Supplemental Material for: Effects of finite non-gaussianity on evolution of a random wind wave field

S.Y. Annenkov, V.I. Shrira

I. NUMERICAL ALGORITHM FOR DNS

The DNS algorithm used in this paper is based on the Zakharov integrodifferential equation for water waves, which was derived in 1968 ([1]), but long considered to be inconvenient and expensive to simulate numerically (e.g. [2]). Nevertheless, it has important advantages from the point of view of numerics, due to the elimination of non-resonant interaction terms by canonical transformation, and complete freedom in the choice of a computational grid. Since there is no need to perform a Fourier transform during time stepping, there is no restriction to a regular grid of wavenumbers, which is an important advantage over spectral methods for the simulation of wave interactions over a range of spatial scales.

An algorithm based on the Zakharov equation was proposed in [3] (see also [4]) and successfully applied to the study of a number of physical problems with a relatively small number of degrees of freedom. The essence of the method is in the efficient computational strategy, where all the coefficients are computed by a preprocessing routine and stored in a way that facilitates all subsequent operations of integration in time, as well as optimizing them for parallel computation. A wave field $b(\boldsymbol{k},t)$ is considered as an ensemble of discrete harmonics of the form

$$b(\boldsymbol{k},t)=\sum_{j=1}^{N} b_j(\boldsymbol{k}_j,t), \tag{1}$$

position of points $\boldsymbol{k}_j$ in Fourier space being arbitrary.

However, application of this algorithm to the evolution of continuous random wave fields is not straightforward. Any dynamical algorithm for the simulation of a wave field has to include a discretisation of the field. The difficulty is that such a discretisation produces a discrete wave system, with properties qualitatively different from those of a continuous field.

In order to model a continuous wave field correctly, a discretised wave field must be capable of representing interactions between all degrees of freedom, which is, strictly speaking, not possible. The usual remedy is to assume that a sufficiently large and refined regular grid, with its high density of resonant and non-resonant interactions, has properties similar to those of a continuous field. However, as noted above, a regular grid is rather inconvenient, since it has inhomogeneous properties at different scales. In addition, it can manifest undesirable artefacts of a resonator. It has been long known that the use of regular grids for the numerical simulation of wave turbulence leads to "frozen turbulence" effects (see discussion in [5]).

An alternative way to model a continuous wave field was first suggested by [6], and used to study the adjustment of a wave field to instantly changing [7] or rapidly fluctuating [8] forcing, and to simulate numerically the evolution of higher statistical moments [9]. The idea behind the algorithm is to perform coarse-graining of a continuous wave field, retaining fundamental properties of nonlinear interactions.

A wave field is represented by a grid consisting of wave packets, coupled through exact and approximate resonant interactions. A wave packet, centred at $\mathbf{k}_0$, is characterised by one amplitude and one phase, but has finite bandwidth in Fourier space, and is allowed to enter into nonlinear interactions with other wavepackets, provided that the wavevector mismatch

$$\Delta\mathbf{k}=\mathbf{k}_0+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3$$

does not exceed a certain threshold (the coarse-graining parameter). Thus, the standard resonance condition $\mathbf{k}_0+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3=0$ is relaxed. As in the standard discretisation case, we need to consider only resonant and approximately resonant interactions, prescribing a similar condition on the frequency mismatch $\Delta\omega$, where

$$\Delta\omega=\omega_0+\omega_1-\omega_2-\omega_3.$$

The values of $\Delta\mathbf{k}$ and $\Delta\omega$ should ensure the homogeneity of interactions on different scales (that is, the number of interactions should not change if all wavenumbers are rescaled). For this purpose, a quartet of grid points is assumed to be in approximate resonance if its wavevector and frequency mismatch satisfies

$$\Delta\omega/\omega_{min}<\lambda_\omega, \qquad |\Delta\mathbf{k}|/k_{min}<\lambda_k\bar{\omega}/\omega_{min}, \tag{2}$$

where $\Delta\omega$ and $|\Delta\mathbf{k}|$ are the frequency and wavevector mismatch in the quartet, $\omega_{\min}$ and $k_{\min}$ are the minimum values of frequency and wavenumber in the quartet, $\bar{\omega}$ is the mean frequency, and λ_ω and λ_k are the detuning parameters, chosen to ensure that the total number of resonances is $O(N^2)$, where N is the number of grid points. The resulting system of N discrete equations can be integrated in time by a standard timestepping scheme.

The construction of the algorithm depends on the choice of two parameters: the coarse-graining parameter λ_k and the detuning parameter λ_ω. For consistency and efficiency of the algorithm λ_k should be kept small, and, ideally, there should be no dependence on its value. However, a zero value of λ_k would mean the absence of evolution on a generic non-regular grid, since the condition $\mathbf{k}_0+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3=0$ is unlikely to be satisfied at all.

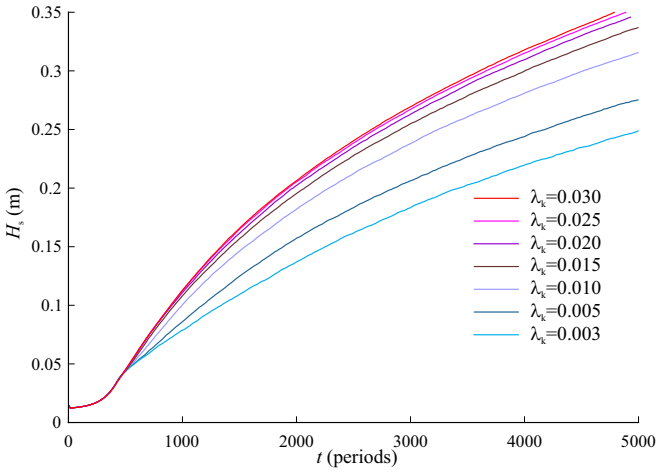

FIG. S1. Evolution of significant wave height H_s on 321×71 grid for different values of λ_k

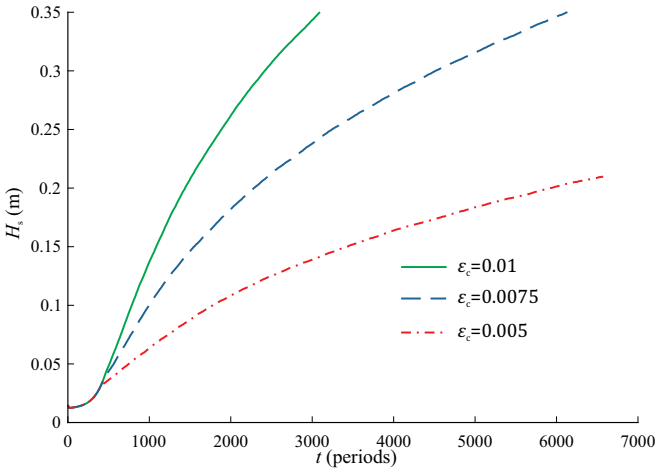

FIG. S2. Evolution of significant wave height H_s on 321×71 grid with $\lambda_k = 0.1$ and different values of ε_c

The algorithm was used by the authors in a number of studies (e.g. [7–9]), and was described in detail in [5]. In the latter work, the choice of the grid resolution and of λ_k and λ_ω parameters was made after extensive numerical trials. It that paper, we used a non-regular grid with 161 logarithmically spaced points in the range $\omega_p/2 \le \omega \le 3\omega_p$, where $\omega_p = 1$, and 41 uniformly spaced angles in the range $-4\pi/9 \le \theta \le 4\pi/9$. For this grid resolution, the trials showed that the rate of the spectral evolution was dependent on the value of λ_k as it was increased from zero, but this dependence approached saturation close to $\lambda_k = 0.03$. This value was chosen for all computations in [5]. It was also found that there was practically no dependence on the specific value of λ_ω, provided that $\lambda_\omega = O(10^{-2})$ or above. Thus, $\lambda_\omega = 0.01$ was set.

In the present work, we leave unchanged the parameters used in [5], except for extending the range of angles in the grid, while retaining the same angular resolution. That is, the current grid has 161 logarithmically spaced frequencies in the range $\omega_p/2 \le \omega \le 3\omega_p$, and 71 uniformly spaced angles in the range $-7\pi/9 \le \theta \le 7\pi/9$. We also use the refined 321×71 resolution, and vary parameter λ_k as a method to create wave fields with the same wave steepness and different nonlinear energy. Again, we find that the dependence on λ_k approaches saturation for $\lambda_k > 0.02$. In figure S1, we plot the evolution in time of the significant wave height for different values of λ_k on the 321×71 grid. This figure, which can be seen as complementary to Figure 4 of the main text of the paper, shows that there is little dependence on λ_k above 0.02.

Next, we will discuss the important modification of the DNS algorithm presented in [5]. This modification is the inclusion of wind forcing and a simple parameterization of wave breaking.

II. WIND FORCING AND WAVE BREAKING

The DNS algorithm described in [5] did not include wind forcing or wave breaking, but only dissipation at high frequencies ($\omega \ge 2.5\omega_p$). In the present paper, the same dissipation is used, and wind forcing is applied to $\omega \le 2.2\omega_p$, using the formula by [10]. However, a DNS algorithm with wind forcing must also include a certain parameterization of wave breaking. The reason for this is not the intention to model wave breaking as a physical process, but simply due to the fact that without such a parameterization the algorithm would not be functional. A wave field in a DNS algorithm is averaged over realizations, and amplitudes of individual harmonics in each realization are random and can become large. If these harmonics are also forced by wind, the randomly attained high amplitudes will grow fast and eventually will destroy the spectrum, so a certain procedure of limiting the fast growth is necessary.

Wind forcing is applied in the standard way by adding S_{in},

$$S_{in} = \beta(\mathbf{k}) b(\mathbf{k}) \tag{3}$$

to the right-hand side of the Zakharov equation. On every time step, and in each realization, the non-dimensional quantity

$$\varepsilon_k = \frac{\sqrt{2E_k}|\mathbf{k}|}{2\pi} \tag{4}$$

is calculated, where $E_k = \omega |b_k|^2/g$ is the energy of the (discrete) harmonic with wavevector $\mathbf{k}$. For convenience, ε_k is normalized as the steepness of the harmonic when it is taken separately. If ε_k exceeds the critical value ε_c, forcing (3) for the harmonic is replaced by strong damping with $\beta = -0.15\omega$. When, after a few periods, ε_k drops below $0.1\varepsilon_c$, the forcing is resumed. A number of numerical tests did not show any noticeable dependence

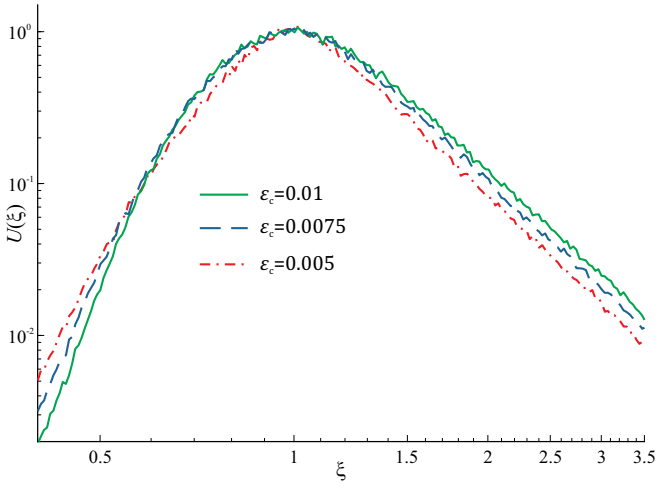

FIG. S3. Normalized spectral shape at $k_p = 1.0$, for 321×71 grid with $\lambda_k = 0.1$ and different values of ε_c

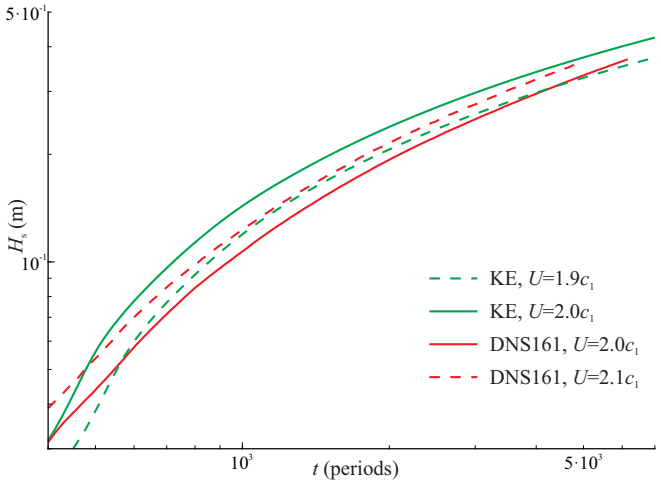

FIG. S4. Evolution of significant wave height H_s: DNS on 161×71 grid with $\lambda_k = 0.03$ and KE, for values of wind speed at $U/c_1 = 2.0$, lower by 5% (for KE) and higher by 5% (for DNS)

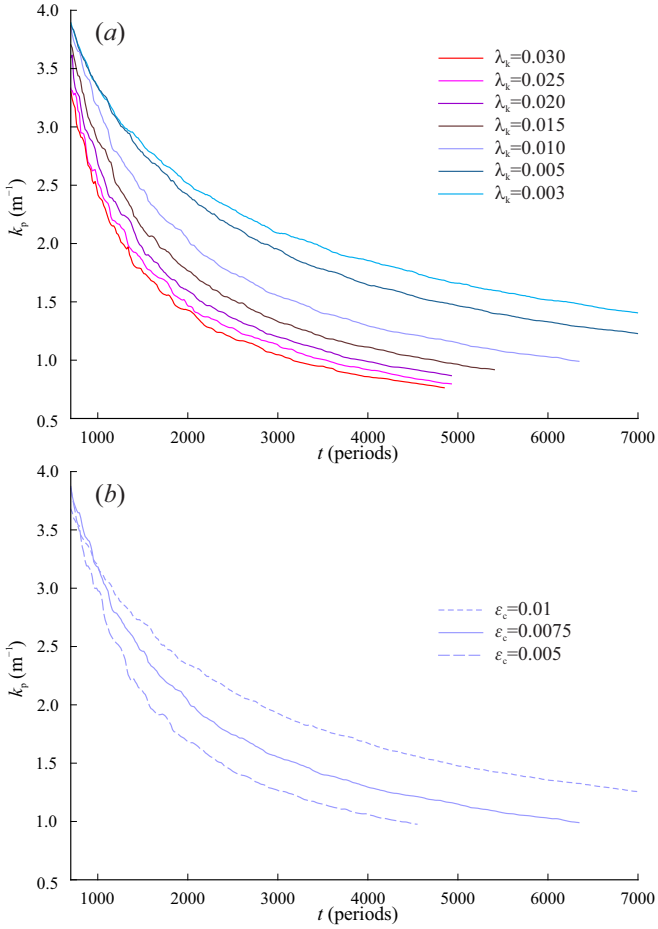

FIG. S5. Evolution of the spectral peak k_p vs time in simulations on 321×71 grid for different values of λ_k and ε_c (a) $\varepsilon_c = 0.0075$, $0.003 \le \lambda_k \le 0.03$, (b) $\lambda_k = 0.01$, three different values of ε_c

on the damping level, or on the lower critical value of the non-dimensional amplitude.

Although there is only a small number of "breaking events" (usually about 3–6 harmonics are affected at any one time, randomly and, generally speaking, differently chosen in each realization, out of $O(10^4)$ total harmonics), they strongly affect the position of the spectral slope and, thus, the total energy. Figure S2 shows evolution of significant wave height for moderate value of λ_k and different values of parameter ε_c.

Thus, the evolution of a wave field has a strong dependence on a free parameter of the numerical scheme, which, at first glance, looks like a major shortcoming of the DNS algorithm. The optimal value of this parameter is grid-dependent and cannot be deduced from physical considerations. However, if ε_c is set to a sufficiently high value, it affects only the position of the spectral slope, not the value of the slope or the shape of the spectral peak. In figure S3, we plot the self-similar spectral shape for $\lambda_k = 0.01$ and the same values of ε_c as in the previous figure. When ε_c is set at 0.005, the "breaking events" occur around the peak and even on the spectral front, distorting the shape of the peak. For higher values of ε_c, there is no difference in the peak shape, so that changing the value of ε_c has virtually the same effect as slightly changing the growth rate due to wind. Since this growth rate is known only approximately, the choice of ε_c is not very significant, provided that it is set to a value that is not too low.

In Figure S4, we plot the evolution of significant wave height obtained with the KE and the DNS (cf. figure 4 in the text of the paper), and the same evolution for wind speed that is different only by 5%. Although the difference between the KE and the DNS results appears to be significant, in reality only a small change of wind speed compensates for it.

In addition to the discussion of the growth of signif-

icant wave height H_s under constant wind forcing, we show the evolution of another important integral characteristics of a wave field, the wavenumber of the spectral peak. Figure S5 shows the dependence of the evolution of the peak wavenumber on parameters of the DNS numerical scheme. First, we fix ε_c and plot the peak wavenumber vs time for different values of λ_k. As it was the case with the significant wave height, with the increase of λ_k the evolution becomes faster, up to a certain, grid-dependent value where it approaches near-saturation. Second, we plot the evolution for the fixed moderate value of λ_k and different ε_c. Since this parameter controls the total amount of energy in the system, the evolution of the peak wavenumber shows dependence on it, although this dependence can be absorbed by only a small variation of wind forcing

We conclude our discussion of the DNS parameters by summarising the parameters used in the main text of the paper in Table I.

Grid	λ_k	λ_ω	ε_c	Number of realizations
161x71	0.03	0.01	0.01	100
321x71	various, up to 0.03	0.01	0.0075	20

TABLE I. Parameters of the simulations used in this study

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