Book Reviews

Enrico Martino, **Intuitionistic Proof Versus Classical Truth: The Role of Brouwer's Creative Subject in Intuitionistic Mathematics**, Springer, 2018. Logic, Epistemology, and the Unity of Science, vol. 42, pp. 170 + XIII. ISBN 978-3-319-74356-1 (hardcover) EUR 93,59, 978-3-030-08971-9 (softcover) EUR 93,59, ISBN 978-3-319-74357-8 EUR 74,96 (eBook).

This is a collection of reprinted papers, some by Martino alone and some written in collaboration with P. Giaretta or G. Usberti, that appeared between 1979 and 2002; two are previously unpublished. It covers a wide range of topics in intuitionistic logic, semantics and analysis; it is broader than the subtitle suggests.

It is preceded by a brief preface, whose bad English may deter some readers from reading further; but the rest of the book is much better written. Anyone with an interest in the murkier aspects of intuitionism will find this book well worth reading.

The papers seem to be reprinted as they were written, in chronological order, with no attempt to group them by subject matter, to add connecting material, or to revise them to incorporate later developments or second thoughts. For example, the paper that is chapter 4 is a correction to the paper that is chapter 2; it would have been better to present us with a rewritten version of chapter 2 instead. For the purposes of review I shall rearrange them (as the author should have) into sections by topic.

The foundations of intuitionistic logic (chapters 11, 13, 14)

Martino begins from the premise that 'Intuitionism identifies mathematical truth with provability'. He mentions an alternative approach to intuitionism, in which 'to know the meaning of a proposition is exactly to know its *proof-conditions*, that is to grasp what counts as a proof of it' (unfortunately, he conflates the two views); but it is the former that he takes as his starting point.

Following Martin-Löf (1991) [10], he distinguishes in chapter 11 between two versions of his premise: (i) a proposition is true iff a proof of it could in principle be constructed; (ii) a proposition is true iff a proof of it has been constructed. Version (i), which is espoused by Prawitz (1987) [11], leads to a rather platonistic conception of an objective realm of proofs; Martino argues that this leads not to intuitionistic logic but to classical epistemic logic.

In chapter 13 Martino argues that, less obviously, version (ii) also presupposes classical logic. He imagines an *ideal agent* who can carry out an infinite calculation. For any decidable predicate P of natural numbers, the ideal agent can test $P(n)$ for each n in turn. 'Then, it is perfectly determined whether in the course of the process he will always find, for every $P(n)$, the value "*true*" or, for some, the value "*false*"; so the truth value of the sentence $[\forall n\, P(n)]$ is perfectly determined.' He admits that this is a 'trivial' argument but finds it 'very cogent' (p. 140); obviously, for an intuitionist, it is nothing more than an unjustified use of the principle of excluded middle.

Personally, I have to say that I found his discussion of possible and actual existence, the objectivity of intuitionistic truth, and what it means to adopt a realist view, rather hazy and unsatisfactory; and I felt he did not distinguish well enough the different meanings of the word 'proof' used by intuitionists. Let me home in the most interesting point of his argument (pp. 142–3). Consider Heyting's (1956) [5] and Kolmogorov's (1932) [6] standard explanation of the meaning of the intuitionistic logical constants, taking the formula $\forall n\, P(n)$ as our example (where P is a decidable predicate of natural numbers): a proof of $\forall n\, P(n)$ is a method of producing, for every number n, a proof of $P(n)$. But since $P(n)$ is decidable, a proof of $P(n)$ is just a calculation. In the case where $\forall n\, P(n)$ is an unsolved problem, such as Goldbach's conjecture, we can specify the *method* (the calculation) now, but we do not know whether it *works* (whether it will always output 'true'). Martino says, 'So, what such a method has to yield is nothing but the *knowledge* that the deciding procedure, applied to any $P(n)$, always gives the value "true".' He concludes that this is just knowledge of the classical truth of $\forall n\ P(n)$. More generally, he decides that intuitionistic proof presupposes classical truth: a proof of a proposition is simply something that establishes that it is true.

Martino is certainly right that Heyting's account of what it takes to prove $\forall n\, P(n)$ is inadequate. The glib talk of a 'method' needs to be unpacked. One might think that to prove $\forall n\, P(n)$ is to provide an algorithm f accompanied by something extra that assures us that, for any n, $f(n)$ really is a proof of $P(n)$. This line of thought leads to the 'second clauses' in the proof-explanation, introduced by Kreisel (1962) [7]. Unfortunately Martino has already dismissed second clauses (rather too hastily) in chapter 11 (p.104). Martino is led therefore to a view that he calls 'computational realism', which amounts to accepting classical logic in arithmetic.

He returns to the question of the intended intuitionistic meaning of the logical constants in chapter 14. It is commonly suspected that the standard intuitionistic explanation of logical implication contains a vicious circle, but the precise nature of the circularity is hard to pin down. The meaning of $A \Rightarrow B$ is given by specifying what counts as a proof of it: a proof of $A \Rightarrow B$ is a method for transforming any given proof of A to a proof B. It appears then that in order to understand 'proof of $A \Rightarrow B$' we must first be able to 'survey' (whatever that means) the class of proofs of A. This is problematic if one thinks (i) that the class of proofs of A may grow over time, as new sorts of proof are invented, (ii) that a proof of A may itself somehow involve the formula $A \Rightarrow B$, in which case the meaning of $A \Rightarrow B$ must first be understood, or (iii) that a 'method' is not just an algorithm but implicitly includes a notion of 'evidence', and so itself depends on an understanding of proof. Martino believes all three of these, so he finds the Heyting-Kolmogorov proof explanations deeply problematic.

Martino turns to Martin-Löf's intuitionistic type theory for clarification (identifying methods with Martin-Löf's functions), looking especially at the way in which equality is understood. He does not believe that this resolves the impredicativity either.

Metamathematics of intuitionistic logic (chapters 3, 8, 10, 15)

Intuitionistic predicate calculus can be given a formal semantics using Beth models; validity of a formula in Beth models is equivalent to validity in 'natural' models (which is the intended meaning of intuitionistic logic). Hence the question of completeness can be investigated from an intuitionistic metamathematical perspective. Completeness turns out to be unprovable without resorting to Markov's principle, which of course we intuitionists do not accept. To overcome this problem, Beth models have been generalised by Veldman and de Swart to *generalised*, or *fallible*, Beth models, in which $\perp$ (which denotes a standard falsehood such as $0 = 1$) may hold at some nodes. ($\perp$ is used to define intuitionistic negation: $\neg A$ is $A \Rightarrow \perp$.) Completeness results can now be obtained; but what on earth does this strange trick mean? Dummett (1977, §5.7) [4] suggests that it throws doubt on the concept of intuitionistic negation.

Martino takes this line of thought further. It is best to start reading at chapter 15. He gives proofs of the incompleteness of intuitionistic logic relative to natural models, using a lawless sequence to construct a set of sentences that is consistent but has no model. To try to make sense of this he introduces a notion of 'truth conditional on a set of hypotheses', which is not the usual one but which is related to truth in fallible Beth models (chapter 8). This yields a proof of strong completeness ($\Gamma \models A$ implies $\Gamma \vdash A$).

He generalises the concept of a natural model to a *fallible* natural model, in which $\perp$ may hold. Validity in fallible natural models is equivalent to validity in fallible Beth models (chapter 3).

Although this works as metamathematics, it does seem far removed from the normal intended meaning of intuitionistic logic. Martino sheds some light on this in chapter 15. He points out that the axioms and rules of intuitionistic logic fail to capture the full intended meaning of $\perp$ (namely that it has no proof); all they tell us about $\perp$ is that it implies everything. To pin down the meaning of $\perp$ exactly Martino defines a *positive* model as like a natural model, except that instead of the usual stipulation that $\perp$ has no proof he stipulates that 'a proof of $\perp$ is a method of proving (the interpretations of) all non-logical atomic sentences'. This provides a well-behaved class of fallible models, relative to which intuitionistic logic is sound and strongly complete.

Martino, like Dummett, thinks that this provides a rationale for Griss' proposal to develop intuitionistic mathematics without reliance on absurdity and negation. This conclusion does not seem to follow. In intuitionistic mathematics, or in any kind of mathematics, we really do want simple falsehoods such as $2 + 2 = 5$ to have no proof. Rather, what this metamathematical work shows is that there is more to the intuitionistic proof-based understanding of logic than is captured in the axioms and rules of predicate calculus.

The theory of the creative subject (chapters 2, 4)

Brouwer's *theory of the creative subject* attempts to describe the way in which an idealised mathematician, Σ, known as the 'creative subject', carries out his or her mathematical activity; Brouwer used the theory to construct 'weak counterexamples' to various classical theorems that are intuitionistically unacceptable. We write $\vdash_n A$ to mean that, at time step n, Σ possesses a proof of the formula A. The usual axioms are as follows.

1. $\vdash_n A \vee \neg \vdash_n A$
2. $\vdash_n A \Rightarrow \vdash_{n+m} A$
3. $(\exists n \vdash_n A) \Leftrightarrow A$

One must distinguish between a *wide* and a *strict* interpretation. According to the wide interpretation, if $\vdash_n A$, and B is an immediate consequence of A, then $\vdash_n B$. According to the strict interpretation, $\vdash_n B$ only holds if Σ has specifically noticed that B is a consequence of A. Those who adopt the wide interpretation would accept the axioms

4. $\vdash_n \forall x\ P(x) \Rightarrow \forall x\ \vdash_n P(x)$
5. $\vdash_n P(m) \Rightarrow \vdash_n \exists x\ P(x)$
6. $\vdash_n (A \vee B) \Rightarrow \vdash_n A \vee \vdash_n B$

Together, however, these axioms lead to intuitionistically incorrect conclusions. Every author who writes on this subject has a different idea of how the axioms ought to be weakened and what $\vdash_n$ actually means. The idealised sense in which Σ may be said to 'know' A at time n becomes more and more obscure, the more one examines it.

Martino adopts an intermediate position between the wide and strict interpretations. His principle is,

'Whenever, for a fixed proposition A, we recognise that if we were in a certain state of knowledge s we could prove A, we can suppose that, whenever Σ is in the state s, he actually deduces A.' (p. 18)

He accepts axioms 1–3, and proposes replacements for axioms 4–6:

4′. $\vdash_n \forall x\ P(x) \Rightarrow \vdash_n P(a)$ (where a is a numeral)
8. $\vdash_n (A \Rightarrow B) \Rightarrow (\vdash_n A \Rightarrow \vdash_n B)$
9. $\vdash_n (A \wedge B) \Rightarrow \vdash_n A \wedge \vdash_n B$

(Note how 'wide' axiom 8 is. Σ's knowledge is assumed to be closed under *modus ponens*, so Σ knows infinitely many theorems of pure implicational logic.)

Martino's interpretation of $\vdash_n$ is highly strained: 'we propose to understand the validity of an axiom scheme in the following sense: for each instance, it is possible "to programme Σ" at the stage 0 so that the instance in question is true. By programming Σ we mean "to instruct Σ so that if he happens to be in certain favourable conditions, he performs certain deductions which interest us".' (p. 18)

His attempts to understand statements of the form $\vdash_n \exists x\, P(x)$ lead him to a second-order *Principle of Inductive Evidence*, which he claims in chapter 2 to be equivalent to monotonic bar induction. The truth however is more complicated, as he admits in chapter 4.

It is hard to believe that this elaborate and obscure concept is really implicit in Brouwer's creative subject arguments. It would be helpful to examine Brouwer's arguments to see what axioms he really needs to construct his weak counter-examples. One would find that he needs only weakened forms of the axioms discussed here, and that the choice between wide and strict interpretation is hardly relevant.

Exegesis of Brouwer's arguments on intuitionistic analysis (chapters 1, 5, 7)

The main proposition of Brouwer's intuitionistic analysis is that every function $f : [0, 1] \to \mathbf{R}$ is uniformly continuous. This is proved using the fan theorem, which depends upon the principle of bar induction. However, Brouwer (1927) [1] also published a weaker theorem that every function $f : [0, 1] \to \mathbf{R}$ is negatively continuous, which is of interest because it does not depend on bar induction but is said to be merely 'an immediate consequence of the intuitionistic point of view'. In chapter 5 Martino provides a useful analysis of Brouwer's sketchy proof. He gives an alternative proof in terms of axioms about lawless sequences; then he traces it back to the theory of the creative subject. He concludes that 'the mere purpose of rendering real numbers and real functions intuitionistically intelligible leads cogently to negative continuity, not to positive continuity' (p. 44). He makes a number of good points about the obscurities in the theory of choice sequences.

The main body of intuitionistic analysis, however, does depend upon the principle of bar induction, which essentially states that if a tree is well-founded (in the sense that every branch is finite, or 'meets a bar') then proof by induction up the tree (from the bar to the root) is valid. Brouwer thought that he could prove this principle, but a counter-example by Kleene showed that it could not be intuitionistically sound unless further conditions were imposed on the tree. Dummett (1977) [4] provided a well-known critique of Brouwer's attempted proof. Martino re-analyses the argument in chapter 1 and comes to a different conclusion about what went wrong.

Brouwer's argument worked by showing how to transform any given proof that the root is barred (i.e., that every branch of the tree is finite) into a proof of induction up the tree. Brouwer began by claiming that the given proof can be analysed into an infinite tree of steps of specified types, called η-inferences, $\mathcal{F}$-inferences and ζ-inferences (Martino calls this claim *Brouwer's Dogma*). In his critique Dummett distinguished between a proof in the usual intuitionistic sense (which is a finite construction of some kind), and the 'fully analysed' proof that Brouwer had in mind here, which is an infinitely branching well-founded proof tree of elementary inference steps. Brouwer further argued that the ζ-inferences are eliminable by a normalisation procedure. The resulting proof tree can be easily converted into a proof that induction up the tree is valid. This is claimed to establish the principle of bar induction.

Dummett argued that a fourth type of elementary inference step, called a θ-inference, needs to be provided. Martino, on the other hand, thinks that η-, $\mathcal{F}$- and ζ-inferences are sufficient but the elimination of ζ-inferences cannot be constructively carried out. Martino finds Brouwer's Dogma quite plausible, provided the bar can be defined without reference to the concept of infinite sequence.

Martino's analysis seems to me an improvement on Dummett's; yet his critique of Dummett, and Dummett's own position, are both frustratingly unclear. Dummett is unclear about whether his θ-inferences are really part of the fully analysed proof or are simply something that we are unable to reduce to fully analysed proof. It is not clear in the end how far apart Martino and Dummett really are. Martino, on pp. 6–7, rejects Dummett's concept of a fully analysed proof and doubts whether it is really to be found in Brouwer; yet Martino goes on to define a concept of infinitely branching proof tree that is very much like a fully analysed proof, and which he relates to a footnote in Brouwer (1927) [1]. In my view Brouwer's own idea of fully analysed proof is most clearly expressed in his Cambridge Lectures on Intuitionism [3] (1981, pp. 46–7), in which every 'mathematical deduction' is said to be isomorphic to a pseudo-well-ordered species, and this is used to prove the fan theorem (pp. 77–9).

In chapter 7 Martino examines another obscure corner of intuitionistic analysis, Brouwer's (1927) [2] claim that any partial order on a set is a virtual order iff it is inextensible. In a manuscript note of 1933 Brouwer thought he had discovered a counter-example, but in later publications he reasserted the theorem. Martino provides a helpful and consistent reading of what Brouwer originally meant, why his counter-example is wrong, and his later discussion of it in the Cambridge Lectures, together with a useful critique of the comments of Heyting, Posy and van Dalen on the subject.

Other topics (chapters 6, 9, 12)

Chapter 9 discusses Martin-Löf's constructivist semantics. Martin-Löf (1985, 1987) [8, 9] has distinguished between *propositions* and *judgements*. A *verification* confers *truth* on a proposition; a *proof* confers *evidence* on a judgement. If A is a proposition then 'A is true' and 'A is a proposition' are judgements. Martino puzzles over the meaning of these distinctions. He shows first that they are not meant in a Fregean sense. He then tries a number of other readings, relating it to Martin-Löf's remarks on different types of realism. Martino finds that Martin-Löf's distinctions do not stand up and that 'propositions can be regarded as the only fundamental entities of logic'.

Chapter 6 summarises Kripke's notion of 'grounded' truth, as a means of resolving the liar paradox in Tarskian semantics. Martino argues that it can also be understood in an intuitionistic framework, with the truth predicate T re-interpreted as provability; this does not however seem to bring anything new to the theory.

Chapter 12 deals with the concept of 'arbitrary reference'. Martino suggests that when we say 'for any x in a set' we are imagining an ideal agent who is able to refer individually to each member of the set. He uses this thought to shed a little new light on questions of impredicativity and plural quantification. He suggests that the concept of a set as constituted by its members (rather than as specified by a membership criterion) rests on the notion of plural reference, which he sees as a process of simultaneous choice of elements of a set by the ideal agent (p. 127). This is intended as an explication of Cantor's concept of set, but he fails to bring in the central Cantorian idea of limitation of size: there is no discussion of why the 'the simultaneous choice of certain individuals' can produce, for example, the set of all real numbers but not the set of all ordinal numbers.

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