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EDGE BENDING WAVES ON AN ORTHOTROPIC ELASTIC PLATE RESTING ON THE WINKLER-FUSS FOUNDATION

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Key words: bending edge waves, Winkler-Fuss elastic foundation, orthotropic plate.
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Բանալի բառեր. ծռման եզրային ալիքներ, Վինկլեր-Ֆուսի առաձգական հիմք, օրթոտրոպ սալ

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Օրթոտրոպ սալի ծռման եզրային ալիքներ Վինկլեր-Ֆուսի առաձգական հիմքի վրա

Դիտարկված է կիսաանվերջ օրթոտրոպ սալի ծռման եզրային ալիքներ տարածումը Վինկլեր-Ֆուսի առաձգական հիմքի վրա ազատ եզրային պայմանների դեպքում: Ստացված դիսպերսիոն առընչության հետազոտումը ցույց է տալիս փակող հաճախության գոյությունը, ինչպես նաև ֆազային արագության լոկալ մինիմումի առկայությունը:

Альтобаити С., Приказчиков Д.А.

Изгибные краевые волны в случае ортотропной упругой пластины на основании Винклера–Фусса

Изучается задача распространения изгибных краевых волн в случае полубесконечной ортотропной пластины на упругом основании Винклера-Фусса со свободными граничными условиями на краю. Анализ полученного дисперсионного соотношения показал наличие частоты запирания, а также локального минимума фазовой скорости. Также получено представление для профиля изгибной краевой волны в терминах произвольной плоской гармонической функции, обобщающее случай стандартного синусоидального профиля.

The propagation of bending edge waves on an orthotropic plate supported by the Winkler-Fuss foundation subject to free edge boundary conditions is investigated. A dispersion relation is derived, with the analysis revealing a cut-off frequency and a local minimum of the phase velocity. The conventional sinusoidal profile of the eigensolution is then extended to a more general form, with the deflection expressed in terms of a single plane harmonic function.

Introduction. Edge waves in semi-infinite thin plates are known since 1960-s. The first contribution on bending edge wave was made by Konenkov [1], followed by a number of studies of edge waves and vibrations in plates and shells see [2-7] including the consideration of 3D edge modes [8-10]. The history of discovery and several re-discoveries of this wave along with an overview of the state-of-art of edge waves and resonances may be found in [11]. This paper aims at extension of the current state of art in two directions. First of all, it generalises the analysis in [2] for a free plate to a slightly more practical case of a plate supported by the Winkler-Fuss elastic foundation in line with a similar analysis for isotropic plate [12]. Secondly, the general profile of the wave is constructed in terms of an arbitrary plane harmonic function, being related to results of Chadwick [13] and a recent chapter [14]. Finally, several illustrative examples are considered including a conventional sinusoidal profile together with the general form of the eigensolution arising for arbitrary initial data. An example of initial conditions corresponding to a point load demonstrates more localized distribution along the longitudinal variable occurring with increase of the transverse variable moving away from the edge.

1. Statement of the problem. Consider an orthotropic elastic plate of thickness $2h$ supported by a Winkler-Fuss foundation, see Figure 1. For a detailed description of the Winkler-Fuss model and historical review of this and other types of foundations reader is referred to [15]. The plate occupies the region $\left(-\infty < x < \infty,\ 0 \le y < \infty,\ 0 \le z \le 2h\right)$, with the foundation domain given by $\left(-\infty < x < \infty,\ 0 \le y < \infty,\ 2h \le z < \infty\right)$.

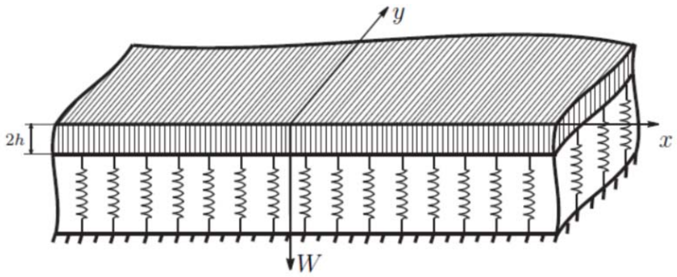

Fig.1. Elastic plate on the Winkler-Fuss foundation.

The deflection of the plate W is described by the plate bending equation

$$D_x \frac{\partial^4 W}{\partial x^4} + \left(2D_1 + 4D_{xy}\right)\frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} + 2\rho h \frac{\partial^2 W}{\partial t^2} + \beta W = 0, \tag{1.1}$$

where β is the Winkler coefficient, ρ is mass density and the bending stiffness moduli D_x, D_y, D_1 and D_{xy} are expressed in terms of the technical constants as

$$D_x = \frac{2E_1 h^3}{3\left(1-\nu_{12}\nu_{21}\right)}, \quad D_y = \frac{2E_2 h^3}{3\left(1-\nu_{12}\nu_{21}\right)}, \quad D_1 = \nu_{21} D_x, \quad D_{xy} = \frac{2G_{xy} h^3}{3}, \tag{1.2}$$

with $\nu_{21}E_1 = \nu_{12}E_2$, for more details see [16].

The free edge boundary conditions at $y = 0$ are imposed, precluding moment and shear force, namely

$$D_1 \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2} = 0, \qquad \left(D_1 + 4D_{xy}\right)\frac{\partial^3 W}{\partial x^2 \partial y} + D_y \frac{\partial^3 W}{\partial y^3} = 0. \tag{1.3}$$

It is noted that the bending stiffness moduli should satisfy the conditions ensuring positive density of the strain energy density

$$D_{xy} > 0, \quad D_x + D_y > 0, \quad D_x + D_y < D_1^2, \tag{1.4}$$

see [2].

2. Dispersion relation. Let us derive the dispersion relation. The deflection is conventionally sought in the form of a harmonic travelling wave of exponential profile

$$W(x,y,t) = A\exp\left[i\left(kx - \omega t\right) - k\lambda y\right], \tag{2.1}$$

where k is wavenumber, ω is frequency and the condition $\operatorname{Re}\lambda > 0$ ensures decay away from the edge. Substituting (2.1) into (1.1), one results in the following bi-quadratic equation

$$\lambda^4 - \frac{2D_1 + 4D_{xy}}{D_y}\lambda^2 + \frac{D_x k^4 + \beta - 2\rho h \omega^2}{D_y k^4} = 0, \tag{2.2}$$

which may be shown to have two roots satisfying the decay condition $\operatorname{Re}\lambda > 0$. Therefore, the deflection is expressed in the form

$$W(x,y,t) = \sum_{j=1}^{2} A_j \exp\left[i\left(kx - \omega t\right) - k\lambda_j y\right], \tag{2.3}$$

with

$$\lambda_1^2 + \lambda_2^2 = \frac{2D_1 + 4D_{xy}}{D_y}, \quad \lambda_1^2\lambda_2^2 = \frac{D_x k^4 + \beta - 2\rho h\omega^2}{D_y k^4}, \quad \left(\operatorname{Re}\lambda_j > 0\right)\cdot \tag{2.4}$$

Substituting (2.3) into the boundary conditions (1.3), one obtains a homogeneous linear system in A_1 and A_2, which possesses non-trivial solutions provided the appropriate determinant vanishes giving

$$\lambda_2\left(\lambda_2^2 D_y - \left(D_1 + 4D_{xy}\right)\right)\left(\lambda_1^2 D_y - D_1\right) - \lambda_1\left(\lambda_1^2 D_y - \left(D_1 + 4D_{xy}\right)\right)\left(\lambda_2^2 D_y - D_1\right) = 0. \tag{2.5}$$

Factorising the last equation and using the definitions (2.4), it is possible to obtain

$$D_y^2\lambda_1^2\lambda_2^2 + 4D_y D_{xy}\lambda_1\lambda_2 - D_1^2 = 0, \tag{2.6}$$

which implies

$$\frac{D_x k^4 + \beta - 2\rho h\omega^2}{D_y k^4} = \frac{\left(\sqrt{D_1^2 + 4D_{xy}^2} - 2D_{xy}\right)^2}{D_y^2}. \tag{2.7}$$

Some small algebraic manipulations lead to the dispersion relation of the form

$$2\rho h\omega^2 - \beta = D_x k^4 c^4, \tag{2.8}$$

generalising the result of [12] to the case of orthotropic elastic plate. Here the constant

$$c = \left(1 - \frac{\left(\sqrt{D_1^2 + 4D_{xy}^2} - 2D_{xy}\right)^2}{D_x D_y}\right)^{1/4} \tag{2.9}$$

appearing first in the paper of Norris [2] generalises the well-known Konenkov constant [1] for isotropic plate

$$c_K = \left(\left(1-\nu\right)\left(3\nu - 1 + 2\sqrt{2\nu^2 - 2\nu + 1}\right)\right)^{1/4}. \tag{2.10}$$

Similarly to [12], the presence of elastic foundation causes a cut-off frequency

$$\omega_0 = \sqrt{\frac{\beta}{2\rho h}}. \tag{2.11}$$

It should be noted that the value of the cut-off frequency is identical to that of the isotropic case due to the fact that the anisotropy is only affecting the coefficient within wave number. Moreover, formal similarity between the dispersion relations implies the critical speed of the associated moving load problem on a supported beam, corresponding to the local minimum of the phase velocity, for which

$$ck = \sqrt[4]{\frac{\beta}{D_x}}, \quad cV^{ph} = \frac{2\sqrt{\rho h}}{\sqrt[4]{\beta D_x}}, \tag{2.12}$$

where V^{ph} denotes the phase velocity, for more details see [KP14]. In view of the boundary conditions (1.3) the deflection profile may be written as

$$W = \exp\left[i\left(kx - \omega t\right) - k\lambda_1 y\right] - \frac{D_1 - \lambda_1^2 D_y}{D_1 - \lambda_2^2 D_y}\exp\left[i\left(kx - \omega t\right) - k\lambda_2 y\right]. \tag{2.13}$$

3. Free wave of arbitrary profile. Let us now extend the exponential profile (2.13) to a more general form following [13] and [14]. On introducing the dimensionless coordinates

$$\xi = \frac{x}{h}, \quad \eta = \frac{y}{h}, \quad \tau = t\sqrt{\frac{D_x}{2\rho h^5}}, \tag{3.1}$$

the boundary value problem (1.1), (1.3) takes the form

$$\frac{\partial^4 W}{\partial \eta^4} + \frac{2D_1 + 4D_{xy}}{D_y}\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{D_x}{D_y}\left(\frac{\partial^4 W}{\partial \xi^4} + \frac{\partial^2 W}{\partial \tau^2} + \beta_1 W\right) = 0, \tag{3.2}$$

where $\beta_1 = \frac{\beta h^4}{D_x}$, subject to the following boundary conditions at the edge $\eta = 0$

$$D_1 \frac{\partial^2 W}{\partial \xi^2} + D_y \frac{\partial^2 W}{\partial \eta^2} = 0, \quad \left(D_1 + 4D_{xy}\right)\frac{\partial^3 W}{\partial \xi^2 \partial \eta} + D_y \frac{\partial^3 W}{\partial \eta^3} = 0. \tag{3.3}$$

Similarly to [14], we adopt an assumption of a beam-like behaviour,

$$\gamma^4 \frac{\partial^4 W}{\partial \xi^4} + \frac{\partial^2 W}{\partial \tau^2} + \beta_1 W = 0, \tag{3.4}$$

corresponding physically to parametric dependence on transverse coordinate η and mirroring the string-like analogy for Rayleigh waves, see [13] and also [17]. As will be shown later, the assumption (3.4) is additionally justified by the fact that the associated dispersion relation of this "effective" beam on the Winkler foundation coincides with the dispersion relation (2.8) of the bending edge wave. Here γ is a constant which will be determined later. In view of assumption (3.4) the plate equation (3.2) transforms to a pseudo-static form

$$\frac{\partial^4 W}{\partial \eta^4} + \frac{2D_1 + 4D_{xy}}{D_y}\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{D_x}{D_y}\left(1 - \gamma^4\right)\frac{\partial^4 W}{\partial \xi^4} = 0, \tag{3.5}$$

which may be shown to be of elliptic type. The fourth order operator in the left hand side of (3.5) may be factorised as

$$\Delta_1 \Delta_2 W = 0, \tag{3.6}$$

where $\Delta_j = \partial_\eta^2 + \lambda_j^2 \partial_\xi^2$, and the constants λ_1 and λ_2 are determined from

$$\lambda_1^2 + \lambda_2^2 = \frac{2D_1 + 4D_{xy}}{D_y}, \quad \lambda_1^2 \lambda_2^2 = \frac{D_x}{D_y}\left(1 - \gamma^4\right), \quad \left(\operatorname{Re}\lambda_j > 0\right). \tag{3.7}$$

The deflection is then expressed as a sum of two arbitrary plane harmonic functions (in the first two arguments)

$$W = \sum_{j=1}^{2} W_j\left(\xi, \lambda_j \eta, \tau\right), \tag{3.8}$$

satisfying the decay conditions at $\eta \to \infty$, generalising the exponential profile (2.3) considered in the previous section and allowing other types of decay. Substituting (3.7) into the boundary conditions (3.3) and employing the Cauchy-Riemann identities for the plane harmonic functions W_j

$$\frac{\partial W_j}{\partial \eta} = -\lambda_j \frac{\partial W_j^*}{\partial \xi}, \quad \frac{\partial W_j}{\partial \xi} = \frac{1}{\lambda_j}\frac{\partial W_j^*}{\partial \eta}, \tag{3.9}$$

with the asterisk denoting the harmonic conjugate function, we deduce

$$\sum_{j=1}^{2}\left(D_1 - \lambda_j^2 D_y\right)\frac{\partial^2 W_j}{\partial \xi^2} = 0, \quad \sum_{j=1}^{2}\lambda_j\left(D_1 + 4D_{xy} - \lambda_j^2 D_y\right)\frac{\partial^3 W_j}{\partial \xi^3} = 0, \tag{3.10}$$

leading to equation (2.5), and therefore, to

$$\lambda_1^2\lambda_2^2 = \frac{\left(\sqrt{D_1^2 + 4D_{xy}^2} - 2D_{xy}\right)^2}{D_y^2}, \tag{3.11}$$

and, finally, to $\gamma = c$. Thus, the physical meaning of the parameter γ introduced in the assumption (3.4) is the coefficient in the dispersion relation (2.8). It is now clear that the dispersion relation corresponding to (3.4), coincides with (2.8). Using the properties of harmonic functions, it is now possible to relate the functions W_1 and W_2 from the boundary conditions (3.10).

Thus, the deflection profile (3.8) may be expressed in terms of an arbitrary single harmonic function as

$$W = W_1(\xi, \lambda_1\eta, \tau) - \frac{D_1 - \lambda_1^2 D_y}{D_1 - \lambda_2^2 D_y} W_1(\xi, \lambda_2\eta, \tau). \tag{3.12}$$

4. Illustrative examples. Let us now present several numerical examples, assuming

$$W_1(\xi, \lambda_1\eta, \tau) = \cos\xi \exp(-\lambda_1\eta - i\omega\tau), \tag{4.1}$$

where the frequency ω is determined from the assumption (3.4). The dependence on time is omitted here, with the curves on Fig. 2 showing the variation of the quantity

$$W_0(\xi, \lambda_1\eta) = \cos\xi \exp(-\lambda_1\eta) - \frac{D_1 - \lambda_1^2 D_y}{D_1 - \lambda_2^2 D_y} \cos\xi \exp(-\lambda_2\eta), \tag{4.2}$$

on the longitudinal coordinate ξ for several values of the transverse coordinate η.

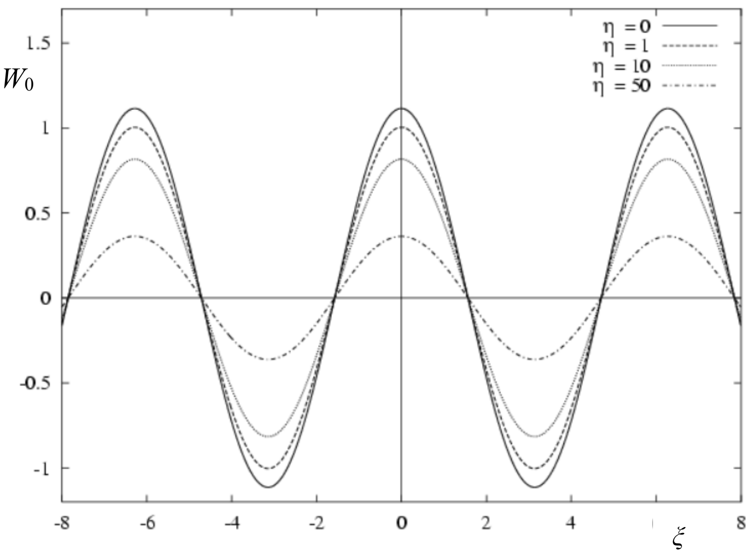

Fig. 2. Sinusoidal profile.

The calculations are performed for the following values of material parameters $E_1 = 54.2\text{GPa}$, $E_1 = 18.1\text{GPa}$, $G_{xy} = 8.96\text{GPa}$, $\nu_{12} = 0.25$, $h = 0.1m$, corresponding to a thin epoxy/glass plate, see [2]. Typically for edge bending waves, one of the attenuation parameters λ_1, λ_2 is close to zero, therefore the exponential decay away from the edge is rather slow, which is clearly confirmed by Fig. 2.

In addition to this expected behaviour the obtained representation (3.12) may allow other types of eigensolutions. Let us consider the wave profile originated by arbitrary initial conditions. According to (3.12) the deflection may be expressed in terms of the function $W_1(\xi, \lambda_1\eta, \tau)$ harmonic in the first two arguments, satisfying

$$\frac{\partial^2 W_1}{\partial \eta^2} + \lambda_1^2 \frac{\partial^2 W_1}{\partial \xi^2} = 0, \tag{4.3}$$

and

$$\gamma^4 \frac{\partial^4 W_1}{\partial \xi^4} + \frac{\partial^2 W_1}{\partial \tau^2} + \beta_1 W_1 = 0, \tag{4.4}$$

subject to the following initial conditions

$$W_1\big|_{\tau=0} = A(\xi, \lambda_1\eta), \qquad \frac{\partial W_1}{\partial \tau}\Big|_{\tau=0} = B(\xi, \lambda_1\eta). \tag{4.5}$$

It is clear from (4.3) that $A(\xi, \lambda_1\eta)$ and $B(\xi, \lambda_1\eta)$ are harmonic functions. Applying the integral Fourier transform with respect to longitudinal variable *ξ* with parameter *k*, we deduce

$$W_1^F = w_1(k, \tau) e^{-\lambda_1 \eta |k|}, \tag{4.6}$$

where W_1^F denotes the Fourier transform

$$W_1^F = \int_{-\infty}^{\infty} W_1(\xi, \lambda_1\eta, \tau) e^{-ik\xi} d\xi.$$

Using (4.4), we have the following initial value problem for $w_1(k, \tau)$

$$\frac{\partial^2 w_1}{\partial \tau^2} + \left(\gamma^4 k^4 + \beta_1\right) w_1 = 0, \tag{4.7}$$

subject to

$$w_1\big|_{\tau=0} = a(k), \qquad \frac{\partial w_1}{\partial \tau}\Big|_{\tau=0} = b(k). \tag{4.8}$$

Therefore the solution is given by

$$W_1(\xi, \lambda_1\eta, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{b(k)}{\delta} \sin(\delta\tau) + a(k)\cos(\delta\tau) \right] e^{-\lambda_1\eta|k| + ik\xi} dk, \tag{4.9}$$

where $\delta = \sqrt{\gamma^4 k^4 + \beta_1}$. Let us specify

$$A(\xi, \lambda_1\eta) = \frac{\lambda_1\eta}{\lambda_1^2\eta^2 + \xi^2}, \qquad B(\xi, \lambda_1\eta) = 0, \tag{4.10}$$

corresponding to point load at the edge

$$W_1\big|_{\tau=\eta=0} = \delta(\xi), \qquad \frac{\partial W_1}{\partial \tau}\Big|_{\tau=\eta=0} = 0. \tag{4.11}$$

The function W_1 is therefore given by

$$W_1(\xi, \lambda_1\eta, \tau) = \frac{1}{\pi} \int_0^{\infty} \cos(\delta\tau)\cos(k\xi) e^{-\lambda_1\eta k} dk, \tag{4.12}$$

with the deflection following from (3.12). In case of absence of the foundation $(\beta = 0)$ the last formula (4.12) may be simplified

$$W_1(\xi, \lambda_1\eta, \tau) = \frac{1}{2\pi} \sum_{m=1}^{2} \mathrm{Re} \int_0^{\infty} \exp\left[i\gamma^2\tau k^2 - \left(\lambda_1\eta + (-1)^m i\xi\right) k \right] dk, \tag{4.13}$$

which may be evaluate exactly through a standard integral

$$\int_0^{\infty} \exp\left(-px^2 - qx\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right) \mathrm{erfc}\left(\frac{q}{2\sqrt{p}}\right), \qquad \left(\mathrm{Re}\, p = 0, \mathrm{Im}\, p \neq 0, \mathrm{Re}\, q > 0\right). \tag{4.14}$$

see [20]. Using the last result (4.14), it is possible to obtain for the scaled deflection $W_s = 4\gamma\sqrt{\pi\tau}W(\xi,\eta,\tau)$ as

$$W_s(\xi,\eta,\tau) = \mathrm{Re}\left[e^{i\pi/4}\sum_{m=1}^{2}\left(f\left(\lambda_1\zeta+(-1)^m i\chi\right)-\frac{D_1-\lambda_1^2 D_y}{D_1-\lambda_2^2 D_y}f\left(\lambda_2\zeta+(-1)^m i\chi\right)\right)\right], \tag{4.15}$$

where $f(z) = e^{z^2}\mathrm{erfc}(z)$, with $\xi = 2\gamma\sqrt{\tau}\chi$ and $\eta = 2\gamma\sqrt{\tau}\zeta$.

The dependence of the scaled deflection W_s given by formula (4.15) for several values of ζ is shown on the next Fig. 3, clearly showing a different type of behaviour compared to Fig. 2. The calculations are performed for epoxy glass, with the values of material parameters are the same as for Figure 2. Indeed, the curves show a more localized type of distribution of the deflection along the longitudinal coordinate. Curiously, the localisation effect increases as we move away from the edge, see for example the curve for $\zeta = 50$. Indeed, as seen from Fig.3, for most of the curves the deflection amplitude decays away from the centre, except for one at the edge $\zeta = 0$. This may be readily explaneed by the result for the integral (4.13) at $\eta = 0$,

$$\int_0^\infty \cos\left(\alpha k^2\right)\cos(\beta k)\,dk = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\sin\left(\frac{\pi}{4}+\frac{\beta^2}{4\alpha}\right), \tag{4.16}$$

see [20]. Using the last result, we deduce for the scaled deflection at the edge $\zeta = 0$

$$W_s(\chi,0,\tau) = \frac{2D_y\left(\lambda_1^2-\lambda_2^2\right)}{D_1-\lambda_2^2 D_y}\sin\left(\frac{\pi}{4}+\chi^2\right). \tag{4.17}$$

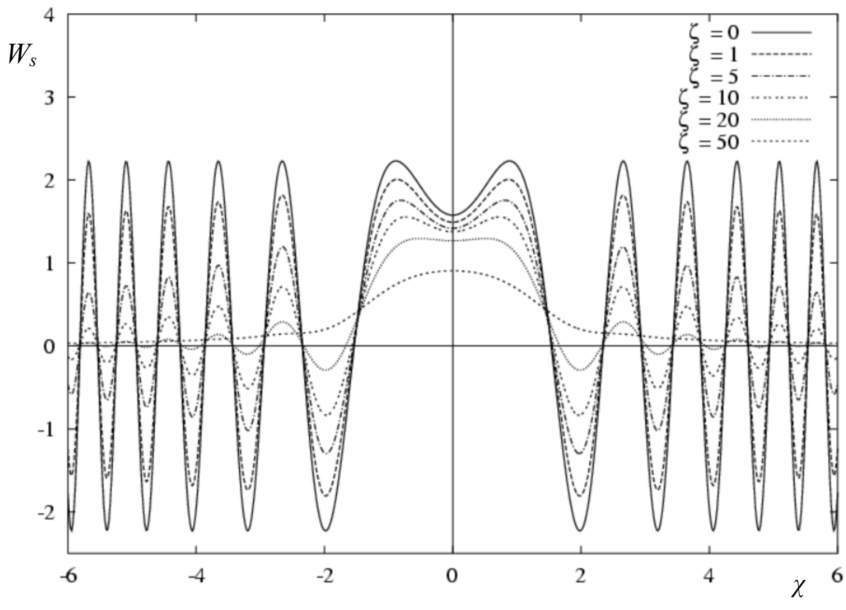

Fig.3. Non-sinusoidal profile.

5. Concluding remarks. Thus, we have considered bending edge waves on a thin orthotropic plate supported by the Winkler-Fuss elastic foundation. The dispersion relation has been analysed revealing similar features with that of isotropic Kirchhoff plate considered in [12]. Clearly, by setting $\beta = 0$ one obtains results for unsupported free plate. Extension of the analysis to different foundation models in line with [18] is possible along with consideration of variable elastic characteristics similarly to [19].

The consideration was then focused on generalisation of the harmonic exponential profile resulting in a general representation of the eigensolution in terms of a single plane harmonic function originating from [13]. The presented examples illustrate theoretical possibilities of not only sinusoidal but also localized profiles.

As one more direction of further development, we note a slow-time perturbation procedure which could be applied to the obtained eigensolution. The results should provide a parabolic-elliptic model for the bending edge wave, extracting its contribution to the overall dynamics response, generalising the results of [14] and [20]. Other plate theories may also be considered, see [21], along with the case of non-classical boundary conditions [3].

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