sciendo
PU. M. A. Vol. **30** (2022), No.1, pp. 31 – 36

Using large random permutations to partition permutation classes

Christian Bean
Department of Computer Science
Reykjavik University
email: christianbean@ru.is

and
Émile Nadeau
Department of Computer Science
Reykjavik University
email: emile19@ru.is

and
Jay Pantone
Department of Mathematical and Statistical Sciences
Marquette University
email: jay.pantone@marquette.edu

and
Henning Ulfarsson
Department of Computer Science
Reykjavik University
email: henningu@ru.is

(Received: March 31, 2022, and in revised form May 15, 2022.)

Abstract. Permutation classes are sets of permutations defined by the absence of certain substructures. In some cases permutation classes can be decomposed as unions of subclasses. We use combinatorial specifications automatically discovered by *Combinatorial Exploration: An algorithmic framework for enumeration*, Albert et al. 2022, to uniformly generate large random permutations in a permutation class, and apply clustering methods to partition them into interesting subclasses. We seek to automate as much of this process as possible.

Mathematics Subject Classification(2020). 05A05.

Keywords: permutation pattern, random sampling, clustering

1 Introduction

Examples of characterizing discrete objects by the absence of substructures abound in mathematics, e.g., graphs missing certain minors, types of polyominoes without given subsets, and the focus of this article, sets of permutations defined by the *avoidance of patterns*. We will consider permutations from two viewpoints, the *one-line notation* where a permutation of length n is written as a word $\pi_1\pi_2\cdots\pi_n$

DOI: 10.2478/puma-2022-0006

indicating that π maps i to π_i, and from a graphical perspective where we place a dot at (i, π_i) in the Cartesian plane. Consider for example the permutation $\pi = 463125$, which maps 1 to 4, 2 to 6, etc; shown in Figure 1. We say that a permutation π *contains* another permutation p (*as a pattern*) if we

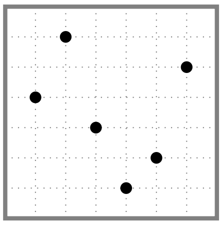

Figure 1: A graph of the permutation 463125.

can find a subsequence in the word $\pi_1\pi_2\cdots\pi_n$, that is order-isomorphic to p. More precisely there is a subsequence $\pi_{i_1}, \ldots, \pi_{i_k}$, the same length as p, such that when we replace the smallest entry in $\pi_{i_1}\cdots\pi_{i_k}$ with 1, the next smallest with 2, etc., we obtain p. Our example permutation $\pi = 463125$ contains $p = 132$ as can be seen from the subsequence 465 which becomes 132 when 4 is replaced with 1, 5 replaced with 2 and 6 with 3. When a permutation π does not contain a pattern p we say that π *avoids* p. The permutation $\pi = 463125$ avoids $p = 4321$.

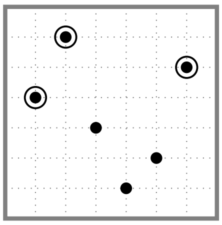

Figure 2: The permutation 463125 with a circled occurrence of the pattern 132.

Several natural sets of permutations can be described by the patterns they avoid. First we define $\mathrm{Av}(P)$ as the set of permutations that avoid every pattern in the set P. For example $\mathrm{Av}(132, 231)$ is the set of permutations that avoid 132 as well as the pattern 231. We call sets of permutations defined in this manner *permutation classes*. Note that they are closed downwards, in the sense that if π belongs to the class then so do all of its patterns.

To name some notable examples we recall that stack-sortable permutations form the class $\mathrm{Av}(231)$, permutations sortable by two queues (in parallel) form $\mathrm{Av}(321)$, the permutations in $\mathrm{Av}(1324, 2143)$ are in correspondence with smooth Schubert varieties, and permutation classes (avoiding too many patterns to list here) appear in some models of genome rearrangements.

When we have chosen a particular permutation class to study, the first question is typically its enumeration, by which we mean how many permutations of length n lie in the class. The answer is often expressed as a generating function, e.g., for the stack-sortable permutations, $\mathrm{Av}(231)$, the enumeration is given by the (ordinary) generating function of the Catalan numbers

$$\frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots .$$

In some cases only recurrences are known, and in others only a finite piece of the enumeration sequence is known, such as for the infamous Av(1324).[1]

When the class is studied further we often want to know how a large permutation in the class behaves, or if there is some average behavior the class displays. A successful answer to those questions sometimes involves breaking the class into smaller, easier to understand, subclasses (sometimes, but not always, defined by the avoidance of additional patterns). This is the focus of the present article, in particular trying to experimentally guess these subclasses, and to automate as much of the process as possible.

Recently Albert et al. [2] introduced an automatic method to find combinatorial specifications for permutation classes, which can then be turned into systems of equations giving the enumeration of the classes. These specifications can in many cases be used to generate large permutations in the classes uniformly at random. We point the reader to Albert et al. [2] for formal definitions of specifications and how they are used for random generation, focusing here on how we can use random large permutations to understand the classes.

2 Average behaviour and heatmaps

By taking several permutations from the same class, and overlaying their graphs, we get a "heatmap" of the class. Consider for example Av(321, 2143), first enumerated by West [4], see also Atkinson [3]. The heatmap is shown in Figure 3.

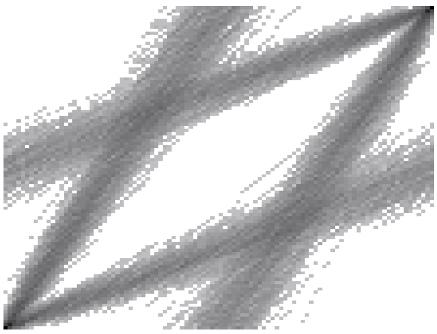

Figure 3: A heatmap of Av(321, 2143) made by overlaying 500 permutations of length 100 from the class.

We might wonder whether the heatmap conveys a good description of random permutations in the class. However, when we view individual permutations this does not appear to be the case. Two such permutations are shown in Figure 4.

If more permutations from the class are examined, they always seem to be of the two "types" shown in Figure 4. We therefore turn to clustering methods, which first require that we define a distance

[1] Technically, recurrences are known, but no polynomial-time algorithm is known to generate the enumeration.

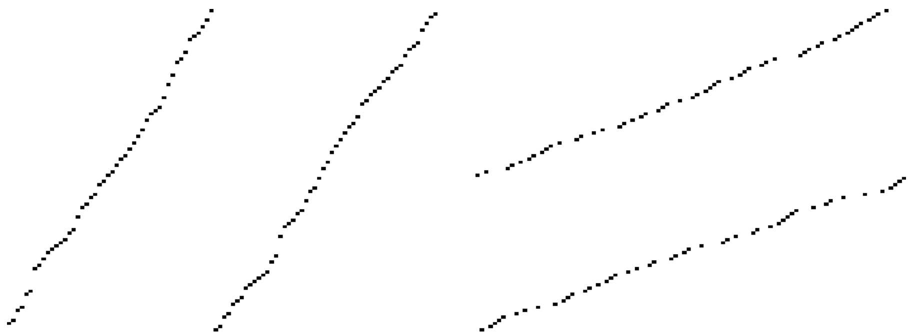

Figure 4: Two permutations of length 100 from $\mathrm{Av}(321, 2143)$.

between two permutations. Preferrably this distance should be small for permutations with graphs that look similar. We start by treating each permutation as a vector $(\pi_1, \pi_2, \ldots, \pi_{100})$. Then for two permutations π and σ, for each point (i, σ_i) we record the distance to the nearest (in the Euclidean distance) point (j, π_j) of π. Let d be the maximum of these point differences. Then the distance between π and σ is $1/(1+d)$.

Next, we choose a clustering method, opting for spectral clustering. Passing our set of 100 permutations through this proceedure gives two clusters with the heatmaps in Figure 5.[2]

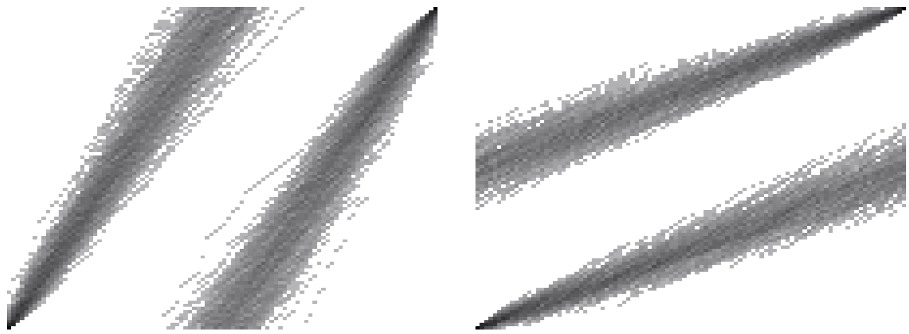

Figure 5: Heatmaps of the two clusters of $\mathrm{Av}(321, 2143)$.

This seems to better capture what we expect after having seen individual permutations from the class.

3 Describing the clusters

We now turn to the question describing the clusters we found in the previous section. Again we use computer experimentation to try and see if these are perhaps best described in terms of the avoidance

[2]The metric is designed to capture when a point in a permutation is close to some point in another permutation, without requiring the points to have the same index or height in the permutations. The formula for the metric and the clustering method is the result of experimentation with several alternatives applied to the permutation class studied here, as well as others. Also, we choose to cluster into two subsets, because experiments revealed that it gave better results than more subsets.

of patterns. To that end, for a fixed pattern p we take 100 subsequences from each of the permutations in a cluster, and record how many of these subsequences are occurrences of p. We scale this statistic by the number of permutations in the cluster.

We apply this to all patterns p up to and including length 4. For most patterns the scores are similar between the clusters, but two patterns stand out: For 3142 the score is 0.0 for cluster 1, vs. 6.2 for cluster 2. Similarly for 2413 the score is 6.3 for cluster 1, vs. 0.0 for cluster 2. We can therefore hope that the two clusters we found can be described as $\mathrm{Av}(321, 2143, 3142)$ and $\mathrm{Av}(321, 2143, 2413)$. Their heatmaps are shown in Figure 6.

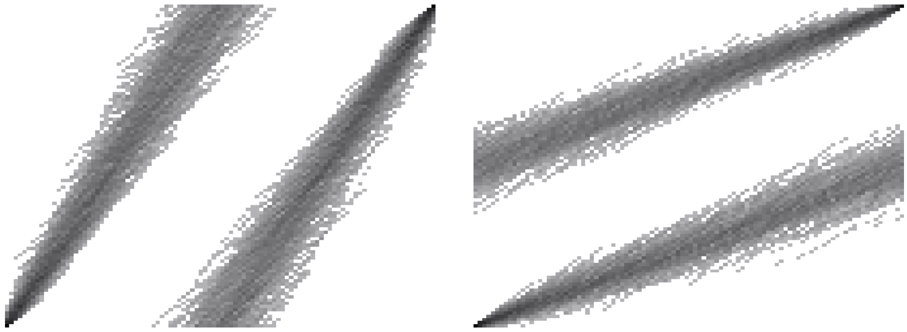

Figure 6: The two subclasses of $\mathrm{Av}(321, 2143)$.

Judging solely from the heatmaps of these subclassses, compared with the heatmaps of the clusters we would tend to think that we have broken the full class $\mathrm{Av}(321, 2143)$ into two subclasses that capture the behaviour we are trying to understand.

Using the PermPal website (`https://permpal.com/`), released with the paper Albert et al. [2] we can see that in terms of enumeration we are extremely close, in the following sense: There are no permutations in the class, outside of the two subclasses, and the number of permutations in intersection of the two subclasses grows polynomially vs. exponentially in the full class.[3]

In the final section we suggest alternative methods to describing the clusters of permutation classes when we are not as successful as for the class investigated here.

4 Looking back at the permutations

The so-called *2x4 classes*, permutation classes avoiding two patterns of length 4, have been intensily studied over the last few decades. For each of them a heatmap is provided at PermPal.com. Some, such as $\mathrm{Av}(1324, 3412)$, have heatmaps indicating the class can be clustered like we did above. See Figure 7.

However, carrying out the analysis as we did above does not appear to be sufficient, in the sense that, although visually we seem to be able to describe the clusters with the avoidance of extra patterns, enumeratively we both miss permutations in the class, and the intersection of the subclasses is larger than we would prefer, in the sense that their number grows exponentially.

[3] This can be seen from the generating functions of the full class, the subclasses, and their intersection, which can be found by searching on the website.

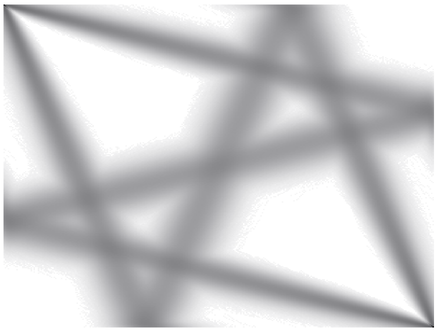

Figure 7: A heatmap of Av(1324, 3412) from PermPal.com created from 1,000,000 permutations of length 300 sampled uniformly at random.

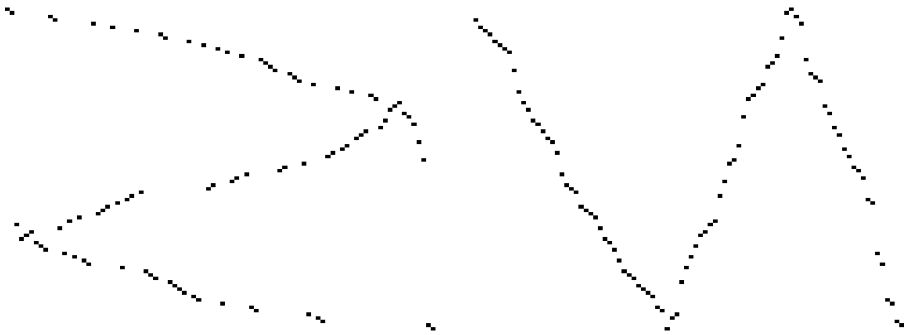

Figure 8: Two permutations of length 100 from Av(1324, 3412).

An alternative would be taking a closer look the large permutations in the class to understand the clusters better, and consider them from the perspective of monotone grid classes. This would involve trying to split the permutations into rectangular regions containing only increasing, or decreasing subsequences of points.

In fact, the enumeration of Av(1324, 3412) was first done by Albert, Atkinson and Brignall [1] by careful case-by-case analysis over several pages, leading to a clustering of the class into two subsets described by monotone grid classes. We could potentially automate this step by guessing the monotone grid classes from the large permutations.

References

[1] M. H. Albert, M. D. Atkinson and R. Brignall, *The enumeration of permutations avoiding 2143 and 4231*, Pure Math. Appl. (PU.M.A), 22 (2011) 87–98.

[2] M. H. Albert, C. Bean, A. Claesson, É. Nadeau, J. Pantone and H. Ulfarsson, *Combinatorial Exploration: An algorithmic framework for enumeration*, arXiv: 2202.07715.

[3] M. D. Atkinson, *Restricted permutations*, Discrete Math., 195 (1999) 27–38.

[4] J. West, *Generating trees and forbidden subsequences*, Discrete Math., 157 (1996) 363–374.