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Edge bending wave on a thin elastic plate resting on a Winkler foundation

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This paper is concerned with elucidation of the general properties of the bending edge wave in a thin linearly elastic plate which is supported by a Winkler foundation. A homogeneous wave of arbitrary profile is considered, and represented in terms of a single harmonic function. This serves as the basis for derivation of an explicit asymptotic model, containing an elliptic equation governing the decay away from the edge, together with a parabolic equation at the edge, corresponding to beam-like behaviour. The model extracts the contribution of the edge wave from the overall dynamic response of the plate, providing significant simplification for analysis of the localised near-edge wave field.

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1. Introduction

The bending Rayleigh-type edge wave has been studied for more than half a century, and has a remarkable history. Konenkov [1] was the first to consider such a wave and derive the associated dispersion relation. However, some initial ideas giving preliminary insight within the framework of stability of elastic plates, had in fact appeared in the earlier work of Ishlinsky [2]. Unfortunately, the contribution of Konenkov [1] was seemingly unnoticed; indeed after fourteen years the bending edge wave was rediscovered independently in [3] and [4]. Some further historical details, and periodic overviews of the state of art, may be found in [5] and in the review [6]. Among other recent contributions on the subject we mention [7]-[10].

It is intuitively clear that edge waves are dispersive analogues of better known surface waves. However, the number of contributions investigating the general properties of the bending edge wave is considerably less. Construction of an appropriate mathematical theory for the Rayleigh wave resulted in the general representation of the wave field in terms of harmonic functions, originated by Friedlander [11] and followed by Chadwick [12]. Recent developments of this approach, including generalisations to anisotropy, laterally dependent surface waves and three-dimensional surface and interfacial waves of arbitrary profile and direction, may be found in [13]-[18]. Alternative approaches to the general description of surface waves were presented in [19] and [20].

A step forward in mathematical modelling of the Rayleigh wave was provided by Kaplunov et al. [21]. In this study a slow time perturbation of the solution previously obtained by Chadwick [12] allowed treatment of non-homogeneous transient boundary conditions. The derived explicit model for the Rayleigh wave uncouples the contribution of the surface wave from the overall dynamic response. In addition to simple approximate formulations for the wave field, some fundamental features were noticed, for example the dual hyperbolic-elliptic nature of the Rayleigh wave. Similar observations have been made in [16] for Schölte waves of arbitrary profile and direction.

The approach of [21] has been extended to the bending edge wave on a free Kirchhoff plate, with some results presented in [22], revealing a parabolic-elliptic formulation for the wave field. Other recent developments in the area of edge waves in elastic plates are related to plates supported by elastic foundations, see [9,10].

In the present paper we extend previous results to the bending edge wave on a plate supported by a Winkler foundation. Our aim is two-fold, including establishment of the general time dependent solution as well as development of a specialized formulation for non-homogeneous edge boundary conditions.

The key to construction of the bending edge wave eigensolution of arbitrary profile is the natural, though not readily straightforward, idea of an effective beam on an elastic foundation acting as a "basic object" for the wave. Some physically intuitive reasoning for this kind of assumption, presented in [22], is clarified in this paper. In particular, it is shown that the promoted assumption of the beam-like behaviour allows construction of the wave field in terms of an arbitrary single harmonic function, actually generalising the known bending edge wave of a sinusoidal profile.

The derived eigensolution of arbitrary shape is then perturbed in slow time, providing a path to an explicit model for the bending edge wave, approximating the wave field. The results contain a parabolic beam-like equation on the edge and an elliptic equation over the interior, governing decay away from the edge. Thus, the solution can be found in the form of a plane harmonic function in spatial coordinates, satisfying the parabolic equation on the edge.

The model is first derived for a particular type of loading evolving in slow time and corresponding to near-resonant behaviour. However, it is shown that the formulation always enables evaluation of the contribution of the bending edge wave to the overall dynamic response. This is demonstrated by calculation of the related residues, presented in the Appendix. Thus, we may expect the model to provide leading order approximation in the far-field near-edge zone for an arbitrary load, similar to the Rayleigh wave. The proposed approach leads to insightful general observations, for example it underlines the dual parabolic-elliptic nature of the dispersive bending edge wave on an elastically supported plate contrasting with the hyperbolic-elliptic nature of the Rayleigh wave.

The paper is organized as follows. A general review of the governing equations and statement of the problem are presented in §2. Bending edge wave of general time dependence is analysed in §3, with the representation of the wave field in terms of a single harmonic function obtained. A multi-scale approach, using slow time perturbation of this solution, is performed in §4, and an explicit model for the bending edge wave is formulated in §5. Both cases of boundary conditions, namely, the prescribed bending moment and shear force, are investigated. In §6 we consider a model example, illustrating the proposed approach and focussing attention on the near-resonant excitation of the bending edge wave by a harmonic moment imposed at the edge.

2. Statement of the problem

Consider a semi-infinite isotropic elastic plate of thickness $2h$, supported by a Winkler foundation, see Figure 1. The plate occupies the region $-\infty < x_1 < \infty, \quad 0 \le x_2 < \infty, \quad 0 \le x_3 \le 2h$. Within

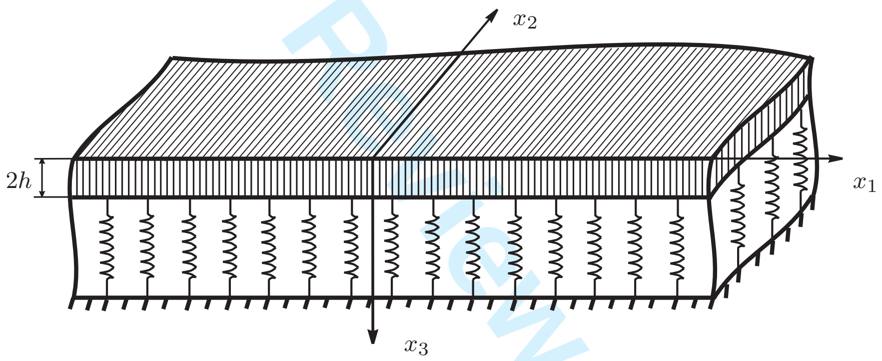

Figure 1. An elastic plate on the Winkler foundation.

the framework of the classical Kirchhoff theory, the approximate 2D equation of bending of an elastic plate, supported by a Winkler foundation, is given by (see e.g. [23])

$$D\Delta^2 W + 2\rho h \frac{\partial^2 W}{\partial t^2} + \beta W = 0, \tag{2.1}$$

where $W(x_1, x_2, t)$ denotes the deflection of the plate, Δ is a 2D Laplace operator in variables x_1 and x_2, β is the Winkler foundation modulus, ρ denotes the volume mass density, D is the bending stiffness given by

$$D = \frac{2Eh^3}{3(1-\nu^2)},$$

and E and ν are the Young's modulus and Poisson's ratio, respectively.

The boundary conditions at the edge $x_2 = 0$ are adopted in the form

$$\begin{aligned} &\frac{\partial^2 W}{\partial x_2^2} + \nu \frac{\partial^2 W}{\partial x_1^2} = -\frac{M_0}{D}, \\ &\frac{\partial^3 W}{\partial x_2^3} + (2-\nu) \frac{\partial^3 W}{\partial x_1^2 \partial x_2} = -\frac{N_0}{D}, \end{aligned} \tag{2.2}$$

where $M_0 = M_0(x_1, t)$ and $N_0 = N_0(x_1, t)$ are the prescribed bending moment and shear force, respectively. We also impose the two initial conditions

$$W(x_1, x_2, 0) = A(x_1, x_2), \qquad \frac{\partial W(x_1, x_2, 0)}{\partial t} = B(x_1, x_2), \tag{2.3}$$

where A and B are given initial data.

3. Homogeneous wave of arbitrary profile

The dispersion relation for the free bending edge wave associated with a Kirchhoff plate resting on the Winkler foundation has the form

$$Dk^4 \gamma_e^4 + \beta = 2\rho h \omega^2, \tag{3.1}$$

see [9], where k and ω denote wave number and frequency, respectively, and the coefficient

$$\gamma_e = \left[(1-\nu)\left(3\nu - 1 + 2\sqrt{2\nu^2 - 2\nu + 1}\right)\right]^{1/4}, \tag{3.2}$$

depending on the Poisson's ratio only, is the material constant introduced in the paper of Konenkov [1]. This parameter lies within the region $0 < \gamma_e < 1$, see Figure 2.

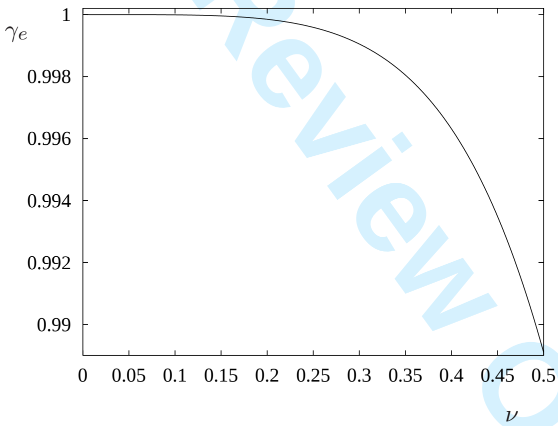

Figure 2. The coefficient γ_e vs. the Poisson's ratio ν.

It has been shown in the aforementioned contribution of Kaplunov *et al.* [9] that the presence of the Winkler foundation brings in a few novel features, in particular leading to a cut-off-frequency, along with a local minimum of the phase velocity. This minimal phase speed coincides with the group velocity and corresponds to the critical speed of an edge moving load.

Still the eigensolution of a sinusoidal profile analysed in [9] is not general enough. Following Chadwick [12], we now derive the sought for ansatz of a homogeneous edge wave of arbitrary shape, presenting the associated wave field in terms of a single plane harmonic function.

We begin by rewriting the plate equation (2.1) in terms of the dimensionless variables

$$\xi = \frac{x_1}{h}, \qquad \eta = \frac{x_2}{h}, \qquad \tau = \frac{t}{\alpha h}, \tag{3.3}$$

with

$$\alpha = \left[3\rho(1-\nu^2)/E\right]^{1/2}, \tag{3.4}$$

giving

$$\Delta^2 W + \frac{\partial^2 W}{\partial \tau^2} + \beta_0 W = 0, \tag{3.5}$$

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where $\varDelta$ now denotes the 2D Laplace operator in the dimensionless variables ξ and η, and $\beta_0 = \beta h^4/D$. Next we proceed with a serendipitous assumption

$$\gamma_e^4 \frac{\partial^4 W}{\partial \xi^4} + \frac{\partial^2 W}{\partial \tau^2} + \beta_0 W = 0, \tag{3.6}$$

where γ_e is defined after (3.2). This is actually the key to generalising the conventional sinusoidal profile

$$W = V(\eta) \exp\left[ih\left(k\xi - \alpha\omega\tau\right)\right], \tag{3.7}$$

see [9].

First of all, it is clear that this sinusoidal solution satisfies (3.6), actually giving the dispersion relation (3.1). On the other hand, intuition suggests an elastically supported beam as a basic object for our plate bending edge wave, in line with a similar analogy between the Rayleigh wave and a string, which was pointed out in [24]. Unfortunately, in contrast to the Rayleigh wave [12], in respect of the edge bending edge wave there is seemingly no explicit functionally invariant solution.

In view of (3.6), equation (3.5) takes the form

$$\left(1 - \gamma_e^4\right) \frac{\partial^4 W}{\partial \xi^4} + 2\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 W}{\partial \eta^4} = 0, \tag{3.8}$$

which may be rewritten in the operator form

$$\varDelta_1 \varDelta_2 W = 0, \tag{3.9}$$

where

$$\varDelta_j = \frac{\partial^2}{\partial \eta^2} + \lambda_j^2 \frac{\partial^2}{\partial \xi^2}, \qquad j = 1, 2 \tag{3.10}$$

and

$$\lambda_j^2 = 1 + (-1)^j \gamma_e^2. \tag{3.11}$$

It should be noted that since $0 < \gamma_e < 1$, λ_j^2 are both positive, and thus equation (3.9) is elliptic. The solution is therefore given by the sum of two arbitrary functions, harmonic in the first two variables, accordingly

$$W = \sum_{j=1}^{2} W_j\left(\xi, \lambda_j \eta, \tau\right). \tag{3.12}$$

Substituting (3.12) into the homogeneous edge boundary conditions (2.2) (with $M_0 = N_0 = 0$), rewritten in terms of the dimensionless variables and employing the Cauchy-Riemann identities (see e.g. [25]), we obtain

$$\begin{gathered}
\left(\nu - \lambda_1^2\right) \frac{\partial^2 W_1}{\partial \xi^2} + \left(\nu - \lambda_2^2\right) \frac{\partial^2 W_2}{\partial \xi^2} = 0, \\
\lambda_1 \left(\lambda_1^2 - 2 + \nu\right) \frac{\partial^3 W_1^*}{\partial \xi^3} + \lambda_2 \left(\lambda_2^2 - 2 + \nu\right) \frac{\partial^3 W_2^*}{\partial \xi^3} = 0,
\end{gathered} \tag{3.13}$$

with the asterisk denoting a harmonic conjugate. These boundary conditions imply

$$\lambda_2(\nu - \lambda_1^2)^2 - \lambda_1(\nu - \lambda_2^2)^2 = 0. \tag{3.14}$$

Due to (3.11), the last equation may be re-cast in the form

$$\lambda_1^2 \lambda_2^2 + 2(1 - \nu)\lambda_1 \lambda_2 - \nu^2 = 0, \tag{3.15}$$

which coincides with the dispersion relation (3.1), see also [9].

The representation for the bending edge wave field in terms of a single harmonic function may now be established by making use of the boundary conditions (3.13), giving

$$W\left(x,y,t\right)=W_j\left(x,\lambda_j y,t\right)-\frac{\nu-\lambda_j^2}{\nu-\lambda_m^2}W_j\left(x,\lambda_m y,t\right),\qquad 1\le j\ne m\le 2. \tag{3.16}$$

The deflection W may thus be expressed through (3.16) as a solution of the following initial value problem for any of the harmonic functions W_j $(j=1,2)$

$$\frac{\partial^2 W_j}{\partial\eta^2}+\lambda_j^2\frac{\partial^2 W_j}{\partial\xi^2}=0, \tag{3.17}$$

demonstrating beam-like behaviour, see (3.6),

$$\gamma_e^4\frac{\partial^4 W_j}{\partial\xi^4}+\frac{\partial^2 W_j}{\partial\tau^2}+\beta_0 W_j=0, \tag{3.18}$$

and satisfying initial conditions

$$W_j\big|_{\tau=0}=A_j(\xi,\lambda_j\eta),\qquad \frac{\partial W_j}{\partial\tau}\bigg|_{\tau=0}=B_j(\xi,\lambda_j\eta). \tag{3.19}$$

It is clear from (2.3) and (3.16) that

$$\begin{aligned}A\left(x,y\right)&=A_j\left(\xi,\lambda_j\eta\right)-\frac{\nu-\lambda_j^2}{\nu-\lambda_m^2}A_j\left(\xi,\lambda_j\eta\right),\\ B\left(x,y\right)&=B_j\left(\xi,\lambda_j\eta\right)-\frac{\nu-\lambda_j^2}{\nu-\lambda_m^2}B_j\left(\xi,\lambda_j\eta\right),\end{aligned} \tag{3.20}$$

with $1\le j\ne m\le 2$. Note that (3.17) necessitates that A_j and B_j are harmonic functions.

Applying the integral Fourier transform with respect to the variable ξ to the elliptic equations (3.17), and imposing the decay conditions $W_j\to 0$ as $\eta\to\infty$, it is possible to obtain for the Fourier transforms

$$W_j^F=f_j(s,\tau)e^{-\lambda_j|s|\eta}, \tag{3.21}$$

with the initial conditions for the functions f_j given by

$$f_j\big|_{\tau=0}=a_j(s),\qquad \frac{\partial f_j}{\partial\tau}\bigg|_{\tau=0}=b_j(s). \tag{3.22}$$

In view of (3.18), the solution for the functions W_j may be expressed as

$$W_j=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{b_j(s)}{\alpha_s}\sin\left(\alpha_s\tau\right)+a_j(s)\cos\left(\alpha_s\tau\right)\right]e^{-\lambda_j|s|\eta+i\xi s}ds, \tag{3.23}$$

where $\alpha_s=\sqrt{\gamma_e^4 s^4+\beta_0}$, with the resulting bending edge wave field given by (3.16).

4. Perturbation scheme

Once representation in terms of a single plane harmonic function is established, we proceed with the development of an explicit model for the bending edge wave. Our intention is to extract the edge wave contribution from the overall dynamic response in a similar manner to [21] for the surface waves. In parallel with the cited paper, our starting point is a multiple scale procedure, perturbing equation (2.1) around the eigensolution constructed in the previous section. Accordingly, fast $(\tau_f=\tau)$ and slow $(\tau_s=\varepsilon\tau)$ time variables are introduced, where $\varepsilon\ll 1$ is a small parameter, indicating the underlying assumption that the deviation of the phase speed from that of the homogeneous edge wave is small. An example of a near-resonant motion evolving in slow time $(\tau_s=\varepsilon\tau)$ is considered later in §6.

We begin the perturbation procedure by representing equation (3.5) in terms of the introduced scaling as

$$\Delta^2 W + \left(\frac{\partial^2 W}{\partial \tau_f^2} + 2\varepsilon \frac{\partial^2 W}{\partial \tau_f \partial \tau_s} + \varepsilon^2 \frac{\partial^2 W}{\partial \tau_s^2}\right) + \beta_0 W = 0. \tag{4.1}$$

The deflection W is then expanded as

$$W = \frac{h^2 P}{\varepsilon D}\left(W^{(0)} + \varepsilon W^{(1)} + ...\right), \tag{4.2}$$

where

$$P = \max_{x,t}\left[M_0(x,t), hN_0(x,t)\right]. \tag{4.3}$$

We now substitute expansion (4.2) into the governing equation (4.1), which at leading order reveals

$$\Delta^2 W^{(0)} + \frac{\partial^2 W^{(0)}}{\partial \tau_f^2} + \beta_0 W^{(0)} = 0, \tag{4.4}$$

which may be transformed to the elliptic equation

$$\left(1-\gamma_e^4\right)\frac{\partial^4 W^{(0)}}{\partial \xi^4} + 2\frac{\partial^4 W^{(0)}}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 W^{(0)}}{\partial \eta^4} = 0, \tag{4.5}$$

through use of assumption (3.6). The solution of (4.5) is then given by a combination of harmonic functions, yielding

$$W^{(0)} = \sum_{j=1}^{2} W_j^{(0)}\left(\xi, \lambda_j \eta, \tau_f, \tau_s\right), \tag{4.6}$$

where the scaling factors λ_j $(j = 1, 2)$ are defined by (3.11) assuming decay as $\eta \to \infty$.

At the next order, we obtain from (4.1) that

$$\Delta^2 W^{(1)} + \frac{\partial^2 W^{(1)}}{\partial \tau_f^2} + \beta_0 W^{(1)} + 2\frac{\partial^2 W^{(0)}}{\partial \tau_f \partial \tau_s} = 0. \tag{4.7}$$

Using assumption (3.6), along with the superposition principle, equation (4.7) may be re-written as

$$\Delta_1 \Delta_2 W_j^{(1)} = -2\frac{\partial^2 W_j^{(0)}}{\partial \tau_f \partial \tau_s} \qquad (j = 1, 2), \tag{4.8}$$

with $W^{(1)} = W_1^{(1)} + W_2^{(1)}$. Let us first consider $j = 1$. Employing the properties of harmonic functions, one may deduce that

$$\Delta_2 W_1^{(0)} = (\lambda_2^2 - \lambda_1^2)\frac{\partial^2 W_1^{(0)}}{\partial \xi^2} = 2\gamma_e^2 \frac{\partial^2 W_1^{(0)}}{\partial \xi^2}. \tag{4.9}$$

Differentiating (4.8) twice with respect to ξ, and using (4.9), we infer that

$$\Delta_1 \Delta_2 \frac{\partial^2 W_1^{(1)}}{\partial \xi^2} = -\frac{1}{\gamma_e^2}\Delta_2 \frac{\partial^2 W_1^{(0)}}{\partial \tau_f \partial \tau_s}, \tag{4.10}$$

from which it is readily deduced that

$$\Delta_1 \frac{\partial^3 W_1^{(1)}}{\partial \xi^2 \partial \eta} = -\frac{1}{\gamma_e^2}\frac{\partial^3 W_1^{(0)}}{\partial \tau_f \partial \tau_s \partial \eta}. \tag{4.11}$$

The solution of (4.10) is found in the form

$$\frac{\partial^3 W_1^{(1)}}{\partial \xi^2 \partial \eta} = \frac{\partial^3 \Phi_1^{(1,0)}}{\partial \xi^2 \partial \eta} - \frac{\eta}{2\gamma_e^2}\frac{\partial^2 W_1^{(0)}}{\partial \tau_f \partial \tau_s}, \tag{4.12}$$

where $\Phi_1 = \Phi_1\left(\xi, \lambda_1 \eta, \tau_f, \tau_s\right)$ is an arbitrary harmonic function of the first two arguments.

Similar consideration for $j=2$ yields

$$\Delta_1 W_2^{(0)} = -2\gamma_e^2 \frac{\partial^2 W_2^{(0)}}{\partial \xi^2}, \tag{4.13}$$

leading to

$$\Delta_2 \frac{\partial^3 W_2^{(1)}}{\partial \xi^2 \partial \eta} = \frac{1}{\gamma_e^2} \frac{\partial^3 W_2^{(0)}}{\partial \tau_f \partial \tau_s \partial \eta}, \tag{4.14}$$

hence

$$\frac{\partial^3 W_2^{(1)}}{\partial \xi^2 \partial \eta} = \frac{\partial^3 \Phi_2^{(1,0)}}{\partial \xi^2 \partial \eta} + \frac{\eta}{2\gamma_e^2} \frac{\partial^2 W_2^{(0)}}{\partial \tau_f \partial \tau_s}, \tag{4.15}$$

where $\Phi_2 = \Phi_2\left(\xi, \lambda_2 \eta, \tau_f, \tau_s\right)$ is also an arbitrary harmonic function.

We may thus obtain the following two-term asymptotic expansion for the third derivative

$$\begin{aligned}\frac{\partial^3 W}{\partial \xi^2 \partial \eta} = \frac{h^2 P}{D} \Bigg[& \varepsilon^{-1} \left(\frac{\partial^3 W_1^{(0)}}{\partial \xi^2 \partial \eta} + \frac{\partial^3 W_2^{(0)}}{\partial \xi^2 \partial \eta} \right) + \frac{\partial^3 \Phi_1^{(1,0)}}{\partial \xi^2 \partial \eta} \\ & + \frac{\partial^3 \Phi_2^{(1,0)}}{\partial \xi^2 \partial \eta} - \frac{\eta}{2\gamma_e^2} \left(\frac{\partial^2 W_1^{(0)}}{\partial \tau_s \partial \tau_f} - \frac{\partial^2 W_2^{(0)}}{\partial \tau_s \partial \tau_f} \right) + \ldots \Bigg].\end{aligned} \tag{4.16}$$

5. Parabolic equation on the edge

We proceed further with the analysis of the non-homogeneous edge boundary conditions (2.2). Due to the linearity of the problem, it may be decomposed into two separate problems, imposing a prescribed edge bending moment or shear force respectively.

(a) Bending moment

Consider the case of an edge bending moment, that is when $N_0 = 0, M_0 \neq 0$. The boundary conditions (2.2) are rewritten in terms of dimensionless variables at $\eta = 0$ as

$$\begin{aligned}\frac{\partial^2 W}{\partial \eta^2} + \nu \frac{\partial^2 W}{\partial \xi^2} &= -\frac{h^2}{D} M_0, \\ \frac{\partial^3 W}{\partial \eta^3} + (2-\nu) \frac{\partial^3 W}{\partial \xi^2 \partial \eta} &= 0.\end{aligned} \tag{5.1}$$

Let the bending moment in the right hand side of (5.1) evolve in slow time as

$$M_0(\xi, \tau_f, \tau_s) = \frac{\partial^2 m_0}{\partial \tau_s \partial \tau_f}, \tag{5.2}$$

with $m_0 = m_0(\xi, \tau_f, \tau_s)$ satisfying the beam-like assumption, i. e.

$$\gamma^4 \frac{\partial^4 m_0}{\partial \xi^4} + \frac{\partial^2 m_0}{\partial \tau_f^2} + \beta_0 m_0 = 0. \tag{5.3}$$

The simplest example of such behavior is provided by

$$m_0 = B \exp\left[ih\left(k\xi + \alpha\left(\omega_0 \tau_f + \omega_1 \tau_s\right)\right)\right], \tag{5.4}$$

where B is a constant related to the amplitude, α is defined by (3.4), ω_0 is an eigenfrequency satisfying the dispersion relation (3.1), and ω_1 is a perturbation term, as we shall see later in §6.

Substituting the asymptotic expansion (4.16) into the (5.1), we obtain at leading order

$$\begin{aligned} &\left(\nu-\lambda_1^2\right)\frac{\partial^2 W_1^{(0)}}{\partial\xi^2}+\left(\nu-\lambda_2^2\right)\frac{\partial^2 W_2^{(0)}}{\partial\xi^2}=0,\\ &\lambda_1\left(\lambda_1^2-2+\nu\right)\frac{\partial^3 W_1^{(0)}}{\partial\xi^3}+\lambda_2\left(\lambda_2^2-2+\nu\right)\frac{\partial^3 W_2^{(0)}}{\partial\xi^3}=0, \end{aligned}\tag{5.5}$$

which is very similar to (3.13) and, hence, leads to the dispersion relation (3.15).

At the next order, the boundary conditions (5.1) yield

$$\begin{aligned} &\frac{\partial^2 W^{(1)}}{\partial\eta^2}+\nu\frac{\partial^2 W^{(1)}}{\partial\xi^2}=-\frac{M_0}{P},\\ &\frac{\partial^3 W^{(1)}}{\partial\eta^3}+(2-\nu)\frac{\partial^3 W^{(1)}}{\partial\xi^2\partial\eta}=0. \end{aligned}\tag{5.6}$$

Differentiating these equations twice with respect to ξ, we deduce, with use of (4.10) and (4.14), that

$$\frac{\partial^4 W_1^{(1)}}{\partial\xi^4}=-\frac{1}{\lambda_1^2}\left(\frac{1}{\gamma_e^2}\frac{\partial^2 W_1^{(0)}}{\partial\tau_s\partial\tau_f}+\frac{\partial^4 W_1^{(1)}}{\partial\eta^2\partial\xi^2}\right),\tag{5.7}$$

and

$$\frac{\partial^4 W_2^{(1)}}{\partial\xi^4}=\frac{1}{\lambda_2^2}\left(\frac{1}{\gamma_e^2}\frac{\partial^2 W_2^{(0)}}{\partial\tau_s\partial\tau_f}-\frac{\partial^4 W_2^{(1)}}{\partial\eta^2\partial\xi^2}\right).\tag{5.8}$$

Therefore, the boundary conditions (5.6) may be transformed to

$$\begin{aligned} &\left(1-\frac{\nu}{\lambda_1^2}\right)\frac{\partial^4 W_1^{(1)}}{\partial\eta^2\partial\xi^2}+\left(1-\frac{\nu}{\lambda_2^2}\right)\frac{\partial^4 W_2^{(1)}}{\partial\eta^2\partial\xi^2}-\frac{\nu}{\gamma_e^2\lambda_1^2}\frac{\partial^2 W_1^{(0)}}{\partial\tau_s\partial\tau_f}\\ &\qquad\qquad+\frac{\nu}{\gamma_e^2\lambda_2^2}\frac{\partial^2 W_2^{(0)}}{\partial\tau_s\partial\tau_f}=-\frac{1}{P}\frac{\partial^2 M_0}{\partial\xi^2},\\ &\frac{\partial^5 W_1^{(1)}}{\partial\eta^3\partial\xi^2}+\frac{\partial^5 W_2^{(1)}}{\partial\eta^3\partial\xi^2}+(2-\nu)\frac{\partial^5 W_1^{(1)}}{\partial\xi^4\partial\eta}+(2-\nu)\frac{\partial^5 W_1^{(1)}}{\partial\xi^4\partial\eta}=0. \end{aligned}\tag{5.9}$$

Substituting solution (4.16) into (5.9), we obtain

$$\begin{aligned} &\left(1-\frac{\nu}{\lambda_1^2}\right)\frac{\partial^4 \Phi_1^{(1,0)}}{\partial\eta^2\partial\xi^2}+\left(1-\frac{\nu}{\lambda_2^2}\right)\frac{\partial^4 \Phi_2^{(1,0)}}{\partial\eta^2\partial\xi^2}-\frac{1}{2\gamma_e^2}\left(1+\frac{\nu}{\lambda_1^2}\right)\frac{\partial^2 W_1^{(0)}}{\partial\tau_s\partial\tau_f}\\ &\qquad\qquad+\frac{1}{2\gamma_e^2}\left(1+\frac{\nu}{\lambda_2^2}\right)\frac{\partial^2 W_2^{(0)}}{\partial\tau_s\partial\tau_f}=-\frac{1}{P}\frac{\partial^2 M_0}{\partial\xi^2},\\ &\left[\frac{\partial^5}{\partial\eta^3\partial\xi^2}+(2-\nu)\frac{\partial^5}{\partial\eta\partial\xi^4}\right]\left(\Phi_1^{(1,0)}+\Phi_2^{(1,0)}\right)-\frac{1}{\gamma_e^2}\frac{\partial^3}{\partial\eta\partial\tau_s\partial\tau_f}\left(W_1^{(0)}-W_2^{(0)}\right)=0. \end{aligned}\tag{5.10}$$

Using the Cauchy-Riemann identities, taking harmonic conjugation of the second equation and integrating with respect to ξ, we may establish that

$$\begin{aligned} &\left(\nu-\lambda_1^2\right)\frac{\partial^4 \Phi_1^{(1,0)}}{\partial\xi^4}+\left(\nu-\lambda_2^2\right)\frac{\partial^4 \Phi_2^{(1,0)}}{\partial\xi^4}=\frac{1}{2\gamma_e^2}\left(1+\frac{\nu}{\lambda_1^2}\right)\frac{\partial^2 W_1^{(0)}}{\partial\tau_s\partial\tau_f}\\ &\qquad\qquad-\frac{1}{2\gamma_e^2}\left(1+\frac{\nu}{\lambda_2^2}\right)\frac{\partial^2 W_2^{(0)}}{\partial\tau_s\partial\tau_f}-\frac{1}{P}\frac{\partial^2 M_0}{\partial\xi^2},\\ &\lambda_1\left(\nu-\lambda_2^2\right)\frac{\partial^4 \Phi_1^{(1,0)}}{\partial\xi^4}+\lambda_2\left(\nu-\lambda_1^2\right)\frac{\partial^4 \Phi_2^{(1,0)}}{\partial\xi^4}=\frac{\lambda_2}{\gamma_e^2}\frac{\partial^2 W_2^{(0)}}{\partial\tau_s\partial\tau_f}-\frac{\lambda_1}{\gamma_e^2}\frac{\partial^2 W_1^{(0)}}{\partial\tau_s\partial\tau_f}. \end{aligned}\tag{5.11}$$

In contrast with (5.5), the system (5.11) is non-homogeneous, with the determinant of the latter vanishing in view of (3.14). Then the compatibility condition gives

$$\left[\lambda_1(\nu-\lambda_2^2)-\frac{\lambda_2(\nu^2-\lambda_1^4)}{2\lambda_1^2}\right]\frac{\partial^2 W_1^{(0)}}{\partial\tau_s\partial\tau_f}$$
$$+\left[\lambda_2(\nu-\lambda_2^2)+\frac{(\nu-\lambda_1^2)(\nu+\lambda_2^2)}{2\lambda_2}\right]\frac{\partial^2 W_2^{(0)}}{\partial\tau_s\partial\tau_f}=-\frac{\gamma_e^2\lambda_2(\nu-\lambda_1^2)}{P}\,\frac{\partial^2 M_0}{\partial\xi^2}. \tag{5.12}$$

Using the general representation (3.16), it is possible to express $W_1^{(0)}$ and $W_2^{(0)}$ through $W^{(0)}$ on the edge $\eta=0$, yielding

$$\frac{\partial^2 W^{(0)}}{\partial\tau_f\partial\tau_s}=\frac{Q}{2P}\,\frac{\partial^2 M_0}{\partial\xi^2}, \tag{5.13}$$

where

$$Q=-\frac{4\gamma_e^4\lambda_2(\nu-\lambda_1^2)}{a_1(\nu-\lambda_1^2)+a_2(\nu-\lambda_2^2)}, \tag{5.14}$$

with

$$a_1=\frac{(\nu+\lambda_2^2)(\nu-\lambda_1^2)}{2\lambda_2}+\lambda_2(\nu-\lambda_2^2),\quad a_2=\frac{\lambda_2(\nu^2-\lambda_1^4)}{2\lambda_1^2}+\lambda_1(\nu-\lambda_2^2).$$

After some rather tedious but straightforward algebra, it is possible to simplify the expression for Q to the form

$$Q=\frac{\chi\,(\nu+\chi)}{1-\nu+\chi}, \tag{5.15}$$

where

$$\chi=\lambda_1\lambda_2=\sqrt{1-\gamma_e^4}.$$

It is noted that the coefficient Q depends on the Poisson's ratio only. The graphical illustration is presented in Figure 3, revealing a monotonic increase of Q with the Poisson's ratio ν. Employing

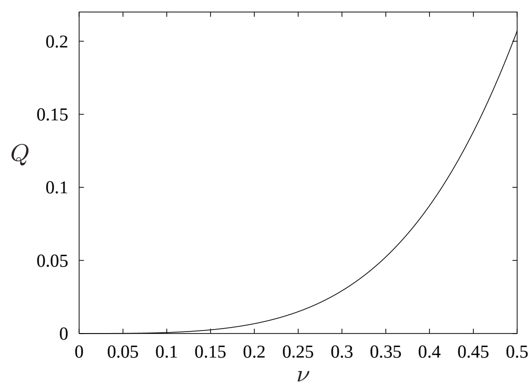

Figure 3. Coefficient Q vs. the Poisson's ratio ν

the leading order approximation

$$W = \frac{h^2 P}{\varepsilon D} W^{(0)}, \tag{5.16}$$

it is possible to rearrange (5.13) as

$$2\varepsilon \frac{\partial^2 W}{\partial \tau_f \partial \tau_s} = \frac{3Q(1-\nu^2)}{2Eh} \frac{\partial^2 M_0}{\partial \xi^2}, \tag{5.17}$$

Then, using the beam-like assumption (3.6), equation (5.17) implies

$$\gamma_e^4 \frac{\partial^4 W}{\partial \xi^4} + \frac{\partial^2 W}{\partial \tau_f^2} + 2\varepsilon \frac{\partial^2 W}{\partial \tau_f \partial \tau_s} + \beta_0 W = \frac{3Q(1-\nu^2)}{2Eh} \frac{\partial^2 M_0}{\partial \xi^2}. \tag{5.18}$$

Neglecting $O(\varepsilon^2)$ terms and returning to the original variables results in the parabolic equation on the edge, given by

$$D\gamma_e^4 \frac{\partial^4 W}{\partial x^4} + 2\rho h \frac{\partial^2 W}{\partial t^2} + \beta W = Q \frac{\partial^2 M_0}{\partial x^2}. \tag{5.19}$$

Within the obtained approximate formulation the decay away from the edge is described by the elliptic equation

$$\Delta_1 \Delta_2 W = 0, \tag{5.20}$$

where

$$\Delta_j = \partial_y^2 + \lambda_j^2 \partial_x^2, \qquad (j = 1, 2) \tag{5.21}$$

which should be solved in conjunction with the parabolic equation (5.19). In fact, the representation in terms of a single harmonic function (3.16) simplifies the formulation even further, since

$$W(x, 0, t) = \frac{\lambda_i^2 - \lambda_j^2}{\nu - \lambda_j^2} W_i(x, 0, t). \tag{5.22}$$

It is also worth noting that for the prescribed edge moment (5.2) evolving in slow time, the solution of (5.13) becomes

$$W^{(0)} = \frac{Q}{2P} \frac{\partial^2 m_0}{\partial \xi^2}. \tag{5.23}$$

At the same time, it is clear that for an arbitrary edge moment M_0 the solution of (5.13), and therefore the parabolic-elliptic model (5.19) and (5.20) provides a correct evaluation of the edge wave contribution to the overall dynamic response. This is not surprising since the procedure does in fact involve approximation in the vicinity of edge wave poles. More details may be found in the Appendix, see (A 9) and (A 10).

We remark that the analysed wave usually dominates in the far-field near-edge zone for a general load, and also in case of near-resonant regimes of moving loads. These have been studied in the context of an explicit hyperbolic-elliptic model for the Rayleigh wave, see [24]. We also note a model example for a moving load on the edge of a Kirchhoff plate resting on the Winkler foundation considered in Kaplunov et al. [9], providing a hint of a beam-like behaviour at the edge.

The explicit model for the bending edge wave for moment edge loading is formulated as a Dirichlet problem for any of the following two pseudo-static elliptic equations

$$\frac{\partial^2 W_j}{\partial y^2} + \lambda_j^2 \frac{\partial^2 W_j}{\partial x^2} = 0, \qquad (j = 1, 2) \tag{5.24}$$

with the deflection at the edge $y = 0$ sought from the parabolic equation (5.19), taking into account (3.16) and (5.22). In other words, the solution of the dynamic parabolic equation (5.19) is used together with relation (5.22) as a boundary condition for the pseudo-static elliptic equation (5.24). The resulting plane harmonic function is then substituted into the relation (3.16) in order to restore the deflection of the plate. Thus, the *dual parabolic-elliptic nature* of the bending edge wave is established.

(b) Shear force

A similar formulation may be derived for the second type of boundary conditions (2.2), with now $M_0 = 0$, $N_0 \neq 0$. This corresponds to shear force excitation, taking the form

$$\begin{aligned} &\frac{\partial^2 W}{\partial \eta^2} + \nu \frac{\partial^2 W}{\partial \xi^2} = 0, \\ &\frac{\partial^3 W}{\partial \eta^3} + (2-\nu)\frac{\partial^3 W}{\partial \xi^2 \partial \eta} = -\frac{h^3 N_0}{D}. \end{aligned} \tag{5.25}$$

The analysis is rather similar to that presented in the previous subsection. The dispersion relation follows from the leading order boundary conditions, with the following system obtained at the next order

$$\begin{aligned} &\left(1-\frac{\nu}{\lambda_1^2}\right)\frac{\partial^4 W_1^{(1)}}{\partial \eta^2 \partial \xi^2} + \left(1-\frac{\nu}{\lambda_2^2}\right)\frac{\partial^4 W_2^{(1)}}{\partial \eta^2 \partial \xi^2} - \frac{\nu}{\gamma_e^2 \lambda_1^2}\frac{\partial^2 W_1^{(0)}}{\partial \tau_s \partial \tau_f} + \frac{\nu}{\gamma_e^2 \lambda_2^2}\frac{\partial^2 W_2^{(0)}}{\partial \tau_s \partial \tau_f} = 0, \\ &\frac{\partial^5 W_1^{(1)}}{\partial \eta^3 \partial \xi^2} + \frac{\partial^5 W_2^{(1)}}{\partial \eta^3 \partial \xi^2} + (2-\nu)\frac{\partial^5 W_1^{(1)}}{\partial \xi^4 \partial \eta} + (2-\nu)\frac{\partial^5 W_1^{(1)}}{\partial \xi^4 \partial \eta} = -\frac{h}{P}\frac{\partial^2 N_0}{\partial \xi^2}. \end{aligned} \tag{5.26}$$

Remarkably, this system does not lead to a parabolic beam equation for the deflection W. Instead it provides an equation for the rotation angle $\theta = \dfrac{\partial W}{\partial y}$, evaluated at the edge $y = 0$, namely,

$$D\gamma_e^4 \frac{\partial^4 \theta}{\partial x^4} + 2\rho h \frac{\partial^2 \theta}{\partial t^2} + \beta\theta = -Q\frac{\partial^2 N_0}{\partial x^2}, \tag{5.27}$$

with the constant Q defined in (5.15).

The resulting explicit model for the shear edge force is similar to that obtained in respect of a bending moment. It contains the elliptic equation,

$$\frac{\partial^2 \theta_j}{\partial y^2} + \lambda_j^2 \frac{\partial^2 \theta_j}{\partial x^2} = 0, \tag{5.28}$$

which is solved in conjunction with the parabolic equation on the edge (5.27) and relations (3.16).

6. Near-resonant harmonic excitation

Let us consider an example illustrating the implementation of the model for a near-resonant edge loading. Consider inhomogeneous boundary conditions, when the bending edge moment is given by

$$M_0 = Ae^{i(kx-\omega t)}, \tag{6.1}$$

and no shear edge force is assumed. This loading allows a particular form of solution of the boundary value problem (2.1), (2.2) which is written as

$$W(x,y,t) = V(y)e^{i(kx-\omega t)}. \tag{6.2}$$

The original plate bending equation (2.1) is then transformed into a secular equation for the function $V(y)$, namely,

$$\frac{d^4 V}{dy^4} - 2k^2 \frac{d^2 V}{dy^2} + \left(k^4 + \frac{\beta - 2\rho h\omega^2}{D}\right)V = 0. \tag{6.3}$$

The decaying solution of (6.3) is written as

$$W(x,y,t) = \sum_{i=1}^{2} C_i e^{i(kx-\omega t) - k\kappa_i y}. \tag{6.4}$$

with

$$\kappa_1^2 + \kappa_1^2 = 2, \qquad \kappa_1^2 \kappa_1^2 = 1 + \frac{\beta - 2\rho h\omega^2}{Dk^4}. \tag{6.5}$$

It is easily verified that the attenuation orders κ_i will coincide with λ_i provided the frequency ω and the wave number k satisfy the dispersion relation (3.1). The constants C_i may be determined from the boundary conditions. The exact solution at the edge is then given by

$$W(x,0,t) = -\frac{A}{Dk^2}\frac{\kappa_1\kappa_2+\nu}{\kappa_1^2\kappa_2^2+2(1-\nu)\kappa_1\kappa_2-\nu^2}\,e^{i(kx-\omega t)}. \tag{6.6}$$

Let us now compare the last formula with that obtained within the approximate formulation derived in §5. In the case of the specified boundary conditions, the related particular solution of equation (5.19) is given by

$$W(x,0,t) = -\frac{AQk^2}{Dk^4\gamma_e^4+\beta-2\rho h\omega^2}\,e^{i(kx-\omega t)}, \tag{6.7}$$

with Q defined in (5.15). It may be observed that both exact and approximate formulae, (6.6) and (6.7), respectively, display resonant behaviour whenever the frequency ω and the wave number k satisfy the dispersion relation (3.1).

We will now compare solutions (6.6) and (6.7) when the wave speed of the excitation is close to that of the bending edge wave. Consider a frequency perturbation of the form

$$\omega = \omega_0 + \varepsilon\omega_1, \qquad |\varepsilon| \ll 1, \tag{6.8}$$

where $\omega_0 = \sqrt{\dfrac{D\gamma_e^2k^4+\beta}{2\rho h}}$, see (3.1). This is exactly the case of the bending edge motion evolving in slow time $t_s = \varepsilon t$ as the asymptotic theory in §4 and §5 requires. Indeed, it may be shown that in view of (6.8) and (5.2) the form of the near-resonant excitation (6.1) will coincide with (5.4) provided that $A = -Bh^2\alpha^2\omega_0\omega_1$. First we obtain

$$\kappa_1\kappa_2 = \sqrt{1+\frac{\beta-2\rho h\omega^2}{Dk^4}} \approx \lambda_1\lambda_2 - \frac{2\rho h}{Dk^4}\frac{\varepsilon\omega_0\omega_1}{\lambda_1\lambda_2}. \tag{6.9}$$

Substituting this into the particular solution (6.6) and making use of the dispersion relation (3.15), we obtain

$$\begin{aligned} W(x,0,t) &\approx -\frac{A}{Dk^2}\frac{(\lambda_1\lambda_2+\nu)\,e^{i(kx-\omega t)}}{\left[\lambda_1^2\lambda_2^2+2(1-\nu)\lambda_1\lambda_2-\nu^2\right]-\dfrac{4\rho h\varepsilon\omega_0\omega_1}{Dk^4}\left(1+\dfrac{1-\nu}{\lambda_1\lambda_2}\right)} \\ &= \frac{Ak^2\lambda_1\lambda_2\,(\nu+\lambda_1\lambda_2)\,e^{i(kx-\omega t)}}{4\rho h\varepsilon\omega_0\omega_1\,(1-\nu+\lambda_1\lambda_2)} = \frac{AQk^2e^{i(kx-\omega t)}}{4\rho h\varepsilon\omega_0\omega_1}. \end{aligned} \tag{6.10}$$

It is readily verified that the last expression coincides with the leading order behaviour of the approximate solution (6.7) obtained within the framework of the the parabolic-elliptic model. Indeed, inserting (6.8), we have

$$W(x,0,t) \approx -\frac{AQk^2\,e^{i(kx-\omega t)}}{\left[Dk^4\gamma_e^4+\beta-2\rho h\omega_0^2\right]-4\rho h\varepsilon\omega_0\omega_1} = \frac{AQk^2e^{i(kx-\omega t)}}{4\rho h\varepsilon\omega_0\omega_1}, \tag{6.11}$$

matching with (6.10).

7. Concluding remarks

Two main goals have been achieved in this paper. First, using the beam-like assumption, a general representation for the bending edge wave field has been obtained in terms of a single harmonic function in §3. Then, perturbing this solution in slow time, an explicit model for the bending edge wave has been constructed in §5. This model consists of a pseudo-static elliptic equation over the interior, governing the decay away from the edge, together with a parabolic equation on the edge describing wave propagation. The model reveals the dual parabolic-elliptic nature of the bending edge wave on a plate supported by a Winker foundation. Considerable simplifications in the analysis of dynamic phenomena associated with edge wave propagation are shown to arise. The model enables the contribution of the bending edge wave field to be separated from the overall dynamic response of the plate. It is therefore envisaged that application of the model would be efficient for analysis of dynamic resonant-type problems when the wave field associated with the bending edge wave is dominant, as is the case of an example considered in §6. The model also provides a leading order approximation in the near-edge far-field region, where the bending edge wave is usually dominant.

The formulation presented in this paper may be developed for bending edge waves in the case of refined plate theories, see e.g. [26], with the approach relying on the plate theories with modified inertia, see [27] and references therein. Another direction of extension is related to edge waves in anisotropic plates [28,29], laminated structures [30] and pre-stressed plates [31]. More elaborate algebra is required to consider curved plates [32,33], shells [34,35], and interfacial edge waves [36]. Finally, we mention considerations of more advanced models of elastic foundation [10]. These problems provide further possible applications of the developed theory.

Appendix. Integral transform solution

The resulting parabolic-elliptic formulation may also be derived through integral transforms. Indeed, applying the Laplace transform to (3.5) with respect to scaled time τ, and the Fourier transform along the scaled longitudinal coordinate ξ (see (3.3)), we have

$$\frac{d^4 W^{FL}}{d\eta^4} - 2p^2 \frac{d^2 W^{FL}}{d\eta^2} + \left(p^4 + s^2 + \beta_0\right) W^{FL} = 0, \tag{A 1}$$

where p and s denote the parameters of Fourier and Laplace transforms, respectively, and W^{FL} is the transformed deflection W. The decaying solution of (A 1) is given by

$$W^{FL} = C_1 e^{-\mu_1 \eta} + C_2 e^{-\mu_2 \eta}, \tag{A 2}$$

where C_1 and C_2 are arbitrary constants, and

$$\mu_{1,2} = p^2 \pm i\sqrt{s^2 + \beta_0}. \tag{A 3}$$

Consider, for example, the case of arbitrary moment excitation at the edge $\eta = 0$, then the boundary conditions (5.1) are transformed to

$$\begin{aligned} &\frac{\partial^2 W^{FL}}{\partial \eta^2} - \nu p^2 W^{FL} = -\frac{h^2}{D} M_0^{FL}, \\ &\frac{\partial^3 W^{FL}}{\partial \eta^3} - (2-\nu)\, p^2 \frac{\partial W^{FL}}{\partial \eta} = 0, \end{aligned} \tag{A 4}$$

where M_0^{FL} is the transformed moment M_0. Substituting the solution (A 2) into the boundary conditions (A 4), it is possible to determine the constants C_1 and C_2. The result for the deflection transform W^{FL} may be expressed as

$$W^{FL} = \frac{K_1}{K} e^{-\mu_1 \eta} + \frac{K_2}{K} e^{-\mu_2 \eta}, \tag{A 5}$$

where after some algebraic manipulations involving (A 3) we establish

$$\begin{aligned} &K_j = (-1)^j \mu_j \left[\mu_j^2 - (2-\nu)p^2\right] \frac{h^2 M_0^{FL}}{D}, \qquad (j = 1, 2) \\ &K = (\mu_1 - \mu_2)\left[\mu_1^2 \mu_2^2 + 2\mu_1 \mu_2 (1-\nu) p^2 - \nu^2 p^4\right]. \end{aligned} \tag{A 6}$$

It should be noted that the term in square brackets in the expression K may be written explicitly as

$$p^4 + s^2 + \beta_0 + 2(1-\nu)p^2\sqrt{p^4 + s^2 + \beta_0} - \nu^2 p^4, \tag{A 7}$$

possessing a zero associated with dispersion relation for bending edge wave (3.15), namely

$$s^2 + \beta_0 = -\gamma_e^4 p^4. \tag{A 8}$$

It may be observed that in view of (A 8) the transformed equation of motion (A 1) corresponds to the elliptic equation (5.20). Let us also illustrate that the solution (A 5) corresponds to the parabolic equation (5.19) at the edge $\eta = 0$. Indeed, using (A 6), it follows from (A 5) that

$$W^{FL}\big|_{\eta=0} = -\frac{h^2 M_0^{FL}\left(\sqrt{p^4+s^2+\beta_0}+\nu p^2\right)}{D\left(p^4+s^2+\beta_0+2(1-\nu)p^2\sqrt{p^4+s^2+\beta_0}-\nu^2 p^4\right)}. \tag{A 9}$$

Approximating the last result around the pole (A 8), we infer

$$W^{FL}\big|_{\eta=0} \approx -\frac{h^2 M_0^{FL} p^2 \chi\,(\chi+\nu)}{D\,(\chi+1-\nu)\,(\gamma_e^4 p^4 + s^2 + \beta_0)}, \tag{A 10}$$

which is a transformed solution of the parabolic equation (5.19) rewritten in terms of ξ and τ.

It is now evident that the presented parabolic-elliptic formulation (5.19) and (5.20) corresponds to the contribution of the bending edge wave field to overall dynamics response. Analogous consideration for a free Kirchhoff plate may be found in [37].

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