Linear and nonlinear wave propagation in coated or uncoated elastic half-spaces

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In these lectures, we discuss the following three closely related topics.

(i) *Unification of different methods for deriving evolution equations for surface acoustic waves.* Early studies on nonlinear surface acoustic waves were thwarted by very complicated derivation of evolution equations. Worse still, different methods seemed to have given different evolution equations. Later on, it became known that all these methods except one yield the same evolution equation, but even at the time when we started to prepare the current lecture notes, there still existed a method that does not agree with the other methods. Such a situation is unsatisfatory since each method has some following. The purpose of the lectures on this topic is three-fold. Firstly, we aim to show that derviation of the evolution eqution for nonlinear surface waves can be carried out in one A4 page even in the most general case. Secondly, we show that the odd method that used to give a different evolution equation can in fact be used to obtain the same evolution equation if it is properly executed. Thus, we set the record straight: all known methods should and do give the same evolution equation! Thirdly, we express our evolution equation in terms of results from the linear surface-wave theory built on the Stroh formulation, and we explain how the coefficients in the evolution equation can be evaluated efficiently.

(ii) *Linear wave propagation in a coated elastic half-space.* This is partly in preparation for our discussion of the third topic, but the problem is of much interest in its own right. We show how the dispersion relation can be expressed elegantly in terms of the surface-impedance matrices associated with the layer and the half-space. We derive a two-term expression for the wave speed in the long-wavelength limit.

(iii) *Periodic and solitary waves in a coated elastic half-space.* An uncoated elastic half-space cannot in general support solitary waves due to lack of dispersion although it has been argued previously that the nonlocal character of nonlinearity may give rise to the existence of steady travelling waves. We derive the nonlinear evolution equation for small-amplitude long-wavelength travelling

waves propagating in a coated elastic half-space where the thin coating induces weak dispersion. When this evolution equation is linearized, we recover the two-term dispersion relation obtained in (ii). We explain a simple method that can be used to compute periodic or solitary travelling-wave solutions.

Keywords: Nonlinear surface waves; elastic half-space; coated half-space; Stroh formalism; nonlinear elasticity; prestress

1. Introduction – background and literature review

Elastic surface waves are travelling waves that can propagate along the surface of an elastic half-space. They satisfy the traction-free boundary condition and decay to zero exponentially away from the surface. Understanding of nonlinear effects on the evolution of surface waves is known to have applications in signal processing, material characterization, and non-destructive evaluation; see, for instance, Parker and Maugin (1987), Mayer (1995), Maugin (1999), Hess (2002) and the references therein. More recently, it has also found applications in nanotechnology; see Hess and Lomonosov (2005), Hess *et al.* (2005). Historically, evolution equations for nonlinear acoustic surface waves have been derived using different procedures and it was very often not immediately clear whether one evolution equation was equivalent to another. It seems that derivation of evolution equations for nonlinear surface waves was actually initiated by Reutov (1973), but this paper remained unnoticed in the West for a considerable period of time. Many researchers thought that Kalyanasundaram's 1981 paper was the first where a multiple scale approach was used to study the evolution of nonlinear surface waves. In both of these two papers, a far-distance variable $X(=\varepsilon x_1)$ and a slow time variable $\tau(=\varepsilon t)$ were introduced, where x_1 is the coordinate along the direction of propagation, t is time and ε is a small parameter characterizing the amplitude of strains. When Kalyanasundaram's (1981) method (hereafter referred to as Method I) is used, secular terms appear in the $O(\varepsilon^2)$ solution. This 'deficiency' was later remedied by Lardner (1983) through the introduction of another far-distance variable $\eta(=\varepsilon x_2)$ where x_2 is the coordinate such that $x_2 > 0$ defines the half-space. We refer to this as Method II. This method was followed by Lardner (1984, 1985, 1986), Lardner and Tupholme (1986), David (1985), Harvey *et al.* (1992), Harvey and Tupholme (1991, 1992), Tupholme and Harvey (1988, 1992). The use of multiple scale η was also independently proposed by Planat (1985) but he assumed that the dependence on η and X was only through a linear combination of X and η with unspecified coefficients. Method III is Parker *et al.*'s (1992) projection method which can immediately be recognized as being equivalent to Method I and the approaches used in Parker (1988) and Hunter (1989). Method IV is Reutov (1973)'s and Zabolotskaya's (1992) Hamiltonian formalism which was followed by Shull *et al.* (1993), Hamilton *et al.* (1995, 1999). Method V is Fu and

Devenish's (1996) virtual-work method which incorporated the best features of the projection method and the Hamiltonian formalism. All the above methods, although seemingly different, work with the frequency domain and are based essentially on the same underlying philosophy. It was explicitly recognized in Fu and Devenish's (1996) and Eckl *et al.* (2004) that Methods I-IV yield the same evolution equation for the surface velocity/elevation although this was not immediately clear from the papers in which these methods were first presented. In Reutov (1973)'s pioneering paper, it was already recognized that the projection method was equivalent to the Hamiltonian formalism.

In contrast with the above methods, Method VI proposed by Gusev *et al.* (1997, 1998) works with the time domain and is recognized as being different from the other methods even if it is translated into the frequency domain; see Meegan *et al.* (1999) and Eckl *et al.* (2004). Method VI was followed by Kolomenskii *et al.* (1997), Kolomenskii and Schuessler (2001), Kolomenskii *et al.* (2003), Jerebtsov *et al.* (2004). It seems that Gusev *et al.* (1997, 1998) did not realize that the evolution equation for the surface velocity is independent of whether the far-distance variable $\eta = \varepsilon x_2$ is introduced or not and they advocated the importance of restoring this variable in deriving the evolution equation for the surface velocity. One of the motivations for the present study is to understand why Method VI gives a different evolution equation. We shall show that Method VI is in fact able to yield the same evolution equation as the other methods if a certain underlying assumption is removed. Thus, all the existing methods can now be said to be equivalent.

Another motivation for the present study is that when studying nonlinear surface waves in generally anisotropic elastic materials, previous investigators seem to have been oblivious of the more recent developments concerning linear surface waves. The many beautiful results concerning linear surface waves based on the Stroh formulation (Stroh 1958) should not only facilitate numerical evaluation of the surface-wave speed and coefficients in the evolution equation, but they also contain a lot of qualitative information about properties of linear surface waves. For instance, it could be misleading not to write the secular equation for the wave speed as a real relation, since otherwise new researchers could question whether it would be possible to find a real root (the wave speed) to satisfy a complex equation (which contains two real equations).

With the aid of the evolution equation derived, it can easily be shown that initially smooth profiles of surface displacement would evolve into shocks within a finite time. This is to be expected since travelling waves propagating in an uncoated elastic half-space are non-dispersive. A natural follow-up problem is the propagation of nonlinear travelling waves, and in particular solitary waves, in a coated elastic half-space where a thin coating gives rise to dispersion. Linear travelling waves in a coated elastic half-space were first studied by Tiersten (1969). Recently, Ogden and Sotiropoulos (1995, 1996) studied linear travelling waves in a pre-stressed, coated, incompressible or compressible, elastic half-space. When a prestress is present, there arises the possibility of existence of static sinusoidal solutions which has stability implications; see also Bigoni *et al.* (1997), Cai and Fu (1999, 2000) for further details and additional references. However, our attention here will be focussed on travelling wave solutions. We shall demonstrate that the surface-impedance matrix plays an important role in studying such waves and in terms of it the dispersion relation can be written in a very compact and revealing form.

Nonlinear travelling waves in a coated elastic half-space have previously been studied by Porubov and Samsonov (1995), Eckl and Mayer (1998), and Eckl *et al.* (2004). Porubov and Samsonov (1995) focussed on finding solitary wave solutions that have analytical/explicit expressions and showed that when the coating layer is perfectly bonded to the half-space (as we assume here), such solutions do not exist. Our treatment of the problem follows the spirit of the last two papers where the authors found solitary wave solutions numerically. In these two papers, the authors replaced the action of the coating layer by an effective boundary condition applied to the surface of the half-space. We show how this effective boundary condition can be derived and again how the evolution equation can be derived with the aid of the virtual work method. We explain a simple method that can be used to compute periodic and solitary travelling wave solutions.

2. Unification of different methods for deriving evolution equations for surface acoustic waves

(*a*) *Governing equations and linear theory*

We shall first consider a homogeneous, unstressed, generally anisotropic elastic half-space defined by

$$0 < x_2 < \infty, \quad -\infty < x_1, x_3 < \infty$$

relative to a rectangular coordinate system with coordinates (x_i). Free surface waves are governed by the equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i, \quad 0 < x_2 < \infty, \tag{2.1}$$

the traction-free boundary condition

$$\sigma_{i2} = 0 \quad \text{on} \;\; x_2 = 0, \tag{2.2}$$

and the decay condition

$$u_k \to 0 \quad \text{as } x_2 \to \infty, \tag{2.3}$$

where (σ_{ij}) is the stress tensor, (u_k) the displacement, ρ the material density, a comma and a superimposed dot denote differentiation with respect to the spatial coordinates and time, respectively. The above equations are closed by the constitutive relation given by

$$\sigma_{ij} = c_{ijkl}u_{k,l} + \frac{1}{2}e_{ijklmn}u_{k,l}u_{m,n} + \cdots\,, \tag{2.4}$$

where c_{ijks} and e_{ijklmn} are tensors of first- and second-order elastic moduli. We assume that c_{ijkl} and e_{ijklmn} satisfy the pairwise symmetry relations $c_{ijkl} = c_{klij}$, $e_{ijklmn} = e_{klijmn} = e_{ijmnkl}$, and that c_{ijkl} satisfies the strong convexity condition, but otherwise the material is generally anisotropic. We observe the summation convention whereby all repeated suffices are summed from 1 to 3. At the end of these notes, we shall explain how results obtained for a generally anisotropic material can be applied to a generally prestressed isotropic material.

We now consider the linearized form of (2.1)–(2.3) and look for a travelling-wave solution of the form

$$\mathbf{u} = \mathbf{a}\,\mathrm{e}^{\mathrm{i}\omega p x_2/v}\cdot \mathrm{e}^{-\mathrm{i}\omega\theta}, \quad \theta = t - x_1/v, \tag{2.5}$$

where $\omega > 0$ is the frequency, v the speed and the constant p and amplitude vector $\mathbf{a}$ are to be determined.

On substituting (2.5) into $c_{ijkl}u_{k,jl} = \rho\ddot{u}_i$, the linearized form of (2.1), we find that p and $\mathbf{a}$ are determined by the eigenvalue problem

$$\left(p^2T + p(R+R^T) + Q - \rho v^2 I\right)\mathbf{a} = \mathbf{0}, \tag{2.6}$$

where I is the identity matrix, the superscript "T" denotes matrix transpose, and the components of the three matrices T, R, Q are defined by

$$T_{ik} = c_{i2k2}, \quad R_{ik} = c_{i1k2}, \quad Q_{ik} = c_{i1k1}. \tag{2.7}$$

Since c_{ijks} satisfies the strong convexity condition, the eigenvalues of p in (2.6) cannot be pure real when $v = 0$ and they will remain complex until $v = \hat{v}$ at which at least one pair of eigenvalues first become pure real. The $\hat{v}$ is usually referred to as the limiting speed (Chadwick and Smith 1977) and surface waves with $v < \hat{v}$ are said to be subsonic. An elegant result in anisotropic elasticity is that a unique free-surface wave should normally exist except in some special cases (Barnett and Lothe 1974).

To characterize the linear free-surface wave solution, we assume from now on that $v < \hat{v}$ and denote by $p^{(1)}, p^{(2)}, p^{(3)}$ the three eigenvalues of p with positive imaginary parts and $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ the associated eigenvectors. Then a general solution that satisfies the decaying condition (2.3) is

$$\mathbf{u} = \left(\sum_{j=1}^{3} c_j\mathbf{a}^{(j)}\mathrm{e}^{\mathrm{i}\omega p^{(j)}x_2/v}\right)\mathrm{e}^{-\mathrm{i}\omega\theta} = A\langle \mathrm{e}^{\mathrm{i}\omega p x_2/v}\rangle\,\mathbf{c}\,\mathrm{e}^{-\mathrm{i}\omega\theta}, \tag{2.8}$$

where c_1, c_2, c_3 are constants,

$$A = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(2)}], \quad \mathbf{c} = [c_1, c_2, c_3]^T,$$

and $\langle e^{i\omega p x_2/v} \rangle$ denotes the diagonal matrix

$$\operatorname{diag}\{e^{i\omega p^{(1)} x_2/v}, e^{i\omega p^{(2)} x_2/v}, e^{i\omega p^{(3)} x_2/v}\}.$$

The linearized boundary condition $c_{i2kl}u_{k,l} = 0$ can be written as

$$R^T \mathbf{u}_{,1} + T\mathbf{u}_{,2} = 0. \tag{2.9}$$

On substituting the general solution (2.8) into (2.9), we obtain $B\mathbf{c} = \mathbf{0}$, where

$$B = [\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(2)}] = R^T A + TA\langle p \rangle, \quad \mathbf{b}^{(j)} = (R^T + p^{(j)}T)\mathbf{a}^{(j)} \tag{2.10}$$

without summation over j, and $\langle p \rangle = \operatorname{diag}\{p_1, p_2, p_3\}$.

At this juncture, we introduce the surface-impedance matrix M (Ingebrigtsen and Tonning 1969) through

$$M = -iBA^{-1}. \tag{2.11}$$

In terms of this matrix, the boundary condition $B\mathbf{c} = \mathbf{0}$ may be rewritten as

$$M\mathbf{d} = \mathbf{0}, \quad \text{where } \mathbf{d} = A\mathbf{c}. \tag{2.12}$$

We remark that it is advantageous to use M instead of B. Among its many useful properties, we mention that M is Hermitian so that the secular equation $\det M = 0$ for the surface-wave speed is real even for the most general anisotropic material (Stroh 1962). In the early studies on surface waves, it was not realized that the secular equation for the wave speed could always be written as a real equation, and as a result it was thought that existence of surface waves in anisotropic materials could only be exceptional (see, e.g., Farnell 1970). Also, all the eigenvalues of M are monotone decreasing functions of v (Barnett and Lothe 1985). As a result, $\det M = 0$ will not have any spurious roots, which is a useful property when $\det M = 0$ is solved numerically.

The surface-impedance matrix also has many applications other than in the surface-wave theory (see, e.g., Fu 2005). There now exist very efficient methods for computing this matrix. Firstly, this matrix has an integral representation given by

$$M = \left(\int_0^\pi {T_\theta}^{-1} d\theta \right)^{-1} \left(\pi I - i \int_0^\pi {T_\theta}^{-1} {R_\theta}^{\mathrm{T}} d\theta \right), \tag{2.13}$$

where

$$\begin{aligned} T_\theta &= \cos^2\theta T - \sin\theta\cos\theta(R + R^{\mathrm{T}}) + \sin^2\theta(Q - \rho v^2 I), \\ R_\theta &= \cos^2\theta R - \sin^2\theta R^{\mathrm{T}} + \sin\theta\cos\theta(T - Q + \rho v^2 I), \\ Q_\theta &= \cos^2\theta(Q - \rho v^2 I) + \sin\theta\cos\theta(R + R^{\mathrm{T}}) + \sin^2\theta T. \end{aligned} \tag{2.14}$$

This integral representation was first derived by Barnett and Lothe (1973), and later rederived by Mielke and Fu (2004) using a different procedure. Secondly, the surface-impedance matrix can also be computed with the aid of the matrix Riccati equation

$$(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^{\mathrm{T}}) - Q + \rho v^2 I = 0, \tag{2.15}$$

see Biryukov (1985), Mielke and Sprenger (1998), Fu and Mielke (2002). Finally, when $x_3 = 0$ is a plane of material symmetry, this matrix has a simple and explicit expression (Fu 2005, Fu and Brookes 2006).

A simple method for computing the surface-wave speed v and the corresponding M is as follows. Increase v gradually from $v = 0$ and at each step use (2.13) to evaluate M and hence $\det M$. As soon as $\det M$ changes sign, use the corresponding values of M and v as a initial guess and solve (2.15) and $\det M = 0$ to find M and v accurately.

In the following, we assume that v has been determined as the unique solution of $\det M = 0$ and $\mathbf{d}$ the corresponding non-trivial solution of $(2.12)_1$.

With the use of $(2.12)_2$, the solution (2.8) may be written as

$$\mathbf{u} = \mathbf{u}(\theta, x_2, \omega) = A\langle \mathrm{e}^{\mathrm{i}\omega p x_2/v}\rangle \, A^{-1}\mathbf{d}\, \mathrm{e}^{-\mathrm{i}\omega\theta}. \tag{2.16}$$

We observe that this solution is only valid for $\omega > 0$. When $\omega < 0$, we would need to use $\bar{p}_1, \bar{p}_2, \bar{p}_2$ in the construction of the general decaying solution (2.8), where an overbar denotes complex conjugation. As a result, when $\omega < 0$, we have

$$\mathbf{u}(\theta, x_2, \omega) = \bar{A}\langle \mathrm{e}^{\mathrm{i}\omega \bar{p} x_2/v}\rangle \, \bar{A}^{-1}\bar{\mathbf{d}}\, \mathrm{e}^{-\mathrm{i}\omega\theta}. \tag{2.17}$$

We note that

$$\mathbf{u}(\theta, x_2, \omega) = \overline{\mathbf{u}(\theta, x_2, -\omega)}, \quad \text{for } \omega < 0, \tag{2.18}$$

and we remark that this rule of defining a frequency-dependent function when the frequency is negative in terms of the same function when the frequency is positive applies to all frequency-dependent functions in our subsequent analysis. To facilitate analysis later, we define a new function $\mathbf{z}$ through

$$\mathbf{z}(\omega, x_2) = A\langle \mathrm{e}^{\mathrm{i}\omega p x_2/v}\rangle \, A^{-1}\mathbf{d}, \quad \text{when } \omega > 0. \tag{2.19}$$

As remarked above, we have $\mathbf{z}(\omega, x_2) = \overline{\mathbf{z}(-\omega, x_2)}$ when $\omega < 0$ is negative.

Once the linear solution is known, we may construct an asymptotic solution

$$\mathbf{u} = \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + O(\varepsilon^3) \tag{2.20}$$

for the original nonlinear surface-wave problem with

$$\mathbf{u}^{(1)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathbf{u}}^{(1)} \mathrm{e}^{-\mathrm{i}\omega\theta} d\omega, \tag{2.21}$$

$$\tilde{\mathbf{u}}^{(1)} = \mathcal{F}[\mathbf{u}^{(1)}] \equiv \int_{-\infty}^{\infty} \mathbf{u}^{(1)} e^{i\omega\theta} d\theta = f(\omega, \tau, \eta)\, \mathbf{z}(\omega, x_2), \tag{2.22}$$

where the unknown amplitude function $f(\omega, \tau, \eta)$ is assumed to depend on the slow-time variable $\tau = \varepsilon t$ and the far-distance variable $\eta = \varepsilon x_2$, and to satisfy the condition

$$f(-\omega, \tau, \eta) = \overline{f(\omega, \tau, \eta)}.$$

However, we shall show later that the evolution equation for the surface velocity is independent of whether the far-distance variable η is introduced or not. Throughout this paper we use both $\mathcal{F}[g]$ and $\tilde{g}$ to denote the Fourier transform of a function g.

The remaining of this subsection is devoted to explaining why the method used by Gusev *et al.* (1998) yields a different evolution from the other methods. If you simply want to learn the easiest method for deriving the evolution equation for surface acoustic waves, you may skip the rest of this subsection and the next subsection, and go straight to the subsection entitled *Evolution equation using the virtual work method.*

With the aid of (2.19) and (2.22), we deduce that

$$\frac{\partial \tilde{\mathbf{u}}^{(1)}}{\partial x_2} = f(\omega, \tau, \eta)\, \frac{\partial \mathbf{z}}{\partial x_2} = \frac{i\omega}{v} \cdot \begin{cases} A\langle p \rangle\, A^{-1} \tilde{\mathbf{u}}^{(1)}, & \text{when}\, \omega > 0, \\ \bar{A}\langle \bar{p} \rangle\, \bar{A}^{-1} \tilde{\mathbf{u}}^{(1)}, & \text{when}\, \omega < 0. \end{cases}$$

Thus, following Gusev *et al.* (1998) we have

$$\frac{\partial \tilde{\mathbf{u}}^{(1)}}{\partial x_2} = -\frac{1}{v}\, (\mathrm{Re}\, G + i\, \mathrm{sgn}(\omega)\, \mathrm{Im}\, G)\; \mathcal{F}\left[\frac{\partial \mathbf{u}^{(1)}}{\partial \theta}\right], \tag{2.23}$$

where

$$G = A\langle p \rangle A^{-1}, \tag{2.24}$$

Re and Im denote the real and imaginary parts, respectively, and sgn is the sign function. The matrix G corresponds to the matrix β defined in Gusev *et al.* (1998). It follows from (2.10), (2.11) and the Hermitian property $M^T = \overline{M}$ that

$$G^T T = iM^T - R, \quad T\bar{G} = -iM^T - R^T, \tag{2.25}$$

which will be used in our derivations later. We also observe that G is related to the matrix E of Fu and Mielke (2002) by

$$G = iE. \tag{2.26}$$

The matrix E plays an important role in the linear surface-wave theory. It can be shown that

$$E = T^{-1}(M + iR^T), \quad E\mathbf{a}^{(k)} = ip^{(k)}\, \mathbf{a}^{(k)}, \;\; k = 1, 2, 3, \tag{2.27}$$

the second relation showing that the three eigenvalues of E are $\mathrm{i}p_1, \mathrm{i}p_2, \mathrm{i}p_3$. These relations can be used to compute $\mathbf{a}^{(k)}$ and $p^{(k)}$ more efficiently than (2.6) once the surface-impedance matrix M is known (e.g. from (2.13)).

Taking the inverse Fourier transform of (2.23), we have

$$\frac{\partial \mathbf{u}^{(1)}}{\partial x_2} = -\frac{1}{v}\left\{\mathrm{Re}\, G \frac{\partial \mathbf{u}^{(1)}}{\partial \theta} - (\mathrm{Im}\, G)\, H[\frac{\partial \mathbf{u}^{(1)}}{\partial \theta}]\right\}, \tag{2.28}$$

where H denotes the Hilbert transform defined by

$$H[g(\theta)] = \frac{1}{\pi}\,\mathrm{p.v.} \int_{-\infty}^{\infty} \frac{g(y)}{y-\theta} dy = -\frac{1}{\pi\theta} \star g(\theta), \tag{2.29}$$

and use has been made of the basic result that

$$\mathcal{F}[\mathrm{p.v.}\frac{1}{\theta}] = \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{1}{\theta}\, \mathrm{e}^{\mathrm{i}\omega\theta} d\theta = 2\mathrm{i} \lim_{a\to 0} \int_{a}^{\infty} \frac{\sin \omega\theta}{\theta} d\theta = \mathrm{i}\pi\, \mathrm{sgn}(\omega). \tag{2.30}$$

In (2.29) and (2.30), p.v. denotes "principal value" and the star denotes integral convolution.

On differentiating (2.20) with respect to x_2 and making use of (2.28), we obtain

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial x_2} &= -\frac{\varepsilon}{v}\left\{\mathrm{Re}\, G \frac{\partial \mathbf{u}^{(1)}}{\partial \theta} - (\mathrm{Im}\, G)\, H[\frac{\partial \mathbf{u}^{(1)}}{\partial \theta}]\right\} + O(\varepsilon^2), \\ &= -\frac{1}{v}\left\{\mathrm{Re}\, G \frac{\partial \mathbf{u}}{\partial \theta} - (\mathrm{Im}\, G)\, H[\frac{\partial \mathbf{u}}{\partial \theta}]\right\} + O(\varepsilon^2).\end{aligned} \tag{2.31}$$

It can further be deduced with the aid of the property $H[H[g(x)]] = -g(x)$ that

$$\frac{\partial^2 \mathbf{u}}{\partial x_2^2} = \frac{1}{v^2}\left\{\mathrm{Re}(G^2) \frac{\partial^2 \mathbf{u}}{\partial \theta^2} - \mathrm{Im}(G^2)\, H[\frac{\partial^2 \mathbf{u}}{\partial \theta^2}]\right\} + O(\varepsilon^2). \tag{2.32}$$

Gusev *et al.* (1998) made a fundamental assumption that effectively says that the $O(\varepsilon^2)$ terms in (2.31) and (2.32) are identically zero (see their statement in the paragraph between equations (29) and (30)). Thus, a major operation used by Gusev *et al.* (1998) is to use (2.31) and (2.32), with the $O(\varepsilon^2)$ terms neglected, to eliminate derivatives with respect to x_2 in favor of derivatives with respect to θ. When this operation is applied to quadratic terms, it will induce an error of order ε^3 which can be neglected in the derivation of the nonlinear evolution equation. However, they also applied this operation to linear terms and in doing so they neglected some $O(\varepsilon^2)$ terms which, we believe, are not identically zero. Thus, the derivation procedure used in Gusev *et al.* (1998) is asymptotically inconsistent. We believe that this is why their method would gave a different evolution equation from all the other methods. This is elaborated further in the following subsection.

(b) Nonlinear evolution equation

With the variable transformation

$$(x_1, x_2, t) \to (\theta, x_2, \tau, \eta),$$

we have

$$\frac{\partial}{\partial x_1} \to -\frac{1}{v}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x_2} \to \frac{\partial}{\partial x_2} + \varepsilon \frac{\partial}{\partial \eta}.$$

The nonlinear equation of motion (2.1) and the boundary condition (2.2) then become

$$(\rho I - \frac{1}{v^2}Q)\frac{\partial^2 \mathbf{u}}{\partial \theta^2} + \frac{1}{v}(R + R^T)\frac{\partial^2 \mathbf{u}}{\partial \theta \partial x_2} - T\frac{\partial^2 \mathbf{u}}{\partial x_2^2} = -2\rho\varepsilon\frac{\partial^2 \mathbf{u}}{\partial \theta \partial \tau} + 2\varepsilon T\frac{\partial^2 \mathbf{u}}{\partial x_2 \partial \eta}$$

$$-\frac{1}{v}(R + R^T)\varepsilon\frac{\partial^2 \mathbf{u}}{\partial \theta \partial \eta} - \frac{1}{v}\frac{\partial \mathbf{n}^{(1)}}{\partial \theta} + \frac{\partial \mathbf{n}^{(2)}}{\partial x_2} + O(\varepsilon^3), \quad 0 < x_2 < \infty, \tag{2.33}$$

$$-\frac{1}{v}R^T\frac{\partial \mathbf{u}}{\partial \theta} + T\frac{\partial \mathbf{u}}{\partial x_2} = -\varepsilon T\frac{\partial \mathbf{u}}{\partial \eta} - \mathbf{n}^{(2)} + O(\varepsilon^3), \quad x_2 = \eta = 0, \tag{2.34}$$

where the two vector functions $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}$ are defined by

$$n_i^{(1)} = \frac{1}{2}e_{i1klmn}u_{k,l}u_{m,n}, \quad n_i^{(2)} = \frac{1}{2}e_{i2klmn}u_{k,l}u_{m,n}. \tag{2.35}$$

At this stage, Gusev *et al.* (1998) would inconsistently apply the substitutions (2.31) and (2.32) to the linear terms of both order ε and order ε^2 in (2.33) and (2.34) and to obtain

$$2\rho\varepsilon\frac{\partial^2 \mathbf{u}}{\partial \theta \partial \tau} + \frac{1}{v}(R + R^T)\varepsilon\frac{\partial^2 \mathbf{u}}{\partial \theta \partial \eta} + \frac{2}{v}\varepsilon T\left\{\mathrm{Re}\,G\frac{\partial^2 \mathbf{u}}{\partial \theta \partial \eta} - (\mathrm{Im}\,G)\,H[\frac{\partial^2 \mathbf{u}}{\partial \theta \partial \eta}]\right\}$$

$$= -\frac{1}{v}\frac{\partial \mathbf{n}^{(1)}}{\partial \theta} + \frac{\partial \mathbf{n}^{(2)}}{\partial x_2} + O(\varepsilon^3), \quad 0 < x_2 < \infty, \tag{2.36}$$

$$\frac{1}{v}\left\{R^T + T\mathrm{Re}\,G\frac{\partial \mathbf{u}}{\partial \theta} - T(\mathrm{Im}\,G)\,H[\frac{\partial \mathbf{u}}{\partial \theta}]\right\} = \varepsilon T\frac{\partial \mathbf{u}}{\partial \eta} + \mathbf{n}^{(2)} + O(\varepsilon^3), \quad x_2 = \eta = 0, \tag{2.37}$$

where use has been made of the fact that

$$\rho v^2 I - Q = (R + R^T)\mathrm{Re}\,G + T\,\mathrm{Re}(G^2), \quad 0 = (R + R^T)\mathrm{Im}\,G + T\,\mathrm{Im}(G^2), \tag{2.38}$$

which can be deduced from (2.6) with aid of (2.10) and (2.24).

Applying the Fourier transform to (2.36) and (2.37), we would obtain for $\omega > 0$

$$\varepsilon\left\{2\rho\frac{\partial \tilde{\mathbf{v}}}{\partial \tau} + \frac{1}{v}(R + R^T + 2TG)\frac{\partial \tilde{\mathbf{v}}}{\partial \eta}\right\} = \frac{1}{v}\mathrm{i}\omega\tilde{\mathbf{n}}^{(1)} + \frac{\partial \tilde{\mathbf{n}}^{(2)}}{\partial x_2} + O(\varepsilon^3), \quad 0 < x_2 < \infty \tag{2.39}$$

$$\frac{1}{v}\mathrm{i}M\tilde{\mathbf{v}} = \varepsilon T\frac{\partial \tilde{\mathbf{u}}}{\partial \eta} + \tilde{\mathbf{n}}^{(2)} + O(\varepsilon^3), \quad x_2 = \eta = 0, \tag{2.40}$$

where $\mathbf{v} = \partial \mathbf{u}/\partial t = \partial \mathbf{u}/\partial \theta$ and $\tilde{\mathbf{v}}$ denotes the Fourier transform of $\mathbf{v}$ so that

$$\tilde{\mathbf{v}} = -\mathrm{i}\omega \tilde{\mathbf{u}} = -\varepsilon \mathrm{i}\omega f(\omega, \tau, \eta)\, \mathbf{z}(\omega, x_2) + O(\varepsilon^2). \tag{2.41}$$

Contracting (2.40) with the left eigenvector $\boldsymbol{\zeta}$ of M and replacing $\tilde{\mathbf{u}}$ by $\tilde{\mathbf{v}}/(-\mathrm{i}\omega)$, we would eliminate the $O(\varepsilon)$ term on the left and obtain

$$\varepsilon \boldsymbol{\zeta} \cdot T \frac{\partial \tilde{\mathbf{v}}}{\partial \eta} - \mathrm{i}\omega\, \boldsymbol{\zeta} \cdot \tilde{\mathbf{n}}^{(2)} + O(\varepsilon^3) = 0. \tag{2.42}$$

It follows from (2.25) that

$$R + R^T + 2TG = 2\mathrm{i}M + R - R^T.$$

Thus, (2.39) may also be written as

$$\frac{\varepsilon}{v}(2\mathrm{i}M + R - R^T)\frac{\partial \tilde{\mathbf{v}}}{\partial \eta} = -\varepsilon 2\rho \frac{\partial \tilde{\mathbf{v}}}{\partial \tau} + \frac{1}{v}\mathrm{i}\omega \tilde{\mathbf{n}}^{(1)} + \frac{\partial \tilde{\mathbf{n}}^{(2)}}{\partial x_2} + O(\varepsilon^3), \quad 0 < x_2 < \infty. \tag{2.43}$$

The evolution equation of Gusev *et al.* (1998) (in the frequency domain) would be obtained by taking the limit $x_2 \to 0$ in (2.43), solving the resulting equation for $\partial \tilde{\mathbf{v}}/\partial \eta$ and substituting it into the boundary condition (2.42), the derivatives with respect to x_2 being eliminated with the aid of (2.31). However, this operation is illegitimate since $\partial \tilde{\mathbf{v}}/\partial \eta$ becomes parallel to $\mathbf{d}$ as $x_2 \to 0$ so that

$$(2\mathrm{i}M + R - R^T)\left.\frac{\partial \tilde{\mathbf{v}}}{\partial \eta}\right|_{x_2 \to 0} = (R - R^T)\left.\frac{\partial \tilde{\mathbf{v}}}{\partial \eta}\right|_{x_2 \to 0},$$

but the coefficient matrix $R - R^T$ is in general not invertible (an example is when the material is isotropic). Gusev *et al.* (1998) did not seem to realize this fact and wrote down (in our notation)

$$\left.\frac{\partial \tilde{\mathbf{v}}}{\partial \eta}\right|_{x_2 \to 0} = \frac{v}{\varepsilon}(2\mathrm{i}M + R - R^T)^{-1} \times \text{ RHS of } (2.43)|_{x_2 \to 0}.$$

Clearly this expression cannot be expected to be consistent with (2.41) since its right hand will not be parallel to $\mathbf{d}$.

We now return to (2.33) and (2.34) in order to derive the correction evolution equation. On substituting (2.20) into these equations and equating the coefficient of ε^2, we obtain

$$(\rho I - \frac{1}{v^2}Q)\frac{\partial^2 \mathbf{u}^{(2)}}{\partial \theta^2} + \frac{1}{v}(R + R^T)\frac{\partial^2 \mathbf{u}^{(2)}}{\partial \theta \partial x_2} - T\frac{\partial^2 \mathbf{u}^{(2)}}{\partial x_2^2} = -2\rho \frac{\partial^2 \mathbf{u}^{(1)}}{\partial \theta \partial \tau} + 2T\frac{\partial^2 \mathbf{u}^{(1)}}{\partial x_2 \partial \eta}$$

$$-\frac{1}{v}(R + R^T)\frac{\partial^2 \mathbf{u}^{(1)}}{\partial \theta \partial \eta} - \frac{1}{v}\frac{\partial \mathbf{n}^{(1)}}{\partial \theta} + \frac{\partial \mathbf{n}^{(2)}}{\partial x_2}, \quad 0 < x_2 < \infty, \tag{2.44}$$

$$-\frac{1}{v}R^T\frac{\partial \mathbf{u}^{(2)}}{\partial \theta} + T\frac{\partial \mathbf{u}^{(2)}}{\partial x_2} = -T\frac{\partial \mathbf{u}^{(1)}}{\partial \eta} - \mathbf{n}^{(2)}, \quad x_2 = \eta = 0, \tag{2.45}$$

where without introducing extra notation $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are still given by (2.35) but with $\mathbf{u}$ on the right hand sides now replaced by $\mathbf{u}^{(1)}$.

Applying the Fourier transform to (2.44) and (2.45), we obtain

$$(\rho I - \frac{1}{v^2}Q)\omega^2\tilde{\mathbf{u}}^{(2)} + \frac{\mathrm{i}\omega}{v}(R+R^T)\frac{\partial\tilde{\mathbf{u}}^{(2)}}{\partial x_2} + T\frac{\partial^2\tilde{\mathbf{u}}^{(2)}}{\partial x_2^2} = -2\rho\,\mathrm{i}\omega\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\tau} - 2T\frac{\partial^2\tilde{\mathbf{u}}^{(1)}}{\partial x_2\partial\eta}$$

$$-\frac{\mathrm{i}\omega}{v}(R+R^T)\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\eta} - \frac{\mathrm{i}\omega}{v}\tilde{\mathbf{n}}^{(1)} - \frac{\partial\tilde{\mathbf{n}}^{(2)}}{\partial x_2}, \quad 0 < x_2 < \infty, \tag{2.46}$$

$$\frac{\mathrm{i}\omega}{v}R^T\tilde{\mathbf{u}}^{(2)} + T\frac{\partial\tilde{\mathbf{u}}^{(2)}}{\partial x_2} = -T\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\eta} - \tilde{\mathbf{n}}^{(2)}, \quad x_2 = \eta = 0. \tag{2.47}$$

We emphasize that in arriving at the equations (2.46) and (2.47) we have used the same procedure as Gusev *et al.* (1998) except that we have not assumed that $\partial\mathbf{u}/\partial x_2$ is related to $\partial\mathbf{u}/\partial\theta$ by *exactly* the same formula as $\partial\mathbf{u}^{(1)}/\partial x_2$ is related to $\partial\mathbf{u}^{(1)}/\partial\theta$. The latter assumption would force the left hand side of (2.46) to be identically zero and the left hand side of (2.47) to be orthogonal to the left eigenvector $\boldsymbol{\zeta}$ of M.

At this stage, the formulation can be connected to the formulations in Methods I-V. Any of these methods would yield the right evolution equation. Since the governing equation and the boundary condition are already expanded out, the best method for deriving the evolution equation from this point is probably the projection method (Parker *et al.* 1992). In the next subsection, however, we shall show that when the virtual-work method is used, there is in fact no need to expand the boundary condition.

In the project method we contract (2.46) with the linear solution $\mathbf{z}(-\omega, x_2)$ defined by (2.19). We write $\mathbf{z}(-\omega, x_2)$ as $\mathbf{z}^-$ in order to avoid confusion with $\mathbf{z}(\omega, x_2)$ when the arguments are not written out. It can be shown by integrating by parts that

$$\int_0^\infty \mathbf{z}^-\cdot \text{LHS of (2.46)}\, dx_2 = -\mathbf{z}^-\cdot\left\{\frac{\mathrm{i}\omega}{v}R^T\tilde{\mathbf{u}}^{(2)} + T\frac{\partial\tilde{\mathbf{u}}^{(2)}}{\partial x_2}\right\}\bigg|_{x_2=0}$$

$$= \mathbf{z}^-\cdot\left\{T\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\eta} + \tilde{\mathbf{n}}^{(2)}\right\}\bigg|_{x_2=0} = -\int_0^\infty \frac{\partial}{\partial x_2}\mathbf{z}^-\cdot\left\{T\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\eta} + \tilde{\mathbf{n}}^{(2)}\right\}dx_2, \tag{2.48}$$

where in obtaining the second equation above use has been made of (2.47). On replacing the left hand side (LHS) of (2.46) by its right hand side in (2.48), we obtain

$$\int_0^\infty \left\{-2\rho\,\mathrm{i}\omega\mathbf{z}^-\cdot\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\tau} - T\mathbf{z}^-\cdot\frac{\partial^2\tilde{\mathbf{u}}^{(1)}}{\partial x_2\partial\eta} - \frac{\mathrm{i}\omega}{v}\mathbf{z}^-\cdot(R+R^T)\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\eta}\right.$$

$$\left.-\frac{\mathrm{i}\omega}{v}\mathbf{z}^-\cdot\mathbf{n}^{(1)} + \frac{\partial\mathbf{z}^-}{\partial x_2}\cdot\mathbf{n}^{(2)} + \frac{\partial\mathbf{z}^-}{\partial x_2}\cdot T\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial\eta}\right\}dx_2 = 0. \tag{2.49}$$

We assume in the rest of this subsection that $\omega > 0$. Then from (2.23),

$$\frac{\partial\tilde{\mathbf{u}}^{(1)}}{\partial x_2} = \frac{\mathrm{i}\omega}{v}G\tilde{\mathbf{u}}^{(1)}. \tag{2.50}$$

Similarly, we have

$$\frac{\partial \mathbf{z}^-}{\partial x_2} = -\frac{\mathrm{i}\omega}{v}\overline{G}\,\mathbf{z}^-. \tag{2.51}$$

With the aid of (2.50) and (2.51), the partial derivatives with respect to x_2 in the second and sixth terms in (2.49) can be eliminated. As a result, we obtain, after simplifying with the use of (2.25),

$$\int_0^\infty \left\{ -2\rho\,\mathrm{i}\omega\mathbf{z}^- \cdot \frac{\partial \tilde{\mathbf{u}}^{(1)}}{\partial \tau} - \frac{\mathrm{i}\omega}{v}\mathbf{z}^- \cdot \tilde{\mathbf{n}}^{(1)} + \frac{\partial \mathbf{z}^-}{\partial x_2} \cdot \tilde{\mathbf{n}}^{(2)} \right\} dx_2 = 0,$$

or equivalently,

$$\int_0^\infty \left\{ -2\rho\,\mathrm{i}\omega\mathbf{z}^- \cdot \frac{\partial \tilde{\mathbf{u}}^{(1)}}{\partial \tau} + \frac{1}{2}e_{ijklmn}\,\mathcal{F}[z^-_{i,j}]\,\mathcal{F}[u^{(1)}_{k,l}u^{(1)}_{m,n}] \right\} dx_2 = 0. \tag{2.52}$$

It is seen that $\partial\tilde{\mathbf{u}}^{(1)}/\partial\eta$ has dropped out of the evolution equation (2.52) so that the evolution equation obtained from (2.52) for $f(\omega,\tau,0)$ would be independent of whether the far-distance variable η has been introduced or not. The latter fact is of course already known (Fu and Devenish 1997, Eckl *et al.* 2004).

To obtain the final evolution equation from (2.52), we first obtain from (2.22)

$$\mathcal{F}[\frac{\partial \mathbf{u}^{(1)}}{\partial x_1}] = \frac{\mathrm{i}\omega}{v}\tilde{\mathbf{u}}^{(1)} = \frac{\mathrm{i}\omega}{v}\,f(\omega,\tau,\eta)\,A\langle \mathrm{e}^{\mathrm{i}\omega p x_2/v}\rangle\,\mathbf{c},$$

$$\mathcal{F}[\frac{\partial \mathbf{u}^{(1)}}{\partial x_2}] = \frac{\mathrm{i}\omega}{v}\,f(\omega,\tau,\eta)\,A\langle p\mathrm{e}^{\mathrm{i}\omega p x_2/v}\rangle\,\mathbf{c}.$$

Thus, we may write

$$\mathcal{F}[u^{(1)}_{m,n}] = \frac{\mathrm{i}\omega}{v}\,f(\omega,\tau,\eta)\,Q_{mn}(x_2,\omega), \tag{2.53}$$

where

$$Q_{mn}(x_2,\omega) = \sum_{j=1}^{3} A_{mj}\,L^{(j)}_n \mathrm{e}^{\mathrm{i}\omega p^{(j)}x_2/v}c_j, \quad L^{(j)}_n = \delta_{n1} + \delta_{n2}p^{(j)}, \;\; n,j = 1,2,3, \tag{2.54}$$

and we have written out the summation over j explicitly since it is not a standard summation over a suffix repeated only once.

In terms of Q_{mn} defined above, we have

$$z^-_{i,j} = -\frac{\mathrm{i}\omega}{v}\overline{Q_{ij}(x_2,\omega)}, \tag{2.55}$$

and

$$\frac{1}{2}e_{ijklmn}\mathcal{F}[u^{(1)}_{k,l}\,u^{(1)}_{m,n}] = \frac{1}{2}e_{ijklmn}\mathcal{F}[u^{(1)}_{k,l}] \star \mathcal{F}[u^{(1)}_{m,n}]$$

$$= -\frac{1}{2v^2}e_{ijklmn}\int_{-\infty}^{\infty} \omega'(\omega-\omega')f(\omega',\tau)f(\omega-\omega',\tau)Q_{kl}(x_2,\omega')Q_{mn}(x_2,\omega-\omega')d\omega'. \tag{2.56}$$

We note that when ω' and $\omega-\omega'$ are negative, the functions $Q_{kl}(x_2,\omega')$ and $Q_{mn}(x_2,\omega-\omega')$ above are defined by the rule

$$Q_{kl}(x_2,\omega') = \overline{Q_{kl}(x_2,-\omega')}, \quad Q_{mn}(x_2,\omega-\omega') = \overline{Q_{mn}(x_2,\omega'-\omega)},$$

as prescribed below (2.18).

On substituting (2.53), (2.55) and (2.56) into (2.52), we obtain

$$\frac{\partial f}{\partial \tau} = \frac{\mathrm{i}}{4\rho v^2 N(\omega)} \int_{-\infty}^{\infty} \omega'(\omega-\omega') f(\omega') f(\omega-\omega') \mathcal{K}(\omega,\omega') d\omega', \tag{2.57}$$

where

$$N(\omega) = \int_0^\infty \boldsymbol{z}\cdot\bar{\boldsymbol{z}} dx_2 = \int_0^\infty \mathbf{d}\cdot A^{-T} \langle \mathrm{e}^{\mathrm{i}\omega p x_2/v}\rangle\, A^T \bar{A} \langle \mathrm{e}^{-\mathrm{i}\omega \bar{p} x_2/v}\rangle\, \bar{A}^{-1}\bar{\mathbf{d}}\, dx_2 = \frac{1}{\omega} N(1), \tag{2.58}$$

$$\mathcal{K}(\omega,\omega') = -\frac{\mathrm{i}}{v} e_{sjklmn} \int_0^\infty Q_{kl}(x_2,\omega') Q_{mn}(x_2,\omega-\omega') \overline{Q_{sj}(x_2,\omega)} dx_2. \tag{2.59}$$

We note that the pair-wise symmetry of e_{sjklmn} implies that

$$\mathcal{K}(\omega,\omega') = \mathcal{K}(\omega,\omega-\omega'), \quad \mathcal{K}(\omega',\omega) = -\overline{\mathcal{K}(\omega,\omega')}. \tag{2.60}$$

(c) *Evolution equation using the virtual work method*

In this subsection we show how with the use of the virtual work method the same evolution can be obtained in the simplest manner. We assume throughout this subsection that ω_0 is a positive constant and we consider a nonlinear surface-wave solution that is periodic in θ with period $2\pi/\omega_0$. Such a solution can be represented as

$$\mathbf{u} = \varepsilon \sum_{m=1}^{\infty} f_m(\tau) A \langle \mathrm{e}^{\mathrm{i}m\omega_0 p x_2/v}\rangle\, A^{-1}\mathbf{d}\, \mathrm{e}^{-\mathrm{i}m\omega_0\theta} + C.C. + O(\varepsilon^2), \tag{2.61}$$

where C.C. denotes the complex conjugate of the preceding term and $f_m(\tau)$ are amplitude functions to be determined. We have not allowed $f_m(\tau)$ to depend on the far-distance variable $\eta = \varepsilon x_2$ since the evolution equation for the surface elevation or surface velocity is independent of its inclusion.

The virtual work method starts with the following line integral:

$$I = \lim_{h\to\infty} \oint_C \sigma_{ij} n_j \hat{u}_i ds, \tag{2.62}$$

where (n_i) is the outward normal to the path, $\hat{\mathbf{u}}$ is a linear solution given by

$$\hat{\mathbf{u}} = \mathbf{z}(-k\omega_0, x_2)\, \mathrm{e}^{\mathrm{i}k\omega_0\theta} = \overline{\mathbf{z}(k\omega_0, x_2)}\, \mathrm{e}^{\mathrm{i}k\omega_0\theta}, \quad k>0 \text{ an integer}, \tag{2.63}$$

the closed path C is the boundary of the rectangular region S: $[0 \le \theta < 2\pi/\omega_0, 0 \le x_2 \le h]$, and h is a positive constant. Since the integrand in (2.62) vanishes both on $x_2 = 0$ and as $x_2 \to \infty$ and since it takes the same value on the two vertical paths due to periodicity, it is easy to see that I is identically zero. With the use of the divergence theorem and the equation of motion (2.1), we also have

$$I = \int_0^\infty dx_2 \int_0^{2\pi/\omega_0} \left\{ \rho \hat{u}_i \frac{\partial^2 u_i}{\partial \theta^2} + c_{ijkl} u_{k,l} \hat{u}_{i,j} + 2\rho\varepsilon \frac{\partial^2 u_i}{\partial\theta\partial\tau} \cdot \hat{u}_i + \frac{1}{2} e_{ijklmn} u_{k,l} u_{m,n} \hat{u}_{i,j} \right\} d\theta. \tag{2.64}$$

Integrating the first two terms by parts and making use of the fact that $\mathbf{z}$ is a linear solution, we can show that the integral of the first two terms vanishes and (2.64) reduces to

$$I = \int_0^\infty dx_2 \int_0^{2\pi/\omega_0} \left\{ 2\rho\,\varepsilon \frac{\partial^2 u_i}{\partial\theta\partial\tau} \cdot \hat{u}_i + \frac{1}{2} e_{ijklmn} u_{k,l} u_{m,n} \hat{u}_{i,j} \right\} d\theta. \tag{2.65}$$

We note that both terms in (2.65) are of order ε^2, and that in the above derivation we did not have to expand the equation of motion or the boundary condition.

With the use of (2.61) and (2.63), we obtain

$$u_{i,j} = \varepsilon \frac{\mathrm{i}\,\omega_0}{v} \sum_{m=-\infty}^{\infty} m f_m(\tau) Q_{ij}(x_2, m\omega_0) \mathrm{e}^{-\mathrm{i}m\omega_0\theta}, \tag{2.66}$$

$$\hat{u}_{i,j} = -\frac{\mathrm{i}\,k\omega_0}{v} \overline{Q_{ij}(x_2, k\omega_0)} \mathrm{e}^{\mathrm{i}k\omega_0\theta}. \tag{2.67}$$

On substituting (2.66) and (2.67) into (2.65) and then evaluating the integral with respect to x_2, we obtain

$$\frac{df_k(\tau)}{d\tau} = \frac{\mathrm{i}\,k\omega_0^2}{4\rho v^2 N(1)} \sum_{m=-\infty}^{\infty} m(k-m)\mathcal{K}(k,m) f_m f_{k-m}, \quad k = 1, 2, \ldots, \tag{2.68}$$

where $N(\omega)$ and $\mathcal{K}(\omega, \omega')$ are as in the previous subsection. As expected, by taking $\omega_m = m\omega_0$, identifying $f_m(\tau)$ with $\omega_0 f(\omega_m, \eta)$, and then taking the limit $\omega_0 \to 0$ in this subsection, all the sums tend to integrals and the evolution equations (2.68) recover the integral equation (2.57).

(*d*) *Evaluation of coefficients*

With the use of (2.61), the velocity on the surface $x_2 = 0$ is obtained as

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}}{\partial \theta} = \varepsilon \sum_{n_1=1}^{\infty} g_{n_1}(\tau)\, \mathbf{d}\, \mathrm{e}^{-\mathrm{i}n_1\omega_0\theta} + C.C. + O(\varepsilon^2), \tag{2.69}$$

where

$$g_{n_1}(\tau) = -\mathrm{i}n_1\omega_0\, f_{n_1}(\tau). \tag{2.70}$$

In terms of g_{n_1}, the evolution equations (2.68) become

$$\frac{dg_{n_1}(\tau)}{d\tau} = -\frac{{n_1}^2\omega_0}{4\rho v^2 N(1)} \sum_{n_2=-\infty}^{\infty} \mathcal{K}(n_1, n_2) g_{n_2} g_{n_1-n_2}, \quad n_1 = 1, 2, \ldots. \tag{2.71}$$

It can easily be shown with the use of (2.60) that (2.71) can be rewritten as

$$\frac{dg_{n_1}(\tau)}{d\tau} = -\frac{{n_1}^2\omega_0}{4\rho v^2 N(1)}\left\{\sum_{n_2=1}^{n_1-1}\mathcal{K}(n_1,n_2)g_{n_2}g_{n_1-n_2}\right.$$

$$\left. -2\sum_{n_2=n_1+1}^{\infty}\overline{\mathcal{K}(n_2,n_1)}g_{n_2}\overline{g_{n_2-n_1}}\right\}, \quad n_1 = 1,2,\cdots. \tag{2.72}$$

With the aid of the various expressions given in the previous subsection, we obtain for $n_1 > n_2$,

$$\mathcal{K}(n_1,n_2) = e_{ijklmn}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{t=1}^{3}\bar{A}_{is}A_{kt}A_{mr}\frac{c_r\bar{c}_s c_t\, L_n^{(r)}\overline{L_j^{(s)}}L_l^{(t)}}{n_2p^{(t)} + (n_1-n_2)p^{(r)} - n_1\bar{p}^{(s)}}. \tag{2.73}$$

We have checked that the evolution equation (2.72) together with (2.73) is consistent with that given by Hamilton *et al.* (1999).

It is seen that to evaluate the coefficients in the evolution equations, we only need to know the values of $v, \mathbf{c}, p^{(1)}, p^{(2)}, p^{(3)}, A$. To find the linear surface-wave speed v and the corresponding M, we may use the procedure stated in the paragraph below equation (2.15). Once M is known, we may use (2.27) to find E and hence $p^{(j)}, \mathbf{a}^{(j)}, j = 1,2,3,\ A = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}]$, and $\mathbf{c} = A^{-1}\mathbf{d}$.

The evolution equation (2.73) can be integrated numerically subjected to specified initial conditions. The evolution of surface velocity or surface elevation can then be determined. It is known that shocks always form at finite times. Parker and Talbot (1985) argued that the nonlocal character of the kernel in the evolution equation (2.73) might give rise to steady nonlinear surface waves and they calculated such solutions using a certain numerical procedure. Ogden and Fu (1996) questioned the validity of the procedure: they applied the same procedure to a simple example and obtained spurious solutions. Thus, the steady wave solutions obtained in Parker and Talbot (1985) were probably spurious as well.

It is not surprising that steady nonlinear surface waves cannot exist in the absence of any form of dispersion. When dispersion is introduced into the problem, it is expected that weak dispersion would balance the wave steepening effects of nonlinearity and lead to solitary wave solutions. In recent years, the existence of solitary wave solutions has been demonstrated for a variety of dispersion forms. Three major forms of dispersion are: (i) when the wavelength is large compared with the thickness of a coating layer, see Porubov and Samsonov (1995) and Eckl *et al.* (1998), (ii) when the material properties of the coating layer are close to those of the half-space, see Fu and Hill (2001), and (iii) when the half-space has a microstructure, see Porubov and Pastrone (2004) and Porubov *et al.* (2004). In Section 4, we shall illustrate the existence of solitary wave solutions by considering the first type of dispersion.

3. Linear wave propagation in a coated elastic half-space

We now consider a coated half-space in which the half-space is made of the same material as in the previous sections but the coating layer has its constitutive relation given by

$$\sigma_{ij} = \tilde{c}_{ijkl} u_{k,l} + \frac{1}{2} \tilde{e}_{ijklmn} u_{k,l} u_{m,n} + \cdots , \tag{3.1}$$

where $\tilde{c}_{ijks}$ and $\tilde{e}_{ijklmn}$ are tensors of first- and second-order elastic moduli. The coating layer is assumed to occupy the region $-h < x_2 < 0$. With the use of (2.8), we may deduce that such an elastic layer admits a travelling wave solution given by

$$\begin{aligned} \mathbf{u} &= \left(\tilde{A} \langle \mathrm{e}^{\mathrm{i}\omega \tilde{p} x_2/v} \rangle \tilde{A}^{-1} \, \mathbf{d}^{(1)} + \overline{\tilde{A}} \langle \mathrm{e}^{\mathrm{i}\omega \bar{\tilde{p}} x_2/v} \rangle \overline{\tilde{A}}^{-1} \, \mathbf{d}^{(2)} \right) \mathrm{e}^{-\mathrm{i}\omega\theta} \\ &= \left(\mathrm{e}^{-\tilde{E} k x_2} \, \mathbf{d}^{(1)} + \mathrm{e}^{\bar{\tilde{E}} k x_2} \, \mathbf{d}^{(2)} \right) \mathrm{e}^{-\mathrm{i}\omega\theta}, \end{aligned} \tag{3.2}$$

where $k = \omega/v$ is the wavenumber, $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ are constant vectors to be determined, and here and hereafter a superimposed tilde signifies association with the elastic layer. Note that in (3.2) we have also included the solution corresponding to $\bar{\tilde{p}}_1, \bar{\tilde{p}}_2, \bar{\tilde{p}}_3$ since we do not require the solution to decay as $x_2 \to \infty$. Corresponding to this solution, we have

$$\mathbf{u}(0) = (\mathbf{d}^{(1)} + \mathbf{d}^{(2)}) \, \mathrm{e}^{-\mathrm{i}\omega\theta}, \tag{3.3}$$

$$(\sigma_{i2})|_{x_2=0} = \mathrm{i}k(-\tilde{M} \, \mathbf{d}^{(1)} + \overline{\tilde{M}} \mathbf{d}^{(2)}) \, \mathrm{e}^{-\mathrm{i}\omega\theta}, \tag{3.4}$$

$$(\sigma_{i2})|_{x_2=-h} = \mathrm{i}k(-\tilde{M} \mathrm{e}^{\tilde{E}kh} \, \mathbf{d}^{(1)} + \overline{\tilde{M}} \mathrm{e}^{-\bar{\tilde{E}}kh} \, \mathbf{d}^{(2)}) \, \mathrm{e}^{-\mathrm{i}\omega\theta}. \tag{3.5}$$

For the half-space, we have

$$\mathbf{u} = \mathrm{e}^{-Ekx_2} \, \mathbf{d}^{(3)} \, \mathrm{e}^{-\mathrm{i}\omega\theta}, \tag{3.6}$$

$$\mathbf{u}(0) = \mathbf{d}^{(3)} \, \mathrm{e}^{-\mathrm{i}\omega\theta}, \tag{3.7}$$

$$(\sigma_{i2})|_{x_2=0} = -\mathrm{i}kM \, \mathbf{d}^{(3)} \, \mathrm{e}^{-\mathrm{i}\omega\theta}, \tag{3.8}$$

where $\mathbf{d}^{(3)}$ is another constant vector to be determined. Thus, the traction-free boundary condition at $x_2 = -h$ and the displacement and traction continuity at the interface $x_2 = 0$ give

$$-\tilde{M} \mathrm{e}^{\tilde{E}kh} \, \mathbf{d}^{(1)} + \overline{\tilde{M}} \mathrm{e}^{-\bar{\tilde{E}}kh} \, \mathbf{d}^{(2)} = 0, \tag{3.9}$$

$$\mathbf{d}^{(1)} + \mathbf{d}^{(2)} = \mathbf{d}^{(3)}, \tag{3.10}$$

$$-\tilde{M} \, \mathbf{d}^{(1)} + \overline{\tilde{M}} \mathbf{d}^{(2)} = -M \, \mathbf{d}^{(3)}. \tag{3.11}$$

On eliminating $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ from the above equations, we obtain

$$D \mathbf{d}^{(3)} = \mathbf{0}, \tag{3.12}$$

where

$$D = \overline{\tilde{M}}\mathrm{e}^{-\overline{\tilde{E}}kh} - \left(\tilde{M}\mathrm{e}^{\tilde{E}kh} + \overline{\tilde{M}}\mathrm{e}^{-\overline{\tilde{E}}kh}\right)(\tilde{M} + \overline{\tilde{M}})^{-1}(M + \overline{\tilde{M}}). \tag{3.13}$$

Thus, the dispersion relation for the speed as a function of kh is given by

$$\det D = 0. \tag{3.14}$$

For $kh \ll 1$, we obtain from (3.13)

$$D = -M - kh\left\{\overline{\tilde{M}\tilde{E}} + (\tilde{M}\tilde{E} - \overline{\tilde{M}\tilde{E}})(\tilde{M} + \overline{\tilde{M}})^{-1}(M + \overline{\tilde{M}})\right\} + O(k^2h^2),$$

$$= -M + kh(\tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q} + \tilde{\rho}v^2 I - \mathrm{i}\tilde{R}\tilde{T}^{-1}M) + O(k^2h^2), \tag{3.15}$$

where we have made use of the relation

$$\tilde{M}\tilde{E} = \mathrm{i}\tilde{R}\tilde{T}^{-1}\tilde{M} - \tilde{R}\tilde{T}^{-1}\tilde{R}^T + \tilde{Q} - \rho v^2 I,$$

which can be established with the aid of $(2.27)_1$ and (2.15).

Denote by v_R the surface-wave speed associated with the half-space and by $\mathbf{d}$ the right eigenvector of M when $v = v_R$ (so that $\mathbf{d}$ has the same meaning as in previous sections). We look for a solution of the form

$$v = v_R + khv_1 + \cdots, \quad \mathbf{d}^{(3)} = \mathbf{d} + O(kh), \tag{3.16}$$

where v_1 is a constant. We have

$$M = M|_{v=v_R} + khv_1 \left.\frac{dM}{dv}\right|_{v=v_R} + O(k^2h^2). \tag{3.17}$$

It is known (see, e.g., Mielke and Fu 2004) that

$$\left.\frac{dM}{dv}\right|_{v=v_R} = -2\rho v_R \int_0^\infty \mathrm{e}^{-x_2\bar{E}^T}\mathrm{e}^{-x_2 E}dx_2. \tag{3.18}$$

Thus,

$$\overline{\mathbf{d}} \cdot \left.\frac{dM}{dv}\right|_{v=v_R} \mathbf{d} = -2\rho v_R N(v_R) = -2\rho N(1), \tag{3.19}$$

where $N(1)$ is given by (2.58). On substituting (3.15)–(3.18) into $\overline{\mathbf{d}} \cdot D\mathbf{d}^{(3)} = 0$ and equating the coefficients of kh, we obtain

$$v_1 = -\frac{\zeta}{2\rho N(1)}, \quad \text{and so } v = v_R - \frac{\zeta kh}{2\rho N(1)} + O(k^2h^2), \tag{3.20}$$

where

$$\zeta = \overline{\mathbf{d}} \cdot (\tilde{\rho}v_R^2 I + \tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q})\mathbf{d}. \tag{3.21}$$

4. Solitary waves in a coated elastic half-space

We now consider propagation of nonlinear travelling waves in the coated elastic half-space specified in the previous section. We shall consider a small-amplitude travelling wave solution in which the wavelength, L say, is much greater than the plate thickness. To be more precise, we assume that h/L is of the same order as ε where ε has the same meaning as in the previous sections. In order to keep track terms of different orders, we scale x_2 by ε, and still use x_2 for its scaled counterpart. The equation of motion $\sigma_{ij,j} = \tilde{\rho}\ddot{u}_i$, where $\tilde{\rho}$ is the material density of the layer, then becomes

$$\frac{1}{\varepsilon^2}\tilde{T}_{ij}u_{j,22} + \frac{1}{\varepsilon}(\tilde{R}_{ij} + \tilde{R}_{ji})u_{j,12} + \tilde{Q}_{ij}u_{j,11} + \frac{1}{\varepsilon^3}\tilde{e}_{i2k2m2}u_{k,22}u_{m,2}$$

$$+\frac{1}{\varepsilon^2}\left\{\tilde{e}_{i2k2m1}u_{k,22}u_{m,1} + (\tilde{e}_{i1k2m2} + \tilde{e}_{i2k1m2})u_{k,12}u_{m,2}\right\} + O(\varepsilon) = \tilde{\rho}\ddot{u}_i, \tag{4.1}$$

where

$$\tilde{T}_{ik} = \tilde{c}_{i2k2}, \quad \tilde{R}_{ik} = \tilde{c}_{i1k2}, \quad \tilde{Q}_{ik} = \tilde{c}_{i1k1}. \tag{4.2}$$

A similar equation can be written down for the traction-free boundary condition $\sigma_{i2} = 0$ at $x_2 = -h$. We now look for an asymptotic solution of the form

$$\mathbf{u} = \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + \varepsilon^3\mathbf{u}^{(3)} + O(\varepsilon^4). \tag{4.3}$$

On substituting (4.3) into (4.1), equating the coefficients of ε^{-1}, ε^0 and then solving the two equations subjected to the corresponding traction-free boundary conditions at $x_2 = -h$, we find that $\mathbf{u}^{(1)}$ is independent of x_2 and that $\mathbf{u}^{(2)}$ is given by

$$\mathbf{u}^{(2)} = -\tilde{T}^{-1}\tilde{R}^T\frac{\partial \mathbf{u}^{(1)}}{\partial x_1}x_2 + \text{an arbitrary function independent of } x_2. \tag{4.4}$$

With the aid of these results, we may evaluate the traction vector (σ_{i2}) at the interface $x_2 = 0$. It is found that its order ε term is identically zero. To find the $O(\varepsilon^2)$ term, we could carry out the above expansion to the next order, but a simpler method is to rewrite the equation of motion as

$$\sigma_{i1,1} + \sigma_{i2,2} = \tilde{\rho}\ddot{u}_i,$$

and then integrate across the layer thickness to obtain

$$\sigma_{i2}|_{x_2=0} = \int_{-h}^{0} (\hat{\rho}\ddot{u}_i - \sigma_{i1,1})\, dx_2. \tag{4.5}$$

With the use of (3.1) and (4.3), we obtain

$$(\sigma_{i2})|_{x_2=0} = h\left\{\tilde{\rho}\ddot{\mathbf{u}} + (\tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q})\mathbf{u}_{,11}\right\}|_{x_2=0} + O(\varepsilon^3). \tag{4.6}$$

Because of continuity of displacement and traction across the interface, the $\mathbf{u}$ and (σ_{i2}) in (4.6) can be taken to their counterparts in the half-space. Equation (4.6) then becomes an effective boundary condition to be imposed on the surface of the half-space. Such an effective boundary condition was first given by Tiersten (1969) and later assumed in Eckl *et al.* (1998). We observe that this effective boundary condition does not contain nonlinear terms. Since the above asymptotic procedure is the same as that used by Porubov and Samsonov (1995), the latter authors have implicitly used such an effective boundary condition although they did not display this explicitly (they used the method of imposing solvability conditions to derive their evolution equations).

The matrix $\tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q}$ in (4.6) is recognized as the matrix N_3 that appears in the Stroh formulation. This is not surprising since the Stroh formulation gives

$$(\sigma_{i2})_{,2} = \tilde{\rho}\ddot{\boldsymbol{u}} + (\tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q})\boldsymbol{u}_{,11} - \tilde{R}\tilde{T}^{-1}(\sigma_{i2})_{,1}.$$

On integrating this equation from $-h$ to 0 and making use of the traction-free boundary condition at $x_2 = -h$, we recover (4.6) in the limit $h \to 0$.

Since the leading order term in the effective boundary condition (4.6) is of order ε^2, the coated half-space behaves like an uncoated half-space to leading order and the leading-order solution (2.61) is still valid but we now expect that the effective boundary condition will affect the evolution of the amplitude functions $f_m(\tau)$. We have

$$\begin{aligned}(\sigma_{i2})|_{x_2=0} &= h\left\{\tilde{\rho}v_R^2 I + \tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q}\right\}\mathbf{u}_{,11}|_{x_2=0} + O(\varepsilon^3) \\ &= -\frac{\varepsilon h\omega_0^2}{v_R^2}(\tilde{\rho}v_R^2 I + \tilde{R}\tilde{T}^{-1}\tilde{R}^T - \tilde{Q})\mathbf{d}\sum_{m=1}^{\infty} m^2 f_m(\tau)\mathrm{e}^{-\mathrm{i}m\omega_0\theta} + C.C. + O(\varepsilon^3). \qquad (4.7)\end{aligned}$$

The integral I given by (2.62) is no longer zero; instead it is now given by

$$I = -\int_0^{2\pi/\omega_0} (\sigma_{i2}\hat{u}_i)|_{x_2=0}d\theta = \frac{\varepsilon h 2\pi\omega_0\zeta}{v_R^2}k^2 f_k(\tau), \qquad (4.8)$$

where ζ is given by (3.21).

The evolution equation (2.68) is now replaced by

$$\begin{aligned}\frac{df_k(\tau)}{d\tau} &= \frac{h}{\varepsilon}\cdot\frac{\mathrm{i}\omega_0^2\zeta}{2\rho v_R^2 N(1)}k^2 f_k \\ &+ \frac{\mathrm{i}\,k\omega_0^2}{4\rho v_R^2 N(1)}\sum_{m=-\infty}^{\infty} m(k-m)\mathcal{K}(k,m)f_m f_{k-m}, \quad k = 1, 2, \ldots, \qquad (4.9)\end{aligned}$$

On neglecting the nonlinear term above, we find that the solution of (4.9) is given by

$$f_k(\tau) = f_k(0)\exp\left\{k^2 h\cdot\frac{\mathrm{i}\omega_0^2\zeta}{2\rho v_R^2 N(1)}\,t\right\}.$$

On substituting this expression into (2.61), we find that the coating layer gives rise to a small correction to the wave speed and that the total wave speed is now given by

$$v_R - \frac{\zeta}{2\rho N(1)}\tilde{k}h + O(\tilde{k}^2 h^2), \qquad (4.10)$$

where $\tilde{k} = k\omega_0/v$ is the actual wave number of the mode. This expression is consistent with the two-term expansion given by $(3.20)_2$.

The velocity at the interface $x_2 = 0$ is again given by (2.69). In terms of the $g_{n_1}(\tau)$ defined by (2.70), the evolution equation (4.9) becomes

$$\frac{dg_{n_1}(\tau)}{d\tau} = \frac{h}{\varepsilon} \cdot \frac{\mathrm{i}\omega_0^2\zeta}{2\rho v_R^2 N(1)} g_{n_1} - \frac{{n_1}^2\omega_0}{4\rho v_R^2 N(1)} \left\{ \sum_{n_2=1}^{n_1-1} \mathcal{K}(n_1, n_2) g_{n_2} g_{n_1-n_2} \right.$$

$$\left. -2 \sum_{n_2=n_1+1}^{\infty} \overline{\mathcal{K}(n_2, n_1)} g_{n_2} \overline{g_{n_2-n_1}} \right\}, \quad n_1 = 1, 2, \cdots . \tag{4.11}$$

We may look for a solution of the form

$$g_{n_1}(\tau) = \Gamma_{n_1} \mathrm{e}^{\mathrm{i}n_1 c\tau}, \tag{4.12}$$

where c is a real constant and $\Gamma_{n_1}, n_1 = 1, 2, ...$ are complex constants to be determined. On substituting (4.12) into (4.11), we obtain

$$\mathrm{i}n_1 c\Gamma_{n_1} = \frac{h}{\varepsilon} \cdot \frac{\mathrm{i}\omega_0^2\zeta}{2\rho v_R^2 N(1)} \Gamma_{n_1} - \frac{{n_1}^2\omega_0}{4\rho v_R^2 N(1)} \left\{ \sum_{n_2=1}^{n_1-1} \mathcal{K}(n_1, n_2) \Gamma_{n_2} \Gamma_{n_1-n_2} \right.$$

$$\left. -2 \sum_{n_2=n_1+1}^{\infty} \overline{\mathcal{K}(n_2, n_1)} \Gamma_{n_2} \overline{\Gamma_{n_2-n_1}} \right\}, \quad n_1 = 1, 2, \cdots . \tag{4.13}$$

This system of algebraic equation can be solved using the following procedure proposed by Parker and Talbot (1985). We first replace the infinite system by a finite sysmtem of M equations. that is we assume that $\Gamma_k = 0\,(k = M, M+1, ...)$. We start with $M = 2$, in which case the two simultaneous equations can be solved exactly. Using each of these solutions as a starting solution, we increase M in unit steps until a convergence criterion is satisfied. At each step, the finite system of quadratic equations can be solved using Mathematica. As suggested by Eckl and Mayer (1998), solitary wave solutions can be obtained in the limit $\omega_0 \to \infty$.

Finally, we note that for a generally prestressed isotropic elastic half-space, small-amplitude perturbations are governed by the incremental equation of motion

$$\chi_{ij,j} = \rho\ddot{u}_i, \tag{4.14}$$

and the incremental traction-free boundary condition takes the form

$$\chi_{i2} = 0, \quad \text{on } \ x_2 = 0, \tag{4.15}$$

where the incremental stress tensor (χ_{ij}) is given by

$$\chi_{ij} = \mathcal{A}^1_{jilk} u_{k,l} + \frac{1}{2}\mathcal{A}^2_{jilknm} u_{k,l} u_{m,n} + \cdots , \tag{4.16}$$

and expressions for the elastic moduli $\mathcal{A}^1_{jilk}$ and $\mathcal{A}^2_{jilknm}$ in terms of the strain-energy function and the principal stretches can be found in Ogden (1984) or Appendix A of Fu and Ogden (1999). Thus, with c_{ijkl} and e_{ijklmn} identified with $\mathcal{A}^1_{jilk}$ and $\mathcal{A}^2_{jilknm}$, respectively, all the results obtained above are also valid for a generally prestressed isotropic elastic half-space or coated half-space. However, we remark that it would be too restrictive to assume that $\mathcal{A}^1_{jilk}$ satisfies the strong convexity condition. Instead, we assume that $\mathcal{A}^1_{jilk}$ are such that the corresponding surface-impedance matrix is positive definite when $v = 0$. This assumption is made so that the incremental dynamic problem is well-posed.

Acknowledgement: The author thanks Professor V.E. Gusev for reading an earlier version of these notes and for drawing his attention to the work of Professor V.P. Reutov.

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