PHILOSOPHICAL TRANSACTIONS A

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Research

Article submitted to journal

Subject Areas:
Applied Mathematics, Mechanics

Keywords:
Nonlocal elasticity, boundary layer, surface wave, integral and differential models, discrete chain

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On nonlocally elastic Rayleigh wave

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The Rayleigh-type wave solution within a widely used differential formulation in nonlocal elasticity is revisited. It is demonstrated that it does not satisfy the equations of motion for non-local stresses. A modified differential model taking into account a nonlocal boundary layer is developed. Correspondence of the latter model to the original integral theory with the kernel in the form of the zero order modified Bessel function of the second kind is addressed. Asymptotic behaviour of the model is investigated resulting in a leading order nonlocal correction to the classical Rayleigh wave speed due to the effect of the boundary layer. The suitability of a continuous setup for modelling boundary layers in the framework of nonlocal elasticity is analysed starting from a toy problem for a semi-infinite chain.

1. Introduction

The theory of nonlocal elasticity originated from the works of Eringen [7,10] and earlier contributions, e.g. [16] and [17], has attracted significant attention in recent decades in view of applications in high-tech domains, including nano-technology, see [8,15,22] and references therein.

The majority of numerous publications on the subject uses differential formulations, e.g see [2,13,18] which appear to be better suited for analytical treatment than initial integral ones. Moreover, due to the nature of nonlocal models governed by integral equations, the issue of their solvability naturally emerges. In particular, integral settings were shown to lead to ill-posed problems for beam bending, see e.g. [23,24,26], and also an earlier analysis for a rod [3]. The 1D problems for beams and rods can seemingly be regularized by adapting the so-called "two-phase" formulations, see [4, 19,20], although a further analysis of the purely nonlocal

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limit is still required, see [26].

The recent contribution [14] questioned in more general context the widely accepted expectation of equivalence between the integral nonlocal model and the associated differential one, suggested by [10], for an elastic halfspace subject to antiplane shear. Straightforward analysis for a 1D exponential kernel revealed that a time-harmonic solution of the differential model does not satisfy the equation of motion for nonlocal shear stresses. Moreover, it was shown that the differential and integral theories are only equivalent provided extra conditions hold on the surface. However, all of the extra conditions could not be satisfied, since one of the shear stresses was already constrained through a boundary condition on the surface. A differential model embedding an extra condition on the non-constrained shear stress component was developed. It was also shown that such a differential model supports a shear surface wave, contrary to the classical 'local' elasticity. This observation seems to be in line with recent considerations dealing with shear surfaced waves, e.g. see [9].

In this paper, we extend previous findings to comparative analysis of integral and differential nonlocal models for a plane-strain problem for an elastic half-space with traction free faces. The adapted integral model relies on the 2D kernel in the form of a zero-order modified Bessel function of the second kind introduced in [10]. A typical wave length along the surface is assumed to be much greater than the internal size. In this case the double integrals can be expanded into asymptotic series in terms of single integrals. Throughout the paper we start from two-term expansions involving integrals with the exponential kernel. Using the derived expansions for integral relations, the Eringen's solution [10] for a nonlocal surface wave obtained within the differential model is revisited. Similarly to [14], it is demonstrated that this solution does not satisfy the equations of motion in integral nonlocal stresses resulting in a discrepancy corresponding to a nonlocal boundary layer localised near the surface.

The equivalence between the integral and differential formulations is then further investigated exploiting a similarity with the limiting 1D integral setup for the exponential kernel studied in [14]. However, now we have to deal with pseudo-differential equations instead of differential ones in [14]. In addition, three extra conditions (not two as in [14]) have to be imposed on the surface. Obviously, all of them cannot be satisfied in view of two constraints already prescribed in the form of the boundary conditions on a traction-free surface. In the paper such an extra condition is imposed on the remaining non-constrained normal stress.

The related improved differential model, including singularly perturbed differential equations of motion, traction-free boundary conditions, as well as the aforementioned extra boundary condition for the non-constrained stress, is examined using a conventional asymptotic procedure. Specifying fast and slow variables, see e.g. [6], a three-term asymptotic expansion is derived, once again illustrating that the correction arising from the boundary conditions is asymptotically greater than that due to the nonlocal corrections in the equations of motion, as it was first noticed in [5]. A leading-order nonlocal correction to the Rayleigh wave speed is calculated and compared to the results in [5], relying on the Gaussian kernel. Finally, we address the validity of continuous models for analysing nonlocal boundary layers of the width of a microscale. As an example, a model problem for a semi-infinite chain is presented. An exponential decay rate is taken as an independent problem parameter, along with the step of the chain. The parameter range suitable for homogenisation is evaluated. Outside this range, a discrete boundary layer is determined.

2. Problem statement

Consider a nonlocally elastic half-space, defined by $-\infty < x_1, x_3 < \infty$ and $x_2 \geq 0$. Below we adopt the plane-strain assumption, within which the displacement components are of the form $u_m = u_m(x_1, x_2, t), \ \ (m = 1, 2)$, and $u_3 \equiv 0$. The equations of motion are given by

$$\frac{\partial t_{m1}}{\partial x_1} + \frac{\partial t_{m2}}{\partial x_2} = \rho \frac{\partial^2 u_m}{\partial t^2}. \tag{2.1}$$

Here ρ is volume mass density, and t_{mn} $(m,n=1,2)$ are the nonlocal symmetric stresses (i.e. $t_{21}=t_{12}$). The latter are expressed through the conventional local stresses σ_{mn} as

$$t_{mn}=\frac{1}{2\pi a^2}\int\limits_0^\infty dx_2'\int\limits_{-\infty}^\infty K_0\left(\frac{\sqrt{(x_1'-x_1)^2+(x_2'-x_2)^2}}{a}\right)\sigma_{mn}\left(x_1',x_2'\right)\,dx_1'. \tag{2.2}$$

In the above K_0 is a modified Bessel function of the second kind of order zero, with a denoting the internal characteristic length, see e.g. [10]. For isotropic elastic solids, the local stresses are written conventionally in terms of the displacements as

$$\sigma_{mn}=\lambda\,\delta_{mn}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)+\mu\left(\frac{\partial u_m}{\partial x_n}+\frac{\partial u_n}{\partial x_m}\right),\qquad m,n=1,2, \tag{2.3}$$

where δ_{mn} is the Kronecker symbol, and λ and μ are the Lamé constants.

We consider the boundary $x_2=0$ to be traction-free, i.e.

$$t_{21}=t_{22}=0. \tag{2.4}$$

We will refer to the conditions (2.2) as integral model of nonlocal elasticity with 2D kernel, and aim to compare them with the related differential model suggested by [10] in the form

$$(a^2\nabla^2-1)t_{mn}=-\sigma_{mn}, \tag{2.5}$$

where $\nabla^2=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}$ is the 2D Laplacian in x_1 and x_2. The associated governing differential equations of motion are conventionally taken in the form

$$\frac{\partial\sigma_{m1}}{\partial x_1}+\frac{\partial\sigma_{m2}}{\partial x_2}-\rho\left(1-a^2\nabla^2\right)\frac{\partial^2 u_m}{\partial t^2}=0. \tag{2.6}$$

Note that under the assumption of slow variation of local stresses σ_{mn} along the longitudinal variable x_1, the following table integral (see 6.596 in [12])

$$\int\limits_0^\infty x^{2\kappa}K_0\left(\alpha\sqrt{x^2+z^2}\right)dx=\frac{\pi}{2\alpha^{2\kappa+1}}\left(1+\kappa\alpha z\right)e^{-\alpha z},\quad \kappa=0,1, \tag{2.7}$$

may be employed to reduce the double integral in (2.2) to a single one. Indeed, on introducing the dimensionless spatial coordinates

$$\xi_n=\frac{x_n}{l},\quad n=1,2, \tag{2.8}$$

where l is a typical wavelength, and the related small parameter associated with nonlocality

$$\varepsilon=\frac{a}{l}\ll 1, \tag{2.9}$$

a two-term Taylor expansion in ξ_1' of the local stress components in the integrand around the point $\xi_1'=\xi_1$ may be constructed, which, after using (2.7), gives

$$\begin{aligned} t_{mn}&=\frac{1}{2\pi\varepsilon^2}\int\limits_0^\infty d\xi_2'\int\limits_{-\infty}^\infty K_0\left(\frac{\sqrt{(\xi_1'-\xi_1)^2+(\xi_2'-\xi_2)^2}}{\varepsilon}\right)\left[1+\frac{(\xi_1'-\xi_1)^2}{2}\frac{\partial^2}{\partial\xi_1^2}+\dots\right]\sigma_{mn}(\xi_1,\xi_2')d\xi_1' \\ &=\frac{1}{2\varepsilon}\int\limits_0^\infty\left[1+\frac{\varepsilon^2}{2}\left(1+\frac{|\xi_2'-\xi_2|}{\varepsilon}\right)\frac{\partial^2}{\partial\xi_1^2}+\dots\right]\sigma_{mn}(\xi_1,\xi_2')e^{-\frac{|\xi_2'-\xi_2|}{\varepsilon}}d\xi_2'. \end{aligned} \tag{2.10}$$

In above the terms of $O(\varepsilon^4)$ corresponding to higher order terms for Taylor series for σ_{mn} in (2.10) are omitted.

3. Revisit of Eringen's solution

Following [10], let us consider a simple example of nonlocal time-harmonic Rayleigh waves, starting from the differential equations of motion (2.6), subject to boundary conditions (2.4).

Using the local constitutive relations (2.3), and introducing the conventional Lamé elastic potentials for the displacement field through

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}, \qquad u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}. \tag{3.1}$$

the wave equations for the potentials may be deduced from (2.3) in the form

$$c_1^2\, \nabla^2 \phi - \left(1 - a^2 \nabla^2\right) \frac{\partial^2 \phi}{\partial t^2} = 0, \qquad c_2^2\, \nabla^2 \psi - \left(1 - a^2 \nabla^2\right) \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{3.2}$$

where

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_2 = \sqrt{\frac{\mu}{\rho}}. \tag{3.3}$$

The solutions of (3.2) are now sought for in the form of a travelling harmonic wave of exponential profile, i.e.

$$\phi = A e^{ik(x_1 - ct) - kq_1 x_2}, \qquad \psi = B e^{ik(x_1 - ct) - kq_2 x_2}, \tag{3.4}$$

where $\mathrm{Re}(q_n) > 0$ $(n = 1, 2)$ ensures exponential decay away from the surface. Here we assume a long-wave regime, such that the wave number $k = 1/l$, hence the small parameter becomes $\varepsilon = ka$. Substituting (3.4) into (3.2) implies

$$q_n^2 = 1 - \frac{c^2}{c_n^2 - \varepsilon^2 c^2}, \quad n = 1, 2. \tag{3.5}$$

Combining (3.4) with (3.1) and (2.3), the local stresses σ_{mn} are given explicitly by

$$\begin{aligned}
\sigma_{11} &= e^{ik(x_1 - ct)} k^2 \left[(\lambda q_1^2 - (\lambda + 2\mu)) A e^{-kq_1 x_2} - 2i\mu q_2 B e^{-kq_2 x_2} \right], \\
\sigma_{12} = \sigma_{21} &= e^{ik(x_1 - ct)} k^2 \mu \left[-2iq_1 A e^{-kq_1 x_2} + (q_2^2 + 1) B e^{-kq_2 x_2} \right], \\
\sigma_{22} &= e^{ik(x_1 - ct)} k^2 \left[\left((\lambda + 2\mu) q_1^2 - \lambda \right) A e^{-kq_1 x_2} + 2i\mu q_2 B e^{-kq_2 x_2} \right].
\end{aligned} \tag{3.6}$$

Then, from (2.2) the non-local stresses can be expressed as

$$\begin{aligned}
t_{11} &= k^2 \left[(\lambda q_1^2 - (\lambda + 2\mu)) A M_1 - 2i\mu q_2 B M_2 \right], \\
t_{12} = t_{21} &= \mu k^2 \left[-2iq_1 A M_1 + (q_2^2 + 1) B M_2 \right], \\
t_{22} &= k^2 \left[((\lambda + 2\mu) q_1^2 - \lambda) A M_1 + 2i\mu q_2 B M_2 \right],
\end{aligned} \tag{3.7}$$

where (see [10])

$$M_n = \frac{1}{2\pi a^2} \int_0^{\infty} dx_2' \int_{-\infty}^{\infty} K_0 \left(\frac{\sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2}}{a} \right) e^{ik(x_1' - ct) - kq_n x_2'} dx_1', \quad n = 1, 2. \tag{3.8}$$

Clearly, substitution of the nonlocal stresses (3.7) into the boundary conditions (2.4) gives $c = c_R$ to within the error of $O(\varepsilon^2)$, where c_R satisfies the classical Rayleigh equation

$$4 q_{10}\, q_{20} - \left(1 + q_{20}^2\right)^2 = 0, \tag{3.9}$$

with

$$q_{n0} = \sqrt{1 - \frac{c_R^2}{c_n^2}}, \quad n = 1, 2. \tag{3.10}$$

Now, let us verify that the (3.7) satisfies the equations of motion (2.1). Assuming slow variation along the x_1' direction and using the integral (2.7) the double integrals in (3.8) may be evaluated, giving

$$
\begin{aligned}
M_n e^{ik(ct-x_1)} &= \frac{1}{2\pi a^2}\int_0^\infty e^{-kq_n x_2'} dx_2' \int_{-\infty}^{\infty} K_0\left(\frac{\sqrt{(x_1'-x_1)^2+(x_2'-x_2)^2}}{a}\right)\left[1-\frac{k^2}{2}(x_1'-x_1)^2+\dots\right]dx_1' \\
&= \frac{1}{2a}\int_0^\infty \left[1-\frac{\varepsilon^2}{2}\left(1+\frac{|x_2'-x_2|}{a}\right)+\dots\right]e^{-kq_n x_2' - \frac{|x_2'-x_2|}{a}}dx_2' \\
&= \left[1+\varepsilon^2(q_{n0}^2-1)+\dots\right]e^{-kq_n x_2} - \frac{1}{2}\left[1+\varepsilon q_{n0}+\varepsilon^2\left(q_{n0}^2-1-\frac{x_2}{2a}\right)+\dots\right]e^{-\frac{x_2}{a}}.
\end{aligned}
\tag{3.11}
$$

In the latter, the coefficients at $e^{-\frac{x_2}{a}}$ are clearly associated with the nonlocal boundary layer.

Now we are in position to substitute the nonlocal stresses (3.7), combined with the leading order approximation (3.11), into the equations of motion (2.1), with the constants A and B related as

$$
A = \frac{1+q_{20}^2}{2iq_{10}}\left[1+(q_{10}-q_{20})\left(\varepsilon-\varepsilon^2 q_{20}\right)+\dots\right]B, \tag{3.12}
$$

following from the boundary condition for the shear stress. Consider the first equation in (2.1) in more detail. As a result, we have at leading order

$$
e^{-\frac{x_2}{\varepsilon}}\,B(1+q_{20}^2)\left(1-2q_{10}^2+q_{20}^4\right)\neq 0. \tag{3.13}
$$

The latter corresponds to the nonlocal boundary layer in the near-surface vicinity. Thus, the solution for the surface wave constructed in [10] does not satisfy the original equations of motion (2.1) in nonlocal stresses.

Thus, similarly to the antiplane case [14], it may be concluded that the integral nonlocal model for an elastic half-space is inconsistent, since the nonlocal stresses provided by these theory, do not satisfy the equations of motion. At the same time, the consideration in antiplane shear allowed exact treatment, whereas the current problem of plane-strain elasticity was treated approximately, expanding the integrand as Taylor series along the longitudinal variable.

4. Singularly perturbed differential model

The equivalence between the integral nonlocal model (2.2) and the differential formulation (2.5) is known for the case of unbounded media. At the same time, the 2D differential model (2.5) can be formally represented in 1D format, i.e.

$$
a^2\frac{\partial^2 t_{mn}}{\partial x_2^2} - \left[1-a^2\frac{\partial^2}{\partial x_1^2}\right]t_{mn} = -\sigma_{mn}, \tag{4.1}
$$

implying

$$
\eta^2\frac{\partial^2 t_{mn}}{\partial x_2^2} - t_{mn} = -\sigma'_{mn}, \tag{4.2}
$$

where a pseudo-differential operator η (e.g. see [25]) is defined as

$$
\eta = a\left[1-a^2\frac{\partial^2}{\partial x_1^2}\right]^{-1/2} \tag{4.3}
$$

and

$$
\sigma'_{mn} = \left[1-a^2\frac{\partial^2}{\partial x_1^2}\right]^{-1}\sigma_{mn}. \tag{4.4}
$$

It is worth mentioning that in case of harmonic dependence on the longitudinal coordinate x_1, i.e. of the form e^{ikx_1}, the operator η simplifies to a constant multiple represented by $a(1+\varepsilon^2)^{-1/2}$, where, as before, $\varepsilon = ka$. Hence, we may formally employ the results of the recent analysis of 1D nonlocal models in [14], implying that the nonlocal differential model (4.2) is equivalent to the integral model of the form

$$t_{mn} = \frac{1}{2}\eta^{-1}\int_0^\infty e^{-\eta^{-1}|x_2'-x_2|}\sigma'_{mn}(x_1,x_2')dx_2', \tag{4.5}$$

with σ'_{mn} defined in (4.4), provided that the additional conditions

$$\left(\left[1-\eta\frac{\partial}{\partial x_2}\right]t_{mn}\right)\Big|_{x_2=0} = 0, \tag{4.6}$$

hold. Moreover, expanding the integrand in (4.5), we deduce that in the long-wave range ($\varepsilon \ll 1$)

$$\begin{aligned} t_{mn} &= \frac{1}{2}\eta^{-1}\left[1-a^2\frac{\partial^2}{\partial x_1^2}\right]^{-1}\int_0^\infty\left[1-(\eta^{-1}-a^{-1})|x_2'-x_2|+\dots\right]\sigma_{mn}(x_1,x_2')e^{-\frac{|x_2'-x_2|}{a}}dx_2' \\ &= \frac{1}{2a}\int_0^\infty\left[1+\frac{a^2}{2}\left(1+\frac{|x_2'-x_2|}{a}\right)\frac{\partial^2}{\partial x_1^2}+\dots\right]\sigma_{mn}(x_1,x_2')e^{-\frac{|x_2'-x_2|}{a}}dx_2', \end{aligned} \tag{4.7}$$

which may be recognised as a dimensional form of the two-term representation of the 2D integral model (2.10).

Clearly, all three conditions (4.6) cannot be satisfied, because of the two constraints (2.4) already imposed on t_{21} and t_{22}. Therefore, in what follows we are adopting the extra condition (4.6) for the non-constrained stress t_{11}, considering it together with the traction-free boundary conditions (2.4). In particular, we will be using the two-term expansion of (4.6) for t_{11}, i.e.

$$\left(\left[1-a\frac{\partial}{\partial x_2}-\frac{a^3}{2}\frac{\partial^3}{\partial x_1^2\partial x_2}\right]t_{11}\right)\Big|_{x_2=0} = 0. \tag{4.8}$$

Thus, the differential and integral formulations are equivalent for t_{11} only.

5. Asymptotic analysis

We introduce dimensionless variables as following

$$\zeta_p = \frac{x_2}{l}, \quad \zeta_q = \frac{x_2}{a}, \quad \xi_1 = \frac{x_1}{l}, \quad \tau = t\frac{c_2}{l}, \tag{5.1}$$

where ζ_p and ζ_q are transverse variables elucidating behaviour along the vertical coordinate in a half-space and in the vicinity of the boundary, respectively. Indeed, these two variables are connected through $\zeta_p = \varepsilon\zeta_q$, where ε is defined in (2.9).

The fast variable ζ_q supports boundary layers, localized over a narrow vicinity of the surface of the size of order internal characteristic length a. In this case, the validity of the analysed continuous model in nonlocal elasticity should be further addressed. Ideally, the latter originates from homogenisation of a discrete lattice, see e.g. [11] assuming that a typical wave length is much greater than the microscale a. Peculiarities of such homogenisation procedure are studied in the Appendix for the model example of a semi-infinite regular chain.

Below, the following dimensionless quantities are adopted

$$u_m^* = \frac{u_m}{l}, \quad \sigma_{mn}^* = \frac{\sigma_{mn}}{\mu}, \quad t_{mn}^* = \frac{t_{mn}}{\mu}, \quad m,n = 1,2. \tag{5.2}$$

Then, the non-local stresses are split into slow and fast components, $p^*_{mn} = p^*_{mn}(\xi_1, \zeta_p, \tau)$ and $q^*_{mn} = q^*_{mn}(\xi_1, \zeta_q, \tau)$, as follows

$$\begin{aligned} t^*_{11} &= p^*_{11} + q^*_{11}, \\ t^*_{12} &= p^*_{12} + \varepsilon q^*_{12}, \\ t^*_{22} &= p^*_{22} + \varepsilon^2 q^*_{22}. \end{aligned} \tag{5.3}$$

Substituting these into the governing equations (2.1), (2.3) and (2.5), and splitting into slow and fast varying quantities, we obtain

$$\begin{aligned} &\frac{\partial p^*_{m1}}{\partial \xi_1} + \frac{\partial p^*_{m2}}{\partial \zeta_p} = \frac{\partial^2 u^*_m}{\partial \tau^2}, \\ &p^*_{mn} - \varepsilon^2 \left(\frac{\partial^2 p^*_{mn}}{\partial \xi_1^2} + \frac{\partial^2 p^*_{mn}}{\partial \zeta_p^2} \right) = \sigma^*_{mn}, \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} &\frac{\partial q^*_{m1}}{\partial \xi_1} + \frac{\partial q^*_{m2}}{\partial \zeta_q} = 0, \\ &q^*_{mn} - \frac{\partial^2 q^*_{mn}}{\partial \zeta_q^2} - \varepsilon^2 \frac{\partial^2 q^*_{mn}}{\partial \xi_1^2} = 0, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \sigma^*_{11} &= \kappa^{-2} \frac{\partial u^*_1}{\partial \xi_1} + (\kappa^{-2} - 2) \frac{\partial u^*_2}{\partial \zeta_p}, \\ \sigma^*_{12} &= \frac{\partial u^*_1}{\partial \zeta_p} + \frac{\partial u^*_2}{\partial \xi_1}, \\ \sigma^*_{22} &= (\kappa^{-2} - 2) \frac{\partial u^*_1}{\partial \xi_1} + \kappa^{-2} \frac{\partial u^*_2}{\partial \zeta_p}, \end{aligned} \tag{5.6}$$

and $\kappa = c_2/c_1$. Traction-free boundary conditions on the surface $\zeta_p = \zeta_q = 0$, together with additional condition (4.8) for t_{11} are re-written as

$$\begin{aligned} &p^*_{11} + q^*_{11} - \frac{\partial q^*_{11}}{\partial \zeta_q} - \varepsilon \frac{\partial p^*_{11}}{\partial \zeta_p} - \frac{1}{2} \varepsilon^2 \frac{\partial^3 q^*_{11}}{\partial \xi_1^2 \partial \zeta_q} = 0, \\ &p^*_{12} + \varepsilon q^*_{12} = 0, \\ &p^*_{22} + \varepsilon^2 q^*_{22} = 0. \end{aligned} \tag{5.7}$$

Now, the sought for quantities may be expanded as asymptotic series

$$f^*_{mn} = f^{(0)}_{mn} + \varepsilon f^{(1)}_{mn} + \varepsilon^2 f^{(2)}_{mn} + \ldots, \tag{5.8}$$

where $f^*_{mn} = \{p^*_{mn},\ q^*_{mn},\ \sigma^*_{mn},\ u^*_{mn}\}$.

Next, the problem can be formulated for various orders of magnitude in the form

$$\begin{aligned} &\frac{\partial p^{(r)}_{m1}}{\partial \xi_1} + \frac{\partial p^{(r)}_{m2}}{\partial \zeta_p} = \frac{\partial^2 u^{(r)}_m}{\partial \tau^2}, \\ &p^{(r)}_{mn} - \frac{\partial^2 p^{(r-2)}_{mn}}{\partial \xi_1^2} - \frac{\partial^2 p^{(r-2)}_{mn}}{\partial \zeta_p^2} = \sigma^{(r)}_{mn}, \end{aligned} \tag{5.9}$$

with

$$\begin{aligned} &\frac{\partial q^{(r)}_{m1}}{\partial \xi_1} + \frac{\partial q^{(r)}_{m2}}{\partial \zeta_q} = 0, \\ &q^{(r)}_{ij} - \frac{\partial^2 q^{(r)}_{mn}}{\partial \zeta_q^2} - \frac{\partial^2 q^{(r-2)}_{mn}}{\partial \xi_1^2} = 0, \end{aligned} \tag{5.10}$$

where

$$\sigma_{11}^{(r)} = \kappa^{-2}\frac{\partial u_1^{(r)}}{\partial \xi_1} + (\kappa^{-2} - 2)\frac{\partial u_2^{(r)}}{\partial \zeta_p},$$
$$\sigma_{12}^{(r)} = \frac{\partial u_1^{(r)}}{\partial \zeta_p} + \frac{\partial u_2^{(r)}}{\partial \xi_1}, \tag{5.11}$$
$$\sigma_{22}^{(r)} = (\kappa^{-2} - 2)\frac{\partial u_1^{(r)}}{\partial \xi_1} + \kappa^{-2}\frac{\partial u_2^{(r)}}{\partial \zeta_p},$$

at the boundary we also have

$$p_{11}^{(r)} + q_{11}^{(r)} - \frac{\partial q_{11}^{(r)}}{\partial \zeta_q} - \frac{\partial p_{11}^{(r-1)}}{\partial \zeta_p} - \frac{1}{2}\frac{\partial^3 q_{11}^{(r-2)}}{\partial \xi_1^2 \partial \zeta_q} = 0, \tag{5.12}$$

$$p_{12}^{(r)} + q_{12}^{(r-1)} = 0, \tag{5.13}$$

$$p_{22}^{(r)} + q_{22}^{(r-2)} = 0. \tag{5.14}$$

In the above $r = 0, 1, 2, \ldots$, and all the terms with negative superscripts are taken to be zero. Here we restrict ourselves to three-term expansions.

At zero order $(r = 0)$ we have from equations (5.9)

$$p_{mn}^{(0)} = \sigma_{mn}^{(0)}, \tag{5.15}$$

resulting in a classical plane-strain boundary value problem. Using equations (5.10) and boundary condition (5.12), we obtain

$$q_{mn}^{(0)} = Q_{mn}^{(0)}(\xi_1, \tau)e^{-\zeta_q}, \tag{5.16}$$

where

$$Q_{11}^{(0)} = -\frac{1}{2}\,\sigma_{11}^{(0)}\Big|_{\zeta_p=0}, \qquad Q_{12}^{(0)} = -\frac{1}{2}\,\frac{\partial \sigma_{11}^{(0)}}{\partial \xi_1}\Big|_{\zeta_p=0}, \qquad Q_{22}^{(0)} = -\frac{1}{2}\,\frac{\partial^2 \sigma_{11}^{(0)}}{\partial \xi_1^2}\Big|_{\zeta_p=0}. \tag{5.17}$$

Thus, all of the fast quantities $q_{mn}^{(0)}$ are determined through the local stresses $\sigma_{mn}^{(0)}$.

At first order $(r = 1)$, we have

$$p_{mn}^{(1)} = \sigma_{mn}^{(1)}, \tag{5.18}$$

again resulting in classical plane-strain equations of motion. The corresponding boundary conditions, however, contain an additional term

$$\sigma_{12}^{(1)} = \frac{1}{2}\frac{\partial \sigma_{11}^{(0)}}{\partial \xi_1}, \qquad \sigma_{22}^{(1)} = 0 \quad \text{at} \quad \zeta_p = 0, \tag{5.19}$$

where $\sigma_{mn}^{(1)}$ can be found in terms of displacements, see (5.11). For the quantities related to the boundary layer we have

$$q_{mn}^{(1)} = Q_{mn}^{(1)}(\xi_1, \tau)e^{-\zeta_q}, \tag{5.20}$$

where

$$Q_{11}^{(1)} = \frac{1}{2}\left(\frac{\partial \sigma_{11}^{(0)}}{\partial \zeta_p}\Big|_{\zeta_p=0} - \sigma_{11}^{(1)}\Big|_{\zeta_p=0}\right),$$
$$Q_{12}^{(1)} = \frac{1}{2}\left(\frac{\partial^2 \sigma_{11}^{(0)}}{\partial \xi_1 \partial \zeta_p}\Big|_{\zeta_p=0} - \frac{\partial \sigma_{11}^{(1)}}{\partial \xi_1}\Big|_{\zeta_p=0}\right),$$
$$Q_{22}^{(1)} = \frac{1}{2}\left(\frac{\partial^3 \sigma_{11}^{(0)}}{\partial \xi_1^2 \partial \zeta_p}\Big|_{\zeta_p=0} - \frac{\partial^2 \sigma_{11}^{(1)}}{\partial \xi_1^2}\Big|_{\zeta_p=0}\right).$$

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At the second order ($r=2$), we obtain from (5.9)$_2$

$$p_{mn}^{(2)}=\sigma_{mn}^{(2)}+\frac{\partial^2\sigma_{mn}^{(0)}}{\partial\xi_1^2}+\frac{\partial^2\sigma_{mn}^{(0)}}{\partial\zeta_p^2}. \tag{5.21}$$

Then equation of motion (5.9)$_1$ can be recast as

$$\frac{\partial\sigma_{m1}^{(2)}}{\partial\xi_1}+\frac{\partial\sigma_{m2}^{(2)}}{\partial\zeta_p}=\frac{\partial^2u_m^{(2)}}{\partial\tau^2}-\frac{\partial^4u_m^{(0)}}{\partial\tau^2\partial\xi_1^2}-\frac{\partial^4u_m^{(0)}}{\partial\tau^2\partial\zeta_p^2}, \tag{5.22}$$

where (5.15) was used. Combining equations (5.15), (5.18) and (5.22), we obtain the refined equations of motion in terms of the local stresses

$$\frac{\partial\sigma_{m1}^*}{\partial\xi_1}+\frac{\partial\sigma_{m2}^*}{\partial\zeta_p}=\frac{\partial^2u_m^*}{\partial\tau^2}-\varepsilon^2\left(\frac{\partial^4u_m^*}{\partial\tau^2\partial\xi_1^2}+\frac{\partial^4u_m^*}{\partial\tau^2\partial\zeta_p^2}\right), \tag{5.23}$$

where stresses σ_{mn}^* and displacements u_m^* are related through (5.6). Next, using boundary conditions (5.13) and (5.14), we arrive (at $\zeta_p=\zeta_q=0$)

$$\sigma_{12}^{(2)}=\frac{1}{2}\frac{\partial\sigma_{11}^{(1)}}{\partial\xi_1}-\frac{1}{2}\frac{\partial^2\sigma_{11}^{(0)}}{\partial\xi_1\partial\zeta_p}-\frac{\partial^2\sigma_{12}^{(0)}}{\partial\xi_1^2}-\frac{\partial^2\sigma_{12}^{(0)}}{\partial\zeta_p^2}, \tag{5.24}$$

and

$$\sigma_{22}^{(2)}=\frac{1}{2}\frac{\partial^2\sigma_{11}^{(0)}}{\partial\xi_1^2}-\frac{\partial^2\sigma_{22}^{(0)}}{\partial\xi_1^2}-\frac{\partial^2\sigma_{22}^{(0)}}{\partial\zeta_p^2}. \tag{5.25}$$

As a result, the refined second order boundary conditions may be formulated as

$$\sigma_{12}^*-\frac{\varepsilon}{2}\frac{\partial\sigma_{11}^*}{\partial\xi_1}+\varepsilon^2\left(\frac{1}{2}\frac{\partial^2\sigma_{11}^*}{\partial\xi_1\partial\zeta_p}+\frac{\partial^2\sigma_{12}^*}{\partial\xi_1^2}+\frac{\partial^2\sigma_{12}^*}{\partial\zeta_p^2}\right)=0, \tag{5.26}$$

and

$$\sigma_{22}^*-\varepsilon^2\left(\frac{1}{2}\frac{\partial^2\sigma_{11}^*}{\partial\xi_1^2}-\frac{\partial^2\sigma_{22}^*}{\partial\xi_1^2}-\frac{\partial^2\sigma_{22}^*}{\partial\zeta_p^2}\right)=0. \tag{5.27}$$

6. Nonlocal correction to the Rayleigh wave speed

In terms of the original variables, a refined boundary value problem, including (5.23), (5.26) and (5.27) is presented as

$$\frac{\partial\sigma_{i1}}{\partial x_1}+\frac{\partial\sigma_{i2}}{\partial x_2}=\rho\frac{\partial^2u_i}{\partial t^2}-a^2\rho\left(\frac{\partial^4u_i}{\partial t^2\partial x_1^2}+\frac{\partial^4u_i}{\partial t^2\partial x_2^2}\right), \tag{6.1}$$

subject to

$$\sigma_{12}-\frac{a}{2}\frac{\partial\sigma_{11}}{\partial x_1}+a^2\left(\frac{1}{2}\frac{\partial^2\sigma_{11}}{\partial x_1\partial x_2}+\frac{\partial^2\sigma_{12}}{\partial x_1^2}+\frac{\partial^2\sigma_{12}}{\partial x_2^2}\right)=0, \tag{6.2}$$

and

$$\sigma_{22}-a^2\left(\frac{1}{2}\frac{\partial^2\sigma_{11}}{\partial x_1^2}-\frac{\partial^2\sigma_{22}}{\partial x_1^2}-\frac{\partial^2\sigma_{22}}{\partial x_2^2}\right)=0 \quad \text{at} \quad x_2=0, \tag{6.3}$$

where stresses and displacements are connected through (2.3). For simplicity, let us neglect all the terms higher than $O(a)$. Then, in the long-wave limit $\varepsilon\ll 1$ as before, $\varepsilon=ka$, the problem reduces to classical plane-strain equations of motion (cf. (3.2) at $a=0$) subject to

$$\sigma_{12}=2a\mu(1-\kappa^2)\frac{\partial^2u_1}{\partial x_1^2}, \qquad \sigma_{22}=0 \quad \text{at} \quad x_2=0. \tag{6.4}$$

These boundary conditions follow from (6.2)-(6.3), with the second condition used to obtain a simpler expression for σ_{11}.

Using (3.1), the problem is reformulated in terms of the Lamé potentials. Next, taking the travelling harmonic ansatz (3.4), where $q_n = q_{n0}$, see (3.10), the associated dispersion relation can be obtained in the form

$$4q_{10}q_{20} - (1+q_{20}^2)^2 + 2\varepsilon(1-\kappa^{-2})(1-q_{10}^2)q_{20} = 0. \tag{6.5}$$

Not surprisingly, the leading order gives a classical Rayleigh wave equation, see (3.9). Introducing the dimensionless speed

$$C = \frac{c}{c_2}, \tag{6.6}$$

equation (6.5) becomes

$$4\sqrt{1-C^2}\,\sqrt{1-\kappa^2 C^2} - (2-C^2)^2 + 2\varepsilon(\kappa^2-1)\,C^2\,\sqrt{1-C^2} = 0. \tag{6.7}$$

Now, expanding C as asymptotic series

$$C = C^{(0)} - C^{(1)}\varepsilon + \ldots, \tag{6.8}$$

we obtain the first two terms

$$C^{(0)} = C_R = \frac{c_R}{c_2}, \qquad C^{(1)} = \frac{2C_R\left(1-C_R^2\right)(\kappa^2-1)\sqrt{1-\kappa^2 C_R^2}}{\left(2-C_R^2\right)^3 - 4(1+\kappa^2-2\kappa^2 C_R^2)}, \tag{6.9}$$

with $\kappa^2 = (1-2\nu)/(2(1-\nu))$. The variation of the correction coefficient $C^{(1)}$ versus the Poisson's ratio ν is presented in Figure 1.

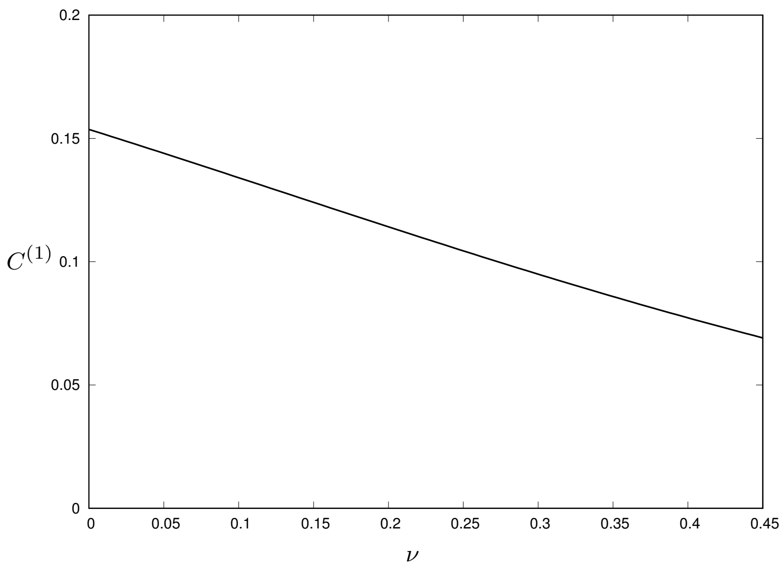

Figure 1. Correction coefficient $C^{(1)}$ to Rayleigh wave speed versus the Poisson's ratio ν, see equation (6.9).

It is worth mentioning, that for the Gaussian kernel, see [5], an effective boundary condition for shearing stress is of the form

$$\sigma_{12} = \frac{2a}{\sqrt{\pi}}\mu(1-\kappa^2)\frac{\partial^2 u_1}{\partial x_1^2} \quad \text{at} \quad x_2 = 0, \tag{6.10}$$

being different from (6.4) only by the factor $1/\sqrt{\pi}$, as it has been also observed within the antiplane problem in [14]. Obviously, the correction coefficient $C^{(1)}$ for the Gaussian kernel will differ from that in (6.9) by the same factor.

7. Conclusion

The original relations for the nonlocal stresses expressed through double integrals with the modified Bessel kernel are reduced to asymptotic expansions in terms of single integrals with the exponential kernel subject to the assumption that a typical wave length along the surface is much greater than the microscale. Two-term expansions are adapted to demonstrate that the well known Eringen's solution for the nonlocal Rayleigh wave, see [10], does not satisfy the equations of motion for nonlocal stresses.

A modified singular perturbed differential model in nonlocal elasticity is developed taking into considerations boundary layers localised near the surface. In this case an extra boundary condition arises as a constraint on the normal stress component that does not enter the traditional boundary conditions along a traction free surface. This constraint ensures the equivalence of integral and differential relations for this normal stress.

An asymptotic solution within the framework of the aforementioned model is derived indicating that the leading-order nonlocal correction to the classical elasticity originates from boundary layers. Thus, the nonlocal corrections to boundary conditions appear to be more significant than those for the equations of motion. The associated refined formula for the surface wave speed is obtained and compared with the results for the Gaussian integral kernel.

The delicate issue of the validity of a continuous setup for modelling boundary layers with the widths of the microscale size is also examined. A toy problem for a semi-infinite chain is considered assuming that the exponential decay rate and the chain step are independent problem parameters. The possibility of asymptotic homogenisation is studied, and discrete boundary layers are calculated.

Analysis of nonlocal near surface effects on Rayleigh wave propagation is related to various modern formulations in the surface wave theory. in particular, we mention mathematical modelling of Rayleigh waves in micro-structured elastic systems, see [21] and references therein.

Appendix A: Discrete nonlocal formulation

Consider a 1D model problem for a semi-infinite chain of particles $x_k = kb, \quad k = 0, 1, 2, \ldots,$ placed evenly with step b, similar to treatment in [11]. Let the discrete counterpart of the nonlocal integral model (2.2) be given by

$$s_k = A \sum_{j=0}^{\infty} \sigma_j e^{-\frac{|x_k - x_j|}{a}}, \tag{A.1}$$

where $s_k = s(x_k)$ is a nonlocal stress, $\sigma_j = \sigma(x_j)$ is a local stress, a is the rate of decay which does not necessarily coincide with step b, and

$$A = \frac{e^{\gamma} - 1}{e^{\gamma} + 1}. \tag{A.2}$$

with $\gamma = \dfrac{b}{a}$. In addition, we introduce a small geometrical parameter $\varepsilon = \dfrac{a}{l}$, where l denotes the typical scale characterising the variation of the local stress.

First, assume that $\gamma \sim 1$. In this case, only the terms in the series (A.1) with $|k - j| \sim 1$ will not be exponentially small. For these terms

$$\sigma_j = \sigma_k \left(1 + O(\varepsilon |k - j|)\right). \tag{A.3}$$

Therefore, we may deduce from (A.1)

$$s_k \approx A\sigma_k \left[\sum_{j=0}^{k-1} e^{\frac{(j-k)b}{a}} + \sum_{j=k}^{\infty} e^{\frac{(k-j)b}{a}} \right] = \sigma_k \left(1 + \frac{A}{1 - e^{\gamma}} e^{-k\gamma} \right). \tag{A.4}$$

This exponential term $e^{-k\gamma}$ in (A.4) corresponds to a boundary layer localised near the origin. It decays at $k \gg 1$, leading to the expected far-field behaviour $s_k \approx \sigma_k$. In fact, the latter justifies a choice of the normalising constant A in the original formula (A.1).

It is worth noting that although no homogenised nonlocal relation corresponding to (A.1) is possible since the decay rate of the boundary layer is of order of the chain step, we still observe a discrete exponential boundary layer. The latter may be combined with a homogenised slow-varying interior solution predicted at leading order within the framework of classical local elasticity.

Next, we consider $\gamma \ll 1$, for which more terms (satisfying $|k-j| \lesssim \gamma^{-1} \gg 1$) should be retained in the sum (A.1). In this case,

$$\sigma_j = \sigma_k\big(1 + O(\varepsilon\gamma|k-j|)\big), \tag{A.5}$$

resulting in

$$s_k \approx \sigma_k\left(1 - \frac{1}{2}e^{-\frac{x_k}{a}}\right), \tag{A.6}$$

which corresponds to the 1D integral homogenised model with exponential kernel, see [14] along with consideration in Section 4, for which, as might be expected, $s_0 \approx \frac{1}{2}\sigma_0$. It can be easily verified that (A.6) describes the limiting behaviour of (A.4) as $\gamma \to 0$.

The range $\gamma \gg 1$ is not of interest because of exponentially small nonlocal interactions, which is also clearly seen from Fig. 2 demonstrating the ratio of local and nonlocal stresses taken in the form (see (A.4))

$$\delta_k(\gamma) = \frac{s_k(\gamma)}{\sigma_k(\gamma)} = 1 + \frac{A}{1-e^{\gamma}}\,e^{-k\gamma}, \quad k = 0, 1, 2, 3, \tag{A.7}$$

This Figure is oriented to elucidating the effect of the distance from the chain origin k and relative decay rate γ on the intensity of the boundary layer.

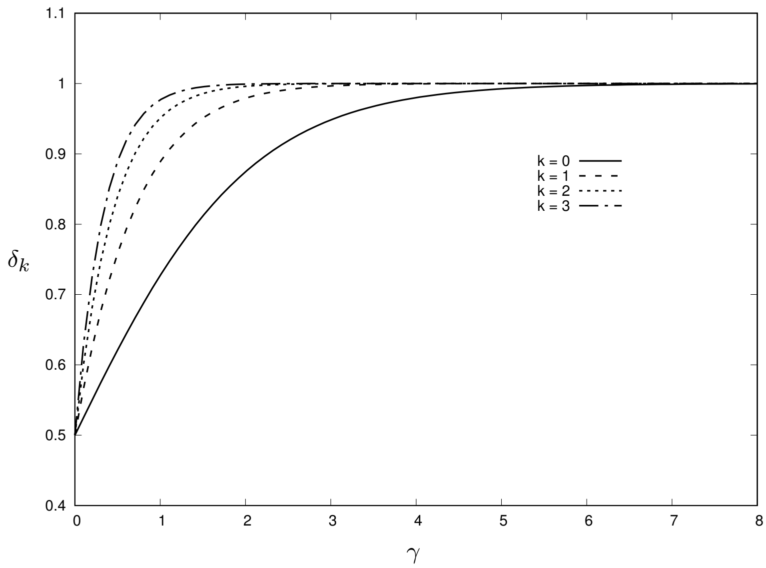

Figure 2. The ratio of local and nonlocal stresses $\delta_k(\gamma)$ for $k = 0, 1, 2, 3$, see equation (A.7).

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