Homogenized equation of second-order accuracy for conductivity of laminates. *†

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Abstract

The high order homogenization techniques potentially generate the so-called infinite order homogenized equations. Since long ago, the coefficients at higher order derivatives in these equations have been calculated within various refined theories for both periodic composites and thin structures. However, it was not always clear, what is a well-posed mathematical formulation for such equations. In the present paper we discuss two techniques for constructing a second order homogenized equation. One of them is concerned with the projection of a weak formulation of the original problem on an "ansatz subspace". The second one corresponds to the traditional two scale asymptotic expansion using representation of a second order corrector via the solution of the classical (leading order) homogenized equation.

1 Introduction

High order homogenized equations often arise both in mechanics of thin structures and composite models as a particular feature of refined theories involving higher derivatives (gradients) of the unknown function, for example, displacement, e.g. see [1, 2, 3, 4, 5, 6]. These theories are usually expected to be more precise than their classical counterparts, since the former

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take into consideration more delicate physical properties, such as dispersion. The aforementioned theories have been initially derived using ad hoc phenomenological approach. Later, substantial efforts have been focused on the asymptotic justification of high order models. For the conductivity and elastostatics of periodic composite materials various homogenized models have been established in [7], see also [8] for more details.

Leading order 2D asymptotic models for thin elastic plates and shells, derived from 3D equations in linear elasticity have been known since long ago [9], [10], [11]; see also [12], [13] for a rigorous mathematical justification and error estimation. Associated higher order 2D models have been developed in [14], [15]. It is crucial, that these have been reduced to the partial differential equations of the same order as those of the leading order approximations. The most accurate high order asymptotic solution of the static 3D boundary value problem for a homogeneous thin elastic plate has been apparently obtained in [16] and [17], see also the previous considerations on the subject in [18], [19] and references therein.

High order asymptotic formulations for thin heterogeneous structures appear to be closely related to homogenization for periodic composites, governed for a broad range of problems by high order homogenized equations proposed in [7], [8]. This approach has been then extended to thin structures in [20], [21], [22] and [23], for which it has common features with the earlier developed asymptotic methodology for plates and shells, e.g. see [18], [24]. More recently, the analogy of the homogenization models for thin structures and periodic media has been addressed in [25] within the general dynamic context.

The approach in [7] leads to infinite order homogenized equations, which formally are pseudo-differential equations with small coefficients at higher order derivatives. This equation has not been interpreted mathematically although there were a number of physical interpretations of infinite order equations and related coefficients. In particular, the variational properties of the infinite order formally homogenized equations have been analysed in [23], numerous publications have been studied the sign of coefficients, e.g. see [26] and [27]. The possibility of truncation of this equation has been addressed in [28]. The point is that the sign of the coefficient at the highest derivative in a truncated equation may appear to be erroneous with regard to the well-posedness of this equation. For the equation defined over the whole space a well-posed truncation procedure was proposed in [29]. It corresponds to a certain projection of the variational formulation on an "ansatz space". This technique automatically brings into the truncated equation small stabilizing terms. Asymptotically justified boundary conditions for high order homogenized equations have been studied in [30], see also [31] using a different methodology.

In the present paper we discuss two techniques for constructing of a second order homogenized equation. The first of them is related to the projection of a weak formulation of the original problem on an "ansatz subspace". The second one is based on the conventional two scale asymptotic expansion in which a second order corrector is expressed through the solution of the leading order homogenized equation. The goal of the paper is to illustrate these two approaches to high order homogenization by analysing a "toy" scalar problem modelling conductivity of a three-layered laminate. A zoom is on the well-posedness of the derived high order equation.

2 Statement of the problem

Consider a 2D scalar equation over a thin strip $G_\varepsilon = \{x = (x_1, x_2) \in \mathbb{R}^2 |\, x_2 \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\}$, see Figure 1, given by

$$\operatorname{div}\Big(A\big(\frac{x_2}{\varepsilon}\big)\nabla u_\varepsilon\Big) = f(x_1), \tag{2.1}$$

with Neumann's boundary condition along $x_2 = \pm\varepsilon/2$

$$A\big(\frac{x_2}{\varepsilon}\big)\frac{\partial u_\varepsilon}{\partial x_2} = 0, \tag{2.2}$$

where ε is a small positive parameter, A is a piecewise constant function, defined as

$$A(\eta) = \begin{cases} 1, & \eta \in (-1/4, 1/4), \\ a, & \eta \in (-1/2, 1/2)\backslash(-1/4, 1/4), \end{cases} \tag{2.3}$$

$a > 0$ and f is a $T-$periodic function from $C^\infty(\mathbb{R})$ with $\int_0^T f(x_1)dx_1 = 0$.

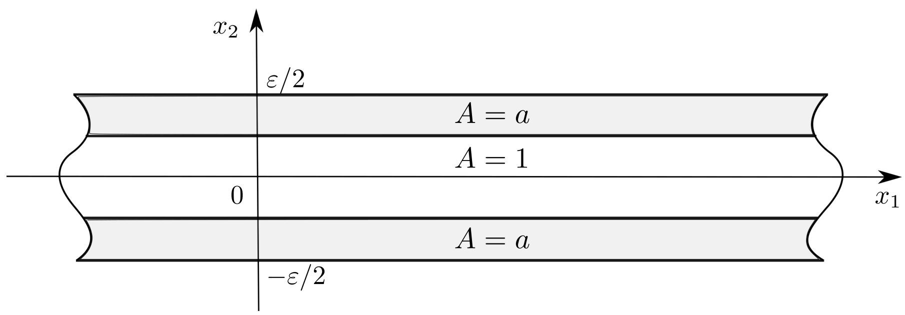

Figure 1: A three-layered laminate.

We seek a $T-$periodic in x_1 solution of problem (2.1) and (2.2). For the formulated toy problem we demonstrate below two methods for deriving high order homogenized equations and compare the obtained approximate solution with the exact one.

3 High order homogenization

Let us consider the ansatz (i.e. asymptotic expansion) of the solution of (2.1)-(2.2), which has been introduced and justified in [22]:

$$u^{(J)}(x) = \sum_{l=0}^{J} \varepsilon^{2l} N_{2l}\left(\frac{x_2}{\varepsilon}\right) D^{2l} v_\varepsilon(x_1), \tag{3.1}$$

where N_{2l} are the solutions of so called cell problems, D^{2l} are derivatives of order $2l$ and $v_\varepsilon(x_1)$ is a 1D approximation, e.g. an averaged characteristic of the solution.

Plug ansatz (3.1) into equation (2.1). Denoting $L_\varepsilon u^{(J)} = \mathrm{div}\Big(A\big(\frac{x_2}{\varepsilon}\big)\nabla u^{(J)}\Big)$ and collecting terms of the same order, we get

$$L_\varepsilon u^{(J)} = \sum_{l=0}^{J} \varepsilon^{2l-2} H_{2l}(\xi_2) D^{2l} v_\varepsilon(x_1) + \varepsilon^{2J} A(\xi_2) N_{2J} D^{2l+2} v_\varepsilon(x_1), \tag{3.2}$$

with $\xi_2 = x_2/\varepsilon$, $\xi_2 \in (-1/2, 1/2)$. In the above

$$H_{2l} = \frac{\partial}{\partial \xi_2}\Big(A(\xi_2)\frac{\partial N_{2l}}{\partial \xi_2}\Big) + A(\xi_2) N_{2l-2}$$

and N_{2l} are the solutions of the cell problems

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_2}\left(A(\xi_2)\frac{\partial N_{2l}}{\partial \xi_2}\right) = -A(\xi_2)N_{2l-2} + \langle AN_{2l-2}\rangle, \\
&\frac{\partial N_{2l}}{\partial \xi_2}(\pm 1/2) = 0,
\end{aligned}
\tag{3.3}
$$

where $\langle F\rangle = \int_{-1/2}^{1/2} F(\xi_2)d\xi_2$ and $N_0 = 1$. Here and below the terms with a negative suffix vanish. In particular, for N_2 ($l = 1$) we have

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_2}\left(A(\xi_2)\frac{\partial N_2}{\partial \xi_2}\right) = -A(\xi_2) + \langle A\rangle, \\
&\frac{\partial N_2}{\partial \xi_2}(\pm 1/2) = 0.
\end{aligned}
\tag{3.4}
$$

Then

$$
L_\varepsilon u^{(J)} = \sum_{l=0}^{J} \varepsilon^{2l-2} h_{2l} D^{2l} v_\varepsilon(x_1) + \varepsilon^{2J} A(\xi_2) N_{2J} D^{2l+2} v_\varepsilon(x_1), \tag{3.5}
$$

where $h_{2l} = \langle AN_{2l-2}\rangle$, $l > 0$, and $h_0 = 0$. In this case, a traditional high order homogenized equation can be obtained by equating a truncated series (3.5) to the right-hand side in (2.1), i.e.

$$
\sum_{l=1}^{J} \varepsilon^{2l-2} h_{2l} D^{2l} v_\varepsilon(x_1) = f(x_1). \tag{3.6}
$$

In particular, taking $J = 1$, we arrive at the homogenized equation

$$
\langle A\rangle v_\varepsilon'' + \varepsilon^2 \langle AN_2\rangle v_\varepsilon'''' = f(x_1). \tag{3.7}
$$

An asymptotic solution of (3.6) can be constructed in the form

$$
v_\varepsilon^{(2J)}(x_1) = \sum_{j=0}^{J} \varepsilon^{2j} v_{2j}(x_1), \tag{3.8}
$$

where v_{2j} satisfy the equations

$$
\langle A\rangle v_{2j}'' = f(x_1)\delta_{j,0} - \sum_{p+q=j, p>0} h_{2p+2} D^{2+2p} v_{2q}. \tag{3.9}
$$

As in [8] it can be proved that for asymptotic expansion (3.8) and

$$u^{(J)}(x) = \sum_{l=0}^{J} \varepsilon^{2l} N_{2l}\big(\frac{x_2}{\varepsilon}\big) D^{2l} v_\varepsilon^{(2J)}(x_1), \tag{3.10}$$

the estimate

$$\|\nabla\Big(u_\varepsilon - u^{(J)}\Big)\|_{L^2((-T/2,T/2)\times(-\varepsilon/2,\varepsilon/2))} = O(\varepsilon^{2J+3/2}) \tag{3.11}$$

holds.

However, within this approach the sign of the coefficient of the senior derivative h_{2J} is not necessarily $(-1)^J$, i.e. the homogenized equation of interest may be not coercive. As each function N_{2l} is defined up to an additive constant, the sign of the coefficient $\langle AN_{2l}\rangle$ depends on the choice of this constant, guided by certain additional reasons. In particular, if v is interpreted as an average temperature, then the constant is determined from the condition

$$\langle N_{2l}\rangle = \delta_{l0}, \tag{3.12}$$

or, if it is the mid-plane temperature, then

$$N_{2l}(0) = \delta_{l0}, \tag{3.13}$$

where δ_{ij} is the Kronecker's delta. We remark that for a similar antiplane problem in elasticity v is a laminate displacement, e.g. see [34] and [35].

At $l = 1$ we have

$$N_2(\xi_2) = \left\{ \begin{array}{c} \frac{a-1}{4}\xi_2^2, \quad |\xi_2| \in [0,1/4], \\ \frac{a-1}{4a}(-\xi_2^2 + |\xi_2| + \frac{a-3}{16}), \quad |\xi_2| \in [1/4,1/2], \end{array} \right. \tag{3.14}$$

provided (3.13) holds. Then, inserting (2.3) and (3.14) into (3.7), we arrive at

$$\frac{a+1}{2} v'' + \varepsilon^2 \frac{a^2-1}{128} v'''' = f(x_1), \tag{3.15}$$

with a positive coefficient at the fourth order derivative at $a > 1$.

It can be also shown that under condition (3.12) problem (3.3) yields at $l = 1$

$$\langle AN_2\rangle = \langle A\rangle\langle N_2\rangle + \langle A\big(\frac{\partial N_2}{\partial \xi_2}\big)^2\rangle > 0, \quad a \neq 1, \tag{3.16}$$

due to vanishing of the first term in the right hand side.

Within the same asymptotic error the fourth order equation (3.15) can be reduced to a second order one, given by

$$v_\varepsilon'' = \frac{2}{a+1} f - \varepsilon^2 \frac{(a-1)}{32(a+1)} f''. \tag{3.17}$$

This approach is widely used in the theory for thin elastic plates and shells, e.g. see [14], [15], [32]. The asymptotic procedure in this theory has a lot in common with the homogenization technique adapted in this paper, see also [25] commenting on the similarities of dynamic homogenization procedures for thin and periodic structures. In fact, they differ from each other in non significant details, in particular condition (3.12) is typical for the homogenization theory, while (3.13) is often adapted in plate and shell theories. In addition, we mention the fundamental contributions [16] and [17], concerned with the derivation of the three-term asymptotic expansions for static boundary value problems in thin elastic plates. Although these papers do not explicitly operate with the differential equations with a perturbed right-hand side, e.g. see (3.17), they incorporate such higher order corrections virtually in the same manner.

However, it is not always possible to reduce the order of the homogenized equation as simple as it has been done for the considered toy problem. Therefore, we often need to deal with non coercive equations like (3.15) at $a > 1$. For making such equation coercive, we develop below an alternative approach based on regularization.

Let us first set a variational formulation for problem (2.1)-(2.2) over the strip G_ε. Denoting by $H^1_{per}(G_\varepsilon)$ the space of $T-$periodic in x_1 functions from $H^1_{per}(G_\varepsilon \cap \{x_1 \in (-R, R)\})$ for any $R > 0$, we seek $u_\varepsilon \in H^1_{per}(G_\varepsilon)$, such that, for all functions $\varphi \in H^1_{per}(G_\varepsilon)$,

$$-\int_{-T/2}^{T/2} \int_{-\varepsilon/2}^{\varepsilon/2} A(\frac{x_2}{\varepsilon}) \nabla u_\varepsilon \cdot \nabla \varphi dx_1 dx_2 = \int_{-T/2}^{T/2} \int_{-\varepsilon/2}^{\varepsilon/2} f(x_1) \varphi dx_1 dx_2. \tag{3.18}$$

Now consider the subspace of functions $H^1_{per}(G_\varepsilon)$ defined as

$$H_{dec} = \{\varphi \in H^1_{per}(G_\varepsilon) | \varphi(x) = \sum_{l=0}^{J} \varepsilon^{2l} N_{2l}(\frac{x_2}{\varepsilon}) D^{2l} \psi(x_1); \;\; \psi \in H^{2J+2}_{per}(\mathbb{R})\}, \tag{3.19}$$

where $H^n_{per}(\mathbb{R})$ is the space of $T-$periodic functions of one variable from H^n_{loc}. It is assumed here that N_{2l} are piecewise polynomial functions of degree $2l$, which is the case iff $a \neq 1$, ensuring that H_{dec} is complete.

Using the ideas of [29] and [33], consider the projection of (3.18) on Hilbert subspace H_{dec}, i.e. find $u_{\varepsilon,dec} \in H_{dec}$, such that, for all functions $\varphi \in H_{dec}$,

$$-\int_{-T/2}^{T/2}\int_{-\varepsilon/2}^{\varepsilon/2} A(\frac{x_2}{\varepsilon})\nabla u_{\varepsilon,dec}\cdot\nabla\varphi dx_1 dx_2 = \int_{-T/2}^{T/2}\int_{-\varepsilon/2}^{\varepsilon/2} f(x_1)\varphi dx_1 dx_2. \tag{3.20}$$

This variational formulation generates the coercive equation for v_ε given by

$$\sum_{l=1}^{2J+2} \varepsilon^{2l-2}\tilde{h}_{2l}D^{2l}v_\varepsilon(x_1) = f(x_1), \tag{3.21}$$

where

$$\tilde{h}_{2l} = \sum_{j+p=l}\langle A(\xi)\Big(\frac{\partial N_{2j}}{\partial\xi_2}\frac{\partial N_{2p}}{\partial\xi_2}+N_{2j-2}N_{2p-2}\Big)\rangle,\ \ l < J+1;\ \ \tilde{h}_{2J+2} = \langle A(\xi)N_{2J}^2\rangle. \tag{3.22}$$

In particular, taking $J = 1$, we get a regularized homogenized equation in the form

$$\langle A\rangle v_\varepsilon'' + \varepsilon^2\langle AN_2\rangle v_\varepsilon'''' + \varepsilon^4\langle AN_2^2\rangle v_\varepsilon^{(6)} = f(x_1), \tag{3.23}$$

where for condition (3.13) the coefficients at the second and fourth derivatives are the same as in the associated equation (3.15) and

$$\langle AN_2^2\rangle = \frac{(a^2-1)(a-1)(15a+8)}{122880a}. \tag{3.24}$$

Below we show that this equation has a solution which is unique up to an additive constant.

4 Comparison of the exact solution and the solution of the second order homogenized equation

Consider first the right-hand side in (2.1) in the form of $f(x_1) = \sin kx_1$, where k ($k \neq 0$) is a multiple of $2\pi/T$, and assume that $J = 1$ and N_2

satisfies condition (3.13). Then, the exact solution of the original problem (2.1)-(2.2) becomes (up to an additive constant)

$$u_\varepsilon(x) = \begin{cases} \Big(\alpha\big(\cosh kx_2 - \tanh(k\varepsilon/2)\sinh k|x_2|\big) - \frac{1}{ak^2}\Big)\sin kx_1, & |x_2| \in [\varepsilon/4, \varepsilon/2], \\ \Big(\alpha a\big(1 - \frac{\tanh(k\varepsilon/2)}{\tanh(k\varepsilon/4)}\cosh kx_2 - \frac{1}{k^2}\Big)\sin kx_1, & |x_2| \in [0, \varepsilon/4], \end{cases} \tag{4.1}$$

where

$$\alpha = \frac{1-a}{k^2 a\cosh(k\varepsilon/4)}\left(1 - \tanh(k\varepsilon/2)\tanh(k\varepsilon/4) - a\left(1 - \frac{\tanh(k\varepsilon/2)}{\tanh(k\varepsilon/4)}\right)\right)^{-1}.$$

The solutions of equations (3.15) and (3.23) are given respectively by formulae (up to an arbitrary additive constant)

$$v_\varepsilon(x_1) = \left(-k^2\frac{(a+1)}{2} + \varepsilon^2 k^4\frac{(a^2-1)}{128}\right)^{-1}\sin kx_1 \tag{4.2}$$

and

$$v_\varepsilon(x_1) = \left(-k^2\frac{(a+1)}{2} + \varepsilon^2 k^4\frac{(a^2-1)}{128} - \varepsilon^4 k^6\frac{(a^2-1)(a-1)(15a+8)}{122880a}\right)^{-1}\sin kx_1. \tag{4.3}$$

It can be easily verified that, to within the error of $O(\varepsilon^4)$, formulae (4.2) and (4.3) coincide with the asymptotic behaviour of the exact solution (4.1) along the midline $x_2 = 0$, calculated by expanding the hyperbolic functions in Taylor series at small ε.

The results of numerical comparison of exact and asymptotic solutions are presented in Figure 2 at $k = 1$ and $a = 0.5$. The function $B(\varepsilon) = -3v_\varepsilon(x_1)/(4\sin x_1)$ is plotted. Along with the formulae (4.2), (4.3) and (4.1), in which we set $u_\varepsilon(x_1, 0) = v_\varepsilon(x_1)$, the leading order asymptotic solution given by $B(\varepsilon) = 1$ is also displayed. Expansions (4.2) and (4.3) are in fact of the same order of accuracy, although the latter one corresponds to a coercive equation.

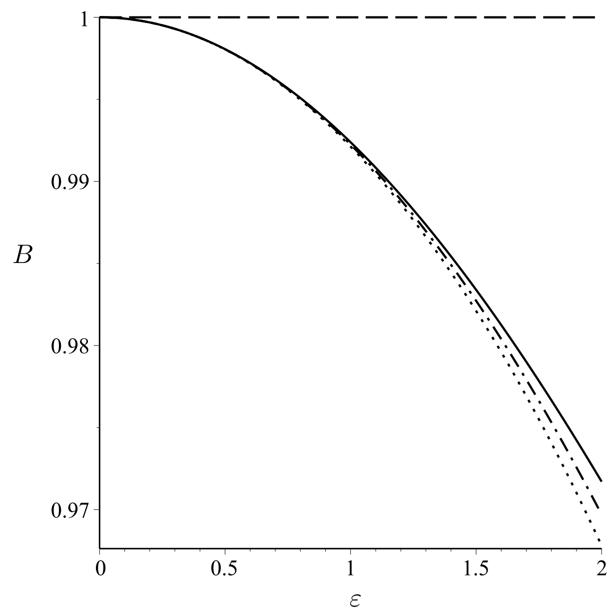

Figure 2: Comparison of exact solution (4.1) (solid line) with approximations (4.2) (dashdotted line) and (4.3) (dotted line). Leading order $B(\varepsilon) = 1$ is shown with a dashed line.

Consider now an arbitrary C^{∞}-smooth 2π-periodic function f with vanishing mean value over the period. Expand f in Fourier series

$$f(x_1) = \sum_{k=1}^{\infty}(a_k \cos kx_1 + b_k \sin kx_1). \tag{4.4}$$

Then, we have from (3.23)

$$v_\varepsilon(x_1) = \sum_{k=1}^{\infty} \Big(-k^2\langle A\rangle + \varepsilon^2 k^4 \langle AN_2\rangle - \varepsilon^4 k^6 \langle AN_2^2\rangle\Big)^{-1})(a_k \cos kx_1 + b_k \sin kx_1). \tag{4.5}$$

Note that due to the Cauchy-Schwarz-Buniakowsky inequality, $\langle AN_2\rangle^2 \leq \langle AN_2^2\rangle\langle A\rangle$. The latter is strict at $a \neq 1$. Thus,

$$\varepsilon^4 k^4 \langle AN_2^2\rangle - \varepsilon^2 k^2 \langle AN_2\rangle + \langle A\rangle \geq (4\langle AN_2^2\rangle\langle A\rangle - \langle AN_2\rangle^2)/(4\langle AN_2^2\rangle) \geq$$

$$3/4\langle A\rangle > 0.$$

As a result, for an infinitely smooth function f we prove the existence and uniqueness of the solution up to an additive constant. In addition, we have a priori estimate

$$\|v_\varepsilon\|_{H^{n+2}((-\pi,\pi))} \leq C_n \|f\|_{H^n((-\pi,\pi))}, \tag{4.6}$$

where constant $C_n > 0$ is independent of ε.

Then, the asymptotic expansion of the solution can be written as

$$v_\varepsilon^{(2N)}(x_1) = \sum_{j=0}^{2N} \varepsilon^{2j} v_{2j}(x_1), \tag{4.7}$$

where v_{2j} are 2π-periodic solutions of the equations

$$\langle A \rangle v''_{2j} + \langle AN_2 \rangle v''''_{2j-2} + \langle AN_2^2 \rangle v^{(6)}_{2j-4} = f(x_1)\delta_{j0}.$$

It should be noted that these equations take the same form for v_0 and v_2 as for the asymptotic solution of equation (3.6). Therefore, estimate (3.11) is still valid for $J = 1$, i.e.

$$\|\nabla \Big(u_\varepsilon - v_\varepsilon - \varepsilon^2 N_2 v''_\varepsilon \Big)\|_{L^2((-\pi,\pi)\times(-\varepsilon/2,\varepsilon/2))} = O(\varepsilon^{7/2}) \tag{4.8}$$

with v_ε denoting the solution of (3.23).

5 Conclusions

Two alternative approaches for formulating well posed high order homogenized equations are illustrated by a toy scalar problem for a thin three-layered laminate. The first of them assumes asymptotic reduction in the order of the derived fourth order homogenized equation (3.7) corresponding to a two-term asymptotic expansion of the original 2D problem, see final second order equation (3.17). Conversely, the second approach suggests the increase of the order of aforementioned equation (3.7), resulting in a sixth order one, see (3.23). In this case, a coercive equation (3.23) follows from a weak formulation (3.20) developed in the paper. The uniqueness and existence of the solution of (3.23) over an infinite strip is also addressed. In addition, the two-term asymptotic formulae derived from equations (3.17) and (3.23) with a sinusoidal right hand side are compared with the associated exact solution on the midline.

In the present paper we consider the periodic boundary conditions. It is worth noting that for other types of boundary conditions even coercive higher order equations, e.g. (3.17) at $a > 1$ and (3.23) have spurious particular solutions, for which asymptotic ansatz (3.1) is violated, see the discussion in [24], [15] for greater detail. In the latter case the treatment of boundary value problems may need a rather delicate treatment to suppress the contribution of spurious patterns, see also [31],[30].

References

[1] E.Cosserat, F.Cosserat, *Théorie des Corps Déformables*, Hermanns, Paris, 1909.

[2] R.D.Mindlin, Microstructure in linear elasticity.*Arch. Ration. Mech. Anal.,***16**:51-78, 1965.

[3] R.A.Tupin, Elastic materials with couple stresses.*Arch. Ration. Mech. Anal.,***11**:385-414, 1962.

[4] R.D. Mindlin, Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates, *J. Appl. Mech.*, 18(1):31-38, 1951.

[5] E. Reissner, The effect of transverse shear deformation on the bending of elastic plates, *J. Appl. Mech.*, 12(2): A69-A77, 1945.

[6] A.L. Goldenveizer, On Reissner's plate theory, *Izvestia Akademii Nauk SSSR OTN*, 4: 102-109, 1958.

[7] N.S.Bakhvalov, Averaging partial differential equations with rapidly oscillating coefficients. *Dokl. Acad. Nauk SSSR*, **221**:516-519, 1975.

[8] N. S. Bakhvalov and G.P. Panasenko, *Homogenization: Averaging Processes in Periodic Media,* Nauka, Moscow (in Russian), 1984. English translation in: Methamatics and Its Applications (Soviet Series), **36**, Kluwer Academic Publishers, Dordrecht-Boston-London, 1989.

[9] K.O. Friedrichs, R.F. Dressler. A boundary-layer theory for elastic plates. *Communications on Pure and Applied Mathematics.* 14(1):1-33, 1961.

[10] A. L. Goldenveizer. The approximation of the shell theory by the asymptotic analysis of elasticity theory. *Pr. Math. Mech.* , 27(4):593-608, 1963.

[11] A. L. Goldenveizer. The principles of reducing three-dimensional problems of elasticity to two-dimensional problems of the theory of plates and shells. In *Proceedings, Eleventh International Congress of Theoretical and Applied Mechanics* (H. Görtler , editor), pages 306-311. Berlin: Springer-Verlag, 1964.

[12] P.G. Ciarlet, P. Destuynder. A justification of the two-dimensional plate model. *J. Mécanique,* **18**,315-344, 1979.

[13] Gilbert, R. P., Hsiao, G. C., Schneider, M. The two-dimensional, linear orthotropic plate. *Applicable Analysis*, 15(1-4):147-169, 1983.

[14] A.L. Goldenveizer, J.D. Kaplunov, E.V. Nolde, Asymptotic analysis and refinement of Timoshenko-Reisner-type theories of plates and shells, *Trans. Acad. Sci. USSR. Mekhanika Tverd. Tela*, 25(6):126-139, 1990.

[15] A.L. Goldenveizer, J.D. Kaplunov, E.V. Nolde, On Timoshenko-Reissner type theories of plates and shells, *Int. J. Solids Struct.*, 30(5):675-94, 1993.

[16] S.A. Nazarov, and A.S. Zorin, Two-term asymptotics of the problem on longitudinal deformation of a plate with clamped edge. *Computer mechanics of solids,* 2:10-21, 1991 (Russian).

[17] S.A. Nazarov, On the accuracy of asymptotic approximations for longitudinal deformation of a thin plate. *Mathematical Modelling and Numerical Analysis* **30**, 2:185-213, 1996

[18] A.L. Goldenweiser, *The theory of elastically thin shells*, Science, Moscow, 1976.

[19] R.D. Gregory, F.Y.M. Wan, On plate theories and Saint-Venant's principle, *Int. J. Solids Struct.*, 21(10): 1005-1024, 1985.

[20] G.P.Panasenko, Boundary layer in homogenization problems for non-homogeneous media. *In Proceedings of the Fourth International Conference on Boundary and Interior Layers- Computational and Asymptotic Methods(BAIL 4).* Edited by S.K.Godunov, J.J.H.Miller, V.A.Novikov. Boole Press, Dublin,1986,pp.398-402.

[21] G.P. Panasenko, M.V. Reztsov, Averaging the 3-D elasticity problem in non homogeneous plates. *Doklady Akademii Nauk SSSR*, **294**, 5, 1061-1065, 1987 (in Russian); English transl. in *Soviet Math. Dokl.*, **35**, 3, 630-636, 1987 .

[22] G.P. Panasenko. *Multi-scale Modelling for Structures and Composites*, Springer, 2005.

[23] N.S.Bakhvalov, M.E.Eglit. Variational properties of averaged equations for periodic mediaI. *Proceedings of the Steklov Institute of Mathematics (Trudy Mat.Inst.Steklov)*, **192,**1990 English version**192,**:3-18,1992 .

[24] J.D. Kaplunov, L.Yu. Kossovitch, E.V. Nolde, *Dynamics of thin walled elastic bodies*, Academic Press, 1998.

[25] R.V. Craster, L.M. Joseph, J. Kaplunov, Long-wave asymptotic theories: the connection between functionally graded waveguides and periodic media, *Wave Motion*, 51(4): 581-588, 2014.

[26] N.S.Bakhvalov, M.E.Eglit. Investigation of the effective equations with dispersion for wave propagation in stratified media and thin plates. Doklady RAN, **383,** 6, 2002.

[27] N.S.Bakhvalov, K.Yu.Bogachev, M.E.Eglit. Investigation of the effective equations with dispersion for wave propagation in heterogeneous thin rods. Doklady RAN, **387,** 6:749-753, 2002.

[28] M.B.Panfilov, *Averaged Models of Flow Processes with Heterogeneous Internal Structure,* Doctor Sci. Thesis, 1992.

[29] V.P. Smyshlyaev, K.D. Cherednichenko. On derivation of "strain gradient" effects in the overall behaviour of periodic heterogeneous media. *J. Mech. Phys. Solids*, 48:1325-1357, 2000.

[30] G.P.Panasenko, Boundary conditions for the high order homogenized equation: laminated rods, plates and composites, *C.R. Mécanique*, **337**, 1, 8-14, 2009.

[31] J.D. Kaplunov, A.V. Pichugin, On rational boundary conditions for higher-order long-wave models. *In IUTAM Symposium on Scaling in Solid Mechanics*, pp. 81-90, Springer, Dordrecht, 2009.

[32] J.D. Kaplunov, E.V. Nolde, B.F. Shorr, A perturbation approach for evaluating natural frequencies of moderately thick elliptic plates, *J. Sound Vib.*, 281(3-5):905-19, 2005.

[33] G.P. Panasenko, Asymptotic partial decomposition of variational problems,*C. R. Acad. Sci. Paris*, **327**, Série IIb, 1185-1190, 1999.

[34] L. Prikazchikova, Y. Ece Aydin, B. Erbas, J. Kaplunov, Asymptotic analysis of an anti-plane dynamic problem for a three-layered strongly inhomogeneous laminate, *Math. Mech. Solids*, 25(1):3-16, 2020.

[35] J. Kaplunov, L. Prikazchikova, M. Alkinidri, Antiplane shear of an asymmetric sandwich plate, *Contin. Mech. Thermodyn.*, 33:1247-1262, 2021.