$$

Now, integrate $(29)_1$ along the ζ, taking into account Eqs. (34) and (35). As a result, we have

$$
\frac{2}{1-\nu}\frac{\partial^2 V_1^{(0)}}{\partial \xi^2} - \frac{\partial^2 V_1^{(0)}}{\partial \tau^2} + \frac{\nu}{1-\nu}\frac{\partial S_{22}^{(0)}}{\partial \xi} = 0. \tag{36}
$$

We can also determine the leading order fluid potential from the equation

$$
\frac{\partial^2 \varphi^{(0)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(0)}}{\partial \zeta_f^2} - \delta^2 \frac{\partial^2 \varphi^{(0)}}{\partial \tau^2} = 0, \tag{37}
$$

governing compressible fluid. In addition, inserting (33) into (32) gives the boundary and contact conditions at $\zeta = -1$ and $\zeta = 1$, respectively. They are

$$
\begin{aligned}
&\sigma_{21}^{(0)}\Big|_{\zeta=1} = 0, \quad \sigma_{22}^{(0)}\Big|_{\zeta=1} = r\frac{\partial^2 \varphi^{(0)}}{\partial \tau^2}\Big|_{\zeta_f=\epsilon}, \quad \frac{\partial \varphi^{(0)}}{\partial \zeta_f}\Big|_{\zeta_f=\epsilon} = v_2^{(0)}\Big|_{\zeta=1}, \quad \text{at } \zeta = 1, \\
&\sigma_{21}^{(0)}\Big|_{\zeta=-1} = 0, \qquad \sigma_{22}^{(0)}\Big|_{\zeta=-1} = F^*, \qquad \text{at } \zeta = -1.
\end{aligned} \tag{38}
$$

Then, we have from $(38)_2$ and $(38)_5$

$$
F^* = r\frac{\partial^2 \varphi^{(0)}}{\partial \tau^2}\Big|_{\zeta_f=\epsilon}. \tag{39}
$$

Therefore, Eq. (36) reduces to

$$
\frac{2}{1-\nu}\frac{\partial^2 V_1^{(0)}}{\partial \xi^2} - \frac{\partial^2 V_1^{(0)}}{\partial \tau^2} + \frac{\nu}{1-\nu}\frac{\partial F^*}{\partial \xi} = 0. \tag{40}
$$

We also have from $(38)_3$

$$
V_2^{(0)} = \frac{\partial \varphi^{(0)}}{\partial \zeta_f}\Big|_{\zeta_f=\epsilon}. \tag{41}
$$

At the next order we get

$$\frac{\partial^2\varphi^{(1)}}{\partial\xi^2}+\frac{\partial^2\varphi^{(1)}}{\partial\zeta_f^2}-\delta^2\frac{\partial^2\varphi^{(1)}}{\partial\tau^2}=0, \tag{42}$$

and

$$\begin{aligned}
\sigma_{11}^{(1)} &= -\zeta\frac{2}{1-\nu}\frac{\partial^2 V_2^{(0)}}{\partial\xi^2}+\frac{2}{1-\nu}\frac{\partial V_1^{(1)}}{\partial\xi}+\zeta\frac{\nu}{1-\nu}\frac{\partial^2 V_2^{(0)}}{\partial\tau^2}+\frac{\nu}{1-\nu}S_{22}^{(1)},\\
\sigma_{33}^{(1)} &= -\zeta\frac{2\nu}{1-\nu}\frac{\partial^2 V_2^{(0)}}{\partial\xi^2}+\frac{2\nu}{1-\nu}\frac{\partial V_1^{(1)}}{\partial\xi}+\zeta\frac{\nu}{1-\nu}\frac{\partial^2 V_2^{(0)}}{\partial\tau^2}+\frac{\nu}{1-\nu}S_{22}^{(1)},\\
v_2^{(1)} &= \zeta\frac{(1-2\nu)F^*}{2(1-\nu)}-\zeta\frac{\nu}{1-\nu}\frac{\partial V_1^{(0)}}{\partial\xi}+V_2^{(1)},\\
v_1^{(1)} &= -\zeta\frac{\partial V_2^{(0)}}{\partial\xi}+V_1^{(1)},\\
\frac{\partial\sigma_{22}^{(1)}}{\partial\zeta} &= \frac{\partial^2 V_2^{(0)}}{\partial\tau^2},\\
\frac{\partial\sigma_{21}^{(0)}}{\partial\zeta} &= \zeta\frac{2}{1-\nu}\frac{\partial^3 V_2^{(0)}}{\partial\xi^3}-\frac{2}{1-\nu}\frac{\partial^2 V_1^{(1)}}{\partial\xi^2}-\zeta\frac{1}{1-\nu}\frac{\partial^3 V_2^{(0)}}{\partial\xi\,\partial\tau^2}-\frac{\nu}{1-\nu}\frac{\partial S_{22}^{(1)}}{\partial\xi}+\frac{\partial^2 V_1^{(1)}}{\partial\tau^2},
\end{aligned} \tag{43}$$

where $S_{22}^{(1)}=S_{22}^{(1)}(\xi,\tau)$, $V_1^{(1)}=V_1^{(1)}(\xi,\tau)$ and $V_2^{(1)}=V_2^{(1)}(\xi,\tau)$ are arbitrary functions. The conditions along the faces of the layer now take the form

$$\begin{aligned}
&\sigma_{21}^{(1)}\Big|_{\zeta=1}=0,\quad \sigma_{22}^{(1)}\Big|_{\zeta=1}=r\frac{\partial^2\varphi^{(1)}}{\partial\tau^2}\Big|_{\zeta_f=\epsilon},\quad \frac{\partial\varphi^{(1)}}{\partial\zeta_f}\Big|_{\zeta_f=\epsilon}=v_2^{(1)}\Big|_{\zeta=1},\quad \text{at } \zeta=1,\\
&\sigma_{21}^{(1)}\Big|_{\zeta=-1}=0,\qquad \sigma_{22}^{(1)}\Big|_{\zeta=-1}=0,\qquad\qquad \text{at } \zeta=-1.
\end{aligned} \tag{44}$$

First, we derive from (44)$_3$ and (43)$_3$

$$V_2^{(1)}=\frac{\nu}{1-\nu}\frac{\partial V_1^{(0)}}{\partial\xi}+\frac{(2\nu-1)F^*}{2(1-\nu)}+\frac{\partial\varphi^{(1)}}{\partial\zeta_f}\Big|_{\zeta_f=\epsilon}. \tag{45}$$

Then, integrating (43)$_5$ we derive

$$\sigma_{22}^{(1)}=\zeta\frac{\partial^2 V_2^{(0)}}{\partial\tau^2}+S_{22}^{(1)}. \tag{46}$$

and

$$\sigma_{22}^{(1)}\big|_{\zeta=1}-\sigma_{22}^{(1)}\big|_{\zeta=-1}=2\frac{\partial^2 V_2^{(0)}}{\partial\tau^2}, \tag{47}$$

leading to

$$r\frac{\partial^2\varphi^{(1)}}{\partial\tau^2}\Big|_{\zeta_f=\epsilon}=2\frac{\partial^2 V_2^{(0)}}{\partial\tau^2}, \tag{48}$$

after making use of (44)$_2$ and (44)$_5$. Next, we obtain from (46) taking into account (44)$_2$

$$S_{22}^{(1)}=\frac{\partial^2 V_2^{(0)}}{\partial\tau^2}. \tag{49}$$

Similarly, integrating (43)$_5$ and using (49) we arrive at

$$\sigma_{21}^{(0)} = \frac{\zeta^2}{1-\nu}\frac{\partial^3 V_2^{(0)}}{\partial \tau^3} - \frac{(\zeta^2+2\nu\zeta)}{2(1-\nu)}\frac{\partial^3 V_2^{(0)}}{\partial \xi\,\partial \tau^2} - \zeta\left(\frac{2}{1-\nu}\frac{\partial^2 V_1^{(1)}}{\partial \xi^2} - \frac{\partial^2 V_1^{(1)}}{\partial \tau^2}\right) + S_{21}^{(0)}. \tag{50}$$

Also, integrating (43)$_5$ we obtain

$$\frac{2}{1-\nu}\frac{\partial^2 V_1^{(1)}}{\partial \xi^2} - \frac{\partial^2 V_1^{(1)}}{\partial \tau^2} + \frac{\nu}{1-\nu}\frac{\partial^3 V_2^{(0)}}{\partial \xi\,\partial \tau^2} = 0, \tag{51}$$

by using the boundary conditions (44)$_1$ and (44)$_4$. Hence, we apply (44)$_1$ (or (44)$_4$) to (50), taking into account (51), to get

$$S_{21}^{(0)} = \frac{1}{2(1-\nu)}\frac{\partial^3 V_2^{(0)}}{\partial \xi\,\partial \tau^2} - \frac{1}{1-\nu}\frac{\partial^3 V_2^{(0)}}{\partial \xi^3}. \tag{52}$$

At the second order, we restrict ourselves to the calculation of $\sigma_{22}^{(2)}$ and $\varphi^{(2)}$, for which we have

$$\frac{\partial^2 \varphi^{(2)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(2)}}{\partial \zeta_f^2} - \delta^2\frac{\partial^2 \varphi^{(2)}}{\partial \tau^2} = 0, \tag{53}$$

and

$$\frac{\partial \sigma_{22}^{(2)}}{\partial \zeta} = \frac{(2\nu-1)(1-\zeta)}{2(1-\nu)}\frac{\partial^2 F^*}{\partial \tau^2} + \frac{\nu(1-\zeta)}{1-\nu}\frac{\partial^3 V_1^{(0)}}{\partial \xi\,\partial \tau^2} + \left.\frac{\partial^3 \varphi^{(1)}}{\partial \tau^2\,\partial \zeta_f}\right|_{\zeta_f=\epsilon}, \tag{54}$$

together with

$$\begin{aligned} \left.\sigma_{33}^{(2)}\right|_{\zeta=1} &= r\left.\frac{\partial^2 \varphi^{(2)}}{\partial \tau^2}\right|_{\zeta_f=\epsilon}, \quad && \text{at } \zeta = 1, \\ \left.\sigma_{33}^{(2)}\right|_{\zeta=-1} &= 0, && \text{at } \zeta = -1. \end{aligned} \tag{55}$$

As above, we integrate (54) across the thickness and apply (55) to obtain

$$r\frac{\partial^2 \varphi^{(2)}}{\partial \tau^2} = \frac{2\nu-1}{1-\nu}\frac{\partial^2 F^*}{\partial \tau^2} + \frac{2\nu}{1-\nu}\frac{\partial^3 V_1^{(0)}}{\partial \xi\,\partial \tau^2} + 2\left.\frac{\partial^3 \varphi^{(1)}}{\partial \tau^2\,\partial \zeta_f}\right|_{\zeta_f=\epsilon}. \tag{56}$$

To establish a refined asymptotic model, we consider the sum of (39), $\epsilon\times$ (48) and $\epsilon^2\times$ (56), having

$$\begin{aligned} & r\frac{\partial^2}{\partial \tau^2}\left.\left(\varphi^{(0)} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)}\right)\right|_{\zeta_f=\epsilon} = F^* + 2\epsilon\frac{\partial^3}{\partial \tau^2\,\partial \zeta_f}\left.\left(\varphi^{(0)} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)}\right)\right|_{\zeta_f=\epsilon} \\ & + \epsilon^2\frac{1}{1-\nu}\frac{\partial^2}{\partial \tau^2}\left((2\nu-1)F^* + 2\nu\frac{\partial V_1^{(0)}}{\partial \xi}\right) + O(\epsilon^3) = 0. \end{aligned} \tag{57}$$

Likewise, we obtain from Eqs. (37), (42) and (53)

$$\frac{\partial^2}{\partial \xi^2}\left(\varphi^{(0)} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)}\right) + \frac{\partial^2}{\partial \zeta_f^2}\left(\varphi^{(0)} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)}\right) - \delta^2\frac{\partial^2}{\partial \tau^2}\left(\varphi^{(0)} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)}\right) = 0. \tag{58}$$

Finally, rewrite the last two equations as well as the leading order Eq. (40) in the original dimensional variables and neglect $O(\epsilon^3)$ terms to obtain

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\rho(1-\nu)}{2\mu}\frac{\partial^2 u}{\partial t^2} = -\frac{\nu}{2\mu}\frac{\partial F}{\partial x_1} \tag{59}$$

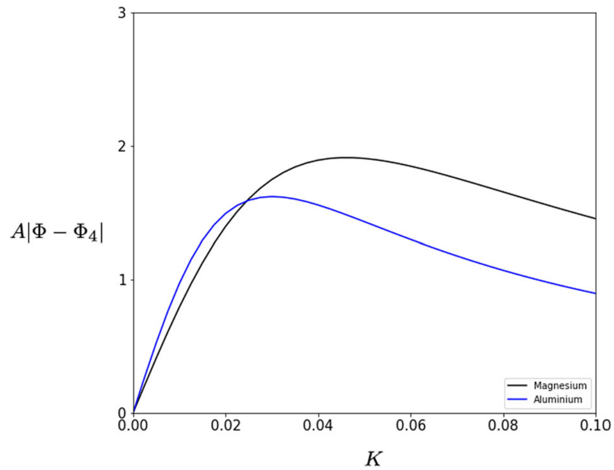

Fig. 5 Comparison showing the absolute error of the exact Φ (17) with the asymptotically consistent solution (61), solid black line for magnesium-water pair and solid blue line for aluminium-water pair

and

$$\left(\rho_0 \frac{\partial^2 \varphi}{\partial t^2} - 2\rho h \frac{\partial^3 \varphi}{\partial t^2 \partial x_2}\right)\Bigg|_{x_2=h} - \frac{2\nu\rho h^2}{1-\nu}\frac{\partial^3 u}{\partial t^2 \partial x_1} = F + \frac{\rho h^2}{1-\nu}\frac{\partial^2}{\partial t^2}\left(\frac{(2\nu-1)F}{\mu}\right), \tag{60}$$

together with the wave Eq. (3) for the acoustic potential. In the above equations, we denote $\varphi = L^2\left(\varphi^{(0)} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)}\right)$ and $u = LV_1^{(0)}$. Thus, a consistent fluid–structure interaction model appears to be based on the Eq. (59) governing transverse plate compression, caused by acoustic pressure, e.g., see [1,36] and references therein. In this case, the Kirchhoff plate bending theory, see Eq. (5), does not seem to be robust for studying forced vibrations over the domain $\Omega \sim K$.

Let us now test the derived model for the travelling wave (16), setting $\Phi = \Phi_4$. Omitting intermediate calculations, we get

$$\Phi_4 = \frac{F_0 h^2}{\mu}\frac{1}{\Omega_0^2 K^2 (2iK\sqrt{\Omega_0^2\delta^2 - 1} - r)}\left(1 - \frac{\Omega_0^2 K^2}{1-\nu}\left(2\nu - 1 - \frac{2\nu^2}{2-(1-\nu)\Omega_0^2}\right)\right). \tag{61}$$

It can be easily verified, that the three-term long-wave asymptotic expansion of Φ_4 is identical to that, coming from exact solution, given by formula (24). Thus, the developed model is consistent in a sense that it can catch a remarkable third-order term involving the effect of the stiffness of the layer.

The Fig. 5 demonstrates that in contrast to the similar Fig. 4 in Sect. 3 the curves corresponding to (61) begin from the origin. It means that the developed asymptotic approximation catches a third-order term in the long-wave expansion improving previously known formulations in fluid–structure interaction.

5 Conclusion

Comparison with the exact solution of a plane strain problem (1)–(4) over the parametric range (20) demonstrates that the approximation (5) using the classical Kirchhoff theory is not robust for studying forced vibrations, since it cannot capture the effect of the plate stiffness on the acoustic radiation. The consistent low-frequency approximation (61), derived in the paper, governs plate transverse compression rather than bending in contrast to Kirchhoff theory.

The developed approach may be readily extended to the whole two-parametric low-frequency domain, associated with the forced vibrations of interest. This seems to be useful for a critical re-assessment of the existing predictions of the radiation using the Kirchhoff theory.

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Declarations

Conflicts of Interest The authors declare no conflict of interest.

Author contributions All authors have contributed equally to this work.

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