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Asymptotic analysis of 3D dynamic equations in linear elasticity for a thin layer resting on a Winkler foundation

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A 3D dynamic problem for a thin elastic layer resting on a Winkler foundation is considered. The stiffness of the layer is assumed to be much greater than that of the foundation in order to allow low-frequency bending motion. The leading long-wave approximation valid outside the vicinity of the cut-off frequency is derived. It is identical to the classical Kirchhoff plate theory. A novel near cut-off 2D approximation is also established. It involves both bending and extension motions which are not separated from each other due to the effect of the foundation. The associated dispersion relation appears to be non-uniform over the small wavenumber domain.

Keywords: low-frequency; long-wave; cut-off; asymptotic; Winkler; elastic; plate.

1. Introduction

Dynamic and static analysis of thin plates and beams resting on an elastic foundation is a major domain in structural mechanics since long ago, e.g. see general reference works Wang *et al.* (2005), Dillard *et al.* (2018), Younesian *et al.* (2019) and also journal papers Tanahashi (2004), Metrikine (2004), Dumir (1988), Chien & Chen (2006), Zhang & Liu (2019), Auersch (2008), Chen & Chen (2011), Froio *et al.* (2018), Ghannadiasl & Mofid (2015) to name but a few.

Recently, *ad hoc* approximations broadly accepted by the interdisciplinary audience started to find mathematical justification. In particular, it was shown that the famous Winkler model, Winkler (1867), see also the historical survey in Kuznetsov (1952), coincide with a leading order behaviour of a thin elastic layer interacting with a stiff environment along one of its faces, Kaplunov *et. al.* (2018), Aghalovyan (2015), Kudish *et al.* (2021).

Strictly speaking, the well-known asymptotic results for thin elastic structures with traction-free or loaded faces, e.g. see books by Goldenveizer (1976), Kossovich (1986), Ciarlet (1990), Kaplunov *et. al.* (1998), Le (2012), cannot immediately be extended to the case of mixed boundary conditions along the faces. The reason is that the latter is not governed by Neumann-type boundary conditions due to the presence of a foundation. To the best of our knowledge, the only effort of justifying an *ad hoc* model of an elastically supported beam model using asymptotic analysis of a 2D plane stress

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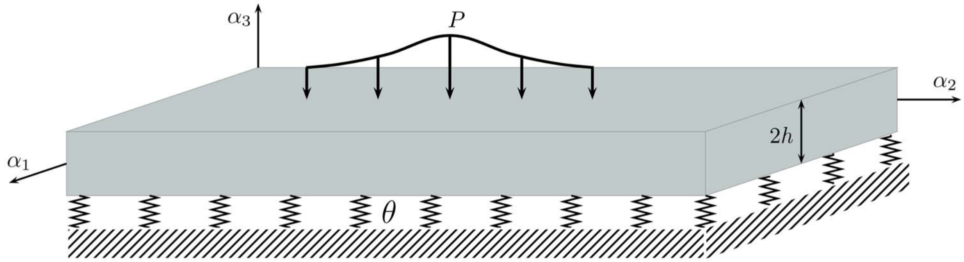

FIG. 1. An elastic layer on a Winkler foundation.

formulation was reported in Erbaş *et al.* (2022). The cited paper implements the scaling corresponding to bending vibrations of a layer with traction-free faces, e.g. see Kaplunov *et. al.* (1998). Both leading and first-order approximations were derived. The former is identical to the classical Euler–Bernoulli theory adapted for a beam lying on a Winler foundation.

In what follows, we consider a 3D dynamic problem in linear elasticity for a thin layer supported by a Winkler foundation. Long-wave low-frequency motion is analysed under the assumption that the stiffness of the layer is much greater than that of the foundation. As might be expected, the leading order approximation generally coincides with the Kirchhoff equation governing bending vibrations of a plate resting on a Winkler foundation. The exception is a narrow vicinity of the associated cut-off frequency where the asymptotic scaling underlying plate bending deformation is violated. Similar situation occurs for a thin cylindrical shell, for which the low-frequency near cut-off behaviour assumes a special treatment, see Strozzi *et al.* (2014), Kaplunov & Nobili (2017), Kaplunov *et al.* (2016).

The near cut-off 2D approximation derived in the paper takes a rather sophisticated form and does not allow simple separation of bending and extension deformations due to the asymmetry caused by the effect of foundation. The asymptotic behaviour of the related dispersion relation appears to be non-uniform near the cut-off frequency because of veering of bending and extension dispersion curves earlier noted in Erbaş *et al.* (2018). This is not usually the feature of the dispersion relations for thin structures with traction-free faces, Kaplunov *et. al.* (1998).

The paper is organized as follows: the studied 3D problem is formulated in Section 2. The leading order approximation for plate bending vibration is derived in Section 3. The near cut-off degeneration of the asymptotic scaling related to plate bending and the derivation of an alternative low-frequency approximation is addressed in Section 4 which is central to the paper. The developed near cut-off model is discussed in greater detail in Section 5.

2. Statement of the Problem

Consider an infinite elastic layer of thickness $2h$ supported by a Winkler foundation of stiffness θ along its lower surface and subject to a vertical load P along the upper one, see Fig. 1. Denote Young's modulus, Poisson's ratio, mass density of the elastic material by E, ν and ρ, respectively.

Let the layer motion in the cartesian coordinates $-\infty < \alpha_1, \alpha_2 < \infty$, $-h \leq \alpha_3 \leq h$ be governed by the 3D equations in linear elasticity given by

$$\frac{\partial \sigma_{3i}}{\partial \alpha_3} + \frac{\partial \sigma_{ii}}{\partial \alpha_i} + \frac{\partial \sigma_{ji}}{\partial \alpha_j} - \rho \frac{\partial^2 v_i}{\partial t^2} = 0, \tag{2.1}$$

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$$\frac{\partial \sigma_{33}}{\partial \alpha_3} + \frac{\partial \sigma_{31}}{\partial \alpha_1} + \frac{\partial \sigma_{32}}{\partial \alpha_2} - \rho \frac{\partial^2 v_3}{\partial t^2} = 0, \tag{2.2}$$

with

$$\sigma_{ii} = \frac{E}{1-\nu^2}\left(\frac{\partial v_i}{\partial \alpha_i} + \nu \frac{\partial v_j}{\partial \alpha_j}\right) + \frac{\nu}{1-\nu}\sigma_{33}, \tag{2.3}$$

$$\sigma_{ij} = \frac{E}{2(1+\nu)}\left(\frac{\partial v_i}{\partial \alpha_j} + \frac{\partial v_j}{\partial \alpha_i}\right), \tag{2.4}$$

$$\sigma_{3i} = \frac{E}{2(1+\nu)}\left(\frac{\partial v_3}{\partial \alpha_i} + \frac{\partial v_i}{\partial \alpha_3}\right), \tag{2.5}$$

$$E\frac{\partial v_3}{\partial \alpha_3} = \sigma_{33} - \nu(\sigma_{11} + \sigma_{22}), \tag{2.6}$$

where σ_{kl}, $k, l = 1, 2, 3$ are stresses, v_m, $m = 1, 2, 3$ are displacements, t is time; throughout the paper, we also consider $i \neq j = 1, 2$.

The boundary conditions along the faces of the layer are written as

$$\sigma_{3i}\Big|_{\alpha_3=\pm h} = 0, \qquad i = 1, 2 \tag{2.7}$$

$$\sigma_{33}\Big|_{\alpha_3=h} = P, \qquad \sigma_{33}\Big|_{\alpha_3=-h} = \theta v_3. \tag{2.8}$$

We assume that the stiffness of the layer is much greater than that of the Winkler foundation such that

$$\varepsilon = \sqrt{\frac{2(1+\nu)\theta h}{E}} \ll 1 \tag{2.9}$$

is a small parameter. The latter condition supports long-wave, low-frequency bending vibrations of a layer, see the plane strain analysis in Erbaş *et al.* (2022) and references therein. For these vibrations typical wavelength L and time scale T may be defined as

$$L \sim \frac{h}{\sqrt{\varepsilon}}, \qquad T \sim \frac{h}{c_2 \varepsilon}, \tag{2.10}$$

implying that $(L/h)^2 \sim Tc_2/h$. The standard in the plate theory, scaling (2.10) (see also Kaplunov *et. al.* (1998) for greater detail), is adopted below for the derivation of the well-known 2D equations governing Kirchhoff plate motion supported by a Winkler foundation. It is demonstrated in the paper that the aforementioned scaling is not valid near the cut-off frequency of a bending wave. An alternative scaling specifically oriented to near cut-off behaviour is then introduced.

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3. Conventional 2D plate model

Let us derive the leading order 2D equations of bending motion starting from 3D relations above. First, according to the scaling (2.10), specify the dimensionless coordinates by

$$\alpha_i = \frac{h}{\sqrt{\varepsilon}}\xi_i \qquad \alpha_3 = h\zeta, \qquad t = \frac{h}{c_2\varepsilon}\tau. \tag{3.1}$$

Also, scale the displacements, stresses and the prescribed load by

$$v_i = \sqrt{\varepsilon}hv_i^*, \quad v_3 = hv_3^*, \tag{3.2}$$

$$\sigma_{ii} = E\varepsilon\sigma_{ii}^*, \quad \sigma_{ij} = E\varepsilon\sigma_{ij}^*, \quad \sigma_{3i} = E\varepsilon^{3/2}\sigma_{3i}^*, \quad \sigma_{33} = E\varepsilon^2\sigma_{33}^*, \tag{3.3}$$

and

$$P = E\varepsilon^2 P^*. \tag{3.4}$$

Remark that the last set of formulae is identical to that for dynamic bending of a plate in the absence of a foundation, e.g. see, Kaplunov *et. al.* (1998).

In dimensionless variables (3.1), the equations of motion in the previous section, rewritten through the starred quantities defined by (3.2)–(3.3), become

$$\frac{\partial\sigma_{3i}^*}{\partial\zeta} = -\left(\frac{\partial\sigma_{ii}^*}{\partial\xi_i} + \frac{\partial\sigma_{ij}^*}{\partial\xi_j}\right) + \frac{\varepsilon}{2(1+\nu)}\frac{\partial^2 v_i^*}{\partial\tau^2} \tag{3.5}$$

$$\frac{\partial\sigma_{33}^*}{\partial\zeta} = -\left(\frac{\partial\sigma_{3i}^*}{\partial\xi_i} + \frac{\partial\sigma_{3j}^*}{\partial\xi_j}\right) + \frac{1}{2(1+\nu)}\frac{\partial^2 v_3^*}{\partial\tau^2} \tag{3.6}$$

with

$$\sigma_{ii}^* = \frac{1}{1-\nu^2}\left(\frac{\partial v_i^*}{\partial\xi_i} + \nu\frac{\partial v_j^*}{\partial\xi_j}\right) + \varepsilon\frac{\nu}{1-\nu}\sigma_{33}^* \tag{3.7}$$

$$\sigma_{ij}^* = \frac{1}{2(1+\nu)}\left(\frac{\partial v_i^*}{\partial\xi_j} + \frac{\partial v_j^*}{\partial\xi_i}\right), \tag{3.8}$$

$$\frac{\partial v_i^*}{\partial\zeta} = -\frac{\partial v_3^*}{\partial\xi_i} + 2\varepsilon(1+\nu)\sigma_{3i}^*, \tag{3.9}$$

$$\frac{\partial v_3^*}{\partial\zeta} = \varepsilon^2\sigma_{33}^* - \varepsilon\,\nu\left(\sigma_{ii}^* + \sigma_{jj}^*\right). \tag{3.10}$$

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The boundary conditions are now written as

$$\sigma^*_{3i}\Big|_{\zeta=\pm 1} = 0, \tag{3.11}$$

$$\sigma^*_{33}\Big|_{\zeta=1} = \varepsilon^2 P^*, \qquad \sigma^*_{33}\Big|_{\zeta=-1} = \frac{1}{2(1+\nu)} v^*_3. \tag{3.12}$$

The starred quantities may now be expanded in an asymptotic series as

$$f^* = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots . \tag{3.13}$$

First, we derive from formula (3.10), at leading order,

$$v^{(0)}_3 = V^{(0)}_3(\xi_i, \tau). \tag{3.14}$$

Then, substituting (3.14) into (3.9), we obtain, integrating in ζ,

$$v^{(0)}_i = -\zeta \frac{\partial V^{(0)}_3}{\partial \xi_i} + V^{(0)}_i. \tag{3.15}$$

Now, continue with the leading order terms for the stress components σ_{ii} and σ_{ij}. On inserting (3.15) into (3.7) and (3.8) we have, respectively,

$$\sigma^{(0)}_{ii} = -\frac{\zeta}{1-\nu^2}\left(\frac{\partial^2 V^{(0)}_3}{\partial \xi_i^2} + \nu \frac{\partial^2 V^{(0)}_3}{\partial \xi_j^2}\right) + \frac{1}{1-\nu^2}\left(\frac{\partial V^{(0)}_i}{\partial \xi_i} + \nu \frac{\partial V^{(0)}_j}{\partial \xi_j}\right), \tag{3.16}$$

$$\sigma^{(0)}_{ij} = -\frac{\zeta}{(1+\nu)} \frac{\partial^2 V^{(0)}_3}{\partial \xi_i \partial \xi_j} + \frac{1}{2(1+\nu)}\left(\frac{\partial V^{(0)}_i}{\partial \xi_j} + \frac{\partial V^{(0)}_j}{\partial \xi_i}\right). \tag{3.17}$$

Next, insert the latter equations into (3.5) and integrate across the thickness in ζ, taking into account the boundary conditions (3.11). The result is

$$\frac{\partial^2 V^{(0)}_i}{\partial \xi_i^2} + \frac{1+\nu}{2} \frac{\partial^2 V^{(0)}_j}{\partial \xi_i \partial \xi_j} + \frac{1-\nu}{2} \frac{\partial^2 V^{(0)}_i}{\partial \xi_j^2} = 0. \tag{3.18}$$

The last formulae correspond to static plane stress problem which is separated from the dynamic bending problem of interest.

We also have, from (3.5), the expression

$$\sigma^{(0)}_{3i} = \frac{\zeta^2 - 1}{2(1-\nu^2)}\left(\frac{\partial^3 V^{(0)}_3}{\partial \xi_i^3} + \frac{\partial^3 V^{(0)}_3}{\partial \xi_i \partial \xi_j^2}\right) \tag{3.19}$$

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satisfying the homogeneous boundary conditions (3.11).

Equation (3.6) at leading order takes the form

$$\frac{\partial \sigma_{33}^{(0)}}{\partial \zeta} = -\left(\frac{\partial \sigma_{3i}^{(0)}}{\partial \xi_i} + \frac{\partial \sigma_{3j}^{(0)}}{\partial \xi_j}\right) + \frac{1}{2(1+\nu)}\frac{\partial^2 v_3^{(0)}}{\partial \tau^2}.$$

On account of (3.19), it becomes

$$\frac{\partial \sigma_{33}^{(0)}}{\partial \zeta} = -\frac{\zeta^2 - 1}{2(1-\nu^2)}\varDelta^2 V_3^{(0)} + \frac{1}{2(1+\nu)}\frac{\partial^2 V_3^{(0)}}{\partial \tau^2}. \tag{3.20}$$

Integrating the latter through the thickness, and taking into account the inhomogeneous boundary conditions (3.12), we arrive at the sought for 2D equation

$$\frac{2}{3(1-\nu)}\varDelta^2 V_3^{(0)} + \frac{1}{2}V_3^{(0)} + \frac{\partial^2 V_3^{(0)}}{\partial \tau^2} = (1+\nu)P^*. \tag{3.21}$$

In the original variables, the latter may be written as

$$D\varDelta^2 w + 2\rho h \frac{\partial^2 w}{\partial t^2} + \theta w = P, \tag{3.22}$$

where $w = hV_3^{(0)}$, $\varDelta^2 = \partial^4/\partial\alpha_1^4 + 2\partial^4/\partial\alpha_1^2\partial\alpha_2^2 + \partial^4/\partial\alpha_2^4$ and the bending rigidity is given by

$$D = \frac{2Eh^3}{3(1-\nu^2)}. \tag{3.23}$$

The derived leading order equation coincides with the traditional engineering set-up associated with the vibrations of a Kirchhoff plate supported by a Winkler foundation, e.g. see Wang *et al.* (2005) and references therein.

4. Near cut-off behaviour

Taking the solution to equation (3.21) in the form of a plane travelling wave $w = We^{i(k\alpha_1 - \omega t)}$, where k and ω are the wavenumber and frequency, respectively, we arrive at the dispersion relation

$$\varOmega^2 - \frac{1}{2} = \frac{2}{3(1-\nu)}\varepsilon^2 K^4, \tag{4.1}$$

where

$$K = \frac{kh}{\varepsilon}, \quad \varOmega = \frac{\omega h}{c_2 \varepsilon}. \tag{4.2}$$

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The original scaling in the previous section corresponding to bending vibration is based on the assumption $T \sim h/c_2\varepsilon$ for which $\Omega \sim \varepsilon K^2$. This assumption is violated near the cut-off frequency $\Omega = 1/\sqrt{2}$. This motivates to look for an alternative scaling. For the sake of simplicity, we assume below that $\left|\Omega^2 - 1/2\right| \sim \varepsilon^2$. We also restrict ourselves to the domain $K \sim 1$. In this case, however, we cannot rely on the previous scaling obtained under the assumption $K \sim \varepsilon^{-1/2}$ at $\Omega \sim 1$.

Thus, we are aiming at establishing a 2D near cut-off formulation supporting a two-term asymptotic behaviour in the form

$$\Omega^2 = 1/2 + \varepsilon^2 P(K), \tag{4.3}$$

where $P(K)$ is a function to be determined. In doing so, due to the reasoning above, we should not expect to obtain $P(K) = 2K^4/(3(1-\nu))$ according to (4.1).

Let us now introduce the non-dimensional variables

$$\alpha_i = \frac{h}{\varepsilon}\xi_i, \qquad \alpha_3 = h\zeta, \qquad t = \frac{h}{c_2\varepsilon}\tau \tag{4.4}$$

in the 3D equations of motion (2.1)–(2.6) and boundary conditions (2.7)–(2.8) and also dimensionalize the sought for displacements and stresses as well as the external load by

$$v_i = h\varepsilon v_i^*, \qquad v_3 = hv_3^*, \tag{4.5}$$

$$\sigma_{ii} = E\varepsilon^2\sigma_{ii}^*, \quad \sigma_{ij} = E\varepsilon^2\sigma_{ij}^*, \qquad \sigma_{3i} = E\varepsilon^3\sigma_{3i}^*, \qquad \sigma_{33} = E\varepsilon^2\sigma_{33}^* \tag{4.6}$$

and

$$P = E\varepsilon^4 P^*, \tag{4.7}$$

where, as usual, the starred quantities are assumed of order unity. The near cut-off scaling given by formulae (4.4)–(4.7) is inspired by asymptotic analysis of the dispersion relation for the problem in question presented in Erbaş *et al.* (2018).

On employing the scaled coordinates (4.4) together with non-dimensional quantities (4.5)–(4.6), we may rewrite equations of motion (2.1)–(2.2) and the stress-displacement relations (2.3)–(2.6) as

$$\frac{\partial\sigma_{3i}^*}{\partial\zeta} = -\left(\frac{\partial\sigma_{ii}^*}{\partial\xi_i} + \frac{\partial\sigma_{ij}^*}{\partial\xi_j}\right) + \frac{\varepsilon^2}{2(1+\nu)}e_i^* - \frac{1}{4(1+\nu)}v_i^* \tag{4.8}$$

$$\frac{\partial\sigma_{33}^*}{\partial\zeta} = -\varepsilon^2\left(\frac{\partial\sigma_{3i}^*}{\partial\xi_i} + \frac{\partial\sigma_{3j}^*}{\partial\xi_j}\right) + \frac{\varepsilon^2}{2(1+\nu)}e_3^* - \frac{1}{4(1+\nu)}v_3^*, \tag{4.9}$$

where

$$e_k^* = \frac{1}{\varepsilon^2}\left(\frac{\partial^2 v_{k^*}}{\partial\tau^2} + \frac{1}{2}v_k^*\right), \qquad k = 1, 2, 3 \tag{4.10}$$

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and

$$\sigma_{ii}^* = \frac{1}{1-\nu^2}\left(\frac{\partial v_i^*}{\partial \xi_i} + \nu \frac{\partial v_j^*}{\partial \xi_j}\right) + \frac{\nu}{1-\nu}\sigma_{33}^* \tag{4.11}$$

$$\sigma_{ij}^* = \frac{1}{2(1+\nu)}\left(\frac{\partial v_i^*}{\partial \xi_j} + \frac{\partial v_j^*}{\partial \xi_i}\right), \tag{4.12}$$

$$\frac{\partial v_i^*}{\partial \zeta} = -\frac{\partial v_3^*}{\partial \xi_i} + 2\varepsilon^2(1+\nu)\sigma_{3i}^*, \tag{4.13}$$

$$\frac{\partial v_3^*}{\partial \zeta} = \varepsilon^2\left[\sigma_{33}^* - \nu\left(\sigma_{ii}^* + \sigma_{jj}^*\right)\right]. \tag{4.14}$$

The boundary conditions (2.7)–(2.8) become

$$\sigma_{3i}^*\Big|_{\zeta=\pm 1} = 0, \tag{4.15}$$

$$\sigma_{33}^*\Big|_{\zeta=1} = \varepsilon^2 P^*, \qquad \sigma_{33}^*\Big|_{\zeta=-1} = \frac{1}{2(1+\nu)} v_3^*. \tag{4.16}$$

The starred unknown quantities will now have a different expansion than (3.13) and is given by

$$f^* = f^{(0)} + \varepsilon^2 f^{(1)} + \varepsilon^4 f^{(2)} + \cdots . \tag{4.17}$$

First, equations (4.14) and (4.13) at leading order result in the same as above displacement components given, respectively, by (3.14) and (3.15). Then, from equation (4.9), we write down the solvability condition in the form

$$\int_{-1}^{1} \frac{\partial \sigma_{33}^{(0)}}{\partial \zeta}\, d\zeta = -\frac{1}{4(1+\nu)}\int_{-1}^{1} v_3^{(0)}\, d\zeta, \tag{4.18}$$

which is satisfied identically due to boundary conditions (4.16) taken at leading order. It also follows from (4.9), taking into account (4.18), that

$$\sigma_{33}^{(0)} = \frac{1-\zeta}{4(1+\nu)} V_3^{(0)}. \tag{4.19}$$

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We continue by inserting (3.15) and (4.19) into (4.11) and (4.12) to obtain, respectively,

$$\sigma_{ii}^{(0)} = -\frac{\zeta}{1-\nu^2}\left(\frac{\partial^2 V_3^{(0)}}{\partial \xi_i^2} + \nu \frac{\partial V_3^{(0)}}{\partial \xi_j^2}\right) + \frac{1}{1-\nu^2}\left(\frac{\partial V_i^{(0)}}{\partial \xi_i} + \nu \frac{\partial V_j^{(0)}}{\partial \xi_j}\right) + \frac{(1-\zeta)\nu}{4(1-\nu^2)} V_3^{(0)} \tag{4.20}$$

and

$$\sigma_{ij}^{(0)} = -\frac{\zeta}{1+\nu}\frac{\partial^2 V_3^{(0)}}{\partial \xi_i \partial \xi_j} + \frac{1}{2(1+\nu)}\left(\frac{\partial V_i^{(0)}}{\partial \xi_j} + \frac{\partial V_j^{(0)}}{\partial \xi_i}\right). \tag{4.21}$$

Next, we obtain the solvability condition for the leading order equation following from (4.8), which is written as

$$\int_{-1}^{1} \frac{\partial \sigma_{3i}^{(0)}}{\partial \zeta} d\zeta = -\int_{-1}^{1}\left(\frac{\partial \sigma_{ii}^{(0)}}{\partial \xi_i} + \frac{\partial \sigma_{ij}^{(0)}}{\partial \xi_j} + \frac{1}{4(1+\nu)} v_i^{(0)}\right) d\zeta. \tag{4.22}$$

After substituting (4.20), (4.21) and (3.15) in the latter and satisfying the homogeneous boundary conditions along the faces, see (4.15), we get

$$-\frac{\nu}{4}\frac{\partial V_3^{(0)}}{\partial \xi_i} = \frac{\partial^2 V_i^{(0)}}{\partial \xi_i^2} + \frac{1+\nu}{2}\frac{\partial^2 V_j^{(0)}}{\partial \xi_i \partial \xi_j} + \frac{1-\nu}{2}\frac{\partial^2 V_i^{(0)}}{\partial \xi_j^2} + \frac{1-\nu}{4} V_i^{(0)}. \tag{4.23}$$

It also follows from (4.8) that

$$\sigma_{3i}^{(0)} = \frac{\zeta^2-1}{2(1-\nu^2)}\left(\frac{\partial^3 V_3^{(0)}}{\partial \xi_i^3} + \frac{\partial^3 V_3^{(0)}}{\partial \xi_i \partial \xi_j^2}\right) + \frac{\zeta^2 - 2\nu\zeta - (1-2\nu)}{8(1-\nu^2)}\frac{\partial V_3^{(0)}}{\partial \xi_i} - \\ - \frac{\zeta-1}{1-\nu^2}\left(\frac{\partial^2 V_i^{(0)}}{\partial \xi_i^2} + \frac{1+\nu}{2}\frac{\partial^2 V_j^{(0)}}{\partial \xi_i \partial \xi_j} + \frac{1-\nu}{2}\frac{\partial^2 V_i}{\partial \xi_j^2}\right) - \frac{\zeta-1}{4(1+\nu)} V_i^{(0)}. \tag{4.24}$$

Now, we have from (4.14), at first order,

$$\frac{\partial v_3^{(1)}}{\partial \zeta} = \sigma_{33}^{(0)} - \nu\left(\sigma_{ii}^{(0)} + \sigma_{jj}^{(0)}\right).$$

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Then, inserting the stresses from (4.19) and (4.20) into the right-hand side of the last formula and integrating with respect to ζ, we obtain

$$v_3^{(1)} = \frac{\zeta^2\nu}{2(1-\nu)}\Delta V_3^{(0)} - \frac{(\zeta^2-2\zeta)(1-2\nu)}{8(1-\nu)}V_3^{(0)} - \frac{\nu\zeta}{1-\nu}\left(\frac{\partial V_i^{(0)}}{\partial \xi_i} + \frac{\partial V_j^{(0)}}{\partial \xi_j}\right) + V_3^{(1)}. \tag{4.25}$$

Similarly, we deduce from (4.9) at first order that

$$\int_{-1}^{1} \frac{\partial \sigma_{33}^{(1)}}{\partial \zeta}\, d\zeta = -\int_{-1}^{1}\left(\frac{\partial \sigma_{3i}^{(0)}}{\partial \xi_i} + \frac{\partial \sigma_{3j}^{(0)}}{\partial \xi_j}\right) d\zeta + \frac{1}{2(1+\nu)}\int_{-1}^{1} e_3^{(0)}\, d\zeta - \frac{1}{4(1+\nu)}\int_{-1}^{1} v_3^{(1)}\, d\zeta. \tag{4.26}$$

Then, taking into account formulae (4.24) and (4.25) along with the boundary conditions (4.16), we arrive at

$$\begin{aligned} e_3^{(0)} = &-\frac{2}{3(1-\nu)}\Delta^2 V_3^{(0)} - \frac{1-2\nu}{6(1-\nu)}\Delta V_3^{(0)} + \frac{1-2\nu}{6(1-\nu)}V_3^{(0)} \\ &+ \frac{2}{1-\nu}\left(\frac{\partial^3 V_i^{(0)}}{\partial \xi_i^3} + \frac{\partial^3 V_j^{(0)}}{\partial \xi_i^2 \partial \xi_j} + \frac{\partial^3 V_i^{(0)}}{\partial \xi_i \partial \xi_j^2} + \frac{\partial^3 V_j^{(0)}}{\partial \xi_j^3}\right) \\ &+ \frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial V_i^{(0)}}{\partial \xi_i} + \frac{\partial V_j^{(0)}}{\partial \xi_j}\right) + (1+\nu)P^* \end{aligned} \tag{4.27}$$

leading to

$$\begin{aligned} \sigma_{33}^{(1)} = &-\frac{\zeta^3-3\zeta+2}{6(1-\nu^2)}\Delta^2 V_3^{(0)} - \frac{\zeta^3(1+\nu)-3\nu\zeta^2-3\zeta(1-2\nu)+2(1-2\nu)}{24(1-\nu^2)}\Delta V_3^{(0)} \\ &+ \frac{\zeta-1}{2(1+\nu)}e_3^{(0)} - \frac{(\zeta^3-3\zeta+2)(1-2\nu)}{96(1-\nu^2)}V_3^{(0)} - \frac{\zeta-1}{4(1+\nu)}V_3^{(1)} \\ &+ \frac{(\zeta-1)^2}{2(1-\nu^2)}\left(\frac{\partial^3 V_i^{(0)}}{\partial \xi_i^3} + \frac{\partial^3 V_j^{(0)}}{\partial \xi_i^2 \partial \xi_j} + \frac{\partial^3 V_i^{(0)}}{\partial \xi_i \partial \xi_j^2} + \frac{\partial^3 V_j^{(0)}}{\partial \xi_j^3}\right) \\ &+ \frac{\zeta^2-2\zeta(1-\nu)+(1-2\nu)}{8(1-\nu^2)}\left(\frac{\partial V_i^{(0)}}{\partial \xi_i} + \frac{\partial V_j^{(0)}}{\partial \xi_j}\right) + P^*. \end{aligned} \tag{4.28}$$

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Finally, expressing the left-hand side in (4.27) in terms of $V_3^{(0)}$ we have

$$\frac{\partial^2 V_3^{(0)}}{\partial \tau^2} + \frac{1}{2} V_3^{(0)} = \varepsilon^2 \Bigg(- \frac{2}{3(1-\nu)} \Delta^2 V_3^{(0)} - \frac{1-2\nu}{6(1-\nu)} \Delta V_3^{(0)} + \frac{1-2\nu}{6(1-\nu)} V_3^{(0)} + \\ + \frac{2}{1-\nu} \left(\frac{\partial^3 V_i^{(0)}}{\partial \xi_i^3} + \frac{\partial^3 V_j^{(0)}}{\partial \xi_i^2 \partial \xi_j} + \frac{\partial^3 V_i^{(0)}}{\partial \xi_i \partial \xi_j^2} + \frac{\partial^3 V_j^{(0)}}{\partial \xi_j^3} \right) \\ + \frac{1-2\nu}{2(1-\nu)} \left(\frac{\partial V_i^{(0)}}{\partial \xi_i} + \frac{\partial V_j^{(0)}}{\partial \xi_j} \right) + (1+\nu) P^* \Bigg). \tag{4.29}$$

5. Discussion

Let us present the equations of motion derived in the previous section in terms of the sought for mid-plane displacements presented in the form $u_i = h\varepsilon \left(V_i^{(0)} + \varepsilon^2 V_i^{(1)} \right)$ and $w = h \left(V_3^{(0)} + \varepsilon^2 V_3^{(1)} \right)$. In terms of the original variables equations (4.23) and (4.29), respectively, become

$$-\frac{\nu\theta}{2E} \operatorname{grad} w = \frac{1-\nu}{2(1+\nu)} \Delta \mathbf{u} + \frac{1}{2} \operatorname{grad} \operatorname{div} \mathbf{u} + \frac{(1-\nu)\theta}{2Eh} \mathbf{u} \tag{5.1}$$

and

$$\frac{2Eh^3}{3(1-\nu^2)} \Delta^2 w + \frac{1-2\nu}{3(1-\nu)} \theta h^2 \Delta w + 2\rho h \frac{\partial^2 w}{\partial t^2} + \theta \left(1 - \frac{2(1+\nu)(1-2\nu)}{3(1-\nu)} \frac{\theta h}{E} \right) w \\ - \frac{2Eh^2}{1-\nu^2} \Delta \operatorname{div} \mathbf{u} - \frac{1-2\nu}{1-\nu} \theta h \operatorname{div} \mathbf{u} = P, \tag{5.2}$$

where $\mathbf{u} = (u_1, u_2)$ and all the 2D operators are specified on the mid-plane.

In the one-dimensional case ($u_2 = 0$, $\partial/\partial\alpha_2 = 0$) equation (5.1) and (5.2) become

$$\frac{\nu(1+\nu)\theta}{2E} \frac{\partial w}{\partial \alpha_1} = - \left(\frac{(1-\nu^2)\theta}{2Eh} + \frac{\partial^2}{\partial \alpha_1^2} \right) u_1 \tag{5.3}$$

and

$$\frac{2Eh^3}{3(1-\nu^2)} \frac{\partial^4 w}{\partial \alpha_1^4} + \frac{1-2\nu}{3(1-\nu)} \theta h^2 \frac{\partial^2 w}{\partial \alpha_1^2} + 2\rho h \frac{\partial^2 w}{\partial t^2} + \theta \left(1 - \frac{2(1+\nu)(1-2\nu)}{3(1-\nu)} \frac{\theta h}{E} \right) w \\ - \frac{2Eh^2}{1-\nu^2} \frac{\partial^3 u_1}{\partial \alpha_1^3} - \frac{1-2\nu}{1-\nu} \theta h \frac{\partial u_1}{\partial \alpha_1} = P. \tag{5.4}$$

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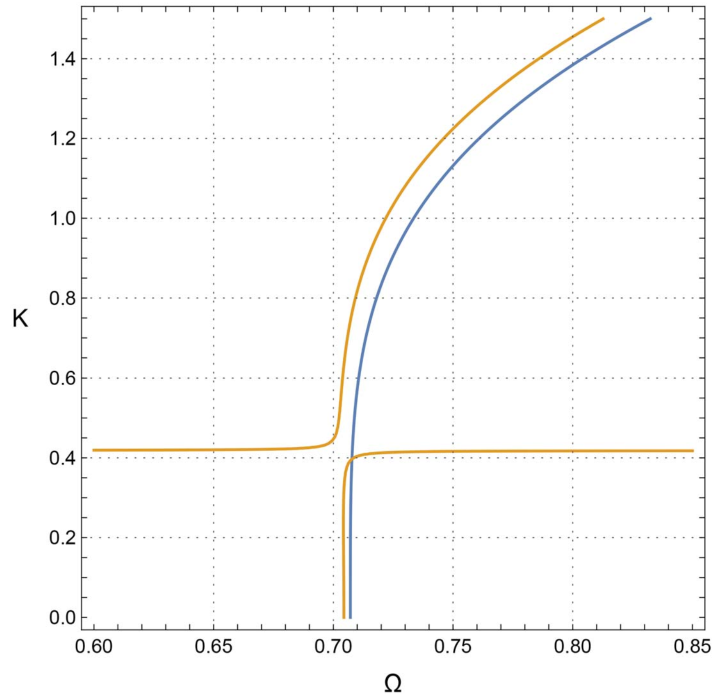

FIG. 2. Comparison of the conventional dispersion relation (4.1) (blue line) against the near cut-off one (5.6) (orange line) with $\varepsilon = 0.2$ and $\nu = 0.3$.

Let us now apply the operator $(h^2/\varepsilon^2)\partial^2/\partial\alpha_1^2 + (1-\nu)/4$ to equation (5.4). Then, expressing the displacement u_1 through (5.3), we arrive at

$$\frac{4Eh^3}{3(1-\nu^2)}\left(1+\frac{Eh}{(1+\nu)\theta}\frac{\partial^2}{\partial\alpha_1^2}\right)\frac{\partial^4 w}{\partial\alpha_1^4}+\left(\frac{2Eh}{1+\nu}-\frac{(1-2\nu)(3-2\nu)}{3(1-\nu)}\theta h^2\right)\frac{\partial^2 w}{\partial\alpha_1^2}$$
$$+(1-\nu)\theta\left(1-\frac{2(1+\nu)(1-2\nu)}{3(1-\nu)}\frac{\theta h}{E}\right)w+2\rho h(1-\nu)\left(1+\frac{2Eh}{(1-\nu^2)\theta}\frac{\partial^2}{\partial\alpha_1^2}\right)\frac{\partial^2 w}{\partial t^2}$$
$$=(1-\nu)\left(1+\frac{2Eh}{(1-\nu^2)\theta}\frac{\partial^2}{\partial\alpha_1^2}\right)P. \tag{5.5}$$

We derive the dispersion relation for the homogeneous equation (5.5) ($P = 0$) setting $w = e^{i(k\alpha_1-\omega t)}$, which results in

$$\left(\Omega^2-\frac{1}{2}\right)\left(K^2-\frac{1-\nu}{4}\right)=\frac{2}{3(1-\nu)}\varepsilon^2\left(K^6-\frac{1}{2}K^4-\frac{(1-2\nu)(3-2\nu)}{16}K^2+\frac{(1-\nu)(1-2\nu)}{16}\right), \tag{5.6}$$

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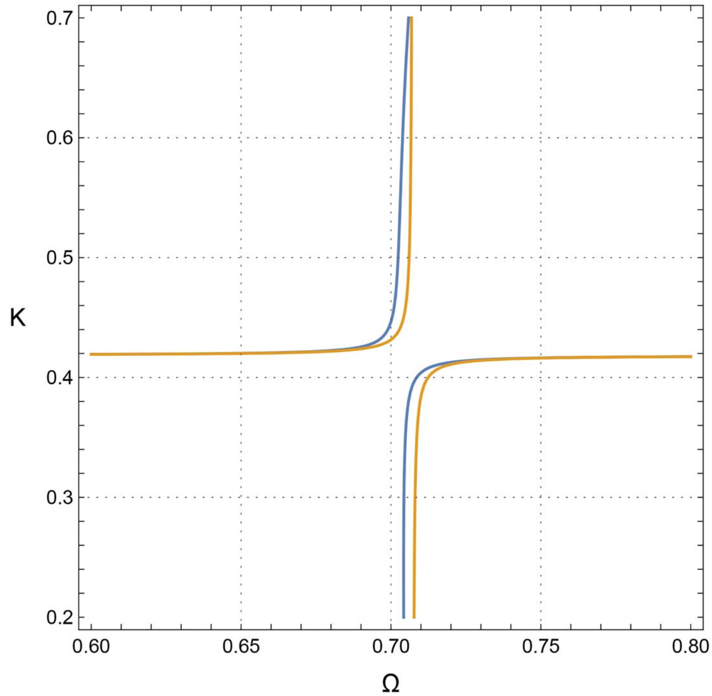

FIG. 3. Comparison of the dispersion relation (5.6) against its approximation around $K = \sqrt{1-\nu}/2$ with $\varepsilon = 0.2$ and $\nu = 0.3$.

where $\varOmega$ and K are same as above in equation (4.2). Remark that the last formula is a near cut-off shortened form of a more general dispersion relation presented in Erbaş *et al.* (2018).

This formula gives

$$P(K) = \frac{1}{24(1-\nu)} \frac{16K^6 - 8K^4 - (1-2\nu)(3-2\nu)K^2 + (1-\nu)(1-2\nu)}{K^2 - (1-\nu)/4} \tag{5.7}$$

in the near cut-off expansion (4.3). It is remarkable that due to the pole $K = \sqrt{1-\nu}/2$, asymptotic behaviour (4.3) appears to be non-uniform in K. The reason is the veering of bending and extension dispersion curves at the point $\varOmega = 1/\sqrt{2}$ and $K = \sqrt{1-\nu}/2$, for which $K = \sqrt{(1-\nu)/2}\,\varOmega$, see Erbaş *et al.* (2018), and also Mace & Manconi (2012) for greater detail.

Over a narrow vicinity of the veering point, the dispersion relation (5.6) may formally be reduced to a simpler form given by

$$\left(\varOmega^2 - \frac{1}{2}\right)\left(K^2 - \frac{1-\nu}{4}\right) = -\frac{1}{32}\varepsilon^2\nu^2. \tag{5.8}$$

We also note that at $K \gg 1$, the asymptotic behaviours of (5.6) and the traditional dispersion relation of a beam on a Winkler foundation, see (4.1), coincide. On the other hand, at $K \ll 1$ we derive from (5.6)

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at leading order $\Omega^2 = 1/2 - \dfrac{(1-2\nu)}{3(1-\nu)}\varepsilon^2$. This gives $O(\varepsilon^2)$ correction to the cut-off value $\Omega^2 = 1/2$ predicted by traditional theory, see (4.1). Numerical illustrations are presented in Figures 2 and 3 for $\epsilon = 0.2$ and $\nu = 0.3$.

6. Concluding Remarks

It is shown that outside the vicinity of the cut-off frequency $\Omega \approx 1/\sqrt{2}$ in Section 4, the traditional model of a Kirchhoff plate resting on a Winkler foundation has an asymptotic justification. The adapted scaling, see (3.1)–(3.4), is identical to that for bending vibrations of a plate with traction-free faces in Kaplunov *et. al.* (1998). Obviously, this scaling is also characteristic of high-order approximations similar to those in Erbaş *et al.* (2022); however, the latter are out of the scope of this paper.

An alternative approximation, (5.1) and (5.2), governing a near cut-off behaviour can hardly be deduced using physical hypothesis, outside the asymptotic framework; see also the scalar one-dimensional equation (5.5). It accommodates a non-trivial interaction of bending and extension motions including veering of the associated dispersion curves, see also Erbaş *et al.* (2018).

The asymptotic near cut-off dispersion relation takes the form of a fraction, see (4.3) and (5.7), instead of a polynomial typical of a plate with free or fixed faces, e.g. see Kaplunov *et. al.* (1998) and also Rogerson *et al.* (2007), Lashhab *et al.* (2015). In this case, the pole $K = \sqrt{1-\nu}/2$ leads to a non-uniform behaviour.

It might be expected that due to the presence of a foundation a plate with thickness imperfections should support low-frequency trapped modes in the vicinity of the aforementioned cut-off frequency. Until now, the focus has been on high-frequency trapping in thin elastic structures with free or fixed faces, see Tovstik (1992), Gridin *et al.* (2005), Kaplunov *et al.* (2005), Postnova & Craster (2008).

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