On Rayleigh wave field induced by surface stresses under the effect of gravity

A. Mubaraki

Department of Mathematics and Statistics, College of Science, Taif University,

P.O. Box 11099, Taif 21944, Saudi Arabia

D. Prikazchikov[1]

School of Computing and Mathematics, Keele University,

Keele, Staffordshire, ST5 5BG, UK

Abstract

The paper is concerned with development of the asymptotic formulation for surface wave field induced by vertical surface stress under the effect of gravity in the short-wave region. The approach relies on the methodology of hyperbolic-elliptic models for the Rayleigh wave and results in a regularly perturbed hyperbolic equation on the surface acting as a boundary condition for the elliptic equation governing decay over the interior. A special value of the Poisson's ratio $\nu = 0.25$ is pointed out, at which the effect of gravity disappears at leading order.

1 Introduction

Surface waves have numerous engineering application, including seismology, aerospace, civil and structural engineering, and more, see e.g. [1], [2],[3] and references therein. We also mention a recent substantial interest in designing seismic meta-barriers and meta-surfaces, see e.g. [4], [5] and [6] to name a few.

After Lord Rayleigh's seminal contribution [7], the area of surface waves has developed immensely, taking into account various advanced properties of material, see e.g. [8], [9] and [10] among many more contributions.

An important sub-area of studies of elastic surface waves is related to accounting for the effect of gravity, starting from the early contributions of Bromwich[11] and Love [12], where gravity is treated as a body force. The approach in the influential work [13] assumes that gravity creates a hydrostatic initial stress, see also [14], extending the consideration to Rayleigh-Lamb waves subject to gravity in an elastic layer. Some of the more recent works on Rayleigh and Stoneley waves under the effect of gravity include [15], [16], [17], [18] and [19], accounting for the effects of anisotropy, initial stress, vertical inhomogeneity and magneto-elasticity.

The majority of the above cited publications are dealing with free surface, analysing the dispersive properties of the wave. In the current paper, we consider a forced problem, aiming at incorporating the effect of gravity within the methodology of explicit asymptotic models for Rayleigh wave fields induced by surface loading, see [20] and references therein. These models are derived as slow-time perturbations of surface waves of arbitrary profile, see e.g. [21] and [22], and hence benefit from reduction of the vector problem in elasticity to a scalar problem for one of the wave potentials. The described model typically contains an elliptic equation governing decay over the interior, along with a hyperbolic equation on the surface, acting as a boundary condition for the elliptic equation. Recent developments of this methodology include incorporation of anisotropy [23], [24], pre-stress [25], as well as development of composite plate models [26] and refined second-order model [27], along with treatment of Rayleigh-type waves on a coated half-space with Dirichlet type boundary condition and ideal contact on the interface [28], and with Neumann type boundary condition and sliding contact on the interface [29].

In the current paper, we are starting from the dispersion relation for a Rayleigh wave under the effect of gravity. The consideration is performed under plane-strain assumption. Introducing a small

[1]**Corresponding author:**
Danila Prikazchikov, School of Computing and Mathematics, Keele University, Staffordshire, ST5 5BG, UK.
Email: d.prikazchikov@keele.ac.uk

parameter, we notice that the equations of motion are weakly coupled, and derive both exact dispersion relation (confirming the result in [14]), along with its three-term approximation. Then, we establish a slow-time perturbation procedure for the forced problem with prescribed surface stresses. Contrary to previous contributions dealing with effective boundary conditions, see e.g. [30], which did not affect the perturbation analysis of wave equations of motion, the aforementioned weak coupling caused by gravity leads to a novel form of two-termed solutions. Consequently, the analysis of boundary conditions implies a regularly perturbed hyperbolic equation on the surface, by means of a pseudo-differential operator. This result is formally similar to that in [28] for a Rayleigh-type wave on a coated half-space with a clamped surface, although the non-uniform nature of the asymptotic approximation in [28] due to presence of thickness resonant frequencies for a coating layer is certainly not the feature of the current manuscript.

The paper is organised as follows. The problem is formulated in Section 2, and a dispersion relation and its three-term approximation are discussed in Section 3. Then, the hyperbolic-elliptic formulation for the Rayleigh wave field induced by surface loading under the effect of gravity is derived. The dependence of the coefficient at the pseudo-differential operator in the perturbed wave equation on the surface on the material parameters is studied, and a special value of the Poisson's ratio ($\nu = 0.25$) at which the effect of gravity disappears, is pointed out. Finally, generalisations to 3D are discussed.

2 Statement of the problem

Consider an elastic, isotropic, compressible half-space occupying the domain $-\infty < x_1, x_3 < \infty$ and $x_2 \geqslant 0$, along with the influence of gravity. Throughout this paper, a plane-strain assumption is adopted, for which $u_3 = 0$, $u_m = u_m(x_1, x_2, t)$, $m = 1, 2$. We focus our attention on surface wave field induced by the prescribed loading on the surface $x_2 = 0$, with $P = P(x_1, t)$ and $Q = Q(x_1, t)$ being its vertical and horizontal components, respectively, see Fig. 1.

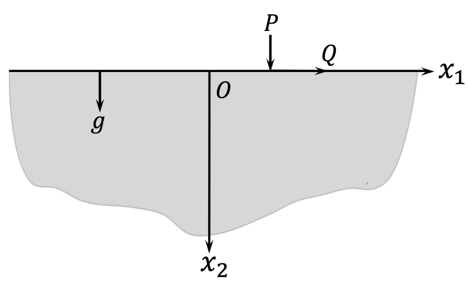

Figure 1: Schematics of an elastic half-space with gravity

The equations of motion accounting for the effect of gravity are given by [13], see also [17] and [16]

$$
\begin{aligned}
&\sigma_{11,1} + \sigma_{12,2} + \rho g\, u_{2,1} = \rho\, u_{1,tt},\\
&\sigma_{21,1} + \sigma_{22,2} - \rho g\, u_{1,1} = \rho\, u_{2,tt},
\end{aligned}
\tag{2.1}
$$

where ρ is volume mass density, σ_{mn} and u_m $(m, n = 1, 2)$ are stress and displacement components, g is acceleration of gravity, and comma indicates differentiation with respect to appropriate spatial or time variable. The constitutive relations for an isotropic elastic solid are taken in conventional form

$$
\sigma_{mn} = \lambda\, \delta_{mn}\, (u_{1,1} + u_{2,2}) + \mu\, (u_{m,n} + u_{n,m})\,,
\tag{2.2}
$$

see e.g. [31], where λ and μ are the Lamé elastic moduli. The boundary conditions at the surface $x_2 = 0$ are

$$
\sigma_{12} = -Q, \qquad \text{and} \qquad \sigma_{22} = -P.
\tag{2.3}
$$

The displacements can be expressed in terms of Lamé elastic potentials ϕ and ψ as

$$
u_1 = \phi_{,1} - \psi_{,2}, \qquad u_2 = \phi_{,2} + \psi_{,1}.
\tag{2.4}
$$

In view of (2.2) and (2.4), equations (2.1) may be rewritten in terms of the wave potentials as

$$\begin{aligned} \phi_{,11} + \phi_{,22} - \frac{1}{c_1^2}\,\phi_{,tt} &= -\frac{g}{c_1^2}\,\psi_{,1}, \\ \psi_{,11} + \psi_{,22} - \frac{1}{c_2^2}\,\psi_{,tt} &= \frac{g}{c_2^2}\,\phi_{,1}, \end{aligned} \tag{2.5}$$

with boundary conditions (2.3) taking the form

$$\begin{aligned} 2\phi_{,12} + \psi_{,11} - \psi_{,22} &= -\frac{Q}{\mu}, \\ \left(\kappa^2 - 2\right)\phi_{,11} + \kappa^2\,\phi_{,22} + 2\psi_{,12} &= -\frac{P}{\mu}, \end{aligned} \tag{2.6}$$

where

$$\kappa = \frac{c_1}{c_2}, \quad \text{and} \quad c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}},$$

with c_1 and c_2 denoting the longitudinal and shear wave speeds, respectively.

3 Dispersion relation

Let us first derive the dispersion relation, assuming $P = Q = 0$. The wave potentials are sought for in the form

$$\phi = f_1\left(x_2\right) e^{ik(x_1 - ct)}, \qquad \psi = f_2\left(x_2\right) e^{ik(x_1 - ct)}, \tag{3.1}$$

where k and c denote wave number and phase speed, respectively.

On inserting (3.1) into (2.5), we obtain

$$f_{1,22} - k^2\,\alpha^2\,f_1 = -ik^2\kappa^{-2}\,\varepsilon\,f_2, \qquad f_{2,22} - k^2\beta^2\,f_2 = ik^2\,\varepsilon\,f_1, \tag{3.2}$$

where

$$\alpha = \sqrt{1 - \frac{c^2}{c_1^2}}, \qquad \beta = \sqrt{1 - \frac{c^2}{c_2^2}}, \qquad \varepsilon = \frac{g}{k\,c_2^2}. \tag{3.3}$$

It is observed from (3.2), that for a range of wave numbers satisfying $k \gg \rho g/\mu$, or, equivalently, $\varepsilon \ll 1$, equations of motion are weakly coupled, which motivates a perturbation approach which will be established later.

Equations (3.2) may then be rearranged as a single fourth order ODE. For example, expressing f_2 from the first of the equations (3.2) and substituting it into the second one, we deduce

$$f_{1,2222} - k^2\left(\alpha^2 + \beta^2\right) f_{1,22} + k^4\left(\alpha^2\,\beta^2 - \varepsilon^2\kappa^{-2}\right) f_1 = 0, \tag{3.4}$$

The solutions for f_m, $(m = 1, 2)$, decaying away from the surface $x_2 = 0$, can be written in the form

$$f_1 = A_1\,e^{-kq_1\,x_2} + A_2\,e^{-kq_2\,x_2}, \qquad f_2 = \gamma_1\,A_1\,e^{-kq\,x_2} + \gamma_2\,A_2\,e^{-kq_2\,x_2} \tag{3.5}$$

where A_1, A_2 being arbitrary constants, with

$$q_m = \sqrt{\frac{\alpha^2 + \beta^2 + (-1)^m\sqrt{\left(\alpha^2 - \beta^2\right)^2 + 4\epsilon^2\kappa^{-2}}}{2}}, \tag{3.6}$$

and

$$\gamma_m = \frac{i\,\kappa^2}{\varepsilon}\left(q_m^2 - \alpha^2\right), \qquad m = 1, 2. \tag{3.7}$$

On inserting (3.5) and (3.1) into the traction-free boundary conditions (2.6) with $P = Q = 0$, we arrive at

$$\sum_{m=1}^{2}\left[2i\,q_m + \left(1+q_m^2\right)\gamma_m\right]A_m = 0,$$
$$\sum_{m=1}^{2}\left[\left(1-q_m^2\right)\kappa^2 + 2\left(i\,\gamma_m\,q_m - 1\right)\right]A_m = 0, \tag{3.8}$$

from which the dispersion relation follows as the solvability condition

$$\frac{2iq_1 + \left(1+q_1^2\right)\gamma_1}{2iq_2 + \left(1+q_2^2\right)\gamma_2} = \frac{\left(1-q_1^2\right)\kappa^2 + 2\left(i\,\gamma_1\,q_1 - 1\right)}{\left(1-q_2^2\right)\kappa^2 + 2\left(i\,\gamma_2\,q_2 - 1\right)}. \tag{3.9}$$

Note that an identical dispersion relation has been obtained in [14] (cf. Eq. (21) in the cited paper).

Assuming $\varepsilon \ll 1$, the dispersion relation (3.9) may be expanded,

$$R_0 + R_1\,\varepsilon + R_2\,\varepsilon^2 + O\left(\varepsilon^3\right) = 0, \tag{3.10}$$

where the leading order

$$R_0 = \left(1+\beta^2\right)^2 - 4\,\alpha\,\beta, \tag{3.11}$$

is associated with the classical Rayleigh wave equation, and the corrector terms are given by

$$K_1 = \frac{4\left(1-\alpha\,\beta\,\kappa^2\right)}{\kappa^2\left(\alpha+\beta\right)}, \tag{3.12}$$

and

$$K_2 = \frac{1}{\kappa^2\left(\alpha^2-\beta^2\right)^2}\left[\frac{2(\alpha^4+\beta^4)}{\alpha\beta} - 12\alpha\beta + \alpha^2\left(\beta^2\left(2\kappa^2-1\right)+1\right) - \kappa^2 - \beta^4\left(\kappa^2-2\right) + 3\beta^2 + 3\right]. \tag{3.13}$$

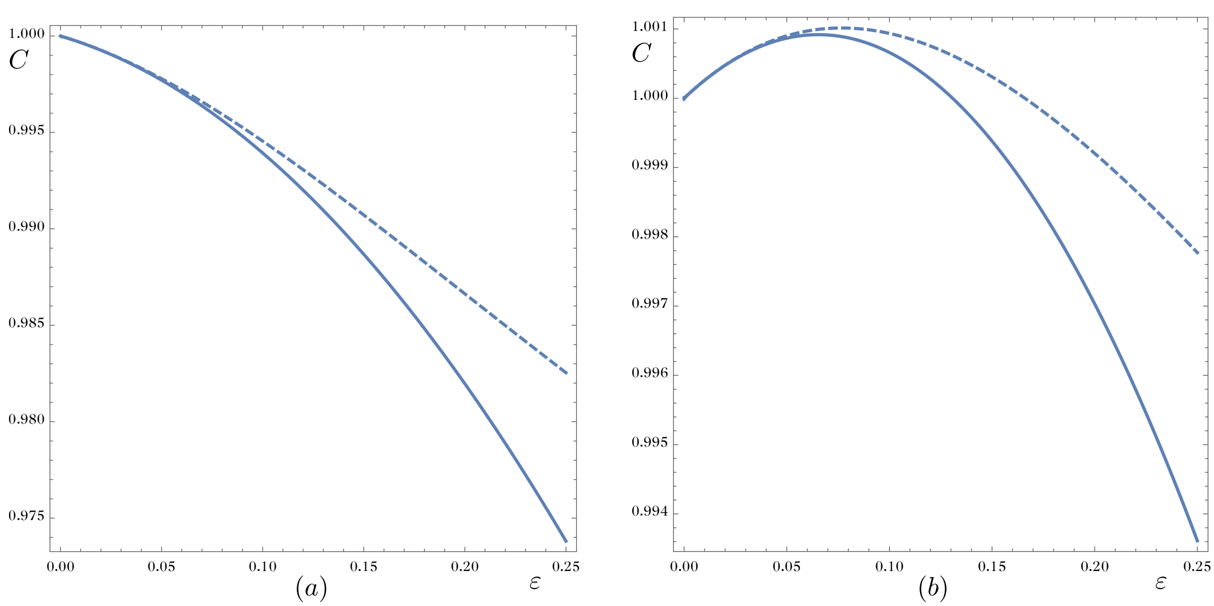

Figure 2: Dispersion relation (3.9) (solid line) and its approximation (3.10) (dashed line) for the Young's modulus $E = 2 \cdot 10^{11}$ Pa, volume mass density $\rho = 8 \cdot 10^3$ kg/m^3 and the Poisson's ratio $(a)\,\nu = 0.2$ and $(b)\,\nu = 0.3$.

The exact secular relation (3.9) and its asymptotic approximation (3.10), illustrating dependence of the dimensionless phase speed $C = c/c_R$ on ε, are depicted by the solid and dashed blue lines, respectively, in Fig. 2, for the Young's modulus $E = 2 \cdot 10^{11}$ Pa, volume mass density $\rho = 8 \cdot 10^3$ kg/m^3 and two values of the Poisson's ratio, namely $(a)\, \nu = 0.2$ and $(b)\, \nu = 0.3$. It may be seen from the latter that the three-term approximation (3.10) works well for $\varepsilon \ll 1$ corresponding to the short-wave region $k \gg \rho g/\mu$. Another distinct feature is a maximum at the Rayleigh wave speed in the short-wave limit $k \to \infty$ (or $\varepsilon \to 0$) in Figure 2(a) and a local minimum at the same point in Figure 2(b), which clearly indicates dependence of the type of extremum on the Poisson's ratio. Indeed, as will be shown later, the the local maximum in the short-wave limit occurs for $0 < \nu < 0.25$, whereas local minimum corresponds to $0.25 < \nu < 0.5$.

4 Asymptotic formulation for the Rayleigh-type wave

Now, we proceed with derivation of an asymptotic model for surface waves induced by applied surface loading ($P \neq 0, Q \neq 0$), with the effect of gravity incorporated. Let us introduce the scaling

$$\xi = \frac{x_1 - c_R\, t}{L}, \qquad \eta = \frac{x_2}{L} \qquad \tau = \frac{\epsilon\, c_R}{L}\, t, \tag{4.1}$$

where L and c_R are a typical wave length and the Rayleigh wave speed, respectively, where ϵ is the small parameter,

$$\epsilon = \frac{g\, L}{c_2^2} \ll 1, \tag{4.2}$$

which may be interpreted physically as the short-wave domain, in which the phase velocity of the studied wave is close to c_R, the classical Rayleigh wave speed.

Then, equations (2.5) and boundary conditions (2.6) become

$$\begin{gathered}\phi_{,\eta\eta} + \alpha_R^2\, \phi_{,\xi\xi} + 2\epsilon\left(1 - \alpha_R^2\right)\phi_{,\xi\tau} - \epsilon^2\left(1 - \alpha_R^2\right)\phi_{,\tau\tau} = -\epsilon\, \kappa^{-2}\, \psi_{,\xi},\\ \psi_{,\eta\eta} + \beta_R^2\, \psi_{,\xi\xi} + 2\epsilon\left(1 - \beta_R^2\right)\psi_{,\xi\tau} - \epsilon^2\left(1 - \beta_R^2\right)\psi_{,\tau\tau} = \epsilon\, \phi_{,\xi},\end{gathered} \tag{4.3}$$

and

$$\begin{gathered}2\phi_{,\xi\eta} + \psi_{,\xi\xi} - \psi_{,\eta\eta} = -\frac{L^2\, Q}{\mu},\\ \left(\kappa^2 - 2\right)\phi_{,\xi\xi} + \kappa^2 \phi_{,\eta\eta} + 2\psi_{,\xi\eta} = -\frac{L^2\, P}{\mu} \qquad \text{at} \qquad \eta = 0,\end{gathered} \tag{4.4}$$

where

$$\alpha_R = \sqrt{1 - \frac{c_R^2}{c_1^2}}, \qquad \text{and} \qquad \beta_R = \sqrt{1 - \frac{c_R^2}{c_2^2}}. \tag{4.5}$$

Expanding the wave potentials ϕ and ψ as asymptotic series

$$\begin{aligned}\phi &= \frac{1}{\epsilon}\left(\phi^{(0)}(\xi, \eta, \tau) + \epsilon\, \phi^{(1)}(\xi, \eta, \tau) + ...\right),\\ \psi &= \frac{1}{\epsilon}\left(\psi^{(0)}(\xi, \eta, \tau) + \epsilon\, \psi^{(1)}(\xi, \eta, \tau) + ...\right).\end{aligned} \tag{4.6}$$

Not surprisingly, the leading order problem for the equations of motion (4.3) gives solutions as harmonic functions in the first two arguments

$$\phi^{(0)} = \phi^{(0)}(\xi, \alpha_R\gamma, \tau), \qquad \psi^{(0)} = \psi^{(0)}(\xi, \beta_R\gamma, \tau). \tag{4.7}$$

Substitution of the latter into the leading order boundary conditions and using the properties of harmonic functions, we have

$$
\begin{aligned}
2\alpha_R\,\phi^{(0)}_{,\xi\xi} + \left(1+\beta_R^2\right)\mathcal{H}\left(\psi^{(0)}_{,\xi\xi}\right) &= 0,\\
\left(1+\beta_R^2\right)\phi^{(0)}_{,\xi\xi} + 2\beta_R\mathcal{H}\left(\psi^{(0)}_{,\xi\xi}\right) &= 0,
\end{aligned}
\tag{4.8}
$$

where $\mathcal{H}$ denotes the Hilbert transform. The solvability of (4.8) implies the classical Rayleigh equation

$$
\left(1+\beta_R^2\right)^2 - 4\,\alpha_R\,\beta_R = 0. \tag{4.9}
$$

In addition, it follows that the leading order potentials are related through Hilbert transform, as noted earlier by [21], see also [20], i.e.

$$
\psi^{(0)}(\xi,\beta_R\eta,\tau) = \vartheta\,\mathcal{H}\left(\phi^{(0)}(\xi,\beta_R\eta,\tau)\right), \qquad \phi^{(0)}(\xi,\alpha_R\eta,\tau) = -\vartheta^{-1}\,\mathcal{H}\left(\psi^{(0)}(\xi,\alpha_R\eta,\tau)\right), \tag{4.10}
$$

with constant ϑ defined by

$$
\vartheta = \frac{2\alpha_R}{1+\beta_R^2} = \frac{1+\beta_R^2}{2\beta_R}. \tag{4.11}
$$

At next order, we have from (4.3)

$$
\begin{aligned}
\phi^{(1)}_{,\eta\eta} + \alpha_R^2\,\phi^{(1)}_{,\xi\xi} &= -2\left(1-\alpha_R^2\right)\phi^{(0)}_{,\xi\tau} - \kappa^{-2}\,\psi^{(0)}_{,\xi},\\
\psi^{(1)}_{,\eta\eta} + \beta_R^2\,\psi^{(1)}_{,\xi\xi} &= -2\left(1-\beta_R^2\right)\psi^{(0)}_{,\xi\tau} + \phi^{(0)}_{,\xi},
\end{aligned}
\tag{4.12}
$$

The structure of the corrector terms $\phi^{(1)}$ and $\psi^{(1)}$ may be presented as

$$
\phi^{(1)} = \phi^{(1,0)} + \eta\,\phi^{(1,1)} + \phi^{(1,2)}, \qquad \text{and} \qquad \psi^{(1)} = \psi^{(1,0)} + \eta\,\psi^{(1,1)} + \psi^{(1,2)}, \tag{4.13}
$$

where $\phi^{(1,0)} = \phi^{(1,0)}\left(\xi,\alpha_R\,\eta,\tau\right)$ and $\psi^{(1,0)} = \psi^{(1,0)}\left(\xi,\beta_R\,\eta,\tau\right)$ are arbitrary plane harmonic functions in the first two arguments. Using the Cauchy-Riemann identities, for the functions $\phi^{(1,1)} = \phi^{(1,1)}\left(\xi,\alpha_R\,\eta,\tau\right)$ and $\psi^{(1,1)} = \psi^{(1,1)}\left(\xi,\beta_R\,\eta,\tau\right)$ we deduce

$$
\phi^{(1,1)} = -\frac{1-\alpha_R^2}{\alpha_R}\,\mathcal{H}\left(\phi^{(0)}_{,\tau}\right), \qquad \psi^{(1,1)} = -\frac{1-\beta_R^2}{\beta_R}\,\mathcal{H}\left(\psi^{(0)}_{,\tau}\right), \tag{4.14}
$$

whereas for $\phi^{(1,2)} = \phi^{(1,2)}\left(\xi,\beta_R\,\eta,\tau\right)$ and $\psi^{(1,2)} = \psi^{(1,2)}\left(\xi,\alpha_R\,\eta,\tau\right)$

$$
\phi^{(1,2)}_{,\xi} = \frac{1}{\kappa^2\left(\beta_R^2-\alpha_R^2\right)}\,\psi^{(0)}, \qquad \psi^{(1,2)}_{,\xi} = \frac{1}{\beta_R^2-\alpha_R^2}\,\phi^{(0)}. \tag{4.15}
$$

At next order, the boundary conditions (4.4) become

$$
\begin{aligned}
2\phi^{(1)}_{,\xi\eta} + \psi^{(1)}_{,\xi\xi} - \psi^{(1)}_{,\eta\eta} &= -\frac{L^2 Q}{\mu},\\
\left(\kappa^2-2\right)\phi^{(1)}_{,\xi\xi} + \kappa^2\,\phi^{(1)}_{,\eta\eta} + 2\psi^{(1)}_{,\xi\eta} &= -\frac{L^2 P}{\mu}.
\end{aligned}
\tag{4.16}
$$

Employing (4.13), (4.14) and (4.15), and applying Hilbert transform to the first equation, we infer

$$
\begin{aligned}
2\alpha_R\,\phi^{(1,0)}_{,\xi\xi} + \left(1+\beta_R^2\right)\mathcal{H}\left(\psi^{(1,0)}_{,\xi\xi}\right) &= \left(\frac{1-\beta_R^4}{\beta_R} - \frac{2(1-\alpha_R^2)}{\alpha_R}\right)\phi^{(0)}_{,\xi\tau}\\
&\quad - \frac{1+\alpha_R^2+\kappa^{-2}(1+\beta_R^2)}{\beta_R^2-\alpha_R^2}\,\mathcal{H}\left(\phi^{(0)}_{,\xi}\right) - \frac{L^2\,\mathcal{H}\left(Q\right)}{\mu},\\
\left(1+\beta_R^2\right)\phi^{(1,0)}_{,\xi\xi} + 2\beta_R\mathcal{H}\left(\psi^{(1,0)}_{,\xi\xi}\right) &= \left(\frac{1-\beta_R^4}{\beta_R^2} - 2(1-\beta_R^2)\right)\phi^{(0)}_{,\xi\tau}\\
&\quad - \frac{4\alpha_R(1+\kappa^2\beta_R^2)}{\kappa(1+\beta_R^2)(\beta_R^2-\alpha_R^2)}\,\mathcal{H}\left(\phi^{(0)}_{,\xi}\right) + \frac{L^2\,P}{\mu}.
\end{aligned}
\tag{4.17}
$$

Then, the solvability of (4.17) gives at $\eta = 0$

$$\left[4\alpha_R\,\beta_R - \left(1+\beta_R^2\right)^2\right]\phi^{(1,0)}_{,\xi\xi} = -\frac{L^2}{\mu}\left[(1+\beta_R^2)\,P + 2\beta_R\mathcal{H}\,(Q)\right]$$
$$-4B\,\phi^{(0)}_{,\xi\tau} + \frac{2}{\kappa^2(\beta_R^2-\alpha_R^2)}\left[2\alpha_R(1+\kappa^2\beta_R^2) - \beta_R\left(\kappa^2(1+\alpha_R^2)+1+\beta_R^2\right)\right]\mathcal{H}\left(\phi^{(0)}_{,\xi}\right), \tag{4.18}$$

where

$$B = \left(1-\alpha_R^2\right)\frac{\beta_R}{\alpha_R} + \left(1-\beta_R^2\right)\frac{\alpha_R}{\beta_R} - 1 + \beta_R^4 \tag{4.19}$$

is a known constant appearing in the hyperbolic-elliptic model for the Rayleigh wave (cf. Eq. (93) in [20]). Since the L.H.S. in (4.18) vanishes due to (4.9), it transforms to

$$2\phi^{(0)}_{,\xi\tau} + B_g\,\mathcal{H}\left(\phi^{(0)}_{,\xi}\right) = -\frac{L^2\beta_R}{\mu B}\left[\vartheta\,P + \mathcal{H}\,(Q)\right], \tag{4.20}$$

where

$$B_g = \frac{2\left(1-\alpha_R^2 - \alpha_R\beta_R(1-\beta_R^2)\right)}{B\kappa^2(\alpha_R+\beta_R)}. \tag{4.21}$$

Returning to original variables and using the operator identity

$$2L^{-2}\epsilon\,\partial^2_{\xi\tau} = \partial^2_{xx} - c_R^{-2}\,\partial^2_{tt} + O(\epsilon^2), \tag{4.22}$$

along with the leading order approximation $\phi \sim \epsilon^{-1}\,\phi^{(0)}$, we may now represent (4.20) as a perturbed hyperbolic equation on the surface $x_2 = 0$,

$$\phi_{,11} - c_R^{-2}\phi_{,tt} + \frac{g}{c_2^2}\,B_g\,\mathcal{H}\,(\phi_{,x}) = -\frac{\beta_R}{\mu B}\left[\vartheta\,P + \mathcal{H}\,(Q)\right]. \tag{4.23}$$

Clearly, the regular perturbation term is associated with the effect of gravity and may be rewritten as a pseudo-differential operator, namely

$$\phi_{,11} - c_R^{-2}\phi_{,tt} + \frac{g}{c_2^2}\,B_g\,\sqrt{-\partial_{,11}}(\phi) = -\frac{\beta_R}{\mu B}\left[\vartheta\,P + \mathcal{H}\,(Q)\right]. \tag{4.24}$$

The obtained equation (4.23) describes wave propagation on the surface $x_2 = 0$ induced by prescribed loading P, Q and serves as a boundary condition for the elliptic equation

$$\phi_{,22} + \alpha_R^2\phi_{,11} = 0, \tag{4.25}$$

governing decay away from the edge. Once the boundary value problem (4.25), (4.23) is solved, shear potential ϕ is found from (4.10) as a Hilbert transform, namely

$$\psi(x - c_R t, \beta_R x_2) = \vartheta\,\mathcal{H}\left(\phi(x - c_R t, \beta_R x_2)\right). \tag{4.26}$$

5 Discussion

In absence of surface loading ($P = Q = 0$) the obtained approximation (4.24) implies

$$\phi_{,11} - c_R^{-2}\phi_{,tt} + \frac{g}{c_2^2}\,B_g\,\sqrt{-\partial_{,11}}(\phi) = 0, \tag{5.1}$$

with the corresponding dispersion relation taking the form

$$C = \sqrt{1 - \frac{B_g}{K}}\,, \tag{5.2}$$

where

$$K = \frac{c_2^2 k}{g} \quad \text{and} \quad C = \frac{c}{c_R} \tag{5.3}$$

are dimensionless wave number and speed, respectively.

The constant B_g corresponds to a leading order influence of gravity on surface wave propagation. Indeed, approximation (5.2) necessitates consideration of the sign of the coefficient B_g defined by (4.21), depending on the Poisson's ratio, see Fig. 3 below.

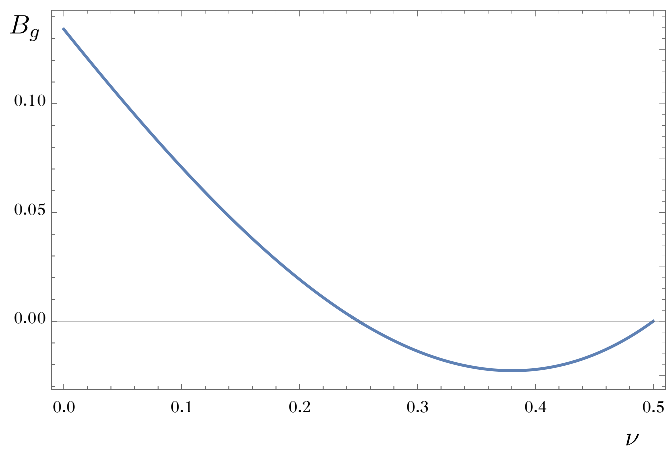

Figure 3: Dependence of the coefficient B_g (4.21) on the Poisson's ratio.

It may be observed from Fig. 3 that the coefficient B_g is positive for $0 \leqslant \nu < 0.25$ and negative for $0.25 < \nu < 0.5$. It is remarkable that $B_g = 0$ for the Poisson's solid ($\nu = 0.25$). In this case, the Rayleigh equation takes a simpler form from (4.21) as

$$\alpha_R \beta_R \kappa^2 = 1, \quad \text{or} \quad 1 - \frac{c_R^2}{2c_2^2} = \frac{c_2}{c_1}, \tag{5.4}$$

implying a well-known exact value for the Rayleigh speed

$$\frac{c_R}{c_2} = \sqrt{2 - \frac{2}{\sqrt{3}}}, \tag{5.5}$$

see e.g. [32]. Moreover, this implies since the coefficient B_g vanishes at $\nu = 0.25$, the known hyperbolic-elliptic model for surface waves in absence of gravity [20] will be valid, with the effect of gravity seemingly appearing only at higher order corrections. The peculiarity of the Poisson's solid has already been noticed by Love [12], who wrote that "when the Poisson's ratio of the material is $\frac{1}{4}$, the wave velocity is not affected by gravity". It is also observed from Fig. 4 that B_g tends to zero as $\nu \to 0.5$, however, treatment of incompressibility constraint requires further investigation.

Let us now illustrate the approximation (5.2) of the dispersion relation (3.9), see Fig. 4. As we know, the short-wave behaviour depends on the Poisson's ratio. Hence, we are presenting variation of the dimensionless phase speed C versus the dimensionless wave number K for three cases, namely for (a) $\nu = 0.2$, (b) $\nu = 0.25$, and (c) $\nu = 0.3$, with the Young's modulus $E = 2 \cdot 10^{11}$ Pa, and volume mass density $\rho = 8 \cdot 10^3$ kg/m^3. In these plots, solid curves indicate the exact dispersion relation (3.9), whereas dashed lines correspond to the approximation (5.2). It can be clearly seen that in case of $\nu = 0.2$ (Fig. 4(a)) the phase speed is monotonously increasing towards the Rayleigh wave speed at short wave limit. At the same time, for $\nu = 0.3$ (Fig. 4(c)) there is a decrease towards the Rayleigh wave speed in the approximation, whereas the exact curve has a turning point (around $K \approx 15$). As for the Poisson's solid when $\nu = 0.25$ (Fig. 4(b)) it is seen that the exact curve is demonstrating a monotonic increase whereas approximation is not capturing the effect of gravity, showing a non-dispersive behaviour, since in this case $B_g = 0$. Apparently, the approximation for $\nu = 0.3$ works slightly better compared to $\nu = 0.2$.

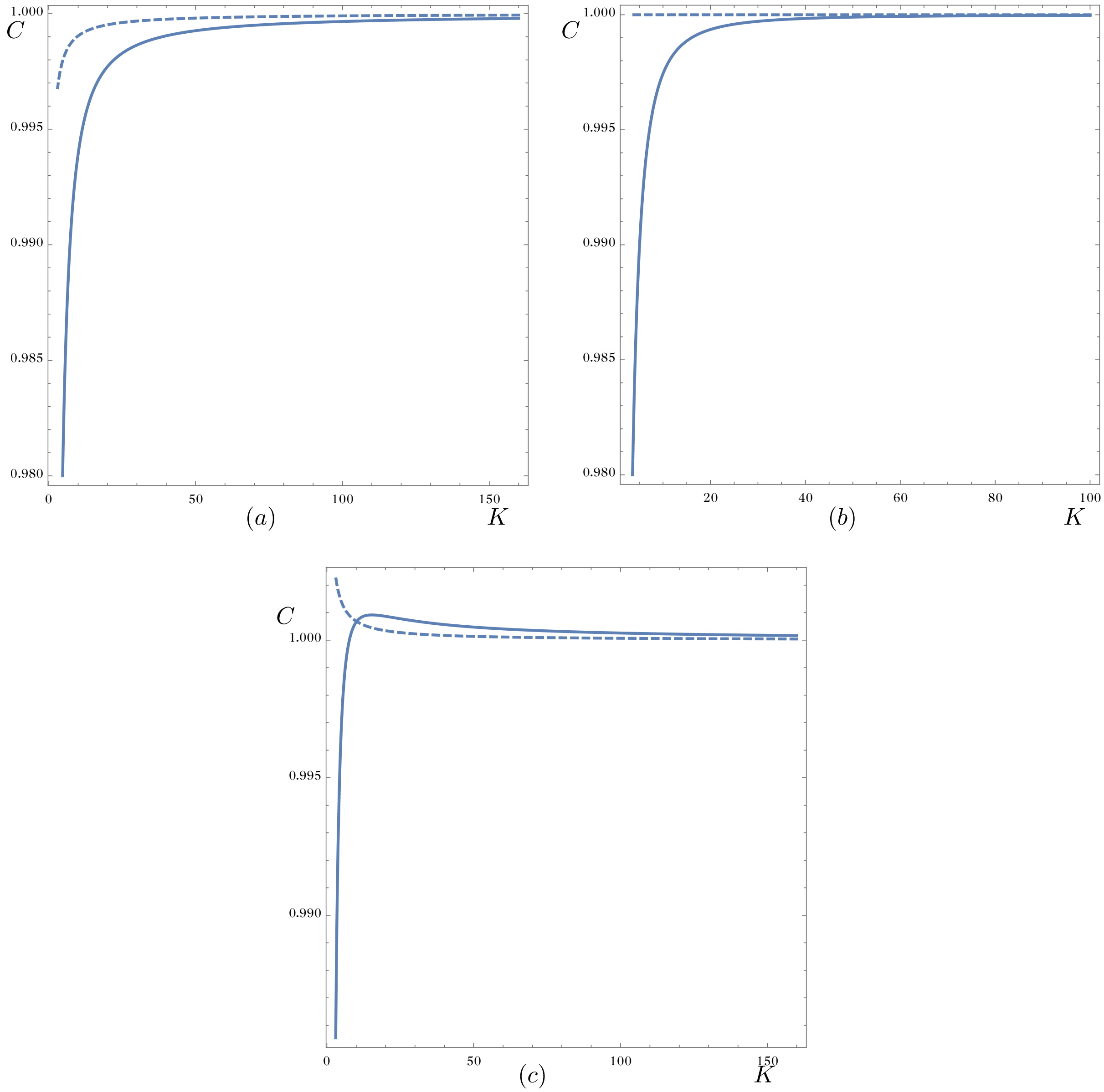

Figure 4: Dispersion relation (3.9) (solid line) and its approximation (5.2) (dashed line) for the Young's modulus $E = 2 \cdot 10^{11}$ Pa, volume mass density $\rho = 8 \cdot 10^3$ kg/m^3, and the Poisson's ratio (a) $\nu = 0.2$, (b) $\nu = 0.25$, and (c) $\nu = 0.3$.

The consideration in this paper was carried out withint the framework of plane-strain regime, however, a generalised 3D result can be predicted at least for the normal surface load. Applying the Radon integral transform and following a procedure similar to that presented in [30], the 3D analogue of (4.24) on the surface $x_3 = 0$ could be derived in the form

$$\Delta_2\phi - c_R^{-2}\phi_{,tt} + \frac{g}{c_2^2}\, B_g \sqrt{-\Delta_2}(\phi) = -\frac{1+\beta_R^2}{2\mu B}\, P, \tag{5.6}$$

serving as a boundary condition for the 3D elliptic equation

$$\phi_{,33} + \alpha_R^2\, \Delta_2\phi = 0, \tag{5.7}$$

where $\Delta_2 = \partial_{11}^2 + \partial_{22}^2$ is a 2D Laplacian in x_1 and x_2, for more details on 3D hyperbolic-elliptic models for the Rayleigh wave see also [20].

6 Conclusions

A hyperbolic-elliptic model for the Rayleigh wave field induced by surface loading, under the effect of gravity, has been derived. The formulation is valid over the short wave region and includes an elliptic equation for the longitudinal Lamé potential, describing decay over the interior, and a hyperbolic equation on the surface, regularly perturbed by a pseudo-differential operator.

It was confirmed that for the Poisson's ratio $\nu = 0.25$ at leading order the effect of gravity vanishes, which is important for a number of applications in civil engineering. In absence of surface loading, the results were also compared to the exact dispersion relation. An insight into generalisation to 3D was also presented.

The results could be further developed to surface-structure interaction and seismic metasurfaces, see e.g. [33], [6], as well as provide a more realistic analytical solution for numerical modelling of seismic metabarriers [34]. It would also be interesting to extend the results to coated structures, having implications for modelling thin layers on foundations, e.g. [35], as well as to derive a refined second-order model for the Rayleigh wave under the effect of gravity. Other, less trivial generalisations are associated with near-resonant solutions to mixed problems, complementing numerical solutions, see e.g. [36], as well as considerations of magneto-elastic surface waves for solids in magnetic fields, with the latter possibly allowing simpler experimental scenarios.

Acknowledgments

Support from the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08857255) is gratefully acknowledged.

References

1. Foti, S, Lai, CG, Rix, GJ, and Strobbia, C. *Surface wave methods for near-surface site characterization.* Boca Raton, CRC Press, 2014.

2. Krylov, V.V. ed., 2001. Noise and vibration from high-speed trains. Thomas Telford.

3. Gupta, M, Khan, MA, Butola, R, and Singari, RM. Advances in applications of Non-Destructive Testing (NDT): A review. *Advances in Materials and Processing Technologies* 2021; doi: 10.1080/2374068X.2021.1909332.

4. Palermo, A, Krödel, S, Marzani, A and Daraio, C. Engineered metabarrier as shield from seismic surface waves. *Scientific Reports* 2016; 6(1): 1–10.

5. D. Colquitt, A. Colombi, R. Craster, P. Roux, S. Guenneau. Seismic metasurfaces: Sub-wavelength resonators and Rayleigh wave interaction. *Journal of the Mechanics and Physics of Solids*, **99** (2017) 379–393.

6. P. Wootton, J. Kaplunov, D. Colquitt. An asymptotic hyperbolic-elliptic model for flexural-seismic metasurfaces. *Proceedings of the Royal Society A*, **475(2227)** (2020) 20190079.

7. Lord Rayleigh, 1885. On waves propagated along the plane surface of an elastic solid. *Proceedings of the London Mathematical Society* 1885; 1(1): 4–11.

8. Chadwick, P., and Smith, GD. Foundations of the theory of surface waves in anisotropic elastic materials. *Advances in Applied Mechanics* 1977; 17: 303–376.

9. Fu, YB. and Mielke, A. 2002. A new identity for the surface–impedance matrix and its application to the determination of surface-wave speeds. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 2002; 458(2026): 2523–2543.

10. Destrade, M. Seismic Rayleigh waves on an exponentially graded, orthotropic half-space. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 2007; 463(2078): 495–502.

11. Bromwich, TIA. On the influence of gravity on elastic waves, and, in particular on the vibrations of an elastic globe. *Proceedings of the London Mathematical Society* 1898; 1(1): 98–165.

12. Love, AEH. *Some problems of geodynamics.* Cambridge University Press, 1911.

13. Biot, MA. *Mechanics of incremental deformations.* New York, John Wiley & Sons, 1965.

14. De, S. N., and Sen-Gupta, P. R. (1974). Influence of gravity on wave propagation in an elastic layer. *The Journal of the Acoustical Society of America* 1974; 55(5): 919–921.

15. Abd-Alla, AM, and Ahmed, SM. Stoneley and Rayleigh waves in a non-homogeneous orthotropic elastic medium under the influence of gravity. *Applied Mathematics and Computation* 2003; 135(1): 187–200.

16. Vinh, PC and Seriani, G. Explicit secular equations of Rayleigh waves in a non-homogeneous orthotropic elastic medium under the influence of gravity. *Wave Motion* 2009; 46(7): 427–434.

17. Ting, TCT. 2011. Surface waves in an exponentially graded, general anisotropic elastic material under the influence of gravity. *Wave Motion* 2011; 48(4): 335–344.

18. Alam, P., Kundu, S., and Gupta, S. Effect of magneto-elasticity, hydrostatic stress and gravity on Rayleigh waves in a hydrostatic stressed magneto-elastic crystalline medium over a gravitating half-space with sliding contact. *Mechanics Research Communications* 2018; 89: 11–17.

19. Sharma, MD. Propagation of Rayleigh waves at the boundary of an orthotropic elastic solid: Influence of initial stress and gravity. *Journal of Vibration and Control* 2020; 26(21-22): 2070–2080.

20. Kaplunov, J. and Prikazchikov, DA. Asymptotic theory for Rayleigh and Rayleigh-type waves. *Advances in Applied Mechanics* 2017; 50: 1–106.

21. Chadwick, P.Surface and interfacial waves of arbitrary form in isotropic elastic media. *Journal of Elasticity* 1976; 6(1): 73–80.

22. Kiselev, AP and Parker, DF. Omni-directional Rayleigh, Stoneley and Schölte waves with general time dependence. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 2010; 466(2120): 2241–2258.

23. Nobili, A, and Prikazchikov, D. Explicit formulation for the Rayleigh wave field induced by surface stresses in an orthorhombic half-plane. *European Journal of Mechanics-A/Solids* 2018; 70: 86–94.

24. Fu, Y., Kaplunov, J. and Prikazchikov, D. Reduced model for the surface dynamics of a generally anisotropic elastic half-space. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 2020; 476(2234): 20190590.

25. Khajiyeva, L. , Prikazchikov, D. and Prikazchikova, L. Hyperbolic-elliptic model for surface wave in a pre-stressed incompressible elastic half-space. *Mechanics Research Communications* 2018; 92: 49–53.

26. Erbaş, B., Kaplunov, J., Nolde, E., and Palsü, M. Composite wave models for elastic plates. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 2018; 474(2214): 20180103.

27. Wootton, P., Kaplunov, J. and Prikazchikov, D. A second-order asymptotic model for Rayleigh waves on a linearly elastic half plane. *IMA Journal of Applied Mathematics* 2020; 85(1): 113–131.

28. Kaplunov, J., Prikazchikov, D., and Sultanova, L. Rayleigh-type waves on a coated elastic half-space with a clamped surface. *Philosophical Transactions of the Royal Society A* 2019; 377(2156): 20190111.

29. Bratov, V., Kaplunov, J., Lapatsin, S., and Prikazchikov, DA. Elastodynamics of a coated half-space under a sliding contact. *Mathematics and Mechanics of Solids.*

30. Dai, H-H., Kaplunov, J. and Prikazchikov, DA. A long-wave model for the surface elastic wave in a coated half-space. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 2010; 466(2122): 3097–3116.

31. Achenbach, JD. *Wave propagation in elastic solids*. Amsterdam, North-Holland, 1973.

32. Malischewsky, P. G., 2000. Comment to ”A new formula for the velocity of Rayleigh waves” by D. Nkemzi [Wave Motion 26 (1997) 199-205]. Wave Motion, 31(1), 93-96.

33. Ege, N., Erbaş, B., Kaplunov, J. and Wootton, P. Approximate analysis of surface wave-structure interaction. *Journal of Mechanics of Materials and Structures* 2018; 13(3): 297–309.

34. Bratov, VA, Ilyashenko, AV, Kuznetsov, SV, Lin, TK, and Morozov, NF. Homogeneous horizontal and vertical seismic barriers: mathematical foundations and dimensional analysis. *Materials Physics and Mechanics* 2020; 44(1): 61–65.

35. Tovstik, PY. The vibrations and stability of a prestressed plate on an elastic foundation. *Journal of Applied Mathematics and Mechanics* 2009; 73(1): 77–87.

36. Bratov, V. Numerical simulations of dynamic fracture. Crack propagation and fracture of initially intact media. *Materials Physics & Mechanics* 2021; 47(3): 455–474.