The Continuous Skolem-Pisot Problem: On the Complexity of Reachability for Linear Ordinary Differential Equations

Paul C. Bell*, Jean-Charles Delvenne, Raphaël M. Jungers, Vincent D. Blondel

Department of Mathematical Engineering, Centre CESAME, Université catholique de Louvain, Bâtiment Euler, Avenue Georges Lemaître, 4, B-1348 Louvain-la-Neuve, Belgium

Abstract

We study decidability and complexity questions related to a continuous analogue of the Skolem-Pisot problem concerning the zeros and nonnegativity of a linear recurrent sequence. In particular, we show that the continuous version of the nonnegativity problem is NP-hard in general and we show that the presence of a zero is decidable for several subcases, including instances of depth two or less, although the decidability in general is left open. The problems may also be stated as reachability problems related to real zeros of *exponential polynomials* or solutions to initial value problems of linear differential equations, which are interesting problems in their own right.

Key words: Skolem-Pisot problem, Exponential polynomials, Continuous time dynamical system, Decidability, Ordinary differential equations

1. Introduction

Skolem's problem (also known in the literature as Pisot's problem) asks whether it is algorithmically decidable if a given linear recurrent sequence (LRS) has a zero or not. A LRS may be written in the form:

$$u_k = a_{n-1}u_{k-1} + a_{n-2}u_{k-2} + \dots + a_0u_{k-n},$$

for $k \geq n$ where $u_0, u_1, \ldots, u_{n-1} \in \mathbb{Z}$ are the initial inputs and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$ are the recurrence coefficients, see also [9]. This forms the infinite sequence

^{*}Corresponding Author

Email addresses: p.c.bell@liverpool.ac.uk (Paul C. Bell), jean-charles.delvenne@uclouvain.be (Jean-Charles Delvenne), raphael.jungers@uclouvain.be (Raphaël M. Jungers), vincent.blondel@uclouvain.be (Vincent D. Blondel)

 $(u_k)_{k=0}^{\infty} \subseteq \mathbb{Z}$. We may assume a_0 is nonzero, otherwise a shorter and equivalent recurrence exists. Such a recurrence sequence is said to be of *depth* n.

For a linear recurrent sequence $u = (u_k)_{k=0}^{\infty} \subseteq \mathbb{Z}$ the zero set of u is given by $Z(u) = \{i \in \mathbb{N} | u_i = 0\}$. One of the first results concerning the zeros of LRS's was by T. Skolem in [23], when he proved that the zero set is semilinear (i.e., the union of finitely many periodic sets and a finite set). This result was also later shown by K. Mahler [18] and C. Lech [17] and is now often referred to as the Skolem-Mahler-Lech theorem. It is known that determining if Z(u) is an infinite set is decidable as was proven by Berstel and Mignotte [5].

It was shown by N. Vereshchagin in 1985 that Skolem's problem (i.e., the problem "is the zero set of a LRS empty?") is decidable when the depth of the linear recurrent sequence is less than or equal to four in [26]. It was also recently shown that Skolem's problem is decidable for depth five in [13], but the general decidability status is open. It is also known that determining if a given linear recurrent sequence has a zero is NP-hard, see [6].

Note that we may always encode a linear recurrent sequence of depth n into an integral matrix $A \in \mathbb{Z}^{(n+1)\times (n+1)}$ such that $u_k = A_{1,n+1}^k$ for $k \geq 1$. This follows since given the initial vector $u = (u_0, u_1, \ldots, u_{n-1})^T$ and the recurrence coefficients, $a_0, a_1, \ldots, a_{n-1}$, we first define matrix $A' \in \mathbb{Z}^{n \times n}$:

$$A' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}.$$

Note that $(A')^k u = (u_k, u_{k+1}, \dots, u_{k+n-1})$. Now we shall extend this matrix by 1 dimension to give:

$$A = \begin{pmatrix} A' & A'v \\ \overline{0} & 0 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)},$$

where $\overline{0}$ is the zero vector of appropriate size. It is not difficult to now see that $u_k = A_{1,n}^k$ for $k \geq 1$ as required. Skolem's problem in this context is therefore to determine if the upper right entry of a positive power of an integral matrix is zero. More generally, one can show that Skolem's problem is equivalent to the following problem: given a matrix $A \in \mathbb{F}^{n \times n}$ and two vectors $c, x_0 \in \mathbb{F}^n$, is there a nonnegative integer t such that $c^T A^t x_0 = 0$? We add that a generalization of this problem where we may take any product of two integral matrices of dimension 10 is known to be undecidable, see [14].

In this paper we shall consider a dynamical system whose updating trajectory is given by $\frac{dx(t)}{dt} = Ax(t)$ where $A \in \mathbb{R}^{n \times n}$ and the initial point $x(0) \in \mathbb{R}^n$ is given. We shall be interested in determining whether this trajectory ever reaches a given hyperplane, thus the problem is equivalent to determining if there exists $t \in \mathbb{R}_{\geq 0}$ such that $c^T \exp(At)x(0) = 0$ where $c \in \mathbb{R}^n$ defines the hyperplane. We consider this as the Skolem-Pisot problem in continuous time. We show that for instances of size two or less this problem is decidable.

We shall also show that determining if $c^T \exp(At)x(0)$ reaches zero is computationally equivalent to determining whether a given real-valued exponential polynomial $f(z) = \sum_{j=1}^m P_j(z) \exp(\theta_j z)$, where each P_j is a polynomial, ever reaches zero for a positive real value. This is also equivalent to determining if the solution y(t) of an ordinary differential equation $y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_0y = 0$ with given initial conditions $y^{(k-1)}(0), y^{(k-2)}(0), \ldots, y(0)$ ever reaches zero.

From 1920, Pólya and others characterized the asymptotic distribution of complex zeros of exponential polynomials [19, 21, 22, 24, 25, 28]. Upper bounds were also found on the number of zeros in a finite region of the complex plane, using the argument principle. Less is known about real zeros. Upper and lower bounds on the number of zeros in a real interval are given in [27]. A formula for the asymptotic density of real zeros for a restricted class of exponential polynomials was found in [15]. Some observations on the first sign change of a sum of cosines are collected in [20]. However, no criterion has been proposed to check the existence of a real zero for a real exponential polynomial.

A related problem, determining whether a given linear recurrent sequence has only nonnegative terms, the *nonnegativity problem*, is decidable for dimension 2, see [12]. The authors note that if the nonnegativity problem is decidable in general, it implies Skolem's problem is decidable. This follows since if $(u_k)_{k=0}^{\infty}$ is recurrent, then so is $(u_k^2 - 1)_{k=0}^{\infty}$.

We may note that using the linear recurrent sequence $(u_k)_{k=0}^{\infty}$ from the proof of NP-hardness of Skolem's problem in [6], and converting it to the form $(u_k^2 - 1)_{k=0}^{\infty}$, allows one to easily derive the following result:

Theorem 1. It is NP-hard to decide if a given linear recurrent sequence is nonnegative, i.e., the nonnegativity problem is NP-hard.

This holds since if $(u_k)_{k=0}^{\infty}$ is represented by a matrix $\mathbb{Z}^{n\times n}$, then $(u_k^2-1)_{k=0}^{\infty}$ may be represented by a matrix $\mathbb{Z}^{(n^2+1)\times(n^2+1)}$ and thus we have a polynomial time reduction. In this paper we show that the nonnegativity problem in the continuous setting is also NP-hard.

Given a matrix $M \in \mathbb{R}^{n \times n}$ and vectors $u, v \in \mathbb{R}^n$, the *orbit problem* asks if there exists a power $k \in \mathbb{N}$ such that $M^k u = v$. Thus it is a type of *reachability problem*, see [7]. This was shown to be decidable even in polynomial time, see [16]. The corresponding version of this problem for continuous time asks whether for a given $M \in \mathbb{R}^{n \times n}$ and vectors $a, b \in \mathbb{R}^n$ there exists some $t \in \mathbb{R}_{\geq 0}$ such that $\exp(Mt)a = b$. This problem was proved to be decidable in [11].

2. Preliminaries

Let $A \in \mathbb{F}^{n \times n}$ denote an $n \times n$ matrix over the field \mathbb{F} and $\sigma(A)$ the set of eigenvalues of A. For a complex number $z \in \mathbb{C}$ we denote by $\Re(z)$ the *real* part of z and by $\Im(z)$ the *imaginary* part of z. We use the notation $\mathbb{R}_{\geq 0}$ to denote the nonnegative real numbers.

We shall denote an exponential polynomial $f: \mathbb{C} \to \mathbb{C}$ by a sum of the form: $f(z) = \sum_{j=1}^{m} P_j(z) \exp(\theta_j z)$, where $P_j \in \mathbb{C}[X]$ and $\theta_j \in \mathbb{C}$.

Given a matrix $A \in \mathbb{C}^{n \times n}$ we shall denote by the dominant eigenvalues of A the set of eigenvalues of A with maximum real part, i.e.,

$$\{\theta \in \sigma(A) | \Re(\theta) \ge \Re(\theta'), \theta' \in \sigma(A) \}.$$

We will later require the following theorem from Diophantine approximation [3]:

Theorem 2. (BAKER) Let $\alpha_1, \ldots, \alpha_k, \beta_0, \ldots, \beta_k$ be algebraic numbers. Then the combination

$$\Lambda = \beta_0 + \sum_i \beta_i \ln \alpha_i$$

is either zero or satisfies $|\Lambda| > h^{-N}$, where h is the largest height of β_1, \ldots, β_k , and N is a computable constant depending only on $\ln \alpha_1, \ldots, \ln \alpha_k$ and the maximum degree of β_0, \ldots, β_k .

Recall that for an algebraic number β with minimal polynomial

$$p(x) = \sum_{0 \le i \le d} a_i x^i,$$

its degree is d and its height is $\max |a_i|$. We shall also use the following theorem regarding the transcendence degree of the field extension of algebraic numbers when considering their exponentials:

Theorem 3. (HERMITE-LINDEMANN) - Let $\alpha_j, \lambda_j \in \mathbb{C}$ for $0 \leq j \leq n-1$ be algebraic numbers such that no $\alpha_j = 0$ and each λ_j is distinct. Then:

$$\sum_{j=0}^{n-1} \alpha_j e^{\lambda_j} \neq 0.$$

The following theorem concerns simultaneous Diophantine approximation of algebraic numbers which are linearly independent over the rationals.

Theorem 4. (KRONECKER, see [8]) Let $1, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ be real algebraic numbers which are linearly independent over \mathbb{Q} . Then for any $\alpha \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $p \in \mathbb{Z}^n$ and $k \in \mathbb{N}$ such that $|(k\lambda_i - \alpha_i - p_i)| < \epsilon$ for all $1 \le i \le n$.

3. Skolem's Problem in Continuous Time

We shall consider continuous time systems governed by the rule $\frac{dx(t)}{dt} = Ax(t)$ where A is a real matrix and x(t) is a real vector ¹. We are interested in the decidability of whether from an initial vector x(0), we cross a given hyperplane.

¹We consider entries to be algebraic so that the input to a problem has a finite description.

We may consider this as a "point-to-set" reachability problem in a dynamical system, see [7] for other examples.

Let $\frac{dx(t)}{dt} = Ax(t)$ where $A \in \mathbb{R}^{n \times n}$ and $x(t) \in \mathbb{R}^n$. Given the initial vector $x(0) \in \mathbb{R}^n$, then x(t) is given by:

$$x(t) = \exp(At) \cdot x(0) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j \cdot x(0).$$

Given a vector $c \in \mathbb{R}^n$ defining a hyperplane, we would like to determine if there exists some $t \in \mathbb{R}_{\geq 0}$ such that $c^T x(t) = 0$. In other words, whether the flow of the point x(0) ever intersects the hyperplane. If such a t exists, we say that there exists a solution to the instance 2 . An instance of Continuous Skolem Problem therefore consists of the matrix $A \in \mathbb{R}^{n \times n}$, the initial point $x(0) \in \mathbb{R}^n$ and the hyperplane vector $c \in \mathbb{R}^n$.

3.1. Equivalent Formulations

To analyze the behaviour of the system, we will convert a given instance of Continuous Skolem Problem into various forms which have different properties but which are essentially equivalent to the original problem.

Given such an instance, the following lemma shows that the problem is equivalent to determining if the upper right entry of the exponential of a matrix equals some constant real. A similar construction is known in the discrete case as shown in Section 1.

Theorem 5. Given an instance of Continuous Skolem Problem defined by $f(t) = c^T \exp(At)x(0)$ where $A \in \mathbb{R}^{n \times n}$ and $c, x(0) \in \mathbb{R}^n$. There exists a polynomial-time computable matrix $B \in \mathbb{R}^{(n+2) \times (n+2)}$ such that $f(t) = \exp(Bt)_{1,n+2} + \lambda$, where $\lambda = c^T x(0) \in \mathbb{R}$ is constant.

Proof. We are given the function $f(t) = c^T \exp(At)x(0)$. Let $B \in \mathbb{R}^{(n+2)\times (n+2)}$ be given by:

$$B \stackrel{\triangle}{=} \left(\begin{array}{ccc} 0 & c^T A & c^T A x(0) \\ \bar{0} & A & A x(0) \\ 0 & \bar{0}^T & 0 \end{array} \right),$$

where $\bar{0} = (0, 0, \dots, 0)^T \in \mathbb{R}^n$, thus:

$$\exp(B) = \begin{pmatrix} 1 & c^T \exp(A) - c^T & c^T \exp(A)x(0) - \lambda \\ \bar{0} & \exp(A) & \exp(A)x(0) - x(0) \\ 0 & \bar{0}^T & 1 \end{pmatrix},$$

where $\lambda = c^T x(0)$ is constant. This can be seen from the power series representation $\exp(tB) = \sum_{j=0}^{\infty} \frac{t^j}{j!} B^j$. Therefore $f(t) = \exp(Bt)_{1,(n+2)} + \lambda$ and thus an

²Note that, in the style of Skolem's problem, we shall be more interested in determining whether any solution exists, rather than trying to find an algebraic description of the solution.

instance of CONTINUOUS SKOLEM PROBLEM can also be given by a single real matrix B and the problem of whether f(t) reaches zero for $t \in \mathbb{R}_{\geq 0}$ is equivalent to whether $\exp(Bt)_{1,(n+2)}$ ever equals $-\lambda$.

Theorem 6. The following problems are computationally equivalent with polynomial time reductions (where all parameters are algebraic numbers):

- (i) Does there exist a solution to a given instance of Continuous Skolem Problem?
- (ii) Determine if a real-valued exponential polynomial:

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\theta_j t},$$

has a nonnegative real zero (where $\theta_j \in \mathbb{C}$ and $P_j \in \mathbb{C}[X]$).

(iii) Determine if a function of the form:

$$f(t) = \sum_{j=1}^{m} e^{r_j t} (P_{1,j}(t) \cos(\lambda_j t) + P_{2,j}(t) \sin(\lambda_j t))$$

has a nonnegative real zero (where $r_j, \lambda_j \in \mathbb{R}$ and $P_{i,j} \in \mathbb{R}[X]$).

(iv) Determining whether the solution y(t) to an ordinary differential equation $y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_0y = 0$ with the given initial conditions $y^{(k-1)}(0), y^{(k-1)}(0), \ldots, y(0)$ reaches zero for a nonnegative real t.

Proof. $(i) \Rightarrow (ii)$: Let $J \in \mathbb{C}^{n \times n}$ be the Jordan matrix for A, thus we may write $A = PJP^{-1}$ for some $P \in GL(n,\mathbb{C})$.³ Since $\exp(PJP^{-1}) = P\exp(J)P^{-1}$, we can ask the equivalent problem, does there exist a time $t \geq 0$ at which:

$$c^{T}y(t) = c^{T} \exp(tA)y(0)$$
$$= u^{T} \exp(tJ)v = 0,$$

where $u, v \in \mathbb{C}^n$ are defined by $u^T = c^T P$ and $v = P^{-1} y(0)$?

Let $J=J_1\oplus J_2\oplus\ldots\oplus J_m$ be a decomposition of J into a direct sum of Jordan blocks with $J_i\in\mathbb{C}^{n_i\times n_i}$ and $\sum_{i=1}^m n_i=n$. Each Jordan block may be written $J_i=\theta_iI_{n_i}+M_i$ where $\theta_i\in\mathbb{C}$ is the associated eigenvalue, $I_{n_i}\in\mathbb{Z}^{n_i\times n_i}$ is the identity matrix and $M_i\in\mathbb{Z}^{n_i\times n_i}$ has 1 on the super-diagonal and 0 elsewhere.

For $1 \leq i \leq m$, we see that $\theta_i I_{n_i}$ and M_i commute and therefore $\exp(tJ_i) = \exp(t\theta_i I_{n_i}) \exp(tM_i)$. The value of $\exp(t\theta_i I_{n_i})$ is $e^{t\theta_i} I_{n_i}$. Let $\exp(tM_i) = [m_{jk}] \in \mathbb{Q}^{n \times n}$, then

$$m_{jk} = \begin{cases} \frac{t^{(k-j)}}{(k-j)!} & ; \text{ if } j \le k\\ 0 & ; \text{ otherwise} \end{cases}$$
 (1)

³These can be effectively found since we only need algebraic descriptions of the Jordan normal form J and the similarity matrix P.

Therefore we may convert our problem equivalently into deciding whether there exists a $t \in \mathbb{R}_{>0}$ such that f(t) = 0 where $f : \mathbb{R} \to \mathbb{C}$ is defined by:

$$f(t) = u^T \exp(Jt)v = \sum_{j=1}^{m} P_j(t)e^{\theta_j t},$$
 (2)

and $P_j \in \mathbb{C}[X]$ are polynomials and whose degree depends upon the size of the corresponding Jordan block and $\theta_j \in \mathbb{C}$. The polynomials P_j can be derived from Equation (1). Note that each of these steps is effective and can be computed in polynomial time for algebraic entries of the initial matrix A.

- $(ii) \Rightarrow (iii)$: This results from Euler's formula for the complex exponential and the fact that f(t) is a real valued function.
 - $(iii) \Rightarrow (iv)$: Functions of the type

$$f(t) = \sum_{j=1}^{m} e^{r_j t} (P_{1,j}(t) \cos(\lambda_j t) + P_{2,j}(t) \sin(\lambda_j t))$$

where $r_j, \lambda_j \in \mathbb{R}$ are fixed and $P_{k,j}$ are arbitrary real polynomials of degree $\leq d_j$ form a real vector space of dimension $k = 2\sum_{j=1}^m (d_j + 1)$. This vector space is closed under differentiation. Hence the first (k+1) derivatives of f are related by $f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0f = 0$ where each a_j can be found in polynomial time. By Cauchy's theorem for ordinary differential equations, a function f is completely determined by the given relation and the initial conditions $f^{(k-1)}(0), f^{(k-2)}(0), \ldots, f(0)$.

 $(iv)\Rightarrow (i)$: The characteristic equation of the linear homogeneous differential equation is given by $z^k+z^{k-1}a_{k-1}+\ldots+a_0=0$. It is well known that we can form the companion matrix of the equation in order to convert the problem into an instance of Continuous Skolem Problem. The initial values are then present in the initial vector x(0).

Lemma 7. Let $A \in \mathbb{R}^{n \times n}$ and $c, x(0) \in \mathbb{R}^n$ form an instance of Continuous Skolem Problem. For any $\lambda \in \mathbb{C}$ we may form a system $f_{\lambda}(t) = u^T \exp(t(A + \lambda I))v$ where $u, v \in \mathbb{C}^n$, $\sigma(A + \lambda I) = \sigma(A) + \lambda$ and f(t) = 0 if and only if $f_{\lambda}(t) = 0$.

Proof. Let $\lambda \in \mathbb{C}$ and define $y(t) = e^{\lambda t}x(t)$, thus:

$$\frac{dy(t)}{dt} = \lambda e^{\lambda t} x(t) + e^{\lambda t} \frac{dx(t)}{dt}$$
$$= e^{\lambda t} (\lambda I + A) x(t)$$
$$= (\lambda I + A) y(t)$$

Define $A_{\lambda} = \lambda I + A$, thus:

$$y(t) = \exp(tA_{\lambda})y(0).$$

Note that there exists $t \geq 0$ such that $c^T x(t) = 0$ if and only if $c^T y(t) = 0$.

As an example, which will be useful later, let us set $\lambda = -\max\{\Re(\theta)|\theta \in \sigma(A)\}$, so that all eigenvalues are shifted to the left complex half-plane or the imaginary axis. This means that we have, in effect, split the set of eigenvalues into two sets, one which decays exponentially with time and one which consists of purely imaginary values.

We now remark that any nontrivial solution to the problem will in fact be transcendental.

Theorem 8. Given an instance of Continuous Skolem Problem, all solutions, if any exist, are transcendental unless the polynomials $P_j(t)$ share a common positive real root.

Proof. The corresponding exponential polynomial formed as in Theorem 6 will be in the form:

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\theta_j t} = 0.$$

We may assume no $P_j \in \mathbb{C}[X]$ is zero otherwise simply remove it from the sum and that each θ_j is distinct, otherwise group them together. Thus, according to Theorem 3 (the Hermite-Lindemann theorem), this exponential polynomial only has solutions for transcendental times t where $t \in \mathbb{R}_{>0}$.

4. Decidable Cases

We shall now investigate some classes of instances for which CONTINUOUS SKOLEM PROBLEM is decidable.

Theorem 9. The Continuous Skolem Problem for depth 2 is decidable.

Proof. Assume we have an instance of Continuous Skolem Problem given by $f(t) = (c_1, c_2) \exp(At)(x_1, x_2)^T$ with $A \in \mathbb{R}^{2 \times 2}$. Let $S \in GL(\mathbb{C}, 2)$ put A into Jordan canonical form. We can rewrite $f(t) = (\alpha_1, \alpha_2) \exp(Jt)(\beta_1, \beta_2)^T$, where $J = S^{-1}AS$ is a Jordan matrix.

If A has one eigenvalue θ , with algebraic multiplicity 2, then $\theta \in \mathbb{R}$. If θ has geometric multiplicity 1 then by Theorem 6 we must solve an equation of the form $(1+xt)ye^{t\theta}$ where $x,y\in\mathbb{R}$, thus the instance has a solution if and only if $-\frac{1}{x}\in\mathbb{R}_{\geq 0}$. If θ has geometric multiplicity 2 then we must solve $e^{\theta t}(\alpha_1\beta_1+\alpha_2\beta_2)=0$ which has a solution if and only if $(\alpha_1\beta_1+\alpha_2\beta_2)=0$.

Otherwise, J is diagonal and we must determine if there exists a $t \in \mathbb{R}_{\geq 0}$ such that $e^{t\theta_1} + \alpha e^{t\theta_2} = 0$ for $\alpha \in \mathbb{R}$. Either $\theta_1, \theta_2 \in \mathbb{R}$ or $\theta_1 = \overline{\theta_2} \in \mathbb{C}$.

If $\theta_1, \theta_2 \in \mathbb{R}$ assume without loss of generality that $\theta_1 < \theta_2$ and we have $g(t) = e^{t\theta_1} + \alpha e^{t\theta_2}$ thus, by taking logarithms, $t = \frac{\ln(-\alpha)}{\theta_1 - \theta_2}$ is a solution of g(t) = 0 and thus there exists a solution if and only if $\frac{\ln(-\alpha)}{\theta_1 - \theta_2} \in \mathbb{R}_{\geq 0}$.

In the other case $\theta_1 = \overline{\theta_2} \in \mathbb{C}$. Since we may therefore shift the real part as allowed by 7, assume that $\theta_1, \theta_2 \in i\mathbb{R}$. At time $t = \frac{\pi}{2\Im(\theta_1)}$ we have

$$\begin{array}{ll} e^{t\theta_1} + \alpha e^{t\overline{\theta_2}} &= e^{\Im(\theta_1)it} + \alpha e^{-\Im(\theta_1)it} \\ &= \cos(\frac{\pi}{2}) + \alpha\cos(-\frac{\pi}{2}) = 0 \end{array}$$

which is a solution, thus we are done.

The following theorem shows that the class of instances where all elements of the input are nonnegative reals in the continuous setting is trivially decidable in polynomial time, whereas in the discrete time case, the problem is NP-hard, as shown in [6]. In fact, using Lemma 7, we see that in the continuous setting the Skolem-Pisot problem is polynomially decidable even where the matrix given is a *Metzler matrix*, meaning only off-diagonal elements need be nonnegative.

Theorem 10. For an instance of CONTINUOUS SKOLEM PROBLEM given by $A \in \mathbb{R}^{n \times n}$ and $c, x(0) \in \mathbb{R}^n_{\geq 0}$ where A is a Metzler matrix (thus all off-diagonal elements are nonnegative) and $f(t) = c^T \exp(At)x(0)$, then we may decide if there exists a solution in polynomial time.

Proof. Let λ be the minimal diagonal element of A. If $\lambda < 0$ then by Lemma 7, we may form an equivalent instance $A' = A + \lambda I$ where $A' \in \mathbb{R}_{\geq 0}^{n \times n}$. Thus assume without loss of generality that A is a nonnegative matrix and c, x(0) are nonnegative vectors.

Note that $\exp(t_2A) > \exp(t_1A)$ for any $t_2 > t_1 \in \mathbb{R}_{\geq 0}$ which is a consequence of the power series representation of $\exp(At) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j$ and the fact that $A \in \mathbb{R}_{\geq 0}^{n \times n}$. We see that $f(0) = c^T x(0) \in \mathbb{R}_{\geq 0}$. Now, if f(0) = 0 then this is a solution, otherwise, since the matrix exponential increases monotonically componentwise with time for a nonnegative matrix, there exists no solution. \square

In some special cases, some eigenvalues of A do not influence the function f(t). This is easily seen when A is put in its Jordan form $J = P^{-1}AP$:

$$f(t) = u^T \exp(tJ)v \tag{3}$$

where $u, v \in \mathbb{C}^n$ are defined by $u^T = c^T P$ and $v = P^{-1}x(0)$. Obviously, if the entries of c or x(0) corresponding to a particular Jordan block are zero, this block does not play any role and one may remove it without changing the function f(t). More generally, from Equation (3) it is easy, as shown in Theorem 6, to write the function f as follows:

$$f(t) = \sum_{i=1}^{m} P_j(t)e^{\theta_j t},\tag{4}$$

where the θ_j are the distinct eigenvalues of A and the P_j are complex polynomials. If no polynomial $P_j(t)$ is identically zero, then we say that the triple (A, c, x(0)) is reduced. If some of the P_j are zero, we can remove the corresponding terms from Equation (4), since it does not change the value of f.

Theorem 6 shows how to build an equivalent instance of the form (A', c', x'(0)) from an instance of the form of Equation (4). One would then obtain a reduced instance of the Continuous Skolem Problem. This can be done in a preprocessing phase.

Theorem 11. Let $\frac{dx(t)}{dt} = Ax(t)$ for $A \in \mathbb{R}^{n \times n}$ and $x(t) \in \mathbb{R}^n$ define an instance of Continuous Skolem Problem given by $f(t) = c^T \exp(At)x(0) = 0$. If (A, c, x(0)) is reduced and none of the dominant eigenvalues of A are real then the problem is decidable.

Proof. By Lemma 7, let us assume all eigenvalues have real part less than or equal to 0. Then, using Theorem 6, we may consider the system as being represented by

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\theta_j t}.$$

We may split this exponential polynomial in two (reordering as necessary) and write $f(t) = f_1(t) + f_2(t)$, where

$$f_1(t) = \sum_{j=1}^k P_j(t) \exp(i\lambda_j t)$$

$$= \sum_{j=1}^k (P_{1,j}(t) \cos(\lambda_j t) + P_{2,j}(t) \sin(\lambda_j t))$$
(5)

are those terms with 0 real part and $f_2(t)$ is the summation of the remaining terms. Equation (5) follows from Theorem 6 since $f_1(t)$ is real valued. Since (A, c, x(0)) is reduced, $f_1(t)$ is not identically zero. Note that $f_2(t)$ tends to zero exponentially fast as t increases.

For a polynomial P of degree n we may use Cauchy's bound on the maximum modulus of any polynomial root to determine that for any root $z \in \mathbb{C}$ of $P(x) = a_n x^n + \ldots + a_1 x + a_0$ we have that:

$$|z| \le 1 + \frac{\max\{|a_0|, |a_1|, \dots, |a_{n_j-1}|\}}{|a_n|},$$

as is easy to prove. Thus define $T \in \mathbb{R}_{\geq 0}$ to be strictly greater than this maximum bound for any $P_{1,j}$ or $P_{2,j}$ in Equation (5) for $1 \leq j \leq m$ and thus for all $t \geq T$, the sign of $P_{1,j}(t)$ and $P_{2,j}(t)$ for each $1 \leq j \leq m$ is fixed.

For each $1 \leq j \leq k$ there exists $t_{j,1}, t_{j,2} > T$ such that

$$P_{1,j}(t_{j,1})\cos(\lambda_j t_{j,1}) + P_{2,j}(t_{j,1})\sin(\lambda_j t_{j,1}) > 0,$$

$$P_{1,j}(t_{j,2})\cos(\lambda_j t_{j,2}) + P_{2,j}(t_{j,2})\sin(\lambda_j t_{j,2}) < 0.$$

Each λ_j is distinct thus we have enough freedom in the choice of these times so that there exists $t_1, t_2 > T$ such that

$$\begin{array}{l} P_{1,j}(t_1)\cos(\lambda_j t_1) + P_{2,j}(t_1)\sin(\lambda_j t_1) > 0 \\ P_{1,j}(t_2)\cos(\lambda_j t_2) + P_{2,j}(t_2)\sin(\lambda_j t_2) < 0 \end{array}; \ 1 \le j \le k. \end{array}$$

We now see that $f_1(t_1)$ is positive and $f_1(t_2)$ is negative thus there exists a solution since $f_2(t)$ decays exponentially fast and there exists an infinite number of solution times.

We now use Theorem 2 (Baker's theorem) to provide bounds on sums of exponentials polynomials. We first start with some lemmata.

Lemma 12. Let ω_1 and ω_2 be different algebraic numbers, linearly independent over \mathbb{Q} , and $e^{i\phi_1}, e^{i\phi_2}$ be algebraic numbers on the unit circle. There exist effective constants C, N, T > 0 such that at any time instant t > T, either $1 - \cos(\omega_1 t + \phi_1) > C/t^N$ or $1 - \cos(\omega_2 t + \phi_2) > C/t^N$.

Proof. We prove that there are C, N such that $|\omega_1 t + \phi_1 - 2k\pi| > C/t^N$ for all integers k or $|\omega_2 t + \phi_2 - 2k\pi| > C/t^N$ for all integers k.

Indeed, suppose that for some t,k,l and $\epsilon>0$, both $|\omega_1t+\phi_1-2k\pi|\leq \epsilon$ and $|\omega_2t+\phi_2-2l\pi|\leq \epsilon$. Then $|t+\phi_1/\omega_1-2k\pi/\omega_1|\leq \epsilon/|\omega_1|$ and $|t+\phi_2/\omega_2-2l\pi/\omega_2|\leq \epsilon/|\omega_2|$. By difference we find $|\phi_1/\omega_1-\phi_2/\omega_2+2l\pi/\omega_2-2k\pi/\omega_1|<\epsilon(\frac{1}{|\omega_1|}+\frac{1}{|\omega_2|})$. Let us introduce $\omega=(\frac{1}{|\omega_1|}+\frac{1}{|\omega_2|})^{-1}$. Then $|k\frac{\omega}{i\omega_1}2\pi i-l\frac{\omega}{i\omega_2}2\pi i+\frac{\omega}{i\omega_1}i\phi_1-\frac{\omega}{i\omega_2}i\phi_2|<\epsilon$.

Observing that $2\pi i, i\phi_1, i\phi_2$ are logarithms of algebraic numbers, we apply Baker's theorem. Note that the height of $k\alpha$, for any algebraic number α of degree d, is at most $|k|^d$ times the height of α , from the definition of height. Thus, $\epsilon > \max \left(C_1|k|^d, C_2|l|^d, C_0\right)^{-N_0}$, for some C_0, C_1, C_2, N_0 not depending on k, l. It is also clear that, given $t, |\omega_1 t + \phi_1 - 2k\pi|$ is closest to zero for some $k < C_3 t$, and similarly for $|\omega_2 t + \phi_2 - 2l\pi|$.

This proves that there exists C', N_0, T such that for every t > T, either $|\omega_1 t + \phi_1 - 2k\pi| > C't^{-N_0}$ for all k, or $|\omega_2 t + \phi_2 - 2l\pi| > C't^{-N_0}$ for all l. Since $1 - \cos(\alpha + 2k\pi) > \alpha^2/3$ for α small enough and some k (from Taylor approximation), the claim follows.

We say that a property T-eventually holds for a function $g: \mathbb{R}_{\geq 0} \to \mathbb{R}$ if it holds for all time instants $t \geq T$. For instance, g is eventually positive if there is a threshold T such that g(t) > 0 for all $t \geq T$. Clearly, if f is the solution of a linear differential equation, then it has finitely many zeros if and only if it is eventually positive or eventually negative.

We say that g_1 is (T, r)-exponentially dominated by g_2 if $|g_1(t)| < e^{-rt}|g_2(t)|$ for r > 0 and all $t \ge T$.

Lemma 13. Let us consider T-eventually nonzero continuous functions g_1, \ldots, g_k and the function $f(t) = g_0(t) + \sum_{j=1}^k g_j(t) \cos(\omega_j t + \phi_j)$, where ω_1, ω_2 are linearly independent positive algebraic numbers, g_0 is (T, r)-exponentially dominated by g_1 and g_2 , and ϕ_1, ϕ_2 are angles such that $e^{i\phi_1}, e^{i\phi_2}$ are algebraic. Then the following T'-eventually holds, for some T':

$$-\sum_{j=1}^{k} |g_j| < f < \sum_{j=1}^{k} |g_j|.$$

Moreover, such T' can be computed as a function of $T, r, \omega_1, \omega_2, \phi_1, \phi_2$.

If all ω_i are linearly independent over the rationals, then for any $\epsilon > 0$, there exist arbitrarily large times t such that

$$f(t) > (1 - \epsilon) \sum_{j=1}^{k} |g_j(t)|$$

and arbitrarily large times t such that

$$f(t) < -(1 - \epsilon) \sum_{j=1}^{k} |g_j(t)|.$$

Proof. It is obvious that $-|g_0| - \sum_{j=1}^k |g_j| < f < |g_0| + \sum_{j=1}^k |g_j|$. To prove that we can get rid of g_0 , we exploit the fact that the cosines $\cos(\omega_1 t + \phi_1)$ and $\cos(\omega_2 t + \phi_2)$ never take the value ± 1 exactly at the same time, except possibly once; this is a consequence of the linear independence of ω_1 and ω_2 . Due to Lemma 12, one of the cosines is Ct^{-N} away from 1, for some C, N and all large enough times. Then for all large enough times t, either $|g_0 + g_1 \cos(\omega_1 t + \phi_1) + g_2 \cos(\omega_2 t + \phi_2)| < |g_1|(1 - Ct^{-N}) + |g_2|$ or $|g_0 + g_1 \cos(\omega_1 t + \phi_1) + g_2 \cos(\omega_2 t + \phi_2)| < |g_1| + |g_2|(1 - Ct^{-N})$. In any case, since g_0 is exponentially dominated by both g_1 and g_2 , we have $|g_0+g_1\cos(\omega_1 t+$ $|\phi_1| + |g_2| \cos(\omega_2 t + \phi_2)| < |g_1| + |g_2|$, for some T', computable as a function of T, r, C, N. Adding all the terms $g_i \cos(\omega_i t + \phi_i)$ proves the first claim of the theorem.

We now prove the second claim. From Kronecker's theorem and the linear independence of frequencies, we have that the set $\Gamma = \{(\cos(\omega_j t + \phi_j))_{1 \le j \le k} | t \ge 1\}$ 0) is dense in $[-1,1]^k$. Hence, Γ will approach all the vertices of $[-1,1]^k$ by less than any $\epsilon > 0$ for arbitrarily large times. For those times such that for all j, $\cos(\omega_j t + \phi_j)$ is close within $\epsilon/2$ to sign g_j , and $|g_0(t)/g_1| < \epsilon/2$, the first part of the second claim holds. The second part is similar.

We now prove the main theorem of this section, which says that in some circumstances, one can reduce the search for a solution to an instance of Con-TINUOUS SKOLEM PROBLEM to a *finite* time interval. Recall that an eigenvalue is nondefective if its algebraic and geometric multiplicities coincide. The frequency of an eigenvalue is the absolute value of the imaginary part. Recall that for a real matrix, complex eigenvalues come in conjugate pairs, determining one equal frequency.

Theorem 14. Given an instance of Continuous Skolem Problem where all dominant eigenvalues are nondefective, at least four in number and such that the set of their distinct nonzero frequencies is linearly independent over the rationals.

Then

• The existence of infinitely many solutions is decidable;

• If there are finitely many solutions, then those solutions are in [0, T], where T is computable.

Note that multiple dominant eigenvalues are allowed.

Proof. As allowed by Lemma 7, we can suppose without loss of generality that the dominant eigenvalues are on the imaginary axis.

Then we are looking for real zeros of a function $f(t) = \gamma_0 + f_1(t) + f_2(t)$, where γ_0 is the contribution of the dominant zero eigenvalue (if any),

$$f_1(t) = \sum_{j=1}^k z_j \exp(i\lambda_j t)$$
$$= \sum_{j=1}^k \alpha_j \cos(\lambda_j t) + \beta_j \sin(\lambda_j t)$$

collects the dominant terms corresponding to dominant complex eigenvalues $\theta_j = i\lambda_j$ and $f_2(t)$ is exponentially decreasing. By elementary trigonometric manipulations, f_1 can be converted into

$$f_1(t) = \sum_{j=1}^k \gamma_j \cos(\lambda_j t + \phi_j),$$

for some ϕ_j such that $\exp{(i\phi_j)}$ is algebraic. Hence f_1 is a linear combination of shifted cosines.

Since there are at least four distinct dominant eigenvalues, f_1 contains at least two different frequencies. We apply Lemma 13, with $g_i = \gamma_i$, $g_0 = f_2$ to obtain that the following eventually holds:

$$-\sum_{j=1}^{k} |\gamma_j| < f - \gamma_0 < \sum_{j=1}^{k} |\gamma_j|.$$

Moreover, the same lemma tells us that for any $\epsilon > 0$ there are arbitrarily large times t such that:

$$-(1-\epsilon)\sum_{j=1}^{k} |\gamma_j| \ge f(t) - \gamma_0$$

and arbitrarily large times t such that:

$$f(t) - \gamma_0 \ge (1 - \epsilon) \sum_{j=1}^k |\gamma_j|.$$

As mentioned above, f has finitely many zeros if and only if f is eventually positive or eventually negative. It results from the above that f < 0

(*T*-eventually, for some *T*) if and only if $\gamma_0 + \sum_{j=1}^k |\gamma_j| \le 0$. Moreover, when f < 0 (*T*-eventually, for some *T*), such a *T* can be computed. A similar argument holds for f > 0. This proves the claim.

Note that in discrete time, checking the existence of a zero in a finite time interval is a trivial task, while in continuous time we do not know how to decide the existence of a zero between time 0 and T.

5. NP-Hardness of Nonnegativity Problem

We now prove the continuous version of Blondel-Portier's result [6].

Theorem 15. The nonnegativity problem for instances of CONTINUOUS SKOLEM PROBLEM given by a skew-symmetric matrix is NP-hard and decidable in exponential time. In particular, the general nonnegativity problem is NP-hard.

Proof. A skew symmetric matrix has only imaginary eigenvalues and Jordan blocks of size one. By Theorem 6 we must find nonnegative real zeros of a function of the form

$$f(t) = \sum_{i} \alpha_{i} \cos(\lambda_{i} t) + \beta_{i} \sin(\lambda_{i} t).$$

We can, in polynomial time, find a basis ξ_1, \ldots, ξ_m over the rationals for the family $\lambda_1, \ldots, \lambda_k$, such that every λ_i is an integral combination of ξ_1, \ldots, ξ_m . For every ξ_i we introduce two variables $x_i = \cos(\xi_i t)$ and $y_i = \sin(\xi_i t)$, which satisfy $x_i^2 + y_i^2 = 1$. Hence f(t) is a polynomial P in x_i, y_i (by elementary trigonometry). From Theorem 4 (Kronecker's theorem), the trajectory $(\xi_1 t, \ldots, \xi_k t)$ is dense in $[0, 2\pi]^k$, from which $\overline{\{f(t)|t \in \mathbb{R}\}} = \{P(x_1, y_1, \ldots, x_m, y_m)|x_i, y_i \in \mathbb{R}\}$ follows. Hence, f is nonnegative if and only if P is, when taken over the set $\{x_1, y_1, \ldots, x_m, y_m | x_j, y_j \in \mathbb{R} \text{ and } x_j^2 + y_j^2 = 1 \text{ for } 1 \leq j \leq m\}$. This problem is solvable in time exponential in m by Tarski's procedure (see for example [4]).

Suppose we are given a polynomial $P(x_1,\ldots,x_k)$. We write $x_i=\cos(\xi_i t)$ for every $1\leq i\leq k$. Every monomial of P can therefore be written as a linear combination of cosines by elementary trigonometry. For instance, $x_1x_2=\cos\xi_1 t\cos\xi_2 t=\frac{\cos(\xi_1-\xi_2)t+\cos(\xi_1+\xi_2)t}{2}$, and so on. In this way, the polynomial $P(x_1,\ldots,x_k)$ can be written as a function $f(t)=\sum_i\alpha_i\cos(\lambda_i t)$, such that $\overline{\{f(t)|t\in\mathbb{R}\}}=\{P(x_1,\ldots,x_k)|x_i\in[-1,1]\}$. Hence f is nonnegative if and only if P is nonnegative on $[-1,1]^k$. Since checking the nonnegativity of a polynomial on $[-1,1]^k$ is NP-hard (which is easily proved via an encoding of the 3-SAT problem, see, e.g., [10]), then the nonnegativity problem for instances of Continuous Skolem Problem is also NP-hard.

Note that physical linear systems that preserve energy can often be modelled by differential equations with a skew-symmetric matrix, because these are precisely, up to a change of variables, the systems for which the energy $1/2x^Tx$ (where x is the state) is constant along the trajectories, (see, e.g., [29]). This case is therefore particularly relevant.

6. Conclusion

In studying this problem, we are not so much interested in exactly describing the solutions to the problem, as determining the *existence* of solutions. For example, if we have algebraic times $t_1, t_2 \in \mathbb{R}_{\geq 0}$ with $t_1 < t_2$ such that $f(t_1)$ and $f(t_2)$ have different signs then there exists $t \in [t_1, t_2]$ such that f(t) = 0 by the intermediate value theorem.

The main problem encountered in solving Continuous Skolem Problem however appears to be that f(t) can reach 0 tangentially, i.e. we may have a solution f(t)=0 where there exists $\varepsilon>0$ such that $f(\tau)\geq 0$ for all $\tau\in [t-\varepsilon,t+\varepsilon]$. Since, by Lemma 8, the solution will, for non trivial cases, be transcendental, it is difficult to determine when such a situation arises. Indeed, given a real valued exponential polynomial, if we take its square then it is positive real valued and reaches zero tangentially if and only if the first exponential polynomial had a zero.

We have therefore attempted to show several instances in which the problem is decidable but the general problem remains open. The equivalent problem of determining if an exponential polynomial has real zeros seems equally interesting. It is surprising that the problem is open even for a finite time interval. Solving Skolem's problem in the discrete case for finite time is obviously decidable since we can simply compute the values in the interval.

Open Problem 16. Is BOUNDED CONTINUOUS SKOLEM'S PROBLEM decidable? I.e. Given a fixed $T \in \mathbb{R}_{\geq 0}$, and an instance of CONTINUOUS SKOLEM PROBLEM, $f(t) = c^T \exp(At)x(0)$, does there exist $t \leq T$ such that f(t) = 0?

We also showed that the nonnegativity problem is NP-hard in the continuous case. It is not clear if a similar technique can be used to show that CONTINUOUS SKOLEM PROBLEM is also NP-hard. In the discrete Skolem's problem it turns out that determining the nonnegativity and positivity of a linear recurrent sequence are equivalent in terms of complexity, however this is not clear in the continuous case.

Open Problem 17. Are Continuous Skolem Problem and the continuous nonnegativity problem computationally equivalent?

7. Acknowledgements

This article presents research results of the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. This research has been also supported by the ARC (Concerted Research Action) "Large Graphs and Networks", of the French Community of Belgium. The scientific responsibility rests with the authors. J.-C. Delvenne and R. Jungers hold FNRS fellowships (Belgian Fund for Scientific Research).

We would like to thank Alexandre Megretski for a useful discussion on this problem. We also would like to thank the two reviewers for their careful reading of this manuscript and helpful comments and suggestions.

References

- [1] T. Apostol, *Liouville's Approximation Theorem*, in: Modular Functions and Dirichlet Series in Number Theory, 2nd ed. New York: Springer-Verlag, pp. 146-148, 1997.
- [2] L. Babai, R. Beals, J. Cai, G. Ivanyos and E. M. Luks, Multiplicative Equations over Commuting Matrices, Proc. 7th ACM-SIAM Symp. on Discrete Algorithms, pp. 498-507, 1996.
- [3] A. Baker, Linear Forms in the Logarithms of Algebraic Numbers I-IV, Mathematika 13 (1966), 204-216; 14 (1967), 102-107 et 220-224; 15 (1968), 204-216.
- [4] S. Basu, R. Pollack, M. Roy, On the Combinatorial and Algebraic Complexity of Quantifier Elimination. J. ACM 43, 6, 1002-1045, 1996.
- [5] J. Berstel, M. Mignotte, Deux Propriétés Décidables des Suites Récurrentes Linéaires, Bull. Soc. Math. France, 104, 175-184, 1976.
- [6] V. Blondel, N. Portier, The Presence of a Zero in an Integer Linear Recurrent Sequence is NP-hard to Decide, Linear Algebra and its Applications, Elsevier, 351-352, pp. 91-98, 2002.
- [7] V. D. Blondel, J. N. Tsitsiklis, A Survey of Computational Complexity Results in Systems and Control, Automatica, Elsevier, 36:9, pp. 1249-1274, 2000.
- [8] J. Cassels, An Introduction to Diophantine Approximation, Cambridge Univ. Press, 1957.
- [9] G. Everest, A. van der Poorten, I. Shparlinski, T. Ward, Recurrence Sequences, AMS Surveys and Monographs, Volume 104, ISBN 0-8218-3387-1, 2003.
- [10] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, New York: W.H. Freeman. ISBN 0-7167-1045-5, 1979.
- [11] E. Hainry, *Reachability in Linear Dynamical Systems*, Computability in Europe 2008, Lecture Notes in Computer Science, LNCS 5028, Springer, 2008.
- [12] V. Halava, T. Harju, M. Hirvensalo, Positivity of Second Order Linear Recurrent Sequences, Discrete Applied Math. 154, Elsevier, pp. 447-451, 2006.
- [13] V. Halava, T. Harju, M. Hirvensalo, J. Karhumäki, Skolem's Problem On the Border between Decidability and Undecidability, TUCS Technical Report Number 683, 2005.

- [14] V. Halava, M. Hirvensalo, Improved Matrix Pair Undecidability Results, Acta Informatica 44, 3:191-205, 2007.
- [15] M. Kac, On the Distribution of Values of Trigonometric Polynomials with Linearly Independent Frequencies, Am. J. Math., 65, 609-615, 1943.
- [16] R. Kannan, R. Lipton, Polynomial-Time Algorithm for the Orbit Problem, Journal of the Association for Computing Machinery, Volume 33, Issue 4, 808 - 821, 1986.
- [17] C. Lech A Note on Recurring Sequences Ark. Mat. 2, 417-421, 1953.
- [18] K. Mahler, Eine Arithmetische Eigenschaft der Taylor-Koeffizienten Rationaler Funktionen, Proc. Akad. Wet., Amsterdam 38, 50-60, 1935.
- [19] C. J. Moreno, The Zeros of Exponential Polynomials, Comp. Math., vol. 26, 69-78, 1973.
- [20] J. D. Nulton, K. B. Stolarsky, The First Sign Change of a Cosine Polynomial, Proc. of the Amer. Math. Soc., 84, No. 1, 55-59, 1982.
- [21] G. Pólya, Geometrisches über die Verteilung der Nullstellen gewisser ganzer Transzendenter Funktionen, Münch. Sitzungsber., 50, 285-290, 1920.
- [22] E. Schwengler, Geometrisches über die Verteilung der Nullstellen Spezieller Ganzer Funktionen, Dissertation, ETH Zürich, 1925.
- [23] T. Skolem, Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen, 8. Skand. Mat. Kongr., Stockholm, 163-188, 1934.
- [24] J. Tamarkin, Some General Problems of the Theory of Ordinary Linear Differential Equations and Expansion of an Arbitrary Function in Series of Fundamental Functions, Mathematische Zeitschrift, 27, Num. 1, 1-54, 1925.
- [25] A. J. van der Poorten, A Note on the Zeros of Exponential Polynomials, Compositio Mathematica, 31 no. 2, 109-113, 1975.
- [26] N. Vereshchagin, Occurrence of Zero in a Linear Recursive Sequence, Math. Notes, 38, 2:609-615, 1985.
- [27] M. Voorhoeve, On the Oscillation of Exponential Polynomials Mat. Z., 151, No. 3, 217-293, 1976.
- [28] C. E. Wilder, Expansion Problems of Ordinary Linear Differential Equations with Auxiliary Conditions at More than Two Points, Trans. of Math. Soc., 18, 415-442, 1917.
- [29] J.C. Willems, Dissipative Dynamical Systems Part II: Linear Systems with Quadratic Supply Rates, Archive for Rational Mechanics and Analysis Vol. 45, 352-393, 1972.