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An edge moving load on an orthotropic plate resting on a Winkler foundation

Saad N. Althobaiti, Julius Kaplunov, Danila A. Prikazchikov[a]

[a] *School of Computing and Mathematics, Keele University, Keele, Staffordshire, ST5 5BG, UK*

Abstract

Steady-state motion of a bending moment along the edge of a semi-infinite orthotropic Kirchhoff plate supported by a Winkler foundation is considered. The analysis of the dispersion relation reveals a local minimum of the phase velocity, coinciding with the value of the group velocity, corresponding to the critical speed of the moving load. In contrast to a free plate, the bending edge wave on an elastically supported plate possesses a cut-off frequency, arising due to the stiffening effect of the foundation. It is shown that the steady-state solution of a moving load problem corresponds to a beam-like edge behaviour. This feature is then confirmed from the specialised parabolic-elliptic formulation, which is oriented to extracting the contribution of the bending edge wave to the overall dynamic response.

Keywords: moving load, edge wave, orthotropic plate, Winkler foundation.

1. Introduction

Bending edge wave on a thin semi-infinite elastic plate have been studied since the well-known contribution of [1], see also [2] and [3] for description of the state of art.

We mention a number of works addressing the effects of anisotropy on propagation of edge waves in elastic plates, see e.g. [4–7], along with several studies within 3D framework [8–10]. Among the recent developments in the field we note [11] and [12], addressing the effect of elastic foundation on the dispersive properties of the bending edge wave.

In this paper we are investigating the effect of anisotropy on the bending edge wave in a plate supported by a Winkler foundation, extending the results in [11]. Similarly to the cited paper, we reveal a cut-off frequency caused by the presence of the foundation, and a local minimum of the phase velocity associated with the resonant regime of the moving load.

* D.A. Prikazchikov. Tel.: +44-178-273-3414 ; fax: +44-178-273-4268.
E-mail address: d.prikazchikov@keele.ac.uk

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Then we analyse steady motion of a point moment along the edge of the plate, observing the resonant regime. In addition, a beam-like behaviour on the edge is revealed, which might be expected from the parabolic-elliptic formulation for the bending edge wave in isotropic plate [14].

Finally, we present briefly the parabolic-elliptic model for the bending edge wave on an orthotropic plate supported by Winkler foundation in case of an arbitrary moment on the edge. The model is oriented to extracting the contribution of the bending edge wave to the overall dynamic response. The formulation includes an elliptic equation associated with decay away from the edge, and a parabolic beam-like equation governing edge wave propagation.

2. Governing relations

Consider an elastic plate of thickness $2h$ supported by a Winkler foundation, occupying the region $-\infty < x < \infty$, $0 \leq y < \infty, -h \leq z \leq h$, with the foundation domain given by $-\infty < x < \infty, 0 < y < \infty, 2h \leqslant z < \infty$, see Fig 1.

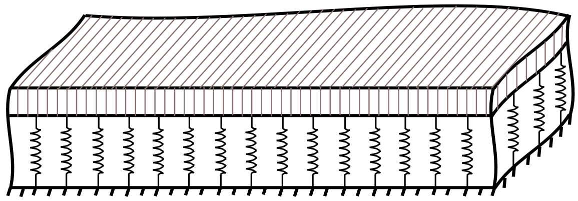

Fig. 1: An elastic plate on the Winkler foundation.

The governing equation for flexural displacement W of an orthotropic, homogeneous, thin elastic plate is written as

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} + 2\rho h \frac{\partial^2 W}{\partial t^2} + \beta W = 0. \tag{1}$$

In above β is the Winkler coefficient, ρ is the mass density, t denotes time, $H = D_1 + 2D_{xy}$, D_x and D_y are bending stiffnesses in the x, y directions respectively, see e.g [4]. Clearly, $\beta = 0$ corresponds to a free Kirchhoff plate.

In absence of prescribed moment and modified shear force on the edge $y = 0$, the boundary conditions are expressed as

$$D_1 \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2} = 0, \quad (D_1 + 4D_{xy}) \frac{\partial^3 W}{\partial x^2 \partial y} + D_y \frac{\partial^3 W}{\partial y^3} = 0, \tag{2}$$

3. Analysis of the dispersion equation

Let us derive the dispersion equation for the bending edge wave. The solution of the plate equation (1) is sought in the form of a travelling harmonic wave, i. e.

$$W(x, y, t) = A e^{i(kx - \omega t) - k\lambda y}, \tag{3}$$

where ω is the frequency, k is wave number, and the condition Re $\lambda > 0$ ensures decay away from the edge $y = 0$. On substituting (3) into (1), we arrive at the following bi-quadratic equation

$$\lambda^4 - \frac{2H}{D_y}\lambda^2 + \frac{D_x k^4 + \beta - 2\rho h \omega^2}{D_y k^4} = 0, \tag{4}$$

which may be shown to possess two roots satisfying the decay condition. Therefore, the deflection W is given by

$$W(x, y, t) = \sum_{j=1}^{2} C_j e^{i(kx - \omega t) - k\lambda_j y}, \tag{5}$$

with the associated attenuation orders, λ_1 and λ_2 specified as

$$\lambda_j = \sqrt{\frac{H}{D_y} + (-1)^j \frac{\kappa}{2}}, \qquad j = 1, 2, \tag{6}$$

where

$$\kappa = 2\sqrt{\frac{H^2}{D_y^2} - \left(\frac{D_x}{D_y} - \gamma^4\right)}. \tag{7}$$

Then, on inserting (5) into the boundary conditions (2), we obtain a set of algebraic equations, yielding non-trivial solutions provided the appropriate determinant is zero, leading to the dispersion equation of the form

$$D_x \gamma^4 k^4 = 2\rho h \omega^2 - \beta. \tag{8}$$

In the above

$$\gamma^4 = 1 - \frac{\left(\sqrt{4D_{xy}^2 + D_1^2} - 2D_{xy}\right)^2}{D_x D_y}$$

is a constant, specific for the bending edge wave on an orthotropic elastic plate, first obtained in [4].
The dispersion relation (8) can be rewritten in a dimensionless form as

$$K^4 = \Omega^2 - 1, \tag{9}$$

where

$$K = k\gamma \sqrt[4]{\frac{D_x}{\beta}}, \quad \Omega = \omega \sqrt{\frac{2\rho h}{\beta}}, \tag{10}$$

hence $\Omega = 1$ is the dimensionless cut-off frequency, similarly to the isotropic case examined in [11].

Following the aforementioned publication, we note that the phase velocity normalized by $\dfrac{\sqrt{2\rho h}}{\gamma \sqrt[4]{\beta D_x}}$,

$$V^{ph} = \frac{\Omega}{K} = \frac{\Omega}{\sqrt[4]{\Omega^2 - 1}}, \tag{11}$$

has the local minimum $V^{ph} = \sqrt{2}$ at $\Omega = \sqrt{2}$, corresponding to $K = 1$. Moreover, at this point V^{ph} coincides with the group velocity

$$V^g = \frac{d\Omega}{dK} = \frac{2\sqrt[4]{(\Omega^2 - 1)^3}}{\Omega}, \tag{12}$$

see Fig. 2, which is in fact almost identical to Fig. 3 of [11].

It is clear that the value $V^{ph} = V^g = \sqrt{2}$, corresponds to the critical speed of a moving load, similarly to the classical 1D problem for a beam on a Winkler foundation, see [13].

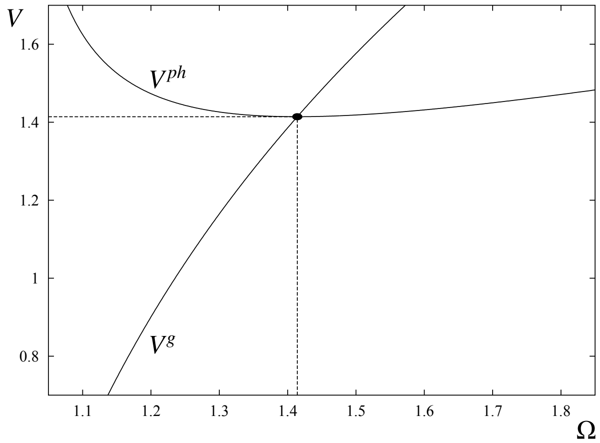

Fig. 2: The phase and group velocities (11) and (12) vs. frequency.

4. Edge Moving Load

Let us replace the first boundary condition in (2) by

$$D_1 \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2} = -M_0 \delta(x - vt), \tag{13}$$

corresponding to a point moment of amplitude M_0 moving along the edge of the plate at a constant speed v. Below we focus only on the steady-state regime in the moving coordinate system $(\xi, y) = (x - vt, y)$.

On applying the Fourier transform with respect to ξ in equation (1) and conditions (13), $(2)_2$, we obtain the following ODE for the transformed deflection W^F

$$D_y \frac{d^4 W^F}{dy^4} - 2Hk^2 \frac{d^2 W^F}{dy^2} + \left(D_x k^4 - 2\rho h v^2 k^2 + \beta\right) W^F = 0, \tag{14}$$

subject to the boundary conditions at the edge $y = 0$

$$D_y \frac{\partial^2 W^F}{\partial y^2} - D_1 k^2 W^F = -M_0, \quad D_y \frac{\partial^3 W^F}{\partial y^3} - (D_1 + 4D_{xy}) k^2 \frac{\partial^3 W^F}{\partial y} = 0, \tag{15}$$

where k is the Fourier parameter.

The decaying solution of (14) is given by

$$W^F = C_1 e^{-\mu_1 y} + C_2 e^{-\mu_2 y} \tag{16}$$

where C_1 and C_2 are arbitrary constants, and

$$\mu_1^2 + \mu_2^2 = \frac{2Hk^2}{D_y}, \qquad \mu_1^2 \mu_2^2 = \frac{D_x k^4 - 2\rho h v^2 k^2 + \beta}{D_y}, \tag{17}$$

On inserting the solution (16) into the transformed boundary conditions (15), the coefficients C_1 and C_2 may be determined. The solution for W^F at the edge $y = 0$ is then given by

$$W^F\Big|_{y=0} = -\frac{M_0^F \left(D_1 k^2 + D_y q\right)}{D_y^2 q^2 + 4D_y D_{xy} q k^2 - D_1^2 k^4}, \tag{18}$$

with $q = \mu_1\mu_2$. The last formula may be rearranged in terms of the dimensionless parameters

$$k = \frac{K}{\gamma}\sqrt{\frac{\beta}{D_x}}, \qquad v = \frac{\gamma\sqrt[4]{\beta D_x}}{\sqrt{2\rho h}}V, \tag{19}$$

as

$$W^F\Big|_{y=0} = -\frac{P}{\beta\,(K^4 - V^2K^2 + 1)}, \tag{20}$$

where

$$P = \sqrt{\frac{\beta^3}{D_y\gamma^4}}\,\frac{M_0\left(Q + \chi_1 K^2\right)\left(Q + \sqrt{1-\gamma^4}K^2\right)}{\left(Q + \left(\sqrt{1-\gamma^4} + 4\chi_2\right)K^2\right)}, \tag{21}$$

with

$$Q = \sqrt{K^4 - \gamma^4\,(K^2V^2 - 1)}, \qquad \chi_1 = \frac{D_1}{\sqrt{D_xD_y}}, \qquad \chi_2 = \frac{D_{xy}}{\sqrt{D_xD_y}}. \tag{22}$$

It is clear from (20) that $V = \sqrt{2}$ corresponds to the resonant speed, confirming the expectations of the previous section. Moreover, it may be shown that in the vicinity of the critical values $K = 1$ and $V = \sqrt{2}$ the resulting transformed deflection (20) corresponds to the moving load problem for an elastically supported beam specified by

$$D_x\gamma^4\frac{\partial^4 W}{\partial x^4} + 2\rho h\frac{\partial^2 W}{\partial t^2} + \beta W = P^*\delta(x - vt), \tag{23}$$

with $P^* = P\Big|_{K=1,V=\sqrt{2}}$.

5. Explicit model for the bending edge wave

Here we present a parabolic-elliptic model for the bending edge wave, extending the results of [15] and [14] to an orthotropic plate. For the sake of brevity, we only consider the boundary conditions

$$D_1\frac{\partial^2 w}{\partial x^2} + D_y\frac{\partial^2 w}{\partial y^2} = -M, \qquad (D_1 + 4D_{xy})\frac{\partial^3 w}{\partial x^2\partial y} + D_y\frac{\partial^3 w}{\partial y^3} = 0, \tag{24}$$

where M is a prescribed edge bending moment.

Similarly to the aforementioned paper, the model below relies on the representation for the eigensolution in terms of a single plane harmonic function derived in [16]. On perturbing this eigensolution in slow time, we deduce a beam-like equation along the edge $y = 0$ which can be written as

$$D_x\gamma^4\frac{\partial^4 W}{\partial x^4} + 2\rho h\frac{\partial^2 W}{\partial t^2} + \beta W = B_K\frac{\partial^2 M}{\partial x^2}. \tag{25}$$

where

$$B_K = \frac{\chi\,(\chi + D_1)}{D_y(\chi + 2D_{xy})}, \tag{26}$$

with

$$\chi = \sqrt{D_xD_y(1-\gamma^4)}.$$

The behaviour over the interior is given by the elliptic equation

$$\frac{\partial^2 W_1}{\partial y^2} + \lambda_1^2\frac{\partial^2 W_1}{\partial x^2} = 0, \tag{27}$$

with the deflection W expressed through the plane harmonic function W_1 as

$$W = W_1(x, \lambda_1 y, t) - \frac{D_1 - \lambda_1^2 D_y}{D_1 - \lambda_2^2 D_y}W_1\,(x, \lambda_2 y, t)\,. \tag{28}$$

In above the parameters λ_1 and λ_2 are the attenuation orders defined by (6).

It may be easily verified that in case of a near-resonant excitation, when $M = M_0\delta(x - vt)$, the Fourier transform of the deflection governed along the edge by (25), coincides with that obtained in the previous section, see (20), where $P = P^*$.

6. Conclusion

This study allowed some qualitative conclusions on the effect of anisotropy on propagation of bending edge waves in a Kirchhoff plate resting on a Winkler foundation. In particular, the previously observed cut-off frequency and local minimum of phase velocity corresponding to the critical regime of the moving edge load, were confirmed. The results for explicit model for bending edge wave were formulated to the case of moment excitation.

The methodology could be extended further to more advanced foundation models, see e.g. [12]. Finally, we note a less trivial extension to interfacial localized waves, curved plates and shells [17–20] .

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