The edge bending wave on a plate reinforced by a beam

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The edge bending wave on a thin isotropic semi-infinite plate reinforced by a beam is considered within the framework of the classical plate and beam theories. The boundary conditions at the plate edge incorporate both dynamic bending and twisting of the beam. A dispersion relation is derived along with its long-wave approximation. The effect of the problem parameters on the cut-offs of the wave in question is studied asymptotically. The obtained results are compared with calculations for the reinforcement in the form of a strip plate.

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I. INTRODUCTION

Thin elastic structures in the shape of a plate reinforced by a beam have various important applications in naval, civil and aerospace engineering, see[1–3]. Static and dynamic behaviour of stiffened plates was intensively studied in numerous publications within the framework of the classical bending theories for plates and beams also taking into consideration beam torsion, see e.g.[4–8]. At the same time, to the best of authors' knowledge edge waves in stiffened plates have only been analysed in two papers[9,10], dealing with a semi-infinite strip with simply supported sides. Bending vibrations of an elastic strip were earlier investigated in various setups, e.g. see[11,12]. We also mention the recent authors' contribution[13] treating a semi-infinite plate reinforced by a strip plate along the edge.

The edge bending wave on an elastic plate has received much attention since long ago, taking into consideration anisotropy, vertical inhomogeneity, contact with elastic foundations, and three dimensional dynamic phenomena, e.g. see the general reference papers[14,15], and also more recent publications[16,17]. In contrast to the Rayleigh wave on an elastic half space, the plate edge bending wave demonstrates dispersion governed by a specialised parabolic-elliptic model[18].

This note is concerned with qualitative analysis of bending vibrations localised along the edge of a semi-infinite plate, stiffened by a beam. Dispersion relation is derived together with its long-wave asymptotic approximations. At the leading order the latter coincides with the dispersion relation for the plate bending wave on a free edge[19]. Next order solution reveals the influence of stiffening on the edge wave localisation. Comparison of the dispersion relation for a plate reinforced by a beam with a narrow rectangular cross-section and for a plate reinforced by a strip plate is demonstrated, justifying the adapted 'plate-beam' formulation.

The effect of material and geometric parameters on edge wave localisation is investigated. A special focus is on the asymptotic evaluation of the cut-offs of the studied edge wave which have been earlier discovered in[9,10]. The possibility of cut-offs over the range of validity of the adapted classical structure theories is addressed.

II. STATEMENT OF THE PROBLEM

Consider a thin isotropic elastic plate stiffened by an elastic beam along the edge. The Cartesian coordinate system is chosen in such a way that x and y are in the midplane of the plate with x going along the interface, see Figure 1.

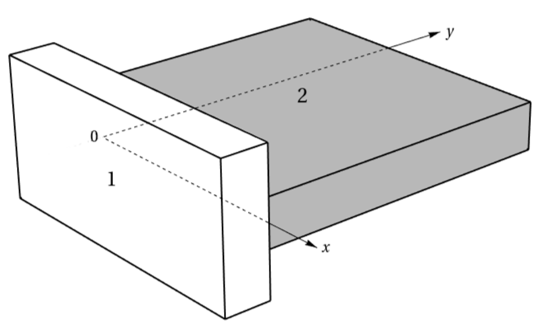

FIG. 1. Plate reinforced by a beam

The equation of motion for the midplane deflection w_2 in the classical theory for plate bending is

$$D_2\left(\frac{\partial^4 w_2}{\partial x^4}+2\frac{\partial^4 w_2}{\partial x^2\partial y^2}+\frac{\partial^4 w_2}{\partial y^4}\right)+2\rho_2 h\frac{\partial^2 w_2}{\partial t^2}=0, \tag{1}$$

where D_2 is bending stiffness of the plate, h is half thickness of the plate, and t is time. Also, in what follows ρ_j are mass densities, E_j are Young's moduli, G_j are shear moduli, ν_j are Poisson's ratios, $j=1,2$. Indexes 1 and 2 correspond to the beam and plate, respectively.

The boundary conditions for the plate edge $y=0$ maybe obtained by considering the beam flexure and twisting, see for example[5], resulting in

$$\begin{aligned} E_1 I_y\frac{\partial^4 w_2}{\partial x^4}+\rho_1 A\frac{\partial^2 w_2}{\partial t^2}&=-D_2\Big(\frac{\partial^3 w_2}{\partial y^3}+(2-\nu_2)\frac{\partial^3 w_2}{\partial x^2\partial y}\Big),\\ G_1 J_t\frac{\partial^3 w_2}{\partial x^2\partial y}-\rho_1 J\frac{\partial^3 w_2}{\partial t^2\partial y}&=-D_2\Big(\frac{\partial^2 w_2}{\partial y^2}+\nu_2\frac{\partial^2 w_2}{\partial x^2}\Big), \end{aligned} \tag{2}$$

where I_y and J are the area and polar moments of inertia of the beam's cross section, J_t is the torsional constant, and A is the area of the beam's cross section.

III. DISPERSION RELATION

The solution of the equation (1) is sought for in the form of a travelling harmonic wave as

$$w_2(x,y,t)=w_2(y)e^{i(kx-\omega t)}, \tag{3}$$

where ω is frequency, and k is wave number. Substituting (3) into (1), we arrive for the edge wave at

$$w_2(y)=C_1 e^{-k\lambda_1 y}+C_2 e^{-k\lambda_2 y},$$

where C_1 and C_2 are arbitrary constants, and

$$\lambda_1 = \sqrt{1+\gamma_2}, \quad \lambda_2 = \sqrt{1-\gamma_2}, \quad \gamma_2 = \frac{\omega}{k^2}\sqrt{\frac{2\rho_2 h}{D_2}}.$$

Now, on substituting (3) into the boundary conditions (2) we arrive at the 2×2 set of linear equations, leading to the general exact dispersion relation

$$\begin{aligned}
&(\lambda_1\lambda_2+1)^2 - \nu_2(\lambda_1+\lambda_2)^2 - (1-\nu_2)^2 \\
&-(\lambda_1+\lambda_2)(\alpha_1\gamma_2^2\rho - \beta_2\lambda_1\lambda_2 - \beta_1)\delta_h \\
&-\beta_2(\alpha_1\gamma_2^2\rho - \beta_1)\delta_h^2 - \alpha_2\gamma_2^2\rho\lambda_1\lambda_2(\lambda_1+\lambda_2)\delta_h^3 \\
&+\alpha_2\gamma_2^2\rho(\alpha_1\gamma_2^2\rho - \beta_1)\delta_h^4 = 0,
\end{aligned} \tag{4}$$

where

$$\alpha_1 = \frac{A}{2h^2}, \quad \alpha_2 = \frac{J}{2h^4}, \quad \beta_1 = \frac{E_1 I_y}{hD_2}, \quad \beta_2 = \frac{G_1 J_t}{hD_2},$$

and $\delta_h = kh$, $\rho = \dfrac{\rho_1}{\rho_2}$.

Setting $\delta_h = 0$ in (4) and returning back to original variables we get the well known relation for a free plate edge, see e.g.[19]

$$D_2 k^4 c^4 = 2\rho_2 h\omega^2,$$

where

$$c = \left[(1-\nu_2)\left(3\nu_2 - 1 + 2\sqrt{2\nu_2^2 - 2\nu_2 + 1}\right)\right]^{1/4}.$$

Let us next introduce a new unknown function by

$$\phi = \frac{\sqrt{1-\gamma_2^2}}{\nu_2^2}, \tag{5}$$

corresponding to the appropriately normalised attenuation rate which is not sensitive to the value of a Poison's ratio. This is seemingly the most relevant characteristic of slowly decaying edge bending waves. Hence, equation (4) can be re-written as

$$\begin{aligned}
&(1+\nu_2^2\phi)^2 - 2\nu_2(1+\nu_2^2\phi) - (1-\nu_2)^2 \\
&- \sqrt{2(1+\nu_2^2\phi)}\left(\alpha_1\rho\left(1-\nu_2^4\phi^2\right) - \beta_2\nu_2^2\phi - \beta_1\right)\delta_h \\
&- \beta_2\left(\alpha_1\rho(1-\nu_2^4\phi^2) - \beta_1\right)\delta_h^2 \\
&- \alpha_2\nu_2^2\rho(1-\nu_2^4\phi^2)\phi\sqrt{2(1+\nu_2^2\phi)}\delta_h^3 \\
&+ \alpha_2\rho\left(\alpha_1\rho(1-\nu_2^4\phi^2) - \beta_1\right)\left(1-\nu_2^4\phi^2\right)\delta_h^4 = 0,
\end{aligned} \tag{6}$$

At $\phi = 0$ ($\gamma_2 = 1$) we have for cut-off values

$$\nu_2^2 + (\alpha_1\rho - \beta_1)\left(\sqrt{2}\delta_h + \beta_2\delta_h^2 - \alpha_2\rho\delta_h^4\right) = 0. \tag{7}$$

Over the range of validity of thin plate theory ($\delta_h \ll 1$) we get at leading order

$$\delta_h^* \approx \frac{\nu_2^2}{\sqrt{2}(\beta_1 - \alpha_1\rho)}. \tag{8}$$

Thus, for a non-contrast setup ($\alpha_1 \sim \beta_1 \sim \rho \sim 1$) the sought for cut-offs belong to the interval $0 < \delta_h^* \ll 1$ provided that $\nu_2 \ll 1$ and $\beta_1 > \alpha_1\rho$. This conclusion agrees with the observations in paper[9], see eq. (23) and Figure 2 therein.

Next, expanding ϕ into a series about $\delta_h = 0$

$$\phi = \phi_0 + \phi_1\delta_h + \ldots \tag{9}$$

and substituting into the dispersion relation (6), we obtain

$$\phi_0 = \frac{\nu_2 - 1 + \sqrt{2\nu_2^2 - 2\nu_2 + 1}}{\nu_2^2}, \tag{10}$$

and

$$\phi_1 = \frac{\left((1-\nu_2^4\phi_0^2)\rho\alpha_1 - \beta_1 - \beta_2\nu_2^2\phi_0\right)\sqrt{2(1+\nu_2^2\phi_0)}}{2\nu_2^4\phi_0 - 2(\nu_2-1)\nu_2^2}. \tag{11}$$

It is worth noting that (8) and (9)-(11) do not contain the parameter α_2 involving rotation inertia of the beam. This is in line with the asymptotic analysis of a similar problem for the edge reinforcement in the form of a plate strip in[13]. In addition, (8) also does not depend on parameter β_2, expressing the effect of torsional rigidity.

IV. EXAMPLE

In this section we present the results of numerical comparison of the dispersion curves for a plate reinforced by a beam and a composite 'plate-plate' structure, in order to validate the 'plate-beam' model in the previous section. To this end, consider bending of a semi-infinite Kirchhoff plate reinforced by a strip plate along the edge as shown in Figure 2, assuming that for the strip plate $H \gg h$. For the latter, the equation of motion follows from (1) by substituting 1 instead of 2 in all the suffices.

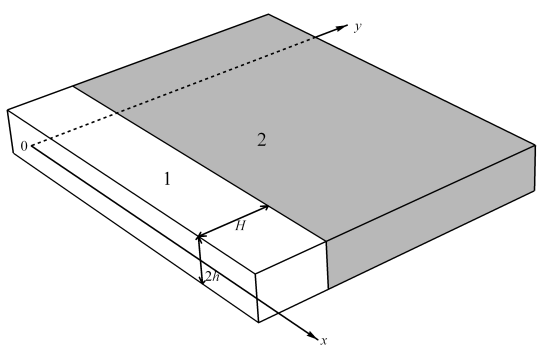

FIG. 2. Plate reinforced by a strip plate

Traction free boundary conditions on the edge $y = 0$ are given by

$$\frac{\partial^2 w_1}{\partial y^2} + \nu_1 \frac{\partial^2 w_1}{\partial x^2} = 0, \quad \frac{\partial^3 w_1}{\partial y^3} + (2 - \nu_1) \frac{\partial^3 w_1}{\partial x^2 \partial y} = 0. \tag{12}$$

The continuity conditions at $y = H$ are

$$
\begin{aligned}
&w_1 = w_2,\\
&\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial y},\\
&D_1\Big(\frac{\partial^2 w_1}{\partial y^2} + \nu_1 \frac{\partial^2 w_1}{\partial x^2}\Big) = D_2\Big(\frac{\partial^2 w_2}{\partial y^2} + \nu_2 \frac{\partial^2 w_2}{\partial x^2}\Big),\\
&D_1\Big(\frac{\partial^3 w_1}{\partial y^3} + (2-\nu_1)\frac{\partial^3 w_1}{\partial x^2 \partial y}\Big) = D_2\Big(\frac{\partial^3 w_2}{\partial y^3} + (2-\nu_2)\frac{\partial^3 w_2}{\partial x^2 \partial y}\Big).
\end{aligned}
$$

The related dispersion equation is

$$\det \mathbf{M} = 0, \tag{13}$$

with the non-zero components of 6×6 matrix $\mathbf{M}$ given in Appendix, where the notation

$$D = \frac{D_1}{D_2}$$

is introduced.

For a plate reinforced by a beam with a narrow rectangular cross section the quantities I_y, J, J_t, and A in (2) are defined as

$$I_y = \frac{2}{3}h^3 H, \quad J = \frac{1}{6}hH^3, \quad J_t = \frac{8}{3}h^3 H, \quad A = 2hH.$$

Taking into account the relations

$$D_j = \frac{2E_j h^3}{3(1-\nu_j^2)}, \qquad G_j = \frac{E_j}{2(1+\nu_j)}, \quad j = 1, 2,$$

we have

$$\alpha_1 = \eta, \quad \alpha_2 = \frac{1}{12}\eta^3, \quad \beta_1 = D(1-\nu_1^2)\eta, \quad \beta_2 = 2D(1-\nu_1)\eta, \tag{14}$$

where $\eta = H/h$. Substituting the above formulae into (6) we obtain dispersion equation

$$\begin{aligned}
&(1+\nu_2^2\phi)^2 - 2\nu_2(1+\nu_2^2\phi) - (1-\nu_2)^2 - \sqrt{2(1+\nu_2^2\phi)}\times \\
&\qquad \left(\rho(1-\nu_2^4\phi^2) - D(1-\nu_1)(1+\nu_1+2\nu_2^2\phi)\right)\delta_H \\
&- 2D(1-\nu_1)\left(\rho(1-\nu_2^4\phi^2) - D(1-\nu_1^2)\right)\delta_H^2 \\
&- \frac{1}{12}(1-\nu_2^4\phi^2)\nu_2^2\rho\phi\sqrt{2(1+\nu_2^2\phi)}\delta_H^3 \\
&+ \frac{1}{12}(1-\nu_2^4\phi^2)\rho(\rho(1-\nu_2^4\phi^2) - D(1-\nu_1^2))\delta_H^4 = 0,
\end{aligned} \tag{15}$$

where $\delta_H = kH \ll 1$. Now, the cut-off at leading order is given by the formula

$$\delta_H^* \approx \frac{\nu_2^2}{\sqrt{2}(D(1-\nu_1^2)-\rho)}, \tag{16}$$

which readily follows from (8) and is valid provided that $\nu_2^2 \ll 1$ and $D(1-\nu_1^2) > \rho$.

Also, the asymptotic expansion for ϕ, analogous to (9), becomes

$$\phi = \widetilde{\phi}_0 + \widetilde{\phi}_1\delta_H + \ldots, \tag{17}$$

where $\widetilde{\phi}_0 = \phi_0$ and

$$\begin{aligned}
\widetilde{\phi}_1 = &\sqrt{2(1+\nu_2^2\phi_0)}\times \\
&\frac{\left(\rho(1-\nu_2^4\phi_0^2) - D(1-\nu_1)(1+\nu_1+\nu_2^2\phi_0)\right)}{2\nu_2^2(1-\nu_2+\nu_2^2\phi_0)}.
\end{aligned}$$

In Figures 3 and 4 the function ϕ is plotted against dimensionless wave number δ_H. In these figures the dispersion curves for a plate reinforced by a beam (15) and by a strip plate (13) are plotted together with those corresponding to the two term asymptotic approximations (17). Numerical examples are presented for $\nu_1 = 0.31$ and $\nu_2 = 0.35$.

As might be expected, both beam approximation (15) and its two-term asymptotics (17) are robust only over the long wave range ($\delta_H \ll 1$), see the curves for $D = 2.3$ in Figure

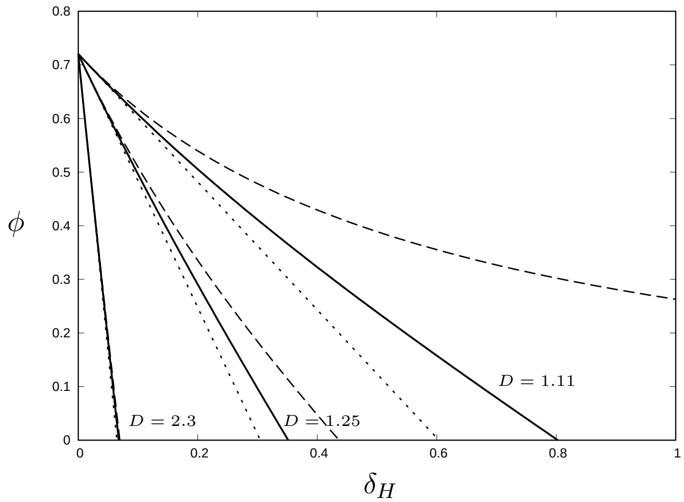

FIG. 3. Comparison of dispersion relations (13) (solid line), (15) (dashed line) and asymptotic expansion (17) (dotted line) for $\rho = 1.0$ and $D = 2.3$, 1.25, 1.11.

3 and $\rho = 0.2$ in Figure 4, for which the asymptotic formulae (16) gives $\delta_H^* = 0.08$ and $\delta_H^* = 0.12$, respectively. Outside the long wave range, the deviation between the results for plate and beam edge reinforcement become more substantial. In particular, as follows from formula (7) with (14) the beam reinforcement does not assume a cut-off under the condition $D(1 - \nu_1^2) - \rho = 0$, which is satisfied for the curves corresponding to $D = 1.11$ in Figure 3 and $\rho = 0.9$ in Figure 4. At the same time, for both of these scenario the strip plate reinforcement predicts cut-offs at $\delta_H^* \sim 1$.

V. CONCLUSION

We studied the edge wave problem for a semi-infinite plate reinforced by a beam taking into account both bending and twisting vibrations of the beam. The explicit asymptotic formulae for the cut-offs of the edge waves are presented. The validity of the chosen approximate formulation starting from the classical plate and beam theories is also addressed. Dis-

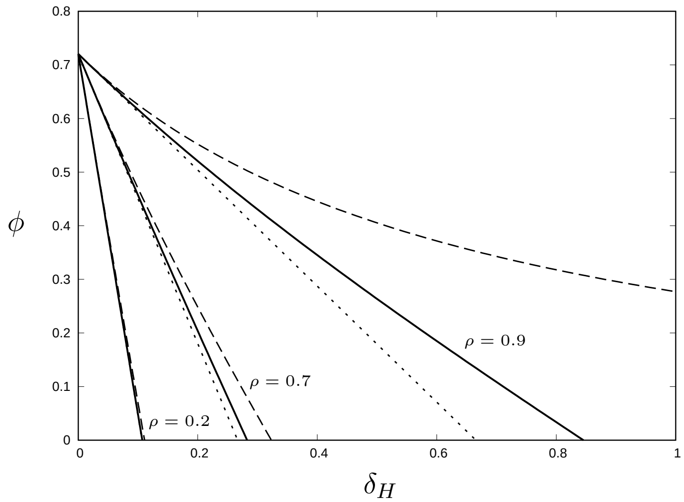

FIG. 4. Comparison of dispersion relations (13) (solid line), (15) (dashed line) and asymptotic expansion (17) (dotted line) for $D = 1.0$ and $\rho = 0.2$, 0.7, 0.9.

persion relation is obtained and long-wave approximation is derived. The numerical results are validated by comparison with the more general dispersion relation for a reinforcement in the form of a strip plate, which is also treated on the basis of the 2D Kirchhoff theory. The developed framework may be extended to more general setups including anisotropic structures as well as more elaborated structure models, e.g. see[14,20].

VI. APPENDIX

The entries of the matrix M in (13) are given by

$$M_{11} = M_{12} = \lambda_{11}^2 - \nu_1, \quad M_{13} = M_{14} = \lambda_{21}^2 - \nu_1$$

$$M_{21} = -M_{22} = \lambda_{11}(\lambda_{11}^2 + \nu_1 - 2),$$

$$M_{23} = -M_{24} = \lambda_{21}(\lambda_{21}^2 + \nu_1 - 2),$$

$$M_{31} = \lambda_{11} e^{\lambda_{11}\delta_H}, \quad M_{32} = -\lambda_{11} e^{-\lambda_{11}\delta_H},$$

$$M_{33} = \lambda_{21} e^{\lambda_{21}\delta_H}, \quad M_{34} = -\lambda_{21} e^{-\lambda_{21}\delta_H},$$

$$M_{35} = \lambda_{12} e^{-\lambda_{12}\delta_H}, \quad M_{36} = \lambda_{22} e^{-\lambda_{22}\delta_H},$$

$$M_{41} = e^{\lambda_{11}\delta_H}, \quad M_{42} = e^{-\lambda_{11}\delta_H}, \quad M_{43} = e^{\lambda_{21}\delta_H},$$

$$M_{44} = e^{-\lambda_{21}\delta_H}, \quad M_{45} = -e^{-\lambda_{12}\delta_H}, \quad M_{46} = -e^{-\lambda_{22}\delta_H},$$

$$M_{51} = D(\lambda_{11}^2 - \nu_1) e^{\lambda_{11}\delta_H}, \quad M_{52} = D(\lambda_{11}^2 - \nu_1) e^{-\lambda_{11}\delta_H},$$

$$M_{53} = D(\lambda_{21}^2 - \nu_1) e^{\lambda_{21}\delta_H}, \quad M_{54} = D(\lambda_{21}^2 - \nu_1) e^{-\lambda_{21}\delta_H},$$

$$M_{55} = -(\lambda_{12}^2 - \nu_2) e^{-\lambda_{12}\delta_H}, \quad M_{56} = -(\lambda_{22}^2 - \nu_2) e^{-\lambda_{22}\delta_H},$$

$$M_{61} = D\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2) e^{\lambda_{11}\delta_H},$$

$$M_{62} = -D\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2) e^{-\lambda_{11}\delta_H},$$

$$M_{63} = D\lambda_{21}(\lambda_{21}^2 + \nu_1 - 2) e^{\lambda_{21}\delta_H},$$

$$M_{64} = -D\lambda_{21}(\lambda_{21}^2 + \nu_1 - 2) e^{-\lambda_{21}\delta_H},$$

$$M_{65} = \lambda_{12}(\lambda_{12}^2 + \nu_2 - 2) e^{-\lambda_{12}\delta_H},$$

$$M_{66} = \lambda_{22}(\lambda_{22}^2 + \nu_2 - 2) e^{-\lambda_{22}\delta_H},$$

where

$$\lambda_{1j} = \sqrt{1+\gamma_j}, \quad \lambda_{2j} = \sqrt{1-\gamma_j},$$

and

$$\gamma_j = \frac{\omega}{k^2}\sqrt{\frac{2\rho_j h}{D_j}}, \quad j = 1, 2.$$

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