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Asymptotic Formulation for the Rayleigh Wave on a Nonlocally Elastic Half-Space

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Abstract: This paper deals with the Rayleigh wave, propagating on a nonlocally elastic, linearly isotropic half-space, excited by a prescribed surface loading. The consideration develops the methodology of hyperbolic–elliptic models for Rayleigh and Rayleigh-type waves, and relies on the effective boundary conditions formulated recently, accounting for the crucial contributions of the nonlocal boundary layer. A slow-time perturbation scheme is established, leading to the reduced model for the Rayleigh wave field, comprised of a singularly perturbed hyperbolic equation for the longitudinal wave potential on the surface, acting as a boundary condition for the elliptic equation governing the decay over the interior. An equivalent alternative formulation involving a pseudo-differential operator acting on the loading terms, with parametric dependence on the depth coordinate, is also presented.

Keywords: Rayleigh wave; nonlocal; boundary layer; asymptotic

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1. Introduction

Surface elastic waves are of primary importance in many engineering applications, e.g., in seismic protection and non-destructive evaluation. Among numerous contributions to the subject, as examples of ongoing interest to the problem, we mention recent publications on seismic meta-surfaces and seismic barriers [1–3], studies of surface waves in micro-structured elastic systems [4], as well as a contribution on surface-mounted sensors used for the inspection of structures via guided waves [5].

The nonlocal theory of elasticity, originating from the contributions in [6,7] and developed substantially by Eringen [8–10], has various high-tech applications in nanotechnology (see [11,12]). There are numerous publications studying nonlocal effects in structural mechanics, as well as in elastic continuum, e.g., see [13–15] to name a few. The nonlocal constitutive models may be classified into integral ones, leading to sophisticated and not always solvable integro-differential equations, differential ones related to gradient elasticity, as well as two-phase models (see the famous contribution [9]), and more recent efforts [16,17]. One of the observations in structural mechanics was the ill-posed nature of the problem for the Euler–Bernoulli beam within the framework of the integral nonlocal model (see [18] and also [19]). A question of equivalence of the differential and integral models has attracted attention recently (see [20,21]). It was found that the differential and integral formulations are equivalent in the case of a nonlocally elastic half-space provided that the additional conditions on the boundary hold, which is in line with the concept of constitutive boundary conditions (e.g., see [18,22]). Moreover, it was shown in [20,21] that the solution of the differential model does not satisfy the equations of motion within the integral formulation, with the discrepancy associated with the nonlocal boundary layer. At the same time, asymptotically consistent differential formulations were proposed, including the effective boundary conditions accounting for the influence of nonlocality.

In this paper, we implement these effective boundary conditions within the methodology of reduced models for Rayleigh-type waves (summarised in [23], see references therein), and derive an asymptotic model for the Rayleigh wave on a nonlocally elastic half-space. The described model presents an asymptotic formulation for the Rayleigh wave field, excited by the prescribed surface loading, and relies on the representation of the eigensolution in terms of a single harmonic function (see [24,25]). The formulation contains a non-homogeneous hyperbolic equation on the surface, with loading terms appearing in the right-hand side, and an elliptic equation over the interior governing attenuation. Recent methodology developments include extensions to pre-stress [26], anisotropy [27], flexural seismic meta-surfaces [28], refined second-order formulation [29], and incorporation of gravity [30].

The presence of the nonlocal boundary layer is modelled via the effective boundary conditions derived in [21]. Following [23], a slow-time near-resonant perturbation scheme is implemented. As a result, a singularly perturbed hyperbolic equation on the surface is obtained, containing the loading terms in its right-hand side. The perturbative term contains a pseudo-differential operator, similarly to the previously known results for a coated half-space. However, the sign of the coefficient in front of this perturbative term admits only one scenario, associated with the local maximum of the phase speed at the Rayleigh wave speed in the long-wave limit. This is in line with the expectations of nonlocality, typically causing a slight decrease in frequency compared with results of conventional local elasticity.

The paper is organised as follows: Section 2 contains the formulation of the problem and development of the slow-time perturbation procedure. In Section 3, the asymptotic formulation for the nonlocally elastic Rayleigh-type wave is obtained and discussed, including an alternative representation involving a pseudo-differential operator acting on the load in the right-hand side, providing the solution for elastic potential at a given depth. Finally, Section 4 is dedicated to concluding remarks and ideas for further progress in the area.

2. Materials and Methods

2.1. Statement of the Problem

Consider a linearly isotropic, nonlocally elastic half-space specified by the inequalities $\{-\infty < x_1, x_3 < \infty,\ x_2 \geq 0\}$ under the effect of a prescribed loading $\mathbf{f} = (f_1(x_1,t), f_2(x_1,t), 0)$ acting along the surface $x_2 = 0$ (Figure 1). Throughout this paper the plane strain assumption is imposed, for which the displacement component $u_3 \equiv 0$, and u_1 and u_2 are independent of x_3.

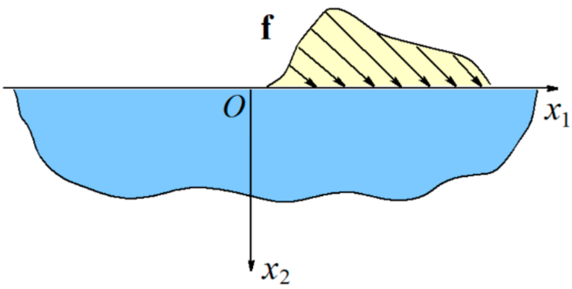

Figure 1. Elastic half-space under the action of a prescribed surface load.

It has been demonstrated in [21] that the nonlocal correction to boundary conditions is by order of magnitude higher than that to the equations of motion (see also an earlier consideration of nonlocal boundary layers in [14]). Thus, the governing equations of motion may be taken in the form of the conventional wave equations for the elastic Lamé potentials $\varphi = \varphi(x_1, x_2, t)$ and $\psi = \psi(x_1, x_2, t)$, namely

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial t^2} - c_1^2 \Delta \varphi = 0, \\ \frac{\partial^2 \psi}{\partial t^2} - c_2^2 \Delta \psi = 0, \end{aligned} \tag{1}$$

where $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$ and $c_2 = \sqrt{\frac{\mu}{\rho}}$ are the longitudinal and transverse wave speeds, respectively, λ and μ are the Lamé parameters, ρ is the volume mass density, and Δ is the Laplace operator in x_1 and x_2. The non-zero displacements are expressed through the Lamé potentials as

$$u_1 = \frac{\partial \varphi}{\partial x_1} - \frac{\partial \psi}{\partial x_2},$$
$$u_2 = \frac{\partial \varphi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}, \tag{2}$$

with the constitutive relations of linear isotropic elasticity taken in the form

$$\sigma_{ij} = \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2, \tag{3}$$

where σ_{ij} are the conventional "local" stresses.

In view of the effective boundary conditions derived in [21], accounting for the presence of a nonlocal boundary layer, the boundary conditions on the surface $x_2 = 0$ are formulated as

$$\sigma_{21} = 2a\mu\left(1-\kappa^2\right)\frac{\partial^2 u_1}{\partial x_1^2} - f_1,$$
$$\sigma_{22} = -f_2. \tag{4}$$

These conditions correspond to excitation of the nonlocally elastic Rayleigh-type wave due to the prescribed surface loading with components $f_1(x_1,t)$ and $f_2(x_1,t)$ along the x_1 and x_2 axis. Here, $\kappa = c_2/c_1$, and the dimensional parameter a is associated with nonlocality (see [21] for more detail). In what follows, a is assumed much smaller than the typical wavelength l, suggesting a natural asymptotic parameter

$$\varepsilon = \frac{a}{l} \ll 1. \tag{5}$$

It should be noted that the effective boundary conditions (4) correspond to the nonlocal kernel in the form of the zeroth-order modified Bessel function of second kind (see also [9]), whereas in case of the Gaussian kernel, an extra constant factor of $\pi^{-1/2}$ appears within the first term in the right-hand side for shear stress component (as shown in [14]).

In view of (2) and (3) the boundary conditions may be rewritten in terms of the Lamé potentials φ and ψ as

$$\mu\left(2\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2}\right) = 2a\mu\left(1-\kappa^2\right)\left(\frac{\partial^3 \varphi}{\partial x_1^3} - \frac{\partial^3 \psi}{\partial x_1^2 \partial x_2}\right) - f_1,$$
$$\mu\left(\left(\kappa^{-2}-2\right)\frac{\partial^2 \varphi}{\partial x_1^2} + \kappa^{-2}\frac{\partial^2 \varphi}{\partial x_2^2} + 2\frac{\partial^2 \psi}{\partial x_1 \partial x_2}\right) = -f_2. \tag{6}$$

2.2. Perturbation Scheme

Following [23], we establish a slow-time perturbation procedure, introducing the scaled variables

$$\xi = \frac{x_1}{l}, \qquad \eta = \frac{x_2}{l}, \qquad \tau_f = \frac{c_R}{l}t, \qquad \tau_s = \varepsilon\tau_f \tag{7}$$

where τ_f and τ_s denote the fast and slow times, respectively, and c_R is the classical Rayleigh wave speed, being the unique non-zero real solution of the Rayleigh equation

$$\left(2 - \frac{c_R^2}{c_2^2}\right)^2 - 4\sqrt{1-\frac{c_R^2}{c_1^2}}\sqrt{1-\frac{c_R^2}{c_2^2}} = 0. \tag{8}$$

It should be noted that the problem essentially has two small parameters, one associated with the nonlocality, and another one corresponding to the near-resonant behaviour, with the phase speed of the wave in question being close to the Rayleigh one. However, for the sake of simplicity these parameters are assumed to be of the same order.

Next, the potentials are expanded as asymptotic series,

$$\varphi = \frac{F_0 l^2}{\mu}\Big(\varepsilon^{-1}\varphi_0 + \varphi_1 + \ldots\Big), \qquad \psi = \frac{F_0 l^2}{\mu}\Big(\varepsilon^{-1}\psi_0 + \psi_1 + \ldots\Big), \tag{9}$$

where $F_0 = \max\limits_{x_1,t}[f_1(x_1,t), f_2(x_1,t)]$, and the coefficient ε^{-1} at leading order is associated with the near-resonant regime under consideration.

Using the string-like assumption associated with the Rayleigh wave

$$\frac{\partial^2\varphi}{\partial\xi^2} - \frac{\partial^2\varphi}{\partial\tau_f^2} = 0, \qquad \frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\tau_f^2} = 0, \tag{10}$$

it is possible to obtain the two-term solutions of the equations of motion (1) in the form

$$\begin{aligned} \frac{\partial\varphi}{\partial\xi} &= \frac{F_0 l^2}{\mu}\left(\varepsilon^{-1}\frac{\partial\varphi_0}{\partial\xi} + \frac{\partial\varphi_{10}}{\partial\xi} + \eta\frac{1-\alpha_R^2}{\alpha_R}\frac{\partial^2\varphi_0^*}{\partial\tau_f\partial\tau_s} + \ldots\right), \\ \frac{\partial\psi}{\partial\xi} &= \frac{F_0 l^2}{\mu}\left(\varepsilon^{-1}\frac{\partial\psi_0}{\partial\xi} + \frac{\partial\psi_{10}}{\partial\xi} + \eta\frac{1-\beta_R^2}{\beta_R}\frac{\partial^2\psi_0^*}{\partial\tau_f\partial\tau_s} + \ldots\right) \end{aligned} \tag{11}$$

(for more details see [23]). Here, the functions $\varphi_0 = \varphi_0\Big(\xi, \alpha_R\eta, \tau_f, \tau_s\Big)$, $\varphi_{10} = \varphi_{10}\Big(\xi, \alpha_R\eta, \tau_f, \tau_s\Big)$, and $\psi_0 = \psi_0\Big(\xi, \beta_R\eta, \tau_f, \tau_s\Big)$, $\psi_{10} = \psi_{10}\Big(\xi, \beta_R\eta, \tau_f, \tau_s\Big)$ are harmonic in the first two arguments, the quantities

$$\alpha_R = \sqrt{1 - \frac{c_R^2}{c_1^2}}, \qquad \beta_R = \sqrt{1 - \frac{c_R^2}{c_2^2}}, \tag{12}$$

are associated with the attenuation factors, and the asterisk denotes a harmonic conjugate quantity.

Next, the two-term solutions (11) are substituted into the boundary conditions (6), rewritten in terms of the scaled variables (8) as

$$\begin{aligned} 2\frac{\partial^2\varphi}{\partial\xi\partial\eta} + \frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} &= 2\varepsilon\Big(1-\kappa^2\Big)\left(\frac{\partial^3\varphi}{\partial\xi^3} - \frac{\partial^3\psi}{\partial\xi^2\partial\eta}\right) - \frac{l^2 f_1}{\mu}, \\ \Big(\kappa^{-2} - 2\Big)\frac{\partial^2\varphi}{\partial\xi^2} + \kappa^{-2}\frac{\partial^2\varphi}{\partial\eta^2} + 2\frac{\partial^2\psi}{\partial\xi\partial\eta} &= -\frac{l^2 f_2}{\mu}. \end{aligned} \tag{13}$$

Then, at leading order we obtain

$$\begin{aligned} 2\frac{\partial^2\varphi_0}{\partial\xi\partial\eta} + \Big(1+\beta_R^2\Big)\frac{\partial^2\psi_0}{\partial\xi^2} &= 0, \\ -\Big(1+\beta_R^2\Big)\frac{\partial^2\varphi_0}{\partial\xi^2} + 2\frac{\partial^2\psi_0}{\partial\xi\partial\eta} &= 0. \end{aligned} \tag{14}$$

Using the Cauchy–Riemann identities for a harmonic function $g(\xi, k\eta)$

$$\frac{\partial g}{\partial\eta} = -k\frac{\partial g^*}{\partial\xi}, \qquad \frac{\partial g}{\partial\xi} = \frac{1}{k}\frac{\partial g^*}{\partial\eta}, \tag{15}$$

the Rayleigh Equation (8) may be deduced as the solvability condition from (14), along with the following relations between the leading-order potentials on the surface $x_2 = 0$

$$\frac{\partial \varphi_0}{\partial \eta} = -\frac{1+\beta_R^2}{2}\frac{\partial \psi_0}{\partial \xi}, \qquad \frac{\partial \varphi_0}{\partial \xi} = \frac{2}{1+\beta_R^2}\frac{\partial \psi_0}{\partial \eta}, \tag{16}$$

originally noted for the classical Rayleigh wave by Sobolev [31] (see also [24]), as well as generalisation to anisotropy [32,33]. Following the reasoning in [24], the leading-order potentials φ_0 and ψ_0 are related as harmonic conjugate functions not only on the boundary but over the interior as well (cf. Equations (38) and (39) in [23]).

At next order, in view of (11) and (16), the boundary conditions imply

$$\begin{aligned}
2\frac{\partial^3 \varphi_{10}}{\partial \xi^2 \partial \eta} + \left(1+\beta_R^2\right)\frac{\partial^3 \psi_{10}}{\partial \xi^3} + \frac{2\left(1-\alpha_R^2\right)}{\alpha_R}\frac{\partial^3 \varphi_0^*}{\partial \tau_f \partial \tau_s \partial \xi} - \frac{2\left(1-\beta_R^2\right)}{\beta_R}\frac{\partial^3 \psi_0^*}{\partial \tau_f \partial \tau_s \partial \eta} \\
= \left(1-\kappa^2\right)\left(1-\beta_R^2\right)\frac{\partial^4 \varphi_0}{\partial \xi^4} - \frac{1}{F_0}\frac{\partial f_1}{\partial \xi}, \\
-\left(1+\beta_R^2\right)\frac{\partial^3 \varphi_{10}}{\partial \xi^3} + 2\frac{\partial^3 \psi_{10}}{\partial \xi^2 \partial \eta} + \frac{2\left(1-\alpha_R^2\right)}{\alpha_R \kappa^2}\frac{\partial^3 \varphi_0^*}{\partial \tau_f \partial \tau_s \partial \eta} + \frac{2\left(1-\beta_R^2\right)}{\beta_R}\frac{\partial^3 \psi_0^*}{\partial \tau_f \partial \tau_s \partial \xi} = -\frac{1}{F_0}\frac{\partial f_2}{\partial \xi}.
\end{aligned} \tag{17}$$

Using the Cauchy–Riemann identities (15), as well as integrating with respect to ξ and making use of the relations (16), we obtain

$$\begin{aligned}
2\alpha_R \frac{\partial^2 \varphi_{10}}{\partial \xi^2} + \left(1+\beta_R^2\right)\frac{\partial^2 \psi_{10}^*}{\partial \xi^2} &= \left(\frac{2\left(1-\alpha_R^2\right)}{\alpha_R} - \frac{1-\beta_R^4}{\beta_R}\right)\frac{\partial^2 \varphi_0}{\partial \tau_f \partial \tau_s} + \left(1-\kappa^2\right)\left(1-\beta_R^2\right)\frac{\partial^3 \varphi_0^*}{\partial \xi^3} - \frac{f_1^*}{F_0}, \\
\left(1+\beta_R^2\right)\frac{\partial^2 \varphi_{10}}{\partial \xi^2} + 2\beta_R \frac{\partial^2 \psi_{10}^*}{\partial \xi^2} &= \left(2\left(1-\beta_R^2\right) - \frac{1-\beta_R^4}{\beta_R^2}\right)\frac{\partial^2 \varphi_0}{\partial \tau_f \partial \tau_s} + \frac{f_2}{F_0},
\end{aligned} \tag{18}$$

where f_1^* is understood in the sense of a Hilbert transform of f_1. The solvability of the latter dictates

$$4B\frac{\partial^2 \varphi_0}{\partial \tau_f \partial \tau_s} + 2\beta_R\left(1-\kappa^2\right)\left(1-\beta_R^2\right)\frac{\partial^3 \varphi_0^*}{\partial \xi^3} - 2\beta_R \frac{f_1^*}{F_0} - \left(1+\beta_R^2\right)\frac{f_2}{F_0} = 0, \tag{19}$$

where

$$B = \frac{\beta_R}{\alpha_R}\left(1-\alpha_R^2\right) + \frac{\alpha_R}{\beta_R}\left(1-\beta_R^2\right) + \beta_R^4 - 1$$

(see [23]). Then, using the string-like assumption (10) along with the leading-order approximation $\mu\varepsilon\varphi \approx F_0 l^2 \varphi_0$ and an operator relation

$$\frac{\partial^2}{\partial \tau_f^2} + 2\varepsilon\frac{\partial^2}{\partial \tau_f \partial \tau_s} \approx \frac{l^2}{c_R^2}\frac{\partial^2}{\partial t^2},$$

returning to the original variables, (19) is re-cast in the form of a singularly perturbed hyperbolic equation on the surface $x_2 = 0$

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \varphi}{\partial t^2} - aN\frac{\partial^3 \varphi^*}{\partial x_1^3} = -\frac{1}{2\mu B}\left(2\beta_R f_1^* + \left(1+\beta_R^2\right)f_2\right), \tag{20}$$

where the coefficient

$$N = \frac{\beta_R}{\mu B}\left(1-\kappa^2\right)\left(1-\beta_R^2\right), \tag{21}$$

accounting for nonlocal effects, may be shown to be positive.

3. Results and Discussion

3.1. Asymptotic Formulation for the Nonlocally Elastic Rayleigh Wave

Thus, given a prescribed loading, one may substitute it into the perturbed hyperbolic Equation (20) and find the value of the longitudinal potential φ at the surface $x_2 = 0$. This may then serve as a boundary value for the elliptic equation

$$\frac{\partial^2 \varphi}{\partial x_2^2} + \alpha_R^2 \frac{\partial^2 \varphi}{\partial x_1^2} = 0, \tag{22}$$

describing the attenuating behaviour over the interior. Then, once the potential φ is determined, its shear counterpart ψ may be restored as a harmonic conjugate by making use of (16) (for more details see [23]). It should be noted that Equation (20) for the potential φ on the surface $x_2 = 0$ may also be written in terms of a pseudo-differential operator, namely

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \varphi}{\partial t^2} - aN\sqrt{-\frac{\partial^2}{\partial x_1^2}}\left(\frac{\partial^2 \varphi}{\partial x_1^2}\right) = -\frac{1}{2\mu B}\left(2\beta_R f_1^* + \left(1+\beta_R^2\right) f_2\right) \tag{23}$$

(for more details on the theory of pseudo-differential operators, readers are referred to [34]).

In the absence of loading (when $f_1 = f_2 \equiv 0$), the associated dispersion relation takes the form

$$C^2 = 1 - NK, \tag{24}$$

where $C = c/c_R$ and $K = ak$ $(k > 0)$ denote the dimensionless phase velocity and wave number, respectively. As might be expected, it is observed that the nonlocality of the media causes a slight decrease in the Rayleigh wave speed. In fact, this local maximum of the phase speed at the long-wave limit implies that in case of the associated Lamb problem the presence of the near-surface nonlocal boundary layer leads to a receding front (see Section 6 in [35]), considering both possibilities of advancing and receding fronts in case of a coated elastic half-space.

3.2. Hyperbolic Equation at a Prescribed Depth

Let us now obtain an equivalent formulation of the derived asymptotic model (22) and (23), combining both in a single equation and keeping the same operator in the left-hand side but having a pseudo-differential operator applied to the loading terms in the right-hand side, following a recent approach in [36]. Indeed, the decaying solution of (22) at a given depth x_2 may be expressed through its value on the surface $x_2 = 0$ as

$$\varphi(x_1, \alpha_R x_2, t) = e^{-\alpha_R x_2 \sqrt{-\frac{\partial^2}{\partial x_1^2}}}[\varphi(x_1, 0, t)]. \tag{25}$$

Here, $e^{-\alpha_R x_2 \sqrt{-\frac{\partial^2}{\partial x_1^2}}}$ is a pseudo-differential operator defined by

$$e^{-\alpha_R x_2 \sqrt{-\frac{\partial^2}{\partial x_1^2}}}[g(x_1,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x_1,t)e^{-isx_1}dx_1\right)e^{-\alpha_R|s|x_2 + isx_1}ds. \tag{26}$$

Therefore, in view of (25), the perturbed hyperbolic Equation (23) may be extended over the interior, yielding

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \varphi}{\partial t^2} - aN\sqrt{-\frac{\partial^2}{\partial x_1^2}}\left(\frac{\partial^2 \varphi}{\partial x_1^2}\right) = -\frac{1}{2\mu B}e^{-\alpha_R x_2 \sqrt{-\frac{\partial^2}{\partial x_1^2}}}\left[2\beta_R f_1^* + \left(1+\beta_R^2\right) f_2\right]. \tag{27}$$

It is worth noting that the vertical coordinate x_2 presents only a parametric dependence in the right-hand side of (27), as observed in [36].

Thus, the obtained reduced model for the nonlocally elastic surface wave field may be presented as a boundary value problem for the elliptic Equation (22), with the boundary condition (23), or, equivalently, as Equation (27) for the longitudinal potential $\varphi(x_1, x_2, t)$.

4. Conclusions

The hyperbolic–elliptic model for the Rayleigh wave on a nonlocally elastic half-space, induced by the prescribed surface loading, is established. The consideration relies on the recently derived formulation for the nonlocal elastic half-space [21], including the conventional equations of motion of local elasticity along with the effective boundary conditions accounting for the presence of the nonlocal boundary layer. The contribution of nonlocality appears to be in a singular perturbative term in the hyperbolic Equation (23), which is formally close to that arising in the asymptotic formulation for the Rayleigh-type wave in case of a coated elastic half-space [34]. As might be expected, the associated dispersion relation reveals a slight decrease in phase velocity due to nonlocality. For a given load, the solution of (23) serves as a boundary value for the elliptic Equation (22). The longitudinal potential φ will then fully determine the displacements (see, e.g., Equation (40) in [23]). An elegant alternative formulation (27) is developed, combining both (22) and (23) in a single hyperbolic equation at a given depth, with the right-hand side containing a pseudo-differential operator acting on surface loading components and having a parametric dependence on the vertical coordinate.

The results could be further generalized to incorporate the effects of anisotropy, pre-stress, and inhomogeneity. The methodology may also be extended to interfacial Stoneley and Schölte waves. It is also worth noting that, as shown in [20], the presence of nonlocality leads to existence of an antiplane shear surface wave (see also [37]). In this scalar case, an analogue of (20) may be derived for the non-zero displacement, with the right-hand side involving a derivative of the prescribed surface stress. Finally, we mention the potential of the presented approach for addressing forced problems for surface water waves (see [38]).

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References

1. Colombi, A.; Colquitt, D.; Roux, P.; Guenneau, S.; Craster, R.V. A seismic metamaterial: The resonant metawedge. *Sci. Rep.* **2016**, *6*, 1–6. [CrossRef] [PubMed]
2. Pu, X.; Palermo, A.; Cheng, Z.; Shi, Z.; Marzani, A. Seismic metasurfaces on porous layered media: Surface resonators and fluid-solid interaction effects on the propagation of Rayleigh waves. *Int. J. Eng. Sci.* **2020**, *154*, 103347. [CrossRef]
3. Bratov, V.; Kuznetsov, S.; Morozov, N. Seismic barriers filled with solid elastic and granular materials: Comparative analysis. *Math. Mech. Solids* **2022**, *27*, 1761–1770. [CrossRef]
4. Nieves, M.J.; Carta, G.; Pagneux, V.; Brun, M. Rayleigh waves in micro-structured elastic systems: Non-reciprocity and energy symmetry breaking. *Int. J. Eng. Sci.* **2020**, *156*, 103365. [CrossRef]
5. El Masri, E.; Waters, T.; Ferguson, N. Guided wave inspection of bars in reinforced-concrete beams using surface-mounted vibration sensors. *Vibration* **2020**, *3*, 343–356. [CrossRef]
6. Kröner, E. Elasticity theory of materials with long range cohesive forces. *Int. J. Solids Struct.* **1967**, *3*, 731–742. [CrossRef]
7. Edelen, D.G.B.; Laws, N. On the thermodynamics of systems with nonlocality. *Arch. Ration. Mech. Anal.* **1971**, *43*, 24–35. [CrossRef]
8. Eringen, A.C.; Edelen, D.G.B. On nonlocal elasticity. *Int. J. Eng. Sci.* **1972**, *10*, 233–248. [CrossRef]
9. Eringen, A.C. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *J. Appl. Phys.* **1983**, *54*, 4703–4710. [CrossRef]
10. Eringen, A.C. *Nonlocal Continuum Field Theories*; Springer: New York, NY, USA, 2002.
11. Elishakoff, I.; Dujat, K.; Muscolino, G.; Bucas, S.; Natsuki, T.; Wang, C.M.; Pentaras, D.; Versaci, C.; Storch, J.; Challamel, N.; et al. *Carbon Nanotubes and Nanosensors: Vibration, Buckling and Balistic Impact*; John Wiley & Sons: New York, NY, USA, 2013.
12. Karlicic, D.; Murmu, T.; Adhikari, S.; McCarthy, M. *Non-Local Structural Mechanics*; John Wiley & Sons: NewYork, NY, USA, 2015.

13. Challamel, N.; Wang, C.M. The small length scale effect for a non-local cantilever beam: A paradox solved. *Nanotechnology* **2008**, *19*, 345703. [CrossRef]
14. Chebakov, R.; Kaplunov, J.; Rogerson, G.A. Refined boundary conditions on the free surface of an elastic half-space taking into account non-local effects. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2016**, *472*, 20150800. [CrossRef] [PubMed]
15. Zhang, Y.P.; Challamel, N.; Wang, C.M.; Zhang, H. Comparison of nano-plate bending behaviour by Eringen nonlocal plate, Hencky bar-net and continualised nonlocal plate models. *Acta Mech.* **2019**, *230*, 885–907. [CrossRef]
16. Pisano, A.A.; Fuschi, P.; Polizzotto, C. Integral and differential approaches to Eringen's nonlocal elasticity models accounting for boundary effects with applications to beams in bending. *ZAMM* **2021**, *101*, e202000152. [CrossRef]
17. Wang, Y.B.; Zhu, X.W.; Dai, H.H. Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. *AIP Adv.* **2016**, *6*, 085114. [CrossRef]
18. Romano, G.; Barretta, R.; Diaco, M.; de Sciarra, F.M. Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. *Int. J. Mech. Sci.* **2017**, *121*, 151–156. [CrossRef]
19. Vaccaro, M.S.; Pinnola, F.P.; de Sciarra, F.M.; Barretta, R. Limit behaviour of Eringen's two-phase elastic beams. *Europ. J. Mech.-A/Solids* **2021**, *89*, 104315. [CrossRef]
20. Kaplunov, J.; Prikazchikov, D.A.; Prikazchikova, L. On integral and differential formulations in nonlocal elasticity. *Eur. J. Mech.-A/Solids* **2022**, 104497. [CrossRef]
21. Kaplunov, J.; Prikazchikov, D.A.; Prikazchikova, L. On non-locally elastic Rayleigh wave. *Philos. Trans. R. Soc. A* **2022**, *380*, 20210387. [CrossRef]
22. Zaera, R.; Serrano, Ó.; Fernández-Sáez, J. Non-standard and constitutive boundary conditions in nonlocal strain gradient elasticity. *Meccanica* **2020**, *55*, 469–479. [CrossRef]
23. Kaplunov, J.; Prikazchikov, D.A. Asymptotic theory for Rayleigh and Rayleigh-type waves. In *Advances in Applied Mechanics*; Bordas, S.P.A., Balint, D.S., Eds.; Elsevier: London, UK, 2017; pp. 1–106.
24. Chadwick, P. Surface and interfacial waves of arbitrary form in isotropic elastic media. *J. Elast.* **1976**, *6*, 73–80. [CrossRef]
25. Kiselev, A.P.; Parker, D.F. Omni-directional Rayleigh, Stoneley and Schölte waves with general time dependence. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2010**, *466*, 2241–2258. [CrossRef]
26. Khajiyeva, L.A.; Prikazchikov, D.A.; Prikazchikova, L.A. Hyperbolic-elliptic model for surface wave in a pre-stressed incompressible elastic half-space. *Mech. Res. Comm.* **2018**, *92*, 49–53. [CrossRef]
27. Fu, Y.; Kaplunov, J.; Prikazchikov, D. Reduced model for the surface dynamics of a generally anisotropic elastic half-space. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2020**, *476*, 20190590. [CrossRef]
28. Wootton, P.T.; Kaplunov, J.; Colquitt, D.J. An asymptotic hyperbolic–elliptic model for flexural-seismic metasurfaces. *Proc. R. Soc. A* **2020**, *475*, 20190079. [CrossRef]
29. Wootton, P.T.; Kaplunov, J.; Prikazchikov, D. A second-order asymptotic model for Rayleigh waves on a linearly elastic half plane. *IMA J. Appl. Math.* **2020**, *85*, 113–131. [CrossRef]
30. Mubaraki, A.; Prikazchikov, D. On Rayleigh wave field induced by surface stresses under the effect of gravity. *Math. Mech. Solids* **2022**, *27*, 1771–1782. [CrossRef]
31. Sobolev, S. Some problems in wave propagation. In *Differential and integral equations of mathematical physics*; Russian translation; Frank, P., von Mises, R., Eds.; ONTI: Moscow, Russia, 1937; pp. 468–617.
32. Prikazchikov, D.A. Rayleigh waves of arbitrary profile in anisotropic media. *Mech. Res. Comm.* **2013**, *50*, 83–86. [CrossRef]
33. Parker, D.F. The Stroh formalism for elastic surface waves of general profile. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2013**, *469*, 20132160. [CrossRef]
34. Shubin, M.A. *Pseudodifferential Operators and Spectral Theory*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2001.
35. Dai, H.H.; Kaplunov, J.D.; Prikazchikov, D.A. A long-wave model for the surface elastic wave in a coated half-space. *Proc. R. Soc.* **2010**, *466*, 3097–3116. [CrossRef]
36. Kaplunov, J.D.; Prikazchikov, D.A.; Sabirova, R.F. On a hyperbolic equation for the Rayleigh wave. *Dokl. Phys.* **2022**, *506*, 63–66.
37. Eremeyev, V. Strongly anisotropic surface elasticity and antiplane surface waves. *Philos. Trans. R. Soc. A* **2020**, *378*, 20190100. [CrossRef] [PubMed]
38. Liam, L.S.; Adytia, D.; Van Groesen, E. Embedded wave generation for dispersive surface wave models. *Ocean Eng.* **2014**, *80*, 73–83. [CrossRef]