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On undecidability bounds for matrix decision problems

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Abstract

In this paper we consider several reachability problems such as vector reachability, membership in matrix semigroups and reachability problems in piecewise linear maps. Since all of these questions are undecidable in general, we work on lowering the bounds for undecidability. In particular, we show an elementary proof of undecidability of the reachability problem for a set of 5 two-dimensional affine transformations. Then, using a modified version of a standard technique, we also prove that the vector reachability problem is undecidable for two (rational) matrices in dimension 11. The above result can be used to show that the system of piecewise linear functions of dimension 12 with only two intervals has an undecidable set-to-point reachability problem. We also show that the "zero in the upper right corner" problem is undecidable for two integral matrices of dimension 18 lowering the bound from 23.

Keywords: Computability theory; Membership; Vector reachability; Matrix semigroups; Piecewise linear maps

1. Introduction

The significant property of iterative maps as well as dynamical systems in general is that the slightest uncertainty about the initial state leads to very large uncertainty after some time. With such initial uncertainties, the system's behaviour can only be predicted accurately for a short amount of time into the future. Many fundamental questions about iterative maps are closely related to reachability problems in different abstract structures. The question of whether these problems have an algorithmic solution or not is important since it gives an answer to many practical questions related to the analysis of model systems, whose components are governed by simple laws, but whose overall behaviour is highly complex.

Let us start with an example of Iterative Function Systems (IFS). IFS can be defined as a set of affine transformations that are iteratively applied (equiprobably or with assigned probability) starting from some initial state. A fascinating property of IFS is that they have the ability to create incredibly complex images by very small sets of functions. A spleenwort fern (Fig. 1, left) can be generated by 4 planar affine transformations and the well-known Sierpinski gasket (Fig. 1, centre) is an attractor for only 3 affine transformations. The "Dragon Fractal" is another example that is generated by affine transformations and is shown in Fig. 1 on the right. Some iterative function systems

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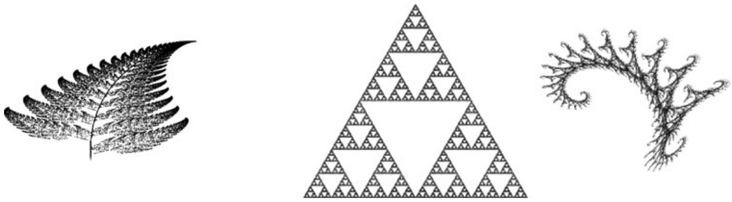

Fig. 1. Fractals generated by two-dimensional affine maps.

may have a very strange combination of both predictable coarse-grain patterns and unpredictable (undecidable) fine-grain questions such as point-to-point reachability.

We show that it is possible to encode Post's Correspondence Problem (PCP) within iterative function systems with 5 rules. This shows that IFSs with 5 transformations are in fact so complex that they can be used as a computational device. This is not a first attempt to encode PCP into an IFS framework. However, in contrast to the previous research [8], which shows only the undecidability of parameterised reachability, we show the undecidability of point-to-point reachability for a set of non-deterministic affine maps.

The question as to whether the reachability problem is decidable for a set of one-dimensional affine transformations applied in a non-deterministic way is currently an open problem. It is also not clear whether there exists any class of more complex functions for which the reachability problem in non-deterministic maps becomes undecidable in one dimension. Similar open problems are stated in [1,14] for *piecewise* affine or polynomial maps. It is known that the reachability question for piecewise maps in the case of elementary [16], rational [17] or analytic maps [15] is undecidable even in one dimension. In the first part of this paper we show an elementary proof of the undecidability of the reachability problem for two-dimensional affine transformations. This leads us to the undecidability result for the vector reachability problem in 3×3 rational matrix semigroups.

We then extend the result to show the undecidability of the vector reachability problem in the case of a matrix semigroup generated by two rational matrices of dimension 11. Using a similar encoding, we improve the bound for the "zero in the upper right corner problem". It was shown that for a semigroup generated by two integral matrices of dimension 24, the problem of determining if some matrix in the semigroup has zero in the upper right corner is undecidable [6]. This bound was later improved to 23 [9]. We show that the problem is still undecidable for two integral matrices of dimension 18.

We also show how to lower the dimensions in related problems concerning control systems that are defined by a system of piecewise transformations. We apply our result for the undecidability of the vector reachability problem in dimension 11 to show that set-to-point reachability is undecidable for piecewise linear functions of dimension 12 with only two intervals. The natural problem we can thus state for iterative maps is to find undecidable problems concerning maps defined by only two transformations in the smallest possible dimensions.

Only a few algorithms for vector reachability or membership problems are known. In particular, the above problems are decidable for semigroups with commutative matrices [2] or row-monomial matrices over commutative elements [18]. The mortality problem (membership of the zero matrix) is known to be decidable only in the case of two 2×2 matrices [4]. So another natural problem is the identification of decidable fragments and the construction of effective semialgorithms that can solve membership and vector reachability problems at least in many cases.

2. Reachability in affine and linear transformations

In this section we show an elementary proof of the undecidability of the reachability problem for two-dimensional affine transformations and the vector reachability problem for 3×3 rational matrix semigroups.

Let us define $\Sigma = \{a, b\}$ to be a binary alphabet and Σ^* to be the free monoid generated by Σ. For any word, $w = w_1 w_2 \cdots w_k$ we denote the reverse of the word by $w^R = w_k \cdots w_2 w_1$. Define a mapping $\psi' : \Sigma \cup \{\epsilon\} \to \mathbb{Q}^{2\times 2}$ by

$$\psi'(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \psi'(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \psi'(b) = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

where $\psi'(\epsilon)$ is the identity matrix I_2. We can now define $\psi : \Sigma^* \to \mathbb{Q}^{2\times 2}$ by:

$$\psi(w) = \psi'(w_1) \cdot \psi'(w_2) \cdot \cdots \cdot \psi'(w_n) \quad | w = w_1 w_2 \cdots w_n \in \Sigma^*.$$

It is easy to check that the matrices $\psi'(a)$ and $\psi'(b)$ generate a free semigroup [5] and the mapping ψ is an isomorphism between Σ^* and the monoid generated by $\{\psi'(\epsilon), \psi'(a), \psi'(b)\}$. We also define the mapping ϕ' using the inverse matrices:

$$\phi'(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \phi'(a) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \qquad \phi'(b) = \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and similarly we have the related morphism $\phi : \Sigma^* \to \mathbb{Q}^{2\times 2}$ given by:

$$\phi(w) = \phi'(w_1) \cdot \phi'(w_2) \cdot \cdots \cdot \phi'(w_n) \quad | w = w_1 w_2 \cdots w_n \in \Sigma^*.$$

Notice that for any word $w \in \Sigma^*$, $\psi(w)$ and $\phi(w)$ will have a matrix representation of the following form:

$$\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \quad | x, y \in \mathbb{Q}.$$

Post's correspondence problem (in short, PCP) is formulated as follows: Given a finite alphabet Γ and a finite set of pairs of words in Γ^*:

$$\{(u_1, v_1), \ldots, (u_k, v_k)\},$$

does there exist a finite sequence of indices $(i_1, i_2, \ldots, i_m)$ with $1 \leq i_j \leq k$ for $1 \leq j \leq m$, such that $u_{i_1} u_{i_2} \cdots u_{i_m} = v_{i_1} v_{i_2} \cdots v_{i_m}$? It can be shown that this problem is undecidable even with a binary alphabet Γ. PCP(n) denotes the problem with a set of n pairs. We write n_p for the minimum size PCP is known to be undecidable (currently 7, see [19]). In some of our reductions we will also use Claus Instances of PCP, where we have the same undecidability result under the assumption that two pairs will be used only once and their positions are fixed (see [7,10]).

Theorem 1 *([7,10]). It is undecidable whether an instance of PCP consisting of 7 pairs of words has a solution $(i_1, i_2, \ldots, i_m)$, where $i_1 = 1$, $i_m = 7$ and $2 \leq i_j \leq 6$ for $1 \leq j \leq m$.*

In other words, the first and last pair of words of a solution is fixed and the remaining 5 pairs occur between these pairs in an arbitrary way. In fact all solutions *must* be of this form. We shall use this theorem to reduce some later undecidability bounds.

Problem 2. Decide whether a point $(x_0, y_0) \in \mathbb{Q}^2$ can be mapped to another point $(x_1, y_1) \in \mathbb{Q}^2$ by non-deterministically applying a sequence of two-dimensional affine transformations from a given finite set.

We shall now show that this problem is undecidable.

Theorem 3. *PCP(n) can be reduced to Problem 2 with a set of n affine transformations.*

Proof. Given a set of pairs of words over a binary alphabet Σ:

$$\{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \qquad | u_i, v_i \in \Sigma^+.$$

Let us construct a set of pairs of 2×2 matrices using the two mappings ϕ and ψ, i.e.: $\{(\phi(u_1), \psi(v_1)), \ldots, (\phi(u_n), \psi(v_n))\}$.

Instead of the equation $u = v$, we consider a concatenation of two words $u^R \cdot v$, which is a palindrome in the case where $u = v$. In fact by using inverse elements for the u word (denoted by $\overline{u}$), then u is equal to v if and only if $\overline{u}^R \cdot v = \epsilon$. We call this an *inverse palindrome* since it equals the identity iff $\overline{u}^R$ is the inverse of v. Initially we take an empty word and for every pair (u_i, v_i) that we use, we concatenate the reverse of word $\overline{u}_i$ (using inverse elements) to the left and word v_i to the right.

Let us consider now a matrix interpretation of the PCP problem. We associate a 2×2 matrix

$$C = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$$

with a word w of the form $u^R \cdot v$. Initially C is an identity matrix corresponding to an empty word. The extension of a word w by a new pair of words (u_i, v_i) (i.e. that gives us $w' = u_i^R \cdot w \cdot v_i$) corresponds to the following matrix multiplication

$$C_{w'} = C_{u_i^R \cdot w \cdot v_i} = \phi(u_i^R) \cdot C_w \cdot \psi(v_i). \tag{1}$$

Eq. (1) is therefore written:

$$\begin{pmatrix} 1 & x' \\ 0 & y' \end{pmatrix} = \begin{pmatrix} 1 & p_1 \\ 0 & p_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \cdot \begin{pmatrix} 1 & q_1 \\ 0 & q_2 \end{pmatrix}. \tag{2}$$

Note that $p_2, q_2 \neq 0$, since they are invertible. If we multiply the matrices in (2) we have a very simple transformation from a word $v_i^R \cdot w \cdot u_i$ to a matrix:

$$\begin{pmatrix} 1 & x' \\ 0 & y' \end{pmatrix} = \begin{pmatrix} 1 & q_2 x + q_2 p_1 y + q_1 \\ 0 & q_2 p_2 y \end{pmatrix}. \tag{3}$$

Thus we can state that PCP has a solution iff we can get the identity matrix by using a set of transformations defined for each pair of words from PCP.

In fact we can rewrite (3) as a two-dimensional affine transformation:

$$\begin{cases} x' = q_2 x + q_2 p_1 y + q_1 \\ y' = p_2 q_2 y \end{cases}$$

where we define one such affine transformation for each pair in the PCP. It can now be seen that the problem is reduced to the question about reaching the point $(x_1, y_1) = (0, 1)$, starting from the point $(x_0, y_0) = (0, 1)$. This follows since $(0, 1)$ corresponds to the identity matrix (see [21]) in the above calculations. □

Corollary 4. *Problem 2 is undecidable for a set of five affine transformations of dimension two.*

Proof. Let us fix the use of transformations associated with the first and the last pair of words in PCP. In order to do so, we will apply the transformation corresponding to the first pair to the original starting point and the inverse of the transformation corresponding to the last pair to the final point. It is straightforward to see that the inverse transformation exists, since its corresponding matrix is invertible. Since PCP is undecidable for Claus instances of size 7 [10] the number of transformations can be reduced to 5. For more details, see the proof of Theorem 6. □

Problem 5 (*Vector Reachability*). Given a semigroup of matrices $S \subset \mathbb{F}^{n \times n}$ and two vectors $x, y \in \mathbb{F}^n$ (where $\mathbb{F}$ is an arbitrary ring), does there exist some $M \in S$ such that $Mx = y$?

The above problem can thus be defined on integer, rational or complex matrices and vectors for example. We shall now show that the vector reachability problem is undecidable for rational matrices of dimension 3.

Theorem 6. *The vector reachability problem in matrix semigroups generated by five rational matrices of dimension* 3 *is undecidable.*

Proof. Let us convert each two-dimensional affine transformation into a three-dimensional linear transformation as follows:

$$\begin{cases} x' = q_2 x + q_2 p_1 y + q_1 \\ y' = p_2 q_2 y \end{cases} \Rightarrow \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} q_2 & q_2 p_1 & q_1 \\ 0 & p_2 q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Thus for a set of n affine functions, this conversion gives us a set of matrices $\{M_1, M_2, \ldots, M_n\}$, where $M_i \in \mathbb{Q}^{3 \times 3}$ for $1 \leq i \leq n$.

From the proof of Theorem 3 follows that the problem to decide whether there exists a product $M = M_{i_1} M_{i_2} \cdots M_{i_k}$, where $1 \leq i_j \leq n$ for $1 \leq j \leq k$ such that $Mv = v$ where $v = (0, 1, 1)^{\mathrm{T}}$ is undecidable. It was stated that PCP for Claus instances of size 7 is undecidable in Corollary 4. Thus applying the same idea of incorporating the matrices corresponding to the first and the last pairs of words to the original and final vectors, we only needed 5 matrices. So fixing the first and the last pair of words in PCP we have to check whether there is a matrix $M = M_{i_1} M_{i_2} \cdots M_{i_k}$, where $2 \leq i_j \leq 6$ for $1 \leq j \leq k$ such that $Mw = s$ where $w = M_7 v$, $s = {M_1}^{-1} v$ and $v = (0, 1, 1)^{\mathrm{T}}$. M_1 is invertible by the construction. $\square$

It is also possible to get a symmetric result by converting the additive form of linear transformations into multiplicative form. In this case we will obtain the undecidability of the reachability problem for two-dimensional transformations of the following form:

$$\begin{cases} x = 2^{q_1} x^{q_2} y^{q_2 p_1} \\ y = y^{p_2 q_2}. \end{cases}$$

Theorem 7. *The vector reachability problem is undecidable for two rational matrices of dimension* $2n_{cp} - 3$ *(where* n_{cp} *is the size of Claus instances for which PCP is undecidable).*

Proof. We use a modification of a standard technique for converting membership problems from a set of matrices into one defined by just two matrices (see, for example, [6] or [3]). We first obtain the undecidability for two matrices of dimension 15, then show how this can be reduced to just 11.

Given a set of matrices $\{M_1, M_2, \ldots, M_n\}$ where $M_i \in \mathbb{Q}^{m \times m}$. Let us define two block diagonal matrices A' and T' by:

$$A' = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & M_n \end{pmatrix}, \qquad T' = \begin{pmatrix} 0 & I_m \\ I_{m(n-1)} & 0 \end{pmatrix}$$

where 0 denotes a submatrix with zero elements. Clearly the dimension of both of A' and T' is nm. Further, it can be seen that for any $1 \leq j \leq n$ then $T'^{n-j+1} A' T'^{j-1}$ cyclically permutes the blocks of A' so that the direct sum of $T'^{n-j+1} A' T'^{j-1}$ is $M_j \oplus M_{j+1} \oplus \cdots \oplus M_n \oplus M_1 \oplus \cdots \oplus M_{j-1}$. We can also note that $A' \sim T'^{n-j+1} A' T'^{j-1}$ (i.e. this is a similarity transform) since $T'^{n-j+1} \cdot T'^{j-1} = T'^n = I_n$. It is therefore apparent that any product of the matrices can thus occur and in fact can appear in the first block of the nm matrix product.

Let us define a vector $x = (v^{\mathrm{T}}, 0, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{Q}^{nm \times 1}$ where $v = (0, 1, 1)^{\mathrm{T}}$ as before and two matrices F_{M_1}, L_{M_n} as follows:

$$F_{M_1} = \begin{pmatrix} M_1 & 0 \\ 0 & I_{m(n-1)} \end{pmatrix}, \qquad L_{M_n} = \begin{pmatrix} M_n & 0 \\ 0 & I_{m(n-1)} \end{pmatrix}.$$

It is easily observed that there exists a matrix product $M = M_{i_1} M_{i_2} \cdots M_{i_t}$ where $2 \leq i_j \leq n - 1$ for $1 \leq j \leq t$ satisfying $M_1 M M_7 v = v$ iff there exists a product $R' = \{A', T'\}^+$ satisfying $R'w = s$, where $w = L_{M_n} x$, $s = {F_{M_1}}^{-1} x$ (F_{M_1} will be invertible by the construction). From Theorem 6 and Corollary 4, this establishes the undecidability of vector reachability over 2 matrices of dimension $3 \cdot 5 = 15$.

We can observe however that $(M_i)_{[3,3]} = 1$ and M_i is upper triangular for all $1 \leq i \leq n$. Let us now construct four new matrices of dimension $2n_{cp} - 3$:

$$A = \begin{pmatrix} (q_2)_1 & (q_2 p_1)_1 & 0 & 0 & \cdots & (q_1)_1 \\ 0 & (p_2 q_2)_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & (q_2)_2 & (q_2 p_1)_2 & \cdots & (q_1)_2 \\ 0 & 0 & 0 & (p_2 q_2)_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & I_2 & 0 \\ I_{2n_{cp}-6} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} (q_2)_1 & (q_2 p_1)_1 & 0 & (q_1)_1 \\ 0 & (p_2 q_2)_1 & 0 & 0 \\ 0 & 0 & & I_{2n_{cp}-5} \end{pmatrix}, \qquad L = \begin{pmatrix} (q_2)_{n_{cp}} & (q_2 p_1)_{n_{cp}} & 0 & (q_1)_{n_{cp}} \\ 0 & (p_2 q_2)_{n_{cp}} & 0 & 0 \\ 0 & 0 & & I_{2n_{cp}-5} \end{pmatrix}$$

where 0 denotes either the number zero or a submatrix with zero elements, I_k is the k-dimensional identity matrix and $(x)_i$ denotes the element x from matrix M_i. Straightforward calculation shows that $T^{n-j+1}AT^{j-1}$ permutes the pairs of *rows* in A and using a similar argument as before, we thus can form any product of matrices in the first two rows of this matrix. We define $(2n_{cp}-3)$-dimensional vectors $w' = L(0,1,0,\dots,0,1)^{\mathrm{T}}$ and $s' = F^{-1}(0,1,0,\dots,0,1)^{\mathrm{T}}$ to act in the same way as w and s did previously (again F is invertible since $q_2, p_2 \neq 0$). Finally we see that there exists a solution to PCP iff there exists a product $R \in \{A, T\}^+$ satisfying $Rw' = s'$. Note that in this construction we have two matrices of dimension $2n_{cp}-3$. □

Since PCP is undecidable for 7 pairs where two of them are fixed ($n_{cp} = 7$), the dimension of the two rational matrices for which the vector reachability is undecidable is therefore $(2 \cdot 7) - 3 = 11$ as required:

Corollary 8. *The vector reachability problem is undecidable for two matrices of dimension* 11.

Theorem 9. *PCP(n) can be reduced to the vector reachability problem for a semigroup generated by n integral matrices of dimension* 4.

Proof. Given a set of pairs of words over a binary alphabet $\Sigma = \{a, b\}$: $\{(u_1, v_1), \dots, (u_n, v_n)\}$. Let us construct the set of pairs of 2×2 matrices using two mappings μ and ξ: $\{(\mu(u_1), \xi(v_1)), \dots, (\mu(u_n), \xi(v_n))\}$ that we define in the same way as we did for ϕ and ψ but using another free semigroup of integer matrices:

$$\xi'(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \xi'(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \xi'(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$\mu'(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mu'(a) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \mu'(b) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

where $\xi'(\epsilon)$ and $\mu'(\epsilon)$ is the identity matrix I_2. We also define two mappings from words to matrices as follows

$$\xi(w) = \xi'(w_1) \cdot \xi'(w_2) \cdots \xi'(w_n) \quad | w = w_1 w_2 \cdots w_n \in \Sigma^*$$
$$\mu(w) = \mu'(w_1) \cdot \mu'(w_2) \cdots \mu'(w_n) \quad | w = w_1 w_2 \cdots w_n \in \Sigma^*.$$

As in Theorem 3, instead of equation $u = v$ we consider a concatenation of two words $u^R \cdot v$ that is a palindrome in the case that $u = v$. We associate 2×2 matrix C with a word w of the form $u^R \cdot v$. Initially C is an identity matrix corresponding to an empty word. The extension of a word w by a new pair of words (u_r, v_r) (i.e. that gives us $w' = u_r^R \cdot w \cdot v_r$) corresponds to the following matrix multiplication

$$C_{w'} = C_{u_r^R \cdot w \cdot v_r} = \mu(u_r^R) \cdot C_w \cdot \xi(v_r). \tag{1}$$

Let us rewrite the operation (1) in more details.

$$\begin{pmatrix} c_{w'}^{11} & c_{w'}^{12} \\ c_{w'}^{21} & c_{w'}^{22} \end{pmatrix} = \begin{pmatrix} u^{11} & u^{12} \\ u^{21} & u^{22} \end{pmatrix} \cdot \begin{pmatrix} c_{w}^{11} & c_{w}^{12} \\ c_{w}^{21} & c_{w}^{22} \end{pmatrix} \cdot \begin{pmatrix} v^{11} & v^{12} \\ v^{21} & v^{22} \end{pmatrix}. \tag{1'}$$

From the injectivity of the mappings, it follows that $u = u_{i_1} u_{i_2} \cdots u_{i_n} = v_{i_1} v_{i_2} \cdots v_{i_n} = v$ for a finite sequence of indices $\{i_j\}$ with $i_j \in \{1..k\}$ if and only if $\mu(u^R) \cdot \xi(v)$ is equal to the identity matrix. So the question of the word equality can be reduced to the problem of finding a sequence of pairwise matrix multiplications that gives us the identity matrix.

Now we show that it is possible to avoid pairwise matrix multiplications by increasing the dimension from 2 to 4. Actually we represent matrices C_w and $C_{w'}$ from (1′) as 4×1 vectors and we unite every pair of matrices $\mu(u_r^R)$ and

$\xi(v_r)$ into 4×4 joint matrix $M_{u_r^R, v_r}$ in the following way:

$$\begin{pmatrix} c_{w'}^{11} \\ c_{w'}^{12} \\ c_{w'}^{21} \\ c_{w'}^{22} \end{pmatrix} = \left(\begin{pmatrix} u^{11} & u^{12} \\ u^{21} & u^{22} \end{pmatrix} \otimes \begin{pmatrix} v^{11} & v^{12} \\ v^{21} & v^{22} \end{pmatrix}^{\mathrm{T}} \right) \cdot \begin{pmatrix} c_{w}^{11} \\ c_{w}^{12} \\ c_{w}^{21} \\ c_{w}^{22} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} u^{11} \cdot v^{11} & u^{11} \cdot v^{21} & u^{12} \cdot v^{11} & u^{12} \cdot v^{21} \\ u^{11} \cdot v^{12} & u^{11} \cdot v^{22} & u^{12} \cdot v^{12} & u^{12} \cdot v^{22} \\ u^{21} \cdot v^{11} & u^{21} \cdot v^{21} & u^{21} \cdot v^{11} & u^{22} \cdot v^{21} \\ u^{21} \cdot v^{12} & u^{21} \cdot v^{22} & u^{21} \cdot v^{12} & u^{22} \cdot v^{22} \end{pmatrix}}_{M_{u_r^R, v_r}} \cdot \begin{pmatrix} c_{w}^{11} \\ c_{w}^{12} \\ c_{w}^{21} \\ c_{w}^{22} \end{pmatrix} \tag{2}$$

where $\otimes$ denotes the tensor product. Note that the Eq. (1′) is equivalent to Eq. (2) in the sense that the expression for computing values of $c_{w'}^{11}$, $c_{w'}^{12}$, $c_{w'}^{21}$ and $c_{w'}^{22}$ in (2) coincide with the corresponding values in (1′).

Thus for every pair of words (u_r, v_r) we construct the matrix $M_{u_r^R, v_r}$. Now PCP(n) can be reduced to the following vector reachability problem: Given a matrix semigroup S generated by a set of matrices $\{M_{u_1^R, v_1}, \ldots, M_{u_n^R, v_n}\}$, decide whether there is a matrix $M \in S$ such that $M \cdot (1, 0, 1, 0)^{\mathrm{T}} = (1, 0, 1, 0)^{\mathrm{T}}$. It follows that if we can solve this problem then we can solve the PCP(n). □

Matiyasevich and Sénizergues proved in [19] that PCP(7) is undecidable. Thus the vector reachability problem is undecidable for semigroups generated by 7 integral matrices of dimension 4. Since all matrices in the generator of the semigroup are invertible, we can apply the "Claus instances" again and incorporate the first and last matrix to the initial and final vectors. Thus, the following corollary of Theorem 9 holds:

Corollary 10. *The vector reachability problem is undecidable for a semigroup generated by* 5 *integer matrices of dimension* ≥ 4.

Next we show a related problem but instead of considering vector reachability in a semigroup, we instead choose the next matrix to apply in a piecewise manner depending upon the current position of an element in the vector. This relates non-deterministic and piecewise reachability questions in a natural way.

Let us consider an arbitrary piecewise linear map $F\colon D \to D$, $D \subseteq \mathbb{Q}^k$ defined by

$$x_{n+1} = M_i x_n, \quad \text{for } x_n \in D_i$$

where $D = D_1 \cup \cdots \cup D_m$; $D_i \cap D_j = \varnothing$ for $i \neq j$; x is a vector in $\mathbb{Q}^k$ and M_i is a matrix in $\mathbb{Q}^{k \times k}$.

An iterative piecewise linear map generates a sequence of points. One of the obvious problems that arises in such systems is a reachability problem that can be formulated as a point-to-point, set-to-point, point-to-set or set-to-set reachability problem. We shall show that the set-to-point reachability problem is undecidable for an iterative piecewise linear map of dimension 12 with only two disjoint partitions.

Theorem 11. *Given a piecewise linear map defined on two domains*

$$F(x_{n+1}) = \begin{cases} M_1 x, & x \in D_1 \\ M_2 x, & x \in D_2 \end{cases}$$

where a pair of matrices $M_1, M_2 \in \mathbb{Q}^{k \times k}$, $D = D_1 \cup D_2 \subseteq \mathbb{Q}^k$ *and a vector* $y \in \mathbb{Q}^k$. *The problem of deciding if there exists any initial point* $w' \in D$ *from which we can reach* y *in the iterative piecewise map is undecidable for* $k = 2n_{cp} - 2$, *(where* n_{cp} *is the smallest size Claus instance known to be undecidable, currently* 7*).*

Proof. We extend the matrices $A, T \in \mathbb{Q}^{m \times m}$ defined in Theorem 7 by 1 dimension therefore $k = m + 1$. Let:

$$\Upsilon = \begin{pmatrix} A & z \\ d_1 & 10 \end{pmatrix}, \qquad W = \begin{pmatrix} T & z \\ d_2 & 10 \end{pmatrix}$$

where $z = (0, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{Q}^{(k-1)}$ is the zero vector, $d_1 = (0, 0, \ldots, -1)$, $d_2 = (0, 0, \ldots, -2)$ and $d_1^{\mathrm{T}}, d_2^{\mathrm{T}} \in \mathbb{Q}^{(k-1)}$. We also extend the vector w used in the previous proof to $w' = (w^{\mathrm{T}}, x)^{\mathrm{T}} = (0, 1, 0, 0, \ldots, 1, x)^{\mathrm{T}} \in \mathbb{Q}^k$ where $x \in \mathbb{Q}$ and $x \in (0, 1]$. We will choose which matrix (Υ or W) to apply at each step depending upon the current x value. Define $y' = (0, 1, 0, 0, \ldots, 1, 0)^{\mathrm{T}} \in \mathbb{Q}^k$.

Let vector w' be given where $0 < x \leq 1$. We choose which matrix to multiply by on the next step by a simple rule. Let $t = (0, 0, \ldots, 1)^{\mathrm{T}} \in \mathbb{Q}^{k \times 1}$. Then at each step of the computation we update w' according to:

$$w' = \begin{cases} \Upsilon w' & ; \text{ if } \frac{1}{10} \leq w'^{\mathrm{T}} \cdot t < \frac{2}{10} \\ W w' & ; \text{ if } \frac{2}{10} \leq w'^{\mathrm{T}} \cdot t < \frac{3}{10} \end{cases}$$

and the next step if undefined outside these regions. Note that $\Upsilon w'$ is the same as in the previous proof except for the last element of the vector. This is equal to $10x - 1$, which is equivalent to shifting the decimal representation of x to the left and subtracting the integer part (since the first decimal digit is a 1). Similarly, $(Ww')_{[k,1]} = 10x - 2$ which is equivalent to shifting the decimal one place left and subtracting the integer part.

This is applied in an iterative manner until either $0 < x < \frac{1}{10}$ or $x \geq \frac{3}{10}$ (which is undefined so we halt) or else $x = 0$ in which case, if $w' = y'$ then there exists a correct solution to the PCP as in the previous proof. If $w' \neq y'$ then x did not correspond to a correct solution and we halt because the next step is undefined (when $x = 0$). $\square$

The previous theorem states that a set-to-point reachability problems is undecidable for piecewise linear map with only two partitions of dimension 12. The initial point defines the order of indices for PCP problem. If there exists some x with the desired property (e.g. of the form $x = 0.1221121\ldots$) then we choose the next matrix depending on whether we have a 1 or a 2 in the next decimal position (e.g. $\Upsilon W W \Upsilon \Upsilon W \Upsilon$). Note that in this case, if a point $y' = (0, 1, 0, \ldots, 1, 0)^{\mathrm{T}} \in \mathbb{Q}^k$ is reachable from a point $w' = (w^{\mathrm{T}}, x)^{\mathrm{T}} = (0, 1, 0, 0, \ldots, 1, x)^{\mathrm{T}} \in \mathbb{Q}^k$ then the decimal part of x at the start of the computation corresponded to a sequence of indices giving a solution to PCP.

3. Zero in the upper right corner problem

We now move to a different problem which has been studied in the literature and is related to Skolem's problem.

Problem 12. Given a set of matrices $M_1, M_2, \ldots, M_n$ of dimension m generating a semigroup S, is it decidable if there exists a matrix $M \in S$ such that $M_{[1,m]} = 0$. I.e. does there exist a matrix $M \in S$ with a 0 in the upper right corner?

This problem has been studied for two main reasons. Firstly it is related to the mortality problem (Does the zero matrix belong to a semigroup?). Actually the upper *left* corner is used in the proof but the upper right corner problem is used in related problems. See for example [12].

Secondly the problem is related to Skolem's problem of finding zeros in linear recurrent sequences. It can be easily shown that Skolem's problem can be reformulated in terms of a single matrix R and the question, "Does there exist a $k \in \mathbb{Z}^+$ such that $(R^k)_{[1,m]} = 0$ where m is the degree of the linear recurrence?". For an overview of this problem and a proof of the decidability for degree 5 linear recurrence sequences, see the recent paper [11].

In terms of undecidability, it was shown that for two integral matrices of dimension 24 the zero upper right corner problem is undecidable [6]. This was improved to dimension 23 in [9]. Using a similar idea to that shown above and the technique used in [6], we show how to improve the bound to two integral matrices of dimension 18.

Theorem 13. *Given two matrices $A, B \in \mathbb{Z}^{n \times n}$ forming a semigroup S, it is undecidable if there exists a matrix $X \in S$ with a zero in the upper right corner for $n = 2n_p + 4$ (currently 18).*

Proof. It can be seen that there exists an injective morphism between words over a binary alphabet $\Gamma = \{L_1, L_2\}$ and three-dimensional matrices. In fact, one such morphism, which was originally used by Paterson to prove the undecidability of the mortality problem [20], is $\lambda' : \Gamma^* \times \Gamma^* \mapsto \mathbb{Z}^{3\times 3}$ defined by:

$$\lambda'(u, v) = \begin{pmatrix} 3^{|u|} & 0 & \sigma(u) \\ 0 & 3^{|v|} & \sigma(v) \\ 0 & 0 & 1 \end{pmatrix}$$

for two words $u = L_{i_1} L_{i_2} \cdots L_{i_r}$ and $v = L_{j_1} L_{j_2} \cdots L_{j_s}$, with $i_1, \ldots, i_r, j_1, \ldots, j_s \in \{1, 2\}$, where we define

$$\sigma(u) = \sum_{k=1}^{|u|} i_k 3^{|u|-k}$$

and similarly for v. This matrix is the unique 3-adic representation of the binary words $u, v \in \Gamma^*$. Now consider the following matrix:

$$H = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which is self-inverse since $HH = I_3$. We can thus define a similarity transform $H\lambda'(u, v)H$ which gives us the alternate (but still injective) morphism:

$$\lambda(u, v) = H\lambda'(u, v)H = \begin{pmatrix} 3^{|u|} & 3^{|v|} - 3^{|u|} & \sigma(v) - \sigma(u) \\ 0 & 3^{|v|} & \sigma(v) \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that $u = v$ iff $\lambda(u, v)_{[1,3]} = 0$ since the top right element of the matrix is the subtraction of the 3-adic representations of u, v.

Using the same idea as before, we can now encode n such matrices into a single matrix, B, of size $2n + 1$ and use a second matrix T which is identical to the permutation matrix defined previously. Therefore, given n pairs of words $\{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ we define:

$$B = \begin{pmatrix} 3^{|u_1|} & 3^{|v_1|} - 3^{|u_1|} & 0 & 0 & \cdots & \sigma(v_1) - \sigma(u_1) \\ 0 & 3^{|v_1|} & 0 & 0 & \cdots & \sigma(v_1) \\ 0 & 0 & 3^{|u_2|} & 3^{|v_2|} - 3^{|u_2|} & \cdots & \sigma(v_2) - \sigma(u_2) \\ 0 & 0 & 0 & 3^{|v_2|} & \cdots & \sigma(v_2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We can see that a product containing *both* B and T has a zero in the upper right corner iff there exists a solution to the PCP however T has a zero upper right corner on its own. We can apply the encoding technique used in [6] so that the case with a power of only T matrices can be avoided. Define:

$$B' = \begin{pmatrix} 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & B & z \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad T' = \begin{pmatrix} 0 & 1 & x & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & T & z \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $x = (1, 0, \cdots, 0)$, $z = (0, 0, \cdots, 1)^{\mathrm{T}}$, with $x \in \mathbb{Z}^{1\times k}$, $z \in \mathbb{Z}^{k\times 1}$ and k is the dimension of matrix B (and T). It is clear that the sub-matrices B, T are multiplied in the same way as before and unaffected by this extension. Notice the $[2, 2]$ element is 0 in B' and 1 in T'. This is used to avoid the pathological case of a matrix product with only T' matrices.

Consider a product of these matrices $Q = Q_1 Q_2 \cdots Q_m$ where $Q_i \in \{B', T'\}$ for $1 \le i \le m$. It is easily seen that if $m \le 2$ then the top right element of Q equals 1 for any Q_1, Q_2. Let us thus assume $m \ge 3$ and write this multiplication as $Q = Q_1 C Q_m$ where $C = Q_2 Q_3 \cdots Q_{m-1}$,

$$C = \begin{pmatrix} 0 & * & * & * \\ 0 & \lambda & 0 & * \\ 0 & 0 & C' & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $*$ denotes unimportant values, $\lambda = \{0, 1\}$ and C' is a submatrix equal to some product of B, T matrices.

Now we will compute the top right element of Q. Let r denote the dimension of matrix C (or Q). The first row of Q_1C equals $(0, \lambda, C'_{1,1}, C'_{1,2}, \ldots, C'_{1,k}, *)$ where again $*$ is unimportant. Note that this vector contains the top row of the C' submatrix. We can now easily see that $Q_{[1,r]} = (Q_1 C Q_m)_{[1,r]}$ equals $(0, \lambda, C'_{1,1}, C'_{1,2}, \ldots, C'_{1,k}, *) \cdot (1, 1, z^{\mathrm{T}}, 0)^{\mathrm{T}} = \lambda + C'_{1,k}$. It is clear that $\lambda = 1$ iff $C = (T')^{m-2}$ i.e. C is a power of only T' matrices. In this case, note that $(C'^{m-2})_{[1,k]} = 0$ since this is a power of matrix T. Thus $Q_{[1,r]} = 1 + 0 = 1$.

In the second case, $\lambda = 0$ whenever C' contains a factor B'. Therefore $Q_{[1,r]} = 0 + C'_{[1,k]} = C'_{[1,k]}$ which is exactly the top right element of C' as required. This equals 0 iff there exists a solution to the PCP.

We require 3 extra rows and columns for this encoding, therefore the problem is undecidable for dimension $2n_p + 1 + 3 = 2n_p + 4$ (currently 18). $\square$

4. Conclusion

We showed that point-to-point reachability is undecidable for n_p (where n_p is the smallest value for which PCP is undecidable, currently 7) two-dimensional affine transformations. A simple extension allowed us to prove that the vector reachability problem for n_p rational matrices of dimension 3 is undecidable and also for 2 matrices of dimension $2n_{cp} - 3$ (currently 11). We then showed an undecidable result for a piecewise vector reachability question where the next matrix applied depends upon the current value of an element in the vector. Finally we improved the bounds on the "zero in the upper right corner problem" from $3n_p + 2$ (currently 23) to $2n_p + 4$ (currently 18) using two integral matrices.

It is interesting that we may get an undecidable problem for semigroups generated by just two integral matrices of dimension 11. Recently, it was shown by Halava and Hirvensalo in [13] that scalar reachability is undecidable even in dimension 9 for two matrices and also that the zero in the upper right corner problem is undecidable for two integral matrices of dimension 10.

It would be interesting to consider whether there exist other undecidable problems related to semigroups generated by pairs of matrices with undecidability over even smaller dimensions. We showed a connection between semigroup membership and piecewise reachability, however we believe this connection can be strengthened to show other undecidability results in piecewise functions.

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