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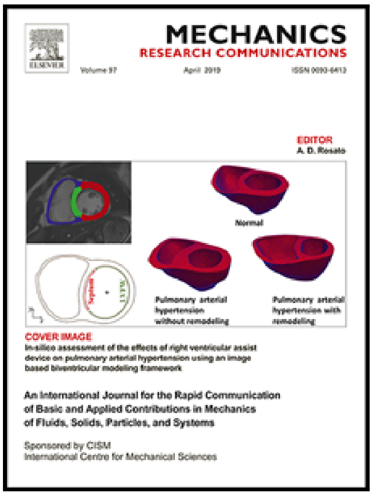

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Highlights

- Fourth-order composite equation for plate extension
- Quasi-front, Rayleigh wave-front
- Comparison with exact plane strain solution
- Pseudo-differential operator in the expression for the external load

A composite hyperbolic equation for plate extension

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Abstract

A fourth-order inhomogeneous hyperbolic equation modeling the symmetric motion of a thin elastic plate subject to shear stresses prescribed along its faces is derived. The shortened forms of this equation govern the quasi-front, i.e. dispersive wave-front of longitudinal waves and the Rayleigh wave front at long-wave, low-frequency and short-wave, high-frequency limits, respectively. Comparison with exact plane strain solutions for both free and forced vibrations demonstrates that the derived equation is also applicable over the intermediate region where a typical wave length is of order the plate thickness.

Keywords:
elasticity, composite equation, asymptotic, plate extension, Rayleigh wave, quasi-front

1. Introduction

It is well known that the 2D hyperbolic theory of plane stress, e.g. see [1], may be treated as the leading order long-wave, low-frequency approximation of the 3D equations in linear elasticity for plate extension. A drawback of this theory is that it distorts the longitudinal wave speed. As a result, a singularly perturbed hyperbolic system arising at next order, cf. [2], supports a dispersive longitudinal wave front, sometimes called quasi-front, corresponding to the wavefront predicted from the degenerated problem. However, as might be expected, neither conventional nor refined plane stress approximations are suited for modeling high-frequency, short-wave behavior. The aforementioned quasi-fronts are also observed for thin elastic rods and shells and have been tackled since long ago using both heuristic and asymptotic arguments, e.g. see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein.

In this paper, we attempt to develop a composite wave model for plate extension supporting not only the long-wave, low-frequency limit associated with the quasi-front, but also the short-wave, high-frequency limit involving surface waves. The latter is incorporated through the specialised formulation for the Rayleigh wave, see [11] and references therein, which includes, in particular, an explicit hyperbolic equation on the surface. The proposed formulation, as a number of composite models, e.g. see [12, 13] for further detail, is not uniformly valid. However, we may expect only qualitative coincide over the intermediate range, where a typical wave length is of order plate thickness.

Analogous composite wave formulations for plate bending have recently been established in [14]. Earlier, known composite dynamic theories for thin elastic structures, e.g. see [15], operated with ad hoc short-wave limits. We also mention composite models for periodic media in [16] demonstrating, again, similarity in asymptotic procedures for thin and periodic wave guides previoulsy noted in [17].

The geometric setup considered in this paper corresponds to a thin elastic strip loaded by shear stresses along its faces. A fourth-order inhomogeneous hyperbolic equation is derived. It is worth mentioning that its right-hand side contains a pseudo-differential operator acting on the prescribed load. The dispersion curve and also the displacement amplitude induced by surface stresses in the form of a travelling harmonic wave predicted from this equation are compared with those calculated from the related plane strain problem.

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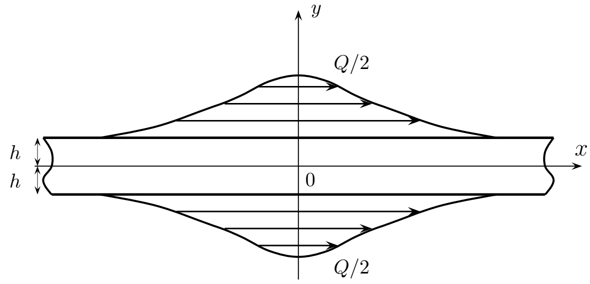

Figure 1: Symmetric deformation of an elastic strip under tangential surface loading.

2. Statement of the Problem

Consider an infinite elastic strip of thickness $2h$ ($-\infty < x < \infty$, $-h \leq y \leq h$) subject to tangential loads $Q/2$ at each of its faces $y = \pm h$, see Figure 1.

Let us first write down the governing equation in the refined asymptotic theory for plate extension [2]

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_3^2}\frac{\partial^2 u}{\partial t^2} + \frac{\nu^2 h^2}{3(1-\nu)^2 c_3^4}\frac{\partial^4 u}{\partial t^4} = -\frac{1-\nu^2}{2Eh}Q \quad (1)$$

with

$$c_3 = \sqrt{\frac{E}{\rho(1-\nu^2)}} \quad (2)$$

where $u(x,t)$ is the longitudinal displacement, t is time, E is Young's modulus, ρ is density and ν is the Poisson's ratio. The differential operator in (1) is the first-order correction to the hyperbolic equation of motion in the theory of generalized plane stress, e.g. see [2]. It contains a fourth–order derivative in time, enabling smoothing of the discontinuity at the quasi-front, i.e. the wave-front predicted by the associated degenerate second-order equation, propagating with speed c_3. The range of validity of equation (1) is given by

$$L \gg h \quad \text{and} \quad T \gg \sqrt{\frac{\rho}{E}}h, \quad (3)$$

where L and T are characteristic wavelength and time scale, respectively.

Over the range

$$L \ll h \quad \text{and} \quad T \ll \sqrt{\frac{\rho}{E}}h \quad (4)$$

we adapt the explicit asymptotic formulation for the surface Rayleigh wave, e.g. see [11] and references therein, starting from the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_R^2}\frac{\partial^2 u}{\partial t^2} = -\frac{(1+\nu)\left(1-k_2^2\right)k_2}{2EB}\sqrt{-\frac{\partial^2}{\partial x^2}}Q \quad (5)$$

specified along each of the faces $y = \pm h$ with

$$B = \frac{k_1}{k_2}\left(1-k_1^2\right) + \frac{k_2}{k_1}\left(1-k_2^2\right) - \left(1-k_2^4\right) \quad (6)$$

and

$$k_i = \sqrt{1 - c_R^2/c_i^2}, \quad i = 1,2. \quad (7)$$

where

$$c_1 = \sqrt{\frac{E}{(1+\nu)(1-2\nu)\rho}}, \qquad c_2 = \sqrt{\frac{E}{2(1+\nu)\rho}}$$

and $c = v_R$ ($v_R = c_R/c_2$) satisfies the equation

$$R(c) = \left(2-c^2\right)^2 - 4\sqrt{1-c^2}\sqrt{1-\chi^2 c^2} = 0 \quad (8)$$

with $\chi = \sqrt{(1-2\nu)/(2-2\nu)}$. Here, c_1, c_2 and c_R stand for compression, shear, and Rayleigh wave speeds, respectively. The pseudo–differential operator in the right-hand side of (5), e.g. see [18], was introduced earlier in [14].

Our aim is to derive a composite equation having the limiting long-wave, low-frequency and short-wave, high-frequency behaviours in the form of equations (1) and (5), respectively.

3. Composite Equation

First, we differentiate (5) twice in time, having

$$\frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{1}{c_R^2}\frac{\partial^4 u}{\partial t^4} = -\frac{(1+\nu)\left(1-k_2^2\right)k_2}{2EB}\sqrt{-\frac{\partial^2}{\partial x^2}}\frac{\partial^2 Q}{\partial t^2}. \quad (9)$$

Then, we find the sum of the degenerate equation (1) and equation (9) for which the fourth–order derivative is neglected, and equation (9) is multiplied by a factor γh^2, where the coefficient γ has to be found. The resulting composite equation becomes

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_3^2}\frac{\partial^2 u}{\partial t^2} + \gamma h^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_R^2}\frac{\partial^2 u}{\partial t^2}\right) = -\frac{(1+\nu)}{E}\left(\frac{1-\nu}{h}Q + \gamma h^2 \frac{\left(1-k_2^2\right)k_2}{2B}\sqrt{-\frac{\partial^2}{\partial x^2}}\frac{\partial^2 Q}{\partial t^2}\right). \quad (10)$$

Let us show that the last equation governs the sought for composite hyperbolic formulation and also determine the factor γ. Introducing nondimensional variables

$$\xi = \frac{x}{L}, \qquad \tau = \frac{c_3 t}{L} \quad (11)$$

and assuming that $\eta = h/L \ll 1$ at $T = L/c_3$ according to (3), we get from (10)

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \tau^2} + \gamma\eta^2 c_3^2 \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial^2 u}{\partial \xi^2} - \frac{c_3^2}{c_R^2}\frac{\partial^2 u}{\partial \tau^2}\right) = -\frac{(1+\nu)L^2}{Eh}\left((1-\nu)Q + \eta^3\gamma c_3^2 \frac{\left(1-k_2^2\right)k_2}{2B}\sqrt{-\frac{\partial^2}{\partial \xi^2}}\frac{\partial^2 Q}{\partial \tau^2}\right). \tag{12}$$

The left hand side of the last equation, within the same truncation error, can be rewritten as

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \tau^2} + \gamma\eta^2 c_3^2\left(1 - \frac{c_3^2}{c_R^2}\right)\frac{\partial^4 u}{\partial \tau^4} = 0. \tag{13}$$

since, at leading order, we have $\dfrac{\partial^2 u}{\partial \xi^2} = \dfrac{\partial^2 u}{\partial \tau^2}$.

Let us now require (13) to coincide with the homogenous equation (1), which takes the form

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \tau^2} + \eta^2\frac{\nu^2}{3(1-\nu)^2}\frac{\partial^4 u}{\partial \tau^4} = 0 \tag{14}$$

in the nondimensional variables (11). This ensures the composite equation picks up quasi-front predicted by (14), see [2] for further details. As a result,

$$\gamma = \frac{\nu^2 c_R^2}{3(1-\nu)^2\left(c_R^2 - c_3^2\right)c_3^2}. \tag{15}$$

Thus, the sought for composite equation is given by (10) with (15). It is constructed in such a way that its long-wave, low-frequency limit coincides with equation (1), see also the right-hand side of (12). At the same time, at the short-wave, high-frequency limit for which $\eta = h/L \gg 1$, see (4), we readily get from (12) at leading order

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{c_3^2}{c_R^2}\frac{\partial^2 u}{\partial \tau^2} = -\frac{(1+\nu)\left(1-k_2^2\right)k_2 L}{2EB}\sqrt{-\frac{\partial^2}{\partial \xi^2}}Q. \tag{16}$$

4. Dispersion Analysis

Consider the dispersion relations corresponding to the derived composite equation (10) with (15) and its limiting forms (1) and (5). They are given, respectively, by

$$K^2 - \frac{c_2^2}{c_3^2}\Omega^2 - \gamma c_2^2\Omega^2\left(K^2 - \frac{1}{\nu_R^2}\Omega^2\right) = 0, \tag{17}$$

$$K^2 - \frac{c_2^2}{c_3^2}\Omega^2 - \frac{\nu^2}{3(1-\nu)^2}\frac{c_2^4}{c_3^4}\Omega^4 = 0, \tag{18}$$

and

$$K^2 - \frac{1}{\nu_R^2}\Omega^2 = 0, \tag{19}$$

with

$$K = kh, \qquad \Omega = \frac{\omega h}{c_2},$$

where k is wavenumber and ω is angular frequency.

As might be expected, at $\Omega \sim K \ll 1$, relation (17) coincides with formula (18) to within higher-order terms. At the same time, at $\Omega \sim K \gg 1$, its leading order part tends to expression (19).

Numerical results are presented for $\nu = 0.25$ for which $\nu_R = 0.9194$. Figure 2 displays the solutions of the limiting dispersion relations (18) and (19) along with composite relation (17) versus the solution of the Rayleigh–Lamb dispersion relation (A.4) for the fundamental mode, see Appendix.

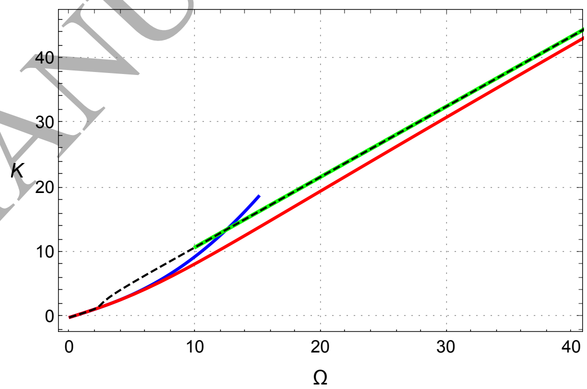

Figure 2: Dispersion curves for the refined theory for plate extension (18) (blue line), the Rayleigh wave (19) (green line) and the composite equations (17) (red line) versus the Rayleigh-Lamb dispersion curve (A.4) (black, dotted line).

The accuracy of the composite relation is tested against the numerical solution of the Rayleigh-Lamb equation. Figure 3 displays the percentage error defined by

$$e_r = \left|\frac{D_{RL} - K}{D_{RL}}\right| \tag{20}$$

where D_{RL} corresponds to the associated Rayleigh-Lamb root. It is clear from the figure that in the lower and higher frequency regimes, corresponding to the classical plate and Rayleigh wave approximations respectively, the relative error is small enough. In the intermediate region, however, we may suggest, based both on Figures 2 and 3, that only a qualitative agreement between the composite model and the exact solution is achieved.

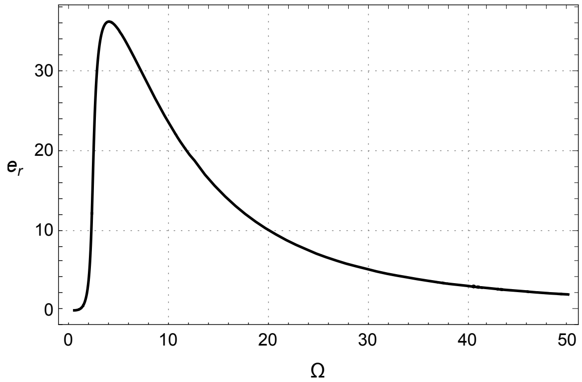

Figure 3: Relative error between composite equation (17) and Rayleigh-Lamb equation (A.6).

5. Forced Problem

Consider surface loading in the form of travelling harmonic waves $Q = Q_0\, e^{i(kx-\omega t)}$ propagating along the faces. Study horizontal displacement $u = 2A(1+\nu)hQ_0/E\, e^{i(kx-\omega t)}$, where A is the dimensionless amplitude of interest. Then, we get from (1), (5), and (10), respectively,

$$A = \frac{1-\nu}{4}\frac{1}{K^2 - \frac{c_2^2}{c_3^2}\Omega^2 - \frac{\nu^2}{3(1-\nu)^2}\frac{c_2^4}{c_3^4}\Omega^4}, \tag{21}$$

$$A = \frac{\left(1-k_2^2\right)k_2}{4B}\frac{K}{K^2 - \frac{c_2^2}{c_R^2}\Omega^2}, \tag{22}$$

and

$$A = \frac{1}{4}\frac{(1-\nu) - \frac{\gamma\left(1-k_2^2\right)k_2 c_2^2}{B}K\Omega^2}{K^2 - \frac{c_2^2}{c_3^2}\Omega^2 - \gamma c_2^2\Omega^2\left(K^2 - \frac{c_2^2}{c_R^2}\Omega^2\right)}, \tag{23}$$

where (21) and (22) correspond to the leading order asymptotic behaviours of (23) at $\Omega \sim K \ll 1$ and $\Omega \sim K \gg 1$. At the same time, (21) and (22) also coincide with the leading order asymptotic behaviours of the exact plane strain solution given by (A.7) and (A.8).

In the following figures, numerical data are demonstrated at $K(\Omega) = (1+\varepsilon)\Omega/v_R$ with $\varepsilon = 0.1$. As above, $\nu = 0.25$ and consequently, $v_R = 0.9194$. Figure 4 illustrates a comparison of the solutions of the limiting equations (21) and (22) versus the solution of the composite equation (23) together with the exact solution given by (23). The relative error, see (20), is plotted in Figure 5.

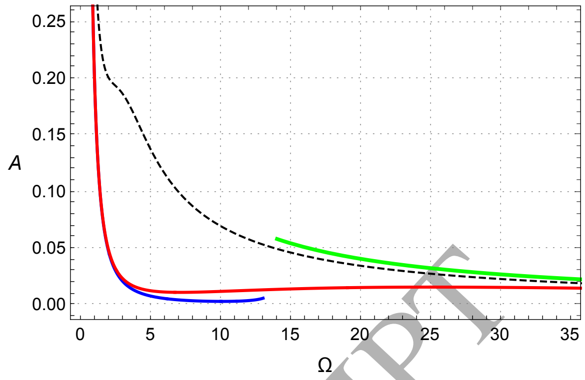

Figure 4: The displacement amplitudes calculated from equation (21) (blue line), equation (23) (red line), equation (22) (green line) and plane elasticity (A.4) (black, dotted line).

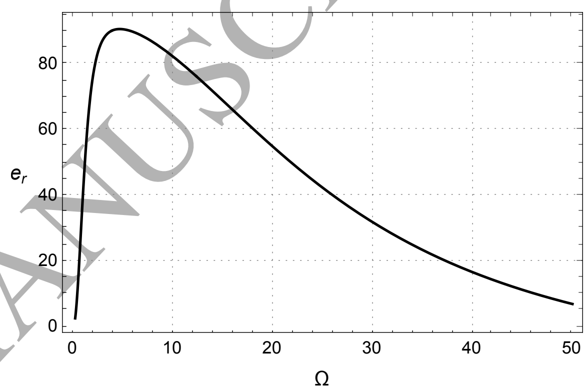

Figure 5: Relative error for composite solution (23) and exact solution (A.8).

Once again, the composite wave model is observed to be in a good agreement with the exact displacement in the extremities of the frequency spectrum. A considerable disprecany over the intermediate region is also due to the small values of the exact solution in Figure 4.

6. Conclusions

A composite wave formulation, see (10) with (15), having as its shortened forms the refined plate equation (1) and the hyperbolic Rayleigh wave operator (5), is derived. It is shown that the associated dispersion curve approximates the limiting behaviours of the the fundamental symmetric Rayleigh-Lamb mode at $\Omega \sim K \ll 1$ and $\Omega \sim K \gg 1$, see Section 4. The acquired composite equation (10) also demonstrates a reasonable accuracy in evaluating forced vibration amplitudes, see Section 5, as it follows from the comparison with the exact solution of the related plane strain problem presented in Appendix.

The developed methodology may readily be extended to the 2D setup and to the analysis of non-symmetric surface loading, when along with the considered in the paper extensional modes, bending modes, studied in [14], are also induced. Further implementation of equation (10), especially in transient problems, appears to be of interest.

Finally, we mention that various comments on peculiarities and limitations of composite wave models for plate bending made in [14], are seemingly relevant for the case of plate extension treated in the paper.

7. Acknowledgements

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Appendix A. Exact plane strain solution

Consider an elastic layer ($-\infty \leq x \leq \infty$, $-h \leq y \leq h$) in the framework of the plane strain theory. The equations of motion and the boundary conditions at $y = \pm h$ expressed through the wave potentials $\phi(x, y, t)$ and $\psi(x, y, t)$ are given respectively by, e.g. see [1],

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{c_1^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{c_2^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{A.1}$$

and

$$\sigma_{31} = \frac{E}{2(1+\nu)}\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} + 2\frac{\partial^2 \varphi}{\partial x \partial y}\right)\Bigg|_{y=\pm h} = \pm Q/2,$$
$$\sigma_{33} = \frac{E}{2(1+\nu)\chi^2}\left(\frac{\nu}{1-\nu}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2\chi^2\frac{\partial^2 \psi}{\partial x \partial y}\right)\Bigg|_{y=\pm h} = 0, \tag{A.2}$$

The solution to the formulated problem for the horizontal displacement along the faces

$$u = \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y}\right)\Bigg|_{y=\pm h}, \tag{A.3}$$

takes the form $u = 2A(1+\nu)hQ_0/E\, e^{i(kx-\omega t)}$ with

$$A = -\frac{\Omega^2 \beta}{2D_{RL}(K, \Omega)}, \tag{A.4}$$

where

$$\alpha = \sqrt{K^2 - \chi^2\Omega^2}, \qquad \beta = \sqrt{K^2 - \Omega^2}, \tag{A.5}$$

and the Rayleigh–Lamb denominator is written as

$$D_{RL}(K, \Omega) = (2K^2 - \Omega^2)^2 \tanh\beta - 4K^2\alpha\beta\tanh\alpha, \tag{A.6}$$

with Ω and K defined in Section 4.

The leading order long–wave, low–frequency expansion of formula (A.4) at $\Omega \ll 1$ and $K \ll 1$ becomes

$$A = \frac{1-\nu}{4}\frac{1}{K^2 - \frac{1-\nu}{2}\Omega^2}. \tag{A.7}$$

At leading order, we also have for the Rayleigh wave contribution at $K \sim \Omega \gg 1$, and $|\Omega/K - v_R| \ll 1$

$$A = -\frac{1}{K}\frac{v_R^3\sqrt{1-v_R^2}}{R'(v_R)\left(\Omega^2/K^2 - v_R^2\right)}, \tag{A.8}$$

where the Rayleigh denominator R is given by equation (8), with prime denoting a differentiation with respect to the argument of the Rayleigh denominator.

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