Nonlinear elastic waves in a fibre-reinforced composite with an imperfect interface

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Abstract: The propagation of nonlinear elastic anti-plane shear waves in a unidirectional fibre-reinforced composite material is studied. A model of structural nonlinearity is considered, for which the nonlinear behaviour of the composite solid is caused by imperfect bonding at the "fibre-matrix" interface. A macroscopic wave equation accounting for the effects of nonlinearity and dispersion is derived using the higher-order asymptotic homogenization method. Explicit analytical solutions for stationary nonlinear strain waves are obtained. This type of nonlinearity has a crucial influence on the wave propagation mode: for soft nonlinearity, localised shock (kink) waves are developed, while for hard nonlinearity localised bell-shaped waves appear. Numerical results are presented and the areas of practical applicability of linear and nonlinear, long- and short-wave approaches discussed.

1. Introduction

Elastic waves propagating in heterogeneous solids can undergo the effects of nonlinearity and dispersion. We study a problem for which the nonlinear behaviour of a composite is associated with imperfect bonding conditions at the interface between constitutive components. This is an example of structural nonlinearity, with the nonlinearity directly related to the presence of a microstructure. Dispersion can be classified as geometrical or structural. Geometrical dispersion is typical for wave-guides and finite-size bodies (e.g., waves in beams and plates). Structural dispersion may be caused by the heterogeneity of a composite solid, with successive reflections and refractions of local waves at the matrix-inclusion interfaces leading to scattering of the overall wave field.

Nonlinearity induces a pumping of energy form the low- to the high-frequency part of the spectrum, with higher-order modes generated and continuous localization of energy occurring, making the wave front steeper. In contrast, dispersion provides scattering of energy and decreases the slope of the wave front. When nonlinearity and dispersion act together, they may balance the influence of each other. In such a case, stationary nonlinear waves of permanent shape and velocity can propagate.

The propagation of nonlinear strain waves in elastic solids has been intensively studied [1–3]. Many authors considered homogeneous systems, with dispersive properties mainly determined by geometrical factors. At the same time, the effects of structural dispersion, related to the scattering of nonlinear waves by the microstructure, were not studied in great detail.

In this present paper, we apply the asymptotic homogenization method (AHM) to the modelling of anti-plane shear waves propagating in a fibre-reinforced composite material with imperfect interface bonding between the matrix and fibres. The effect of imperfect bonding is predicted by assuming that the displacement jump across the interface is related to the interfacial stress by a certain cohesion function. We specifically study a weakly nonlinear interface, with a cohesion function represented by a power series expansion in terms of non-dimensional displacement jumps.

According to the AHM, physical fields in a spatially periodic heterogeneous medium are represented by a two-scale asymptotic expansion in powers of a small parameter $\eta = l / L$, where l is the size of the unit cell and L is the typical wavelength. This leads to a decomposition of the final solution into global and local components; the latter are evaluated from a recurrent sequence of cell boundary value problems (BVPs). Application of the volume-integral homogenizing operator allows us to obtain a homogenized constitutive equation that describes the macroscopic behaviour of the medium. From its conception, the AHM was intended for the determination of quasi-static properties of heterogeneous media and structures [4]. In the last years, taking into account higher-order terms with respect to η extended the area of applicability of the homogenized models and provided a mechanism to predict the effect of structural dispersion [5, 6].

The paper is organized as follows. In Section 2, an asymptotic model of the imperfect bonding is proposed and the input BVP introduced. In Section 3, the higher-order asymptotic homogenization procedure is developed and the macroscopic nonlinear wave equation is obtained. In Section 4, the analytical solution for stationary nonlinear strain waves is derived in terms of elliptic functions. The interplay between the effects of nonlinearity and dispersion is analysed in Section 5. Section 6 is devoted to the conclusions.

2. Asymptotic model of the imperfect bonding and input BVP problem

Let us consider a unidirectional fibre-reinforced composite consisting of an infinite matrix $\Omega^{(1)}$ and a periodic square array of cylindrical inclusions $\Omega^{(2)}$, see figure 1. It is supposed that geometrical and physical nonlinearity can be neglected, with the nonlinear behaviour of the composite caused by imperfect bonding at the matrix-fibres interface $\partial\Omega$.

We study anti-plane shear waves propagating in the plane $x_1 x_2$. The governing wave equation is as follows:

$$\mu^{(n)} \nabla_x^2 u^{(n)} = \rho^{(n)} \frac{\partial^2 u^{(n)}}{\partial t^2}, \tag{1}$$

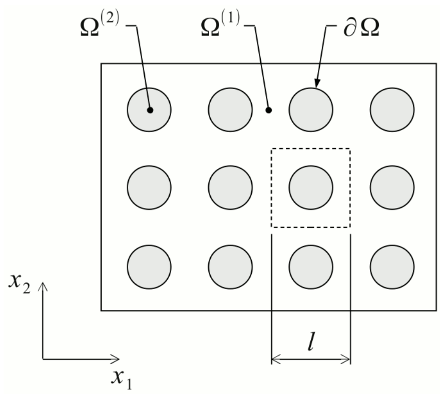

Figure 1. Fibre-reinforce composite structure with a distinguished unit cell.

where $\mu^{(n)}$ is the shear modulus; $\rho^{(n)}$ is the density; $u^{(n)}$ is the displacement in the direction orthogonal to the plane x_1x_2; $\nabla_x = (\partial/\partial x_1)\mathbf{e}_1 + (\partial/\partial x_2)\mathbf{e}_2$; $\mathbf{e}_1$, $\mathbf{e}_2$ are the Cartesian unit vectors. Here, and throughout this paper, the upper index (n) refers to different components of the composite structure, $n = 1,2$.

Let us consider the case of imperfect bonding at the interface $\partial\Omega$. The equilibrium state implies the equality of tangential stresses, thus:

$$\sigma^* = \sigma^{(1)} = \sigma^{(2)} \quad \text{at} \quad \partial\Omega, \tag{2}$$

where $\sigma^{(n)} = \mu^{(n)}\left(\partial u^{(n)}/\partial\mathbf{n}\right)$; $\partial/\partial\mathbf{n}$ is the normal derivative to $\partial\Omega$. Weakening the bonding between the matrix and fibres leads to a jump in the displacement field across the interface. We suppose that the displacement jump $\Delta u^* = u^{(1)} - u^{(2)}$ is related to the interfacial stress σ^* as follows:

$$\sigma^* = f\left(\Delta u^*\right) \quad \text{at} \quad \partial\Omega, \tag{3}$$

where $f\left(\Delta u^*\right)$ is the so-called cohesion function [7, 8]. If the interface exhibits a weakly nonlinear behaviour, the cohesion function can be assumed in the following form:

$$\sigma^* = \mu_1^* \frac{\Delta u^*}{h} + \mu_3^* \left(\frac{\Delta u^*}{h}\right)^3 \quad \text{at} \quad \partial\Omega, \tag{4}$$

where h is the thickness of the interface. From the mathematical point of view, expression (4) can be considered as the first terms of Taylor series expansion of $f\left(\Delta u^*\right)$ in powers of $\Delta u^*/h$. The shear deformation is symmetric, therefore, expansion (4) includes only terms of odd powers. The coefficients μ_1^*, μ_3^* can be interpreted, respectively, as the linear and the nonlinear shear modulus of the interface. We invert series (4), introduce non-dimensional bonding parameters $\alpha = h\mu^{(1)}\big/(l\mu_1^*)$, $\beta = (\mu_3^*\big/\mu_1^*)(\mu^{(1)}\big/\mu_1^*)^2$, and finally let $h \to 0$, $\mu_1^* \to 0$, $\mu_3^* \to 0$. Then expression (4) yields

$$u^{(1)} - u^{(2)} = \alpha l \frac{\sigma^*}{\mu^{(1)}} - \alpha\beta\, l \left(\frac{\sigma^*}{\mu^{(1)}}\right)^3 \quad \text{at} \quad \partial\Omega . \tag{5}$$

The case $\alpha = 0$ corresponds to perfect bonding, $\alpha \to \infty$ – to complete separation of the components. At $\beta = 0$ the interface is purely linear, whilst any increase in $|\beta|$ increases nonlinear effects. The nonlinearity is soft for $\beta < 0$ and hard for $\beta > 0$.

The input BVP includes equations (1), (2), and (5).

3. Higher-order asymptotic homogenization

Let us introduce non-dimensional variables $\overline{u} = u / U$, $\overline{\mathbf{n}} = \mathbf{n} / L$, $\overline{x}_k = x_k / L$, $k = 1,2$, where U is the displacement amplitude and L is the wavelength. The input BVP (1), (2), (5) reads

$$\mu^{(n)} \nabla_{\overline{x}}^2 \overline{u}^{(n)} = \rho^{(n)} L^2 \frac{\partial^2 \overline{u}^{(n)}}{\partial t^2}, \tag{6}$$

$$\mu^{(1)} \frac{\partial \overline{u}^{(1)}}{\partial \overline{\mathbf{n}}} = \mu^{(2)} \frac{\partial \overline{u}^{(2)}}{\partial \overline{\mathbf{n}}} \quad \text{at} \quad \partial\Omega , \tag{7}$$

$$\overline{u}^{(1)} - \overline{u}^{(2)} = \alpha\eta \frac{\partial \overline{u}^{(1)}}{\partial \overline{\mathbf{n}}} - \alpha\eta\delta \left(\frac{\partial \overline{u}^{(1)}}{\partial \overline{\mathbf{n}}}\right)^3 \quad \text{at} \quad \partial\Omega , \tag{8}$$

where $\eta = l / L$, $\delta = \beta \left(U / L\right)^2$, $\nabla_{\overline{x}} = L^{-1} \nabla_x$.

The ratio U / L indicates the magnitude of the elastic strains. We suppose that the size l of the unit cell is smaller than the wavelength L. Hence, the non-dimensional variables η and δ may be considered as natural small parameters characterising, accordingly, the rate of dispersion and the rate of nonlinearity.

Let us introduce so-called *fast* $y_k = \eta^{-1} \overline{x}_k$ and *slow* $\overline{x}_k = \overline{x}_k$ coordinate variables. The spatial derivatives are then given by $\nabla_{\overline{x}} = \nabla_{\overline{x}} + \eta^{-1} \nabla_y$, where $\nabla_y = \left(\partial / \partial y_1\right)\mathbf{e}_1 + \left(\partial / \partial y_2\right)\mathbf{e}_2$. The solution is sought as the asymptotic expansion:

$$\overline{u}^{(n)} = u_0\left(\overline{x}_k\right) + \eta u_1^{(n)}\left(\overline{x}_k, y_k\right) + \eta^2 u_2^{(n)}\left(\overline{x}_k, y_k\right) + \dots .$$

Here the first term u_0 represents the homogenized part of the displacement field; it varies "slowly" on the macrolevel and does not depend on the fast coordinates. The next terms $u_i^{(n)}$, $i = 1,2,3,\dots$, provide order η^i corrections and describe local oscillations of the displacements within each unit cell. Since the composite structure is periodic, the functions $u_i^{(n)}$ satisfy the periodicity condition:

$$u_i^{(n)}\left(\overline{x}_k, y_k\right) = u_i^{(n)}\left(\overline{x}_k, y_k \pm 1\right), \quad i = 1,2,3,\dots \ . \tag{9}$$

Splitting the BVP (6)–(8) with respect to η, we obtain the recurrent sequence of local BVPs:

$$\mu^{(n)}\left(\nabla_{\overline{x}}^2 u_{i-2}^{(n)} + 2\nabla_{\overline{x}} \cdot \nabla_y u_{i-1}^{(n)} + \nabla_y^2 u_i^{(n)}\right) = \rho^{(n)} L^2 \frac{\partial^2 u_{i-2}^{(n)}}{\partial t^2}, \tag{10}$$

$$\mu^{(1)}\left(\frac{\partial u_{i-1}^{(1)}}{\partial \overline{\mathbf{n}}} + \frac{\partial u_i^{(1)}}{\partial \mathbf{m}}\right) = \mu^{(2)}\left(\frac{\partial u_{i-1}^{(2)}}{\partial \overline{\mathbf{n}}} + \frac{\partial u_i^{(2)}}{\partial \mathbf{m}}\right) \quad \text{at} \quad \partial\Omega, \tag{11}$$

$$u_i^{(1)} - u_i^{(2)} = \alpha\left(\frac{\partial u_{i-1}^{(1)}}{\partial \overline{\mathbf{n}}} + \frac{\partial u_i^{(1)}}{\partial \mathbf{m}}\right) - \alpha\delta\left(\frac{\partial u_{i-1}^{(1)}}{\partial \overline{\mathbf{n}}} + \frac{\partial u_i^{(1)}}{\partial \mathbf{m}}\right)^3 \quad \text{at} \quad \partial\Omega, \tag{12}$$

where $i = 1,2,3,\dots$; $u_{-1}^{(n)} = 0$; $\partial / \partial\mathbf{m}$ is the normal derivative to $\partial\Omega$ written in terms of fast variables.

Due to the periodicity condition (9), the local problems (9)–(12) are considered within a distinguished unit cell of the composite structure. Let us replace the outer square contour of the unit cell by a circle of the same area. This simplification is well known in the theory of composites [9]. The accuracy of such an approach is known to be good, when the volume fraction $c^{(2)}$ of the fibres is relatively small. Solutions of the nonlinear local problems (12)–(15) are sought through the asymptotic expansion in powers of δ. The term $u_1^{(n)}$ is evaluated with accuracy $O(\delta)$, the terms $u_2^{(n)}$, $u_3^{(n)}$ – with accuracy $O(\delta^0)$.

Next, we apply to equation (10) at $i = 4$ the homogenizing operator $\iint(\cdot)dy_1 dy_2$ over the unit cell domain. As a result, the macroscopic nonlinear wave equation is obtained. Reverting back to the dimension variables $u = u_0 U$, $x_k = \overline{x}_k L$, the macroscopic wave equation reads

$$\mu_1 \nabla_x^2 u + \frac{1}{3}\beta\mu_2 \nabla_x \cdot (\nabla_x u)^3 + \eta^2 L^2 \mu_3 \nabla_x^4 u + O(\delta^2 + \eta^4) = \rho \frac{\partial^2 u}{\partial t^2}, \tag{13}$$

where μ_1, μ_2, μ_3 are the effective elastic coefficients. The parameter μ_1 is the linear shear modulus; the parameters μ_2 and μ_3 account, respectively, for nonlinearity and for dispersive properties. For μ_1 and μ_2, explicit analytical formulas are derived, while μ_3 is evaluated by numerical integration over the unit cell. It should be noted that μ_1, μ_2, μ_3 are always positive.

The analysis of numerical examples shows that the obtained solutions for μ_1, μ_2, μ_3 provides a good accuracy at $c^{(2)} < 0.5...0.6$. However, for most real composites materials the volume fraction of the fibres does not exceed $0.4...0.5$.

For the composite structure under consideration, the anti-plane shear problem is transversely isotropic in the long-wave limit, when the wavelength is essentially larger than the size of the microstructure, $l / L \to 0$. If the wavelength decreases, the composite material exhibits anisotropic properties and the parameters of elastic waves become dependant on the direction of propagation [5]. In this paper, the simplification of the geometrical shape of the unit cell implies the axial symmetry of the local problems. Therefore, the derived approximate solution is transversely isotropic and the macroscopic wave equation (13) is invariant to the direction of the wave propagation. A comparison of the obtained results with a solution derived by the Floquet-Bloch method [5] has shown that equation (13) provides a good accuracy at $\eta = l/L < 0.4$.

4. Analytical solution for stationary waves

Let us consider a stationary plane wave propagating with a permanent shape and velocity in the direction of the wave vector $\mathbf{k}$. In such a case the solution meets the following condition: $u(\mathbf{x},t) = u(\xi)$, where ξ is the propagation coordinate, $\xi = \mathbf{e}_k \cdot \mathbf{x} - vt$; v is the phase velocity and $\mathbf{e}_k$ is the unit wave vector. Let us define the strain of the wave profile as follows $f = du/d\xi$ and introduce non-dimensional variables $\overline{f} = f / F$, $\zeta = \xi / L$. Here F is the amplitude of the strain wave and L is the wavelength. After routine transformations, equation (13) reads:

$$\frac{d^2 \overline{f}}{d\zeta^2} + a\overline{f} + b\overline{f}^3 = 0, \tag{14}$$

where $a = \mu_1 \left(1 - v^2 / v_0^2\right) / \left(\mu_3 \eta^2\right)$, $b = \beta F^2 \mu_2 / \left(3\mu_3 \eta^2\right)$, v_0 is the effective phase velocity in the linear long-wave limit, $v_0 = \sqrt{\mu_1 / \rho}$.

The type of nonlinearity (soft or hard) has a crucial influence upon the shape and properties of elastic strain waves. For soft nonlinearity ($\beta < 0$), the exact periodical solution of equation (14) is:

$$\bar{f} = \frac{1}{2}\mathrm{sn}\left(\kappa\zeta, s\right), \qquad (15)$$

where κ is the propagation variable, $\kappa = 4K(s)$, $\mathrm{sn}(\cdot)$ is the elliptic sine, $K(s)$ is the complete elliptic integral of the first kind, s is the modulus of the elliptic functions that is determined form the transcendental equation: $s^2 K(s)^2 = -\beta F^2 \mu_2 / \left(384\eta^2 \mu_3\right)$, $0 \le s \le 1$. For the phase velocity v we obtain: $v^2/v_0^2 = 1 - 16\left(1+s^2\right)K(s)^2 \eta^2 \mu_3/\mu_1$.

For hard nonlinearity ($\beta > 0$), the solution takes the following form:

$$\bar{f} = \frac{1}{2}\mathrm{cn}\left(\kappa\zeta, s\right), \qquad (16)$$

where $s^2 K(s)^2 = \beta F^2 \mu_2 / \left(384\eta^2 \mu_3\right)$, $v^2/v_0^2 = 1 - 16\left(1-2s^2\right)K(s)^2 \eta^2 \mu_3/\mu_1$.

The magnitude of the modulus s determines the intensity of nonlinear effects. The limit $s = 0$ corresponds to the purely linear case: $\bar{f} = (1/2)\sin\left(2\pi\zeta\right)$ at $\beta < 0$; $\bar{f} = (1/2)\cos\left(2\pi\zeta\right)$ at $\beta > 0$; $v^2/v_0^2 = 1 - 4\pi^2\eta^2 \mu_3/\mu_1$.

At the opposite limit, $s = 1$, solutions (15) and (16) describe localised solitary waves. In the case of soft nonlinearity, a shock (so-called *kink*) strain wave appears (figure 2, *a*): $\bar{f} = (1/2)\,\mathrm{th}\left(\zeta/\Delta\right)$, $\Delta^2 = -24\mu_3\eta^2 / \left(F^2 \beta\, \mu_2\right)$, $v^2/v_0^2 = 1 - 2\mu_3\eta^2 / \left(\mu_1 \Delta^2\right)$. Here the parameter Δ can be treated as the width of the localised wave. The kink wave propagates with a velocity lower than the velocity v_0 associated with the linear long-wave limit: $v < v_0$. This is the so-called *subsonic* mode. The increase in the amplitude F leads to a decrease in the width Δ and the velocity v of the wave.

If nonlinearity is hard, the localised solution takes the form of a bell-shaped wave (figure 2, *b*): $\bar{f} = \left[2\mathrm{ch}(\zeta/\Delta)\right]^{-1}$, $\Delta^2 = 24\mu_3\eta^2 / \left(F^2 \beta \mu_2\right)$, $v^2/v_0^2 = 1 + \mu_3\eta^2 / \left(\mu_1 \Delta^2\right)$. In this case a *supersonic* propagation mode is realised, i.e. $v > v_0$. When the amplitude F grows, the width Δ decreases and the velocity v increases.

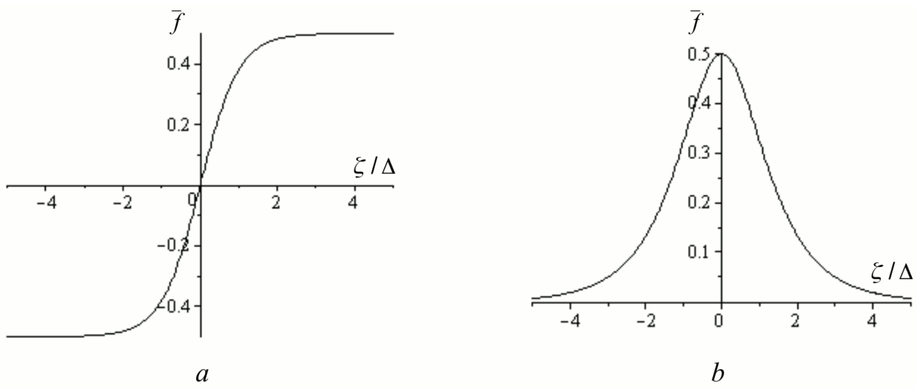

Figure 2. Localised nonlinear elastic strain waves; *a* – soft nonlinearity, *b* – hard nonlinearity.

5. Interplay between nonlinearity and dispersion

As an illustrative example, let us consider a composite material consisting of the aluminium matrix ($\mu^{(1)} = 27.9$ GPa, $\rho^{(1)} = 2700$ kg/m^3) and nickel fibres ($\mu^{(2)} = 75.4$ GPa, $\rho^{(2)} = 8940$ kg/m^3). The volume fraction of the fibres is $c^{(2)} = 0.4$. The following magnitudes of the bonding parameters are assumed: $\alpha = 0.1$, $|\beta| = 10^5$. Basing on the solutions obtained in Section 3, the effective elastic coefficients are evaluated: $\mu_1 = 33.1$ GPa, $\mu_2 = 21.6$ GPa, $\mu_3 = 0.119$ GPa.

Figure 3 displays the parametric dependence of the modulus s on the amplitude F and the dispersion parameter η. The domain of elastic strains is restricted by $F \le 10^{-3}$, a regime typical for most engineering materials.

The presented results show how the phenomena of nonlinearity and dispersion compensate the influence of each other. The increase of the amplitude F (at a fixed value of η) leads to the growing of the modulus s and, therefore, the intensity of nonlinear effects increases. In contrary, the decrease of the wavelength and the increase in η (at a fixed F) is followed by the decrease of the modulus s, so the influence of nonlinearity is reduced.

The numerical analysis of the obtained solutions (15), (16) has shown that nonlinearity has a noticeable influence on both the wave shape and velocity if $s > 0.6$. As follows from figure 3, in this case $\eta < 0.2$ and, consequently, the solution can be evaluated utilizing the long-wave approach. On the other hand, the homogenized equation (13) lacks accuracy for $\eta > 0.4$. Then $s < 0.34$, which means that the wave shape and velocity are very close to the linear case and, consequently, an approximate solution may be found utilising the linear theory. This analysis is particularly important, helping to estimate the domain of practical applicability of linear and nonlinear approaches.

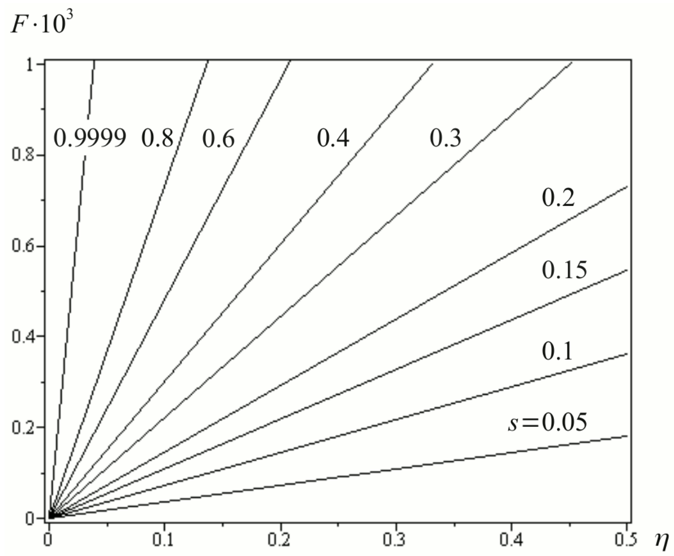

Figure 3. The modulus s characterising the intensity of nonlinear effects.

6. Conclusions

The paper presents analytical solutions that describe the propagation of nonlinear elastic anti-plane shear waves in a unidirectional fibre-reinforced composite material with imperfect bonding between constitutive components. It should be emphasised that this type of nonlinearity has a particularly strong influence upon the propagation mode and the shape of the strain waves. In the case of soft nonlinearity, localised shock (kink) waves appear, while in respect of materials with hard nonlinearity the localised solution takes the form of bell-shaped waves.

The analysis allowed us to estimate the domain of applicability of the different approximate theories used for the modelling of elastic waves in heterogeneous solids. It is shown that nonlinear waves can be adequately described within the long-wave framework (such as the AHM). When dealing with the propagation of short waves, with wave length commensurable with the scale of the microstructure, nonlinear effects become very small. In such a case, an approximate solution may be obtained using the linear Floquet-Bloch theory. This conclusion is true, if the strain amplitude F does not exceed 10^{-3} , which is typical for many solids.

The results presented in the paper can be applied to facilitate the development of new efficient methods of acoustic diagnostic and non-destructive testing in various branches of engineering. Measuring the characteristics of nonlinear waves allows us to receive much more precise information about the internal structure and defects of solids. This is sometimes that may be not possible within a linear framework. We also remark that the propagation of localised nonlinear waves is accompanied by an essential concentration of mechanical energy. The obtained solutions can help in the development of new criteria for the dynamic failure of heterogeneous materials and structures.

Acknowledgments

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