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Unambiguous injective morphisms in free groups

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ABSTRACT

A morphism g is ambiguous with respect to a word u if there exists a second morphism $h \neq g$ such that $g(u) = h(u)$. Otherwise g is unambiguous with respect to u. Thus unambiguous morphisms are those for which the structure of the morphism is preserved in the image. Ambiguity has so far been studied for morphisms of free monoids, where several characterisations exist for the set of words u permitting an (injective) unambiguous morphism. In the present paper, we consider ambiguity of morphisms of free groups, and consider possible analogies to the existing characterisations in the free monoid. While a direct generalisation results in a trivial situation where all morphisms are ambiguous, we discuss some natural and well-motivated reformulations, and provide a characterisation of words in a free group that permit a morphism which is "as unambiguous as possible".

1. Introduction

A morphism g is *ambiguous* with respect to a word u if there exists a morphism $h \neq g$ such that $g(u) = h(u)$. For example the morphism $g : \{\mathtt{a}, \mathtt{b}\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ such that $g(\mathtt{a}) = \mathtt{aba}$ and $g(\mathtt{b}) = \mathtt{b}$ is ambiguous with respect to $\mathtt{abba}$, as the same result may be achieved with the morphism h given by $h(\mathtt{a}) = \mathtt{a}$ and $h(\mathtt{b}) = \mathtt{bab}$. Indeed, we have $g(\mathtt{abba}) = h(\mathtt{abba}) = \mathtt{ababbaba}$. On the other hand, the (identity) morphism $g : \{\mathtt{a}, \mathtt{b}\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ given by $g(\mathtt{a}) = \mathtt{a}$ and $g(\mathtt{b}) = \mathtt{b}$ is unambiguous with respect to $u = \mathtt{abba}$, since no other morphism $g : \{\mathtt{a}, \mathtt{b}\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ produces the same image when applied to u. However, if instead of the monoid $\{\mathtt{a}, \mathtt{b}\}^*$, we consider the free group $\mathcal{F}_{\{\mathtt{a},\mathtt{b}\}}$ generated by $\mathtt{a}$ and $\mathtt{b}$, then the identity morphism $g : \mathcal{F}_{\{\mathtt{a},\mathtt{b}\}} \to \mathcal{F}_{\{\mathtt{a},\mathtt{b}\}}$ becomes ambiguous with respect to $u = \mathtt{abba}$, as verified by, e.g., the morphism $h : \mathcal{F}_{\{\mathtt{a},\mathtt{b}\}} \to \mathcal{F}_{\{\mathtt{a},\mathtt{b}\}}$ given by $h(\mathtt{a}) = \mathtt{abbab}^{-2}\mathtt{a}^{-1}$ and $h(\mathtt{b}) = \mathtt{abbaba}^{-1}\mathtt{b}^{-2}\mathtt{a}^{-1}$. Thus, we see that by moving from the free monoid to the free group, the ambiguity of morphisms can change.

The ambiguity of morphisms can be seen as both a property of the pair of words $(u, g(u))$, and of the morphism itself. Put another way, it provides some measure of (non-)determinism in the process of mapping u to $g(u)$. Similarly to injectivity, the unambiguity of a morphism determines to some extent the information lost when the morphism is applied, and indeed unambiguity can be seen as a dual to injectivity. To substantiate this claim, consider two words u, v and a morphism g such that $g(u) = v$. It is possible to determine u from v and g whenever g is injective. Similarly, it is possible to determine g from u and v whenever g is unambiguous with respect to u. Of course, the final configuration, that v may be determined by u and g, occurs when g is a function.

Previous research addressing the topic of ambiguity directly considers only morphisms of the free monoid, and is surprisingly recent (see e.g., Freydenberger et al. [6], Freydenberger, Reidenbach [5]), although earlier topics such as the Dual Post

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Correspondence Problem (Culik II, Karhumäki [2]) have addressed ambiguity of morphisms indirectly. Due to its fundamental combinatorial nature, the ambiguity of morphisms has implications on many areas such as equality sets (Salomaa [19], Engelfriet, Rozenberg [4], Harju, Karhumäki [7]), the Post Correspondence Problem (Post [16]), and word equations (see, e.g., Lothaire [13]); however, arguably the biggest achievements of research on the ambiguity of morphisms have come in the world of pattern languages (Reidenbach [17]).

In the current work, we extend the study of ambiguity to morphisms of the free group. While ambiguity and indeed pattern languages have generally been studied in the context of a free monoid, or semigroup, many of these ideas overlap with areas of study related to free groups, in which morphisms play a central role. Recent publications have addressed pattern languages in a group setting (Jain et al. [9]), and a group-equivalent of the Post Correspondence Problem (Bell, Potapov [1]), while more established areas include test words for automorphisms (Turner [21]), automorphisms themselves and their fixed subgroups (see e.g., Ventura [22]), equations of the free group (Makanin [15]), and equalisers – all of which are connected to the ambiguity of morphisms.

We shall address in particular the question of whether a given word in a free group permits an unambiguous injective morphism, with particular emphasis on establishing analogies to existing characterisations of such words in the free monoid (cf. Theorem 3). Our first observation is that, due to the existence of non-trivial inner automorphisms, all morphisms are ambiguous with respect to all words in a free group. Of course this appears to be bad news for the concept of ambiguity in a free group and even contradictory to our claim that ambiguity is related to existing research in combinatorial group theory. However, on closer inspection, we show that provided a particular construction based on composition with inner automorphisms is disregarded, we once again have unambiguous morphisms.

Since inner automorphisms are particularly closely related – both combinatorially, and algebraically – to the identity morphism, a given morphism and its composition with an inner automorphism are also very closely related. In terms of the structure preserved, we see that the "unambiguous" morphic images in this context preserve the structure of the morphism up to composition with inner automorphisms, which is demonstrably maximal. We say such morphisms are unambiguous up to inner automorphism. Since the only inner automorphism in a free monoid is the identity morphism, our definition can be considered a direct generalisation. Similarly, since automorphisms are a superset of the inner automorphisms which are also closely related to the identity morphism and of wide interest, we define the slightly weaker notion of unambiguity up to automorphism in the same way.

Our main result is a characterisation of words in a free group for which there exists an injective morphism that is unambiguous up to inner automorphism in terms of fixed points of morphisms, replicating an existing result for words in the free monoid. The proof is purely combinatorial, and in the case that such a morphism exists, provides an explicit construction. The ideas of the proof are also sufficient to reduce the equivalent statement for ambiguity up to automorphism to a conjecture on certain non-trivial fixed points presented in Section 4.2.

We show that this (potential) characterisation for unambiguity up to automorphism is also equivalent to a natural generalisation of the notion of morphic primitivity in a free monoid, and hence that, subject to the correctness of our conjecture, two existing characterisations in the free monoid also hold for unambiguity up to automorphism. Interestingly, the second (potential) characterisation does not hold when considering unambiguity up to inner automorphism, and so in this sense, we see that unambiguity up to automorphism seems to adhere more closely to unambiguity in a free monoid.

The rest of the paper is organised as follows. Firstly, we provide some necessary preliminary definitions and observations in Section 2. In Section 3 we consider generalisations of ambiguity and morphic primitivity. In Section 4 we present and prove our main result(s). Finally, in Section 5, we exploit our constructions from Section 4 to provide some simple proofs of properties of pattern languages over a group alphabet.

2. Preliminaries

An *alphabet* is a set of symbols, called *letters*. A *word* over an alphabet Σ is a string/sequence of letters from Σ, so that, for example, $\mathtt{abaaba}$ is a word over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. The set of letters occurring in a word u is $\mathrm{symb}(u)$. We shall generally use Σ to refer to the specific alphabet $\{\mathtt{a}, \mathtt{b}\}$ unless explicitly stated otherwise. For two words u, v we define the operation *concatenation* $(\cdot)$ such that $u \cdot v = uv$. Hence a word is simply a concatenation of letters from a given alphabet. We shall generally omit the $\cdot$ symbol, and use it only when needed to avoid confusion (so for example when considering words over the alphabet $\mathbb{N}$, so we can distinguish between, e.g., $1 \cdot 1 \cdot 2$ and $11 \cdot 2$). The length of a word u is the number of its letters, and denoted by $|u|$ so that, e.g., $|\mathtt{abaaba}| = 6$. The word of length 0 is called the *empty word* and is denoted by ε. Hence the set of all words over a given alphabet X (including ε) forms a free monoid under the operation of concatenation, which we denote by X^*. Likewise, $X^* \backslash \{\varepsilon\}$ is a free semigroup, which we denote by X^+.

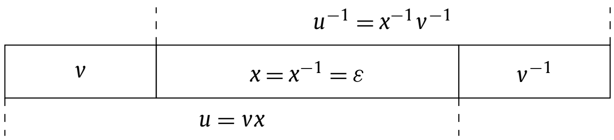
For an alphabet $X = \{\mathtt{a}_1, \mathtt{a}_2, \ldots, \mathtt{a}_n\}$, we define its *inverse* X^{-1} to be an alphabet $\{\mathtt{a}_1^{-1}, \mathtt{a}_2^{-1}, \ldots, \mathtt{a}_n^{-1}\}$ such that $X \cap X^{-1} = \emptyset$. The free monoid $(X \cup X^{-1})^*$, along with additional relations $\mathtt{a}_i \mathtt{a}_i^{-1} = \mathtt{a}_i^{-1} \mathtt{a}_i = \varepsilon$ for each $\mathtt{a}_i \in X$, forms the free group $\mathcal{F}_X$. For example, for the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, we have $\Sigma^{-1} = \{\mathtt{a}^{-1}, \mathtt{b}^{-1}\}$, and e.g., words $u_1 = \mathtt{ba}^{-1}\mathtt{bb}^{-1}$, $u_2 = \mathtt{ba}^{-1}$, and $u_3 = \mathtt{aba}^{-1}\mathtt{b}^{-1}$ are all words belonging to both the free monoid $\{\mathtt{a}, \mathtt{b}, \mathtt{a}^{-1}, \mathtt{b}^{-1}\}^*$ and the free group $\mathcal{F}_\Sigma$, but while u_1 and u_2 are not graphically equal (i.e. they appear differently when written on the page) and therefore not equal in the free monoid, they *are* equivalent in the free group since $\mathtt{bb}^{-1} = \varepsilon$ in $\mathcal{F}_\Sigma$, so $u_1 = \mathtt{ba}^{-1} \cdot \varepsilon = \mathtt{ba}^{-1} = u_2$. In order to avoid confusion, when dealing with words in a free group, we will always assume that we consider the more general equality (i.e., equality in context of the free group) unless specifically stated.[1]

For an alphabet X, let $u \in \mathcal{F}_X$ be the word $u = \mathtt{a}_1^{p_1}\mathtt{a}_2^{p_2}\cdots\mathtt{a}_n^{p_n}$ where $\mathtt{a}_i \in X$ and $p_i \in \{1,-1\}$ for $1 \le i \le n$. The *inverse* of u in $\mathcal{F}_X$ is the word $u^{-1} = \mathtt{a}_n^{-p_n}\mathtt{a}_{n-1}^{-p_{n-1}}\cdots\mathtt{a}_1^{-p_1}$, so that $uu^{-1} = u^{-1}u = \varepsilon$, and it is unique (up to equivalence in the associated free group). Moreover u is *reduced* if $\mathtt{a}_i^{p_i}\mathtt{a}_{i+1}^{p_{i+1}} \neq \varepsilon$ for all $i, 1 \le i < n$. Otherwise u is *unreduced*. For two reduced words $u, v \in \mathcal{F}_X$, $u = v$ if and only if u and v are graphically equal (i.e., equal in the free monoid). For every word $u \in \mathcal{F}_X$, there exists a unique reduced word v such that $u = v$. Two words in a free group are equal if and only if their respective reduced words are equal. An example of two words which are equal in the free group, with their (identical) reduced versions underlined is given below.

$$\underline{\mathtt{a\,a\,b}}\,\mathtt{a\,a^{-1}}\,\underline{\mathtt{b}}\,\mathtt{b\,a^{-1}\,a\,b^{-1}}\,\underline{\mathtt{b}} = \underline{\mathtt{a}}\,\mathtt{b\,b^{-1}}\,\underline{\mathtt{a\,b\,b\,b}} = \underline{\mathtt{a\,a\,b\,b\,b}}$$

The result of concatenating n occurrences of a single word u is denoted by u^n, and a word v is *primitive* if $v = u^n$ implies that $n \in \{1,-1\}$. The word u is a *primitive root* of v if $v = u^n$ for some $n \in \mathbb{Z}$ and u is primitive. The primitive root is unique (up to inverse) for all words except ε. If, for words u, v, w, x, $u = vwx$ (graphical equality), then w is a *factor* of u. It is a *proper* factor if $u \neq w$. If $v = \varepsilon$ then w is a *prefix* and if $x = \varepsilon$ then w is a *suffix* of u. If $u = vwv$ (graphic equality) for some words v and w with $v \neq \varepsilon$, it is *bordered*. Moreover, we say that the factor w *occurs* in u, and we may refer to specific (e.g., leftmost, rightmost, all, etc.) *occurrences* of w in u. For example there are 3 occurrences of the factor $\mathtt{ab}$ in the word $\underline{\mathtt{ab}}\mathtt{ababa}$ and the leftmost occurrence is underlined.

When we refer to occurrences of a factor u in a word in the free group, we mean occurrences of both u and u^{-1}. If we wish to distinguish between the two, we call the former *positive* occurrences and the latter *negative* occurrences. By $|u|_v$, we shall mean the number of positive occurrences of v in u minus the number of negative occurrences, and we shall refer to $|u|_v$ as the *balance* of v in u. For a word $u = \mathtt{a}_1\mathtt{a}_2\ldots\mathtt{a}_n$, two factors $v = \mathtt{a}_i\ldots\mathtt{a}_j$ and $w = \mathtt{a}_k\ldots\mathtt{a}_\ell$, for $1 \le i, j, k, \ell \le n$, *partially overlap* if $k < i$ and $i \le \ell < j$ or if $i < k$ and $k \le j < \ell$. It is straightforward to see that, for any non-empty factor u, no positive occurrence overlaps or partially overlaps with a negative occurrence of u. In particular, it follows from the fact that the only word in the free group satisfying $x = x^{-1}$ is ε. Furthermore, if two occurrences of the same factor u do overlap, then u is bordered.



Two words u, v *commute* if $uv = vu$. The following result is generally regarded as folklore (cf. e.g., [20,12]).

Lemma 1. *Let u, v be words. Then the following conditions are equivalent:*

1. *u and v satisfy a non-trivial equation,*
2. *u and v commute, and*
3. *u, v have the same primitive root.*[2]

A *contraction* is a non-empty factor which is equal to ε. For example, if $u = \mathtt{aabb^{-1}a^{-1}abb^{-1}a^{-1}}$, then *all* the contractions occurring in u are as follows: $\mathtt{bb^{-1}}$ (twice), $\mathtt{a^{-1}a}$, $\mathtt{abb^{-1}a^{-1}}$ (twice), $\mathtt{bb^{-1}a^{-1}a}$, $\mathtt{b^{-1}a^{-1}ab}$, $\mathtt{a^{-1}abb^{-1}}$, $\mathtt{bb^{-1}a^{-1}abb^{-1}}$ and $\mathtt{abb^{-1}a^{-1}abb^{-1}a^{-1}}$. We highlight some examples below.

$$\mathtt{a\;a}\;\overbrace{\mathtt{b\;b^{-1}}}^{\varepsilon}\;\underbrace{\mathtt{a^{-1}\;a\;b\;b^{-1}}}_{\varepsilon}\;\mathtt{a^{-1}}$$

Note that a word is reduced if and only if it contains no contractions. Let X be an alphabet and let $u = \mathtt{a}_1^{p_1}\mathtt{a}_2^{p_2}\cdots\mathtt{a}_n^{p_n}$ where $\mathtt{a}_i \in X$ and $p_i \in \{1,-1\}$ for $1 \le i \le n$. If the factor $v = \mathtt{a}_i^{p_i}\cdots\mathtt{a}_j^{p_j}$ is a contraction, and either $i = 1$, $j = n$, or $\mathtt{a}_{i-1}^{p_{i-1}}\mathtt{a}_{j+1}^{p_{j+1}} \neq \varepsilon$, then v is a *maximal* contraction. The maximal contractions of $u = \mathtt{aabb^{-1}a^{-1}abb^{-1}a^{-1}}$ are: $\mathtt{abb^{-1}a^{-1}}$ (twice), $\mathtt{bb^{-1}a^{-1}a}$, $\mathtt{a^{-1}abb^{-1}}$, and $\mathtt{abb^{-1}a^{-1}abb^{-1}a^{-1}}$. We show two examples below.

[1] If we wish instead to express graphical/monoid equality, then usually we will state this explicitly, or by specifying whether a word should be taken as reduced or unreduced as defined in the next paragraph.

[2] For this final statement to hold, we must consider all primitive words to be primitive roots of the empty word. Note that this fits with our given definition, but is not always the case in the existing literature.

$$\mathtt{a}\ \overbrace{\mathtt{a}\ \mathtt{b}\ \mathtt{b}^{-1}\ \underbrace{\mathtt{a}^{-1}\ \mathtt{a}\ \mathtt{b}\ \mathtt{b}^{-1}}_{\varepsilon}\ \mathtt{a}^{-1}}^{\varepsilon}$$

A *strictly maximal* contraction is a maximal contraction which is not the concatenation of two or more maximal contractions. So, for example $\mathtt{bb}^{-1}\mathtt{a}^{-1}\mathtt{a}$ is not strictly maximal as it is the concatenation of $\mathtt{bb}^{-1}$ and $\mathtt{aa}^{-1}$ which are maximal. On the other hand $u' = \mathtt{aabb}^{-1}\mathtt{a}^{-1}\mathtt{abb}^{-1}\mathtt{a}^{-1}\mathtt{a}^{-1}$ is a strictly maximal contraction.

A *primary* contraction is one which does not have a maximal contraction as a proper factor. For example, the contraction $\mathtt{a}^{-1}\mathtt{abb}^{-1}$ from the example above is not a primary contraction, since e.g., it has the maximal contraction $\mathtt{bb}^{-1}$ as a proper factor, while $\mathtt{abb}^{-1}\mathtt{a}^{-1}$ *is* primary, as the only proper factor which is also a contraction is $\mathtt{bb}^{-1}$, which is not maximal. It is straightforward to see that the reduced version of a word may be obtained by removing a sequence of primary maximal contractions, although the choice of these contractions is not necessarily fixed. For example the word $\mathtt{aa}^{-1}\mathtt{a}$ has two primary maximal contractions: $\mathtt{aa}^{-1}$ and $\mathtt{a}^{-1}\mathtt{a}$, and removing either gives the (same) reduced word $\mathtt{a}$.

For alphabets X, Y and their respective free groups/monoids $\mathcal{A}_X, \mathcal{B}_Y$, a *(homo)morphism* is a mapping $h : \mathcal{A}_X \to \mathcal{B}_Y$ such that, for all $u, v \in \mathcal{A}_X$, $h(uv) = h(u)h(v)$. Hence a morphism preserves the structure of the monoid/group, and is compatible with the associated operation (in our case, concatenation). It follows from this definition that a morphism is fully defined as soon as it is specified for each $x \in X$. Thus we shall usually define morphisms in this manner. If, for a word u and morphism h, $h(u) = u$, then u is a *fixed point* of h, or equivalently, we say that h *fixes* u.

The set $\{y \in \mathcal{B}_Y \mid y = h(x), x \in \mathcal{A}_X\}$ is denoted by $h(\mathcal{A}_X)$. The *composition* of two morphisms $g : \mathcal{A}_X \to \mathcal{B}_Y$, $h : \mathcal{B}_Y \to \mathcal{C}_Z$ is the morphism $h \circ g : \mathcal{A}_X \to \mathcal{C}_Z$ such that $h \circ g(x) = h(g(x))$ for all $x \in X$. For a morphism $h : \mathcal{A}_X \to \mathcal{A}_X$ and $n \in \mathbb{N}$, we define $h^n = \underbrace{h \circ h \circ \ldots \circ h}_{\text{n times}}$. For a subset X' of X, h is *periodic over* X' if there exists some $y \in \mathcal{B}_Y$ such that for every $x \in X'$, $h(x) = y^n$ for some $n \in \mathbb{N}_0$. If $X' = X$ then h is simply *periodic*. By $\mathrm{id}_{\mathcal{A}_X} : \mathcal{A}_X \to \mathcal{A}_X$, we denote the identity morphism on $\mathcal{A}_X$. A bijective morphism $h : \mathcal{A}_X \to \mathcal{A}_X$ is an *automorphism*. It follows from the relationship between the rank of a free group and its (free) generating sets that $h : \mathcal{A}_X \to \mathcal{A}_X$ is an automorphism if and only if $h(\mathcal{A}_X) = \mathcal{A}_X$. In the case of a free group $\mathcal{A}_X$, a word $u \in \mathcal{A}_X$ is called a *test word* if every morphism fixing u is an automorphism.

If there exists some $y \in \mathcal{A}_X$ such that, for every $x \in X$, $h(x) = yxy^{-1}$, then h is an *inner automorphism* generated by y. The only inner automorphism of a free monoid $\mathcal{M}$ is the identity $\mathrm{id}_{\mathcal{M}}$. As a result of Lemma 1, we can infer the following:

Corollary 2. *Let u, v be words in a free group. Then $u = vuv^{-1}$ if and only if u, v share a primitive root. Consequently, for alphabets X, Y, a word $u \in \mathcal{F}_X$, morphism $g : \mathcal{F}_X \to \mathcal{F}_Y$ and inner automorphism $h : \mathcal{F}_X \to \mathcal{F}_X$ generated by $v \in \mathcal{F}_X$, we have $g(u) = g \circ h(u)$ if and only if $g(u)$, $g(v)$ share a primitive root.*

For two morphisms $g, h : \mathcal{A}_X \to \mathcal{B}_Y$, g and h *agree* on a word $u \in \mathcal{A}_X$ if $g(u) = h(u)$. They are *distinct* if there exists $x \in X$ such that $g(x) \neq h(x)$. For a word $u \in \mathcal{A}_X$, and morphism $g : \mathcal{A}_X \to \mathcal{B}_Y$, g is *ambiguous* with respect to u if there exists a morphism $h : \mathcal{A}_X \to \mathcal{B}_Y$ such that g and h agree on u and are distinct. Otherwise g is *unambiguous* with respect to u. If there exists a morphism which is (un)ambiguous with respect to u then we say that u *possesses* an (un)ambiguous morphism.

In order to remain consistent with existing literature on ambiguity of morphisms, we shall borrow some terminology from the theory of pattern languages. In this respect, a *pattern* is simply a word (usually, to which we intend to apply morphisms). The letters of a pattern are called *variables*. Conversely, a word to which we no longer usually intend to apply morphisms is called a *terminal word*, consisting of *terminal symbols* – normally $\mathtt{a}$ and $\mathtt{b}$. We shall generally use $\mathbb{N}$ (or subsets of $\mathbb{N}$) as our set(s) of variables. We denote the set of variables occurring in α by $\mathrm{var}(\alpha)$. The pattern language of a pattern $\alpha \in \mathbb{N}^*$ over the alphabet Σ is the set $L_\Sigma(\alpha) = \{\sigma(\alpha) \mid \sigma : \mathrm{var}(\alpha)^* \to \Sigma^* \text{ is a morphism}\}$. We extend this to patterns in the free group in the natural way: the (group) pattern language of a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$ over the alphabet Σ is the set:

$$L_\Sigma(\alpha) = \{\sigma(\alpha) \mid \sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma \text{ is a morphism}\}.$$

A pattern α with pattern language L is *succinct* if it is the shortest pattern satisfying $L_\Sigma(\alpha) = L$.

We shall often use σ and τ to denote morphisms mapping patterns to terminal symbols, while φ, ψ and ρ shall be used to denote morphisms between patterns. Given patterns α, β and a set of variables $X \subset \mathrm{var}(\alpha)$, if β may be obtained from α by erasing all occurrences of variables in X, then β is a *subpattern* of α. In the free monoid, two patterns α and β are *morphically coincident* if there exist morphisms $\varphi : \mathrm{var}(\alpha)^* \to \mathrm{var}(\beta)^*$ and $\psi : \mathrm{var}(\beta)^* \to \mathrm{var}(\alpha)^*$ such that $\varphi(\alpha) = \beta$ and $\psi(\beta) = \alpha$. A pattern α is *morphically imprimitive* if it is morphically coincident to a strictly shorter pattern β. Otherwise it is *morphically primitive*. We extend the definition of morphic coincidence to free groups in the natural way. On the other hand, we extend the idea of morphic (im)primitivity to a free group in a slightly non-trivial way (cf. Section 3): a pattern $\alpha \in \mathcal{F}_{\mathbb{N}}$ is morphically imprimitive if it is morphically coincident to a pattern $\beta \in \mathcal{F}_{\mathbb{N}}$ such that $|\mathrm{var}(\beta)| < |\mathrm{var}(\alpha)|$. It is not difficult to see that for patterns in the free monoid $\mathbb{N}^*$, these two definitions are equivalent.

Next, we recall the following theorem, which provides several characterisations of those patterns in a free monoid which possess an injective unambiguous morphism,[3] and which forms the starting point of the current paper which we aim to emulate as far as possible in the free group.

Theorem 3 *(Reidenbach, Schneider [18]). Let $\alpha \in \mathbb{N}^+$. The following statements are equivalent:*

(i) α *possesses an injective unambiguous morphism.*
(ii) *The only morphism* $\varphi : \mathrm{var}(\alpha)^* \to \mathrm{var}(\alpha)^*$ *of which* α *is a fixed point is the identity morphism.*
(iii) α *is morphically primitive.*
(iv) α *is succinct.*

Finally, we introduce the following notation for replacing all occurrences of a factor in a word, pattern or morphism which we shall make regular use of in Section 4. Let X, Y be alphabets. Let u, w be words in $\mathcal{F}_X$ such that u is unbordered[4] and let $v \in \mathcal{F}_Y$. Denote by $R[u \to v](w)$ the word obtained by replacing all occurrences of u in w with v. We replace both positive and negative occurrences so that each factor u is replaced with v and each factor u^{-1} is replaced with v^{-1}. For example, we have $R[\mathtt{a} \to \mathtt{b}](\mathtt{abba}^{-1}) = \mathtt{bbbb}^{-1} = \mathtt{bb}$.

For a set of variables Z and a morphism $\sigma : \mathcal{F}_Z \to \mathcal{F}_X$, define the morphism $R[u \to v](\sigma) : \mathcal{F}_Z \to \mathcal{F}_{X \cup Y}$ such that $R[u \to v](\sigma)(x) = R[u \to v]\sigma(x)$ for each $x \in Z$. For example, for the morphism $\sigma : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ given by $\sigma(1) = \mathtt{ab}$ and $\sigma(2) = \mathtt{aab}$, we have that $R[\mathtt{a} \to \mathtt{aa}](\sigma)$ is the morphism $\sigma' : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ such that $\sigma'(1) = \mathtt{aab}$ and $\sigma'(2) = \mathtt{a}^4\mathtt{b}$.

3. Ambiguity and morphic primitivity in a free group

Before we can consider an analogue of Theorem 3, we must first address the task of generalising the notions of ambiguity and morphic primitivity to free groups, which requires some closer attention. While these definitions can be applied in a straightforward way in the free group setting – simply replacing words and morphisms in a free monoid with their free group counterparts – this is not necessarily the most appropriate approach as will become clear from the remainder of the section. We shall also not consider succinctness of patterns in a free group, due to the lack of a clear suitable equivalent definition or generalisation.

We begin with ambiguity, and the observation that, when considering the most straightforward generalisation, we are left with a trivial situation: that all morphisms are actually ambiguous in the free group.

Theorem 4. *Let $\alpha \in \mathcal{F}_\mathbb{N}$ be a pattern with $|\mathrm{var}(\alpha)| > 1$ and let $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a morphism. Then σ is ambiguous w.r.t. α.*

Proof. Suppose that σ is periodic. We shall construct a morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ with $\tau \neq \sigma$ such that $\tau(\alpha) = \sigma(\alpha)$ as follows. Since σ is periodic, there exists a word $w \in \mathcal{F}_\Sigma$ with $w \neq \varepsilon$ such that $\sigma(z) = w^{n_z}$, $n_z \in \mathbb{Z}$, for every $z \in \mathrm{var}(\alpha)$. Let $x, y \in \mathrm{var}(\alpha)$ with $x \neq y$. Let $p = |\alpha|_x$ and $q = |\alpha|_y$. If $p = 0$, then let $\tau : \mathcal{F}_\mathbb{N} \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(x) = w^{n_x+1}$ and $\tau(z) = \sigma(z)$ for all $z \neq x$. The case that $q = 0$ can be treated in the same way. Since $|\alpha|_x = 0$, every extra occurrence of w in $\tau(\alpha)$ will be cancelled out by an extra occurrence of w^{-1} and vice versa. Hence, $\sigma(\alpha) = \tau(\alpha)$, and σ is ambiguous as required. Otherwise, suppose $p \neq 0$ and $q \neq 0$. Let τ be the morphism given by $\tau(x) = w^{n_x+q}$, $\tau(y) = w^{n_y-p}$ and $\tau(z) = \sigma(z)$ otherwise. Let k be the number of occurrences of w in $\sigma(\alpha)$. Then $\tau(\alpha) = w^k w^{p\times q} w^{-q\times p} = w^k = \sigma(\alpha)$, and σ is ambiguous as required.

Assume finally that σ is non-periodic. Let $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the inner automorphism given by $\varphi(x) = \alpha x \alpha^{-1}$ for each $x \in \mathrm{var}(\alpha)$. Then writing $\alpha = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $x_i \in \mathbb{N}$ and $p_i \in \{1, -1\}$ for $1 \leq i \leq n$, we have:

$$
\begin{aligned}
\varphi(\alpha) &= \left(\alpha\, x_1\, \alpha^{-1}\right)^{p_1} \left(\alpha\, x_2\, \alpha^{-1}\right)^{p_2} \cdots \left(\alpha\, x_n\, \alpha^{-1}\right)^{p_n} \\
&= \alpha\, x_1^{p_1}\, \alpha^{-1}\, \alpha\, x_2^{p_2}\, \alpha^{-1} \cdots \alpha\, x_n^{p_n}\, \alpha^{-1} \\
&= \alpha\, x_1^{p_1}\, x_2^{p_2} \cdots x_n^{p_n} \alpha^{-1} \\
&= \alpha\, \alpha\, \alpha^{-1} \\
&= \alpha.
\end{aligned}
$$

Thus $\sigma \circ \varphi(\alpha) = \sigma(\alpha)$. It remains to show that $\sigma \neq \sigma \circ \varphi$. Suppose to the contrary that $\sigma = \sigma \circ \varphi$. Then for each $x \in \mathrm{var}(\alpha)$, $\sigma(x) = \sigma(\alpha x \alpha^{-1})$. Recall from Corollary 2, that this implies $\sigma(x)$ and $\sigma(\alpha)$ share a primitive root. Thus there exists $w \in \mathcal{F}_\Sigma$ such that for every $x \in \mathrm{var}(\alpha)$, $\sigma(x) = w^n$ for some $n \in \mathbb{Z}$, and σ is periodic, which is a contradiction. Hence we have $\sigma \neq \sigma \circ \varphi$ and σ is ambiguous. □

[3] The theorem is actually given in a slightly stronger form which considers non-erasing morphisms (those which do not map any variable to the empty word).

[4] By requiring that the factor u to be replaced is unbordered, no two occurrences can overlap, and thus $R[u \to v]$ is a well-defined function.

The proof of Theorem 4 reveals two distinct causes of ambiguity. The first is periodicity of the morphism, while the second is a specific construction involving composition with inner automorphisms. For periodic morphisms this fits perfectly well with our intuition: the images of periodic morphisms have no distinguishing structural features to restrict the possible alternatives. This is not true of those morphisms which are ambiguous only because of composition with inner automorphisms, however, and closer inspection reveals a structure which is combinatorially trivial. In fact, the second morphism τ is not only combinatorially very close to the original, but algebraically so as well. Thus it makes sense to disregard this particular structure, and instead to consider morphisms which have some combinatorially (or algebraically) significant difference and yet still produce the same morphic image. With this in mind, we define ambiguity up to inner automorphism and ambiguity up to automorphism in the following manner.

Definition 5. Let Δ_1, Δ_2 be alphabets and let $\alpha \in \mathcal{F}_{\Delta_1}$ be a pattern. Let $\sigma : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ be a morphism. Then σ is *unambiguous up to (inner) automorphism* w.r.t. α if, for every morphism $\tau : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_2}$ with $\tau(\alpha) = \sigma(\alpha)$, there exists an (inner) automorphism $\varphi : \mathcal{F}_{\Delta_1} \to \mathcal{F}_{\Delta_1}$ such that $\tau = \sigma \circ \varphi$. Otherwise, σ is *ambiguous up to (inner) automorphism* w.r.t. α.

Of course, Definition 5 actually gives two types of unambiguity: up to inner automorphism and up to automorphism. The former, as discussed above, is essentially the least restrictive definition possible which permits unambiguous non-periodic morphisms. It can be inferred from results on so-called C-test words (cf. Ivanov [8], Lee [11]) that our definition is indeed non-trivial, and in fact, there exist patterns for which all non-periodic morphisms are unambiguous up to automorphism.

Proposition 6 *(Ivanov [8], Lee [11]). For every finite set of variables Δ there exists a pattern α with $\mathrm{var}(\alpha) = \Delta$ such that if $\sigma(\alpha) = \tau(\alpha)$ for two morphisms $\sigma, \tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$, then either:*

(i) σ and τ are periodic, and $\sigma(\alpha) = \tau(\alpha) = \varepsilon$, or
(ii) $\tau = \sigma \circ \varphi$ for some inner automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$.

Corollary 7. *There exist patterns α over every set of variables such that every non-periodic morphism is unambiguous up to inner automorphism w.r.t. α.*

The second type of unambiguity we introduce, namely unambiguity up to automorphism, is a (strictly) weaker, and therefore more common. Since every inner automorphism is by definition an automorphism, it follows immediately that if a morphism is unambiguous up to inner automorphism then it is unambiguous up to automorphism. It can be seen that the converse does not hold however, as demonstrated by the following Proposition 8, whose proof constructs a (periodic) morphism σ which is unambiguous up to automorphism w.r.t. a pattern α but which is nevertheless ambiguous up to inner automorphism.

Proposition 8. *Every periodic morphism is ambiguous up to inner automorphism w.r.t. every pattern α. However, the same does* not *hold for ambiguity up to automorphism.*

Proof. The statement for periodic morphisms follows from the fact that every periodic morphism is ambiguous (in the strict sense, cf. Theorem 4). It can easily be seen that the second, "witness" morphism cannot be obtained by composition with inner automorphisms in this case and thus for each pattern $\alpha \in \mathcal{F}_\mathbb{N}$ and each periodic morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$, there exists a second morphism $\tau : \mathcal{F}_\mathbb{N} \to \mathcal{F}_\Sigma$ with $\sigma(\alpha) = \tau(\alpha)$ and $\sigma \neq \sigma \circ \varphi$ for any inner automorphism φ. Hence σ is ambiguous up to inner automorphism w.r.t. α. To see that the same statement does not hold for ambiguity up to automorphism, we show that the morphism $\sigma : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ given by $\sigma(1) = \mathtt{a}^{-1}$ and $\sigma(2) = \mathtt{aa}$ is unambiguous up to automorphism w.r.t. the pattern $\alpha = 1 \cdot 1 \cdot 2 \cdot 2$.

To show this, we need the following observations. Firstly, that the set of morphisms $\tau : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ such that $\sigma(\alpha) = \tau(\alpha)$ is given by

$$\mathcal{M} = \{\tau \mid \tau(1), \tau(2) \in \mathcal{F}_{\{\mathtt{a}\}}, \text{ and } |\tau(1)| = -|\tau(2)| + 1\}.$$

We substantiate this claim, firstly by noting that any such morphism τ must be periodic, and furthermore have primitive root $\mathtt{a}$ (due to Lyndon, Schützenberger [14]), and secondly, observing that, given τ has primitive root $\mathtt{a}$, we must have $2|\tau(1)| + 2|\tau(2)| = 2$.

Our second claim is that, for any $k \in \mathbb{Z}$, the morphism $\varphi_k : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\{1,2\}}$ given by $\varphi_k(1) = 2^{-1} \cdot 1^{-k-1}$ and $\varphi_k(2) = 1^k \cdot 2$ is an automorphism. We verify this by observing that $\varphi_k(1^{-1} \cdot 2^{-1}) = 1$ and $\varphi_k((2 \cdot 1)^k \cdot 2) = 2$. Hence the image $\mathcal{F}_{\{1,2\}}$ under φ_k is exactly $\mathcal{F}_{\{1,2\}}$ and thus φ_k is an automorphism.

We can now prove our main statement as follows. Let $\tau \in \mathcal{M}$, and let $k = -|\tau(2)| + 2$. Then we have

$$\begin{aligned} \sigma \circ \varphi_k(1) &= \sigma(2^{-1} \cdot 1^{-k-1}) \\ &= \mathtt{a}^{-2}\mathtt{a}^{-1(-k-1)} \end{aligned} \qquad \begin{aligned} \sigma \circ \varphi_k(2) &= \sigma(1^k \cdot 2) \\ &= \mathtt{a}^{-k}\mathtt{a}^2 \end{aligned}$$

$$\begin{aligned} &= \mathtt{a}^{k-1} && = \mathtt{a}^{-(-|\tau(2)|+2)+2} \\ &= \mathtt{a}^{-|\tau(2)|+1} && = \mathtt{a}^{|\tau(2)|} \\ &= \tau(1), && = \tau(2). \end{aligned}$$

Thus $\sigma \circ \varphi_k = \tau$. So, for any morphism τ such that $\tau(\alpha) = \sigma(\alpha)$, there exists an automorphism φ such that $\sigma \circ \varphi = \tau$ and σ is unambiguous up to automorphism with respect to α. □

We also note the following reasonably straightforward observation, which we shall use again later when considering characterisations of which patterns possess morphisms which are unambiguous up to (inner) automorphism.

Proposition 9. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. If α is fixed by a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism, then all injective morphisms are ambiguous up to automorphism with respect to α. Likewise if α is fixed by a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an inner automorphism, then all injective morphisms are ambiguous up to inner automorphism with respect to α.*

Proof. We prove the statement for ambiguity up to automorphism. The proof for ambiguity up to inner automorphism is a straightforward adaptation. Suppose there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism, and such that $\varphi(\alpha) = \alpha$. Let $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ be an injective morphism. Note that $\sigma \circ \varphi(\alpha) = \sigma(\alpha)$. We shall now show that σ is ambiguous up to automorphism with respect to α by showing that, for $\tau = \sigma \circ \varphi$, there does not exist an automorphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\tau \circ \psi = \sigma$. In particular, suppose to the contrary that $\tau = \sigma \circ \varphi = \sigma \circ \psi$ for some automorphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$. Then due to the injectivity of σ, $\sigma(\psi(x)) = \sigma(\varphi(x))$ for all $x \in \mathrm{var}(\alpha)$ implies that $\psi(x) = \varphi(x)$ for all $x \in \mathrm{var}(\alpha)$, and hence that $\psi = \varphi$. However, since ψ is an automorphism and φ is not, this is a contradiction. □

As with unambiguity, morphic primitivity requires some care when being adapted to the free group. In particular (as with succinctness) the problem stems from the fact that it relies heavily on the length of a word – a property which behaves very differently in the free group. However, we propose the following alternative definition which, in the free monoid, is equivalent to the existing one, and which fits more naturally with the free group, giving a closer analogy.

Definition 10. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ be a pattern. If there exists a pattern $\beta \in \mathcal{F}_{\mathbb{N}}$ with $|\mathrm{var}(\beta)| < |\mathrm{var}(\alpha)|$ and morphisms φ, ψ such that $\varphi(\alpha) = \beta$ and $\psi(\beta) = \alpha$, then α is morphically imprimitive. Otherwise, α is morphically primitive.*

For example, the pattern $\alpha_1 = 1 \cdot 2 \cdot 3 \cdot 1^{-1} \cdot 4 \cdot 4 \cdot 1 \cdot 3 \cdot 2 \cdot 1^{-1}$ is morphically imprimitive, since $\varphi(\alpha_1) = \beta_1$ and $\psi(\beta_1) = \alpha_1$ where $\beta_1 = 2 \cdot 3 \cdot 4 \cdot 4 \cdot 3 \cdot 2$, $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ is the morphism erasing 1 and mapping all other variables to themselves, and $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ is the morphism given by $\psi(x) = 1 \cdot x \cdot 1^{-1}$ for $x \in \{2, 3\}$ and $\psi(4) = 4$. On the other hand, any morphism mapping $\alpha_2 = 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$ to a pattern with fewer (i.e., 1) variables is clearly going to map α_2 to the empty word ε. There exists no morphism mapping ε back to α_2, so α_2 is morphically primitive.

Motivating our new definition, and with regards to our aim of providing analogous results to Theorem 3, we are able to give the following statement which generalises the equivalence of Statements (ii) and (iii) of that theorem to a free group. Note that the only automorphisms in a free monoid are renaming morphisms and moreover that any renaming morphism fixing a pattern α must necessarily act as the identity over $\mathrm{var}(\alpha)$. Consequently, in the free monoid, the statement "the only morphisms $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α are automorphisms" is equivalent to the statement "the only morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α is the identity".

Theorem 11. *Let α be a pattern. Then α is morphically primitive if and only if, for every morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\alpha) = \alpha$, φ is an automorphism.*

Proof. We start by showing that if α is morphically imprimitive, it is fixed by a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism. Let $\beta \in \mathcal{F}_{\mathbb{N}}$ with $|\mathrm{var}(\beta)| < |\mathrm{var}(\alpha)|$, and suppose there exist morphisms $\psi_1 : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ and $\psi_2 : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\psi_1(\alpha) = \beta$ and $\psi_2(\beta) = \alpha$. Clearly the morphism $\psi_2 \circ \psi_1$ fixes α. In order to show $\psi_2 \circ \psi_1$ is not an automorphism of $\mathcal{F}_{\mathrm{var}(\alpha)}$, consider the image $\psi_2 \circ \psi_1(\mathcal{F}_{\mathrm{var}(\alpha)})$. In particular, note that $\psi_1(\mathcal{F}_{\mathrm{var}(\alpha)}) \subseteq \mathcal{F}_{\mathrm{var}(\beta)}$ and hence $\psi_2(\psi_1(\mathcal{F}_{\mathrm{var}(\alpha)})) \subseteq \psi_2(\mathcal{F}_{\mathrm{var}(\beta)}) \subset \mathcal{F}_{\mathrm{var}(\alpha)}$. Thus $\psi_2 \circ \psi_1(\mathcal{F}_{\mathrm{var}(\alpha)}) \neq \mathcal{F}_{\mathrm{var}(\alpha)}$, and $\psi_2 \circ \psi_1$ is not an automorphism.

We now prove that if α is fixed by a morphism which is not an automorphism, then it is morphically imprimitive. The main step is to observe that if α is fixed by a morphism which is not an automorphism, then it is fixed by a morphism which is not injective.

For this part of the proof, we need the definition of a *retract*. For a (not necessarily free) group $\mathcal{G}$, a subgroup $\mathcal{H}$ of $\mathcal{G}$ is a retract if there exists a morphism $\sigma : \mathcal{G} \to \mathcal{G}$ (called a retraction) such that

(1) $\sigma(u) = u$ for every $u \in \mathcal{H}$, and
(2) $\sigma(v) \in \mathcal{H}$ for every $v \in \mathcal{G}$.

Turner [21] showed that if a pattern α is not a test word (which is true if it is fixed by a morphism which is not an automorphism), then it belongs to a proper retract R of $\mathcal{F}_{\text{var}(\alpha)}$. That is, a retract R of $\mathcal{F}_{\text{var}(\alpha)}$ with $R \neq \mathcal{F}_{\text{var}(\alpha)}$. Let $\sigma\, \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\text{var}(\alpha)}$ be the associated retraction. Now, since $R \neq \mathcal{F}_{\text{var}(\alpha)}$, there exists $w \notin R$ for some $w \in \mathcal{F}_{\text{var}(\alpha)}$. By Condition (2), there exists $w' \in R$ such that $\sigma(w) = w'$. By Condition (1), $\sigma(w') = w'$. As $w \notin R$ and $w' \in R$, $w \neq w'$ and thus σ is not injective. Furthermore, by Condition (1), and due to the fact that $\alpha \in R$, we have $\sigma(\alpha) = \alpha$.

In the next step, we need some basic notions from group theory regarding generators and generating sets. A subset $S = \{g_1, g_2, \ldots, g_n\}$ of a group $\mathcal{G}$ is a *generating set* for $\mathcal{G}$ if every element of $\mathcal{G}$ can be written as a product of elements of S. The g_is are called *generators*. S generates $\mathcal{G}$ *freely* if every element of $\mathcal{G}$ can be written uniquely as a product of the generators not containing any trivial products $g_i g_i^{-1}$ or $g_i^{-1} g_i$ (sometimes called a reduced product or reduced word over S). A group is free if and only if it has a set of generators which generate it freely. The *rank* of a group $\mathcal{G}$ is the cardinality of the smallest generating set. A well known fact is that if a set S of generators for a free group $\mathcal{F}$ of rank n generates $\mathcal{F}$ freely, then $|S| = n$.

The Nielsen-Schreier Theorem tells us that $\sigma(\mathcal{F}_{\text{var}(\alpha)})$ is a free group, and thus that there exists a set of generators $\{g_1, g_2, \ldots, g_m\}$ which generate $\sigma(\mathcal{F}_{\text{var}(\alpha)})$ freely. Since $S' = \{\sigma(x) \mid x \in \text{var}(\alpha)\}$ is a generating set for $\sigma(\mathcal{F}_{\text{var}(\alpha)})$, the rank of $\sigma(\mathcal{F}_{\text{var}(\alpha)})$ is at most $|\text{var}(\alpha)|$. However, since σ is not injective, the rank of $\sigma(\mathcal{F}_{\text{var}(\alpha)})$ is strictly less than $|\text{var}(\alpha)|$ and thus $m < |\text{var}(\alpha)|$.

Define a morphism $\rho : \mathcal{F}_{\{g_1, g_2, \ldots, g_m\}} \to \mathcal{F}_{\{1,2,\ldots,n\}}$ such that $\rho(g_i) = i$. Since every word in $\sigma(\mathcal{F}_{\text{var}(\alpha)})$ is a unique product of the generators g_i, we can also define the morphism $\varphi : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\{1,2,\ldots,m\}}$ by $\varphi = \rho \circ \sigma$. Let $\psi : \mathcal{F}_{\{1,2\ldots,m\}} \to \mathcal{F}_{\text{var}(\alpha)}$ be the morphism such that $\psi(i) = g_i$. Let $\beta \in \mathcal{F}_{\{1,2,\ldots,m\}}$ such that $\varphi(\alpha) = \beta$ and note that $\psi(\beta) = \alpha$. Since $m < |\text{var}(\alpha)|$, it follows that α is morphically imprimitive. □

Corollary 12. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Then α is morphically primitive if and only if it is a test word of $\mathcal{F}_{\text{var}(\alpha)}$.*

Corollary 13. *The set of patterns for which there exists an injective morphism which is unambiguous up to inner automorphism is a strict subset of the set of morphically primitive patterns.*

Interestingly, we cannot show the same equivalence for ambiguity up to inner automorphism.

Proposition 14. *There exists a morphically primitive pattern α and morphism $\varphi : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\text{var}(\alpha)}$ such that $\varphi(\alpha) = \alpha$ and φ is not an inner automorphism.*

Proof. Recall that the pattern $1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$ is morphically primitive. Let $\varphi : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\{1,2\}}$ be the morphism given by $\varphi(1) = 1 \cdot 2$ and $\varphi(2) = 2$. Then

$$\varphi(1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}) = 1 \cdot 2 \cdot 2 \cdot 2^{-1} \cdot 1^{-1} \cdot 2^{-1} = 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1},$$

and it is clear that φ is not an inner automorphism. □

It can easily be verified that the pattern $\alpha = 1 \cdot 2 \cdot 1^{-1} \cdot 2^{-1}$ is also morphically primitive according to the original (length-based) definition, and thus the above statement holds regardless of which definition we use.

4. Unambiguous injective morphisms

Our main result is a characterisation of when a pattern in a free group possesses an injective morphism which is unambiguous up to inner automorphism. It provides a direct analogy to the equivalence of Statements (i) and (ii) of Theorem 3.

We already know from Proposition 9 that if a pattern has an injective morphism $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which is unambiguous up to (inner) automorphism, then the identity morphism $\text{id}_{\mathcal{F}_{\text{var}(\alpha)}} : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\text{var}(\alpha)}$ is unambiguous up to (inner) automorphism. Our objective in this section is to show that the converse also holds: that if the identity morphism is unambiguous up to (inner) automorphism, then there exists an injective morphism $\sigma : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which is unambiguous up to (inner) automorphism, hence establishing a complete characterisation of when a pattern has an injective morphism which is unambiguous up to inner automorphism.

Our strategy is to construct, for any two patterns $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, a morphism $\sigma_{\alpha,\beta} : \mathcal{F}_{\text{var}(\alpha)} \to \mathcal{F}_{\Sigma}$ which encodes the pre-image α over the alphabet Σ in such a way that any morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ induces a morphism φ_τ mapping β to α which is at least as ambiguous as τ. In the case that $\alpha = \beta$, at least one possibility for τ is the morphism $\sigma_{\alpha,\beta}$ itself and φ_τ fixes α. Thus we get that the identity morphism is as ambiguous as $\sigma_{\alpha,\beta}$ and by ensuring that $\sigma_{\alpha,\beta}$ is injective, we can prove the required statement. Formally, we want to construct $\sigma_{\alpha,\beta}$ such that the following two properties hold.

(P1) If $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ for some morphism $\tau : \mathcal{F}_{\text{var}(\beta)} \to \mathcal{F}_{\Sigma}$, then there exists a morphism $\varphi_\tau : \mathcal{F}_{\text{var}(\beta)} \to \mathcal{F}_{\text{var}(\alpha)}$ mapping β to α.

(P2) If φ_τ is unambiguous up to inner automorphism w.r.t. β, then τ is unambiguous up to inner automorphism w.r.t. β.

$$\begin{tikzcd}
 & \sigma_{\alpha,\beta}(\alpha) & \\
\beta \arrow[ur, "\tau"] \arrow[rr, dashed, "\varphi_\tau"'] & & \alpha \arrow[ul, "\sigma_{\alpha,\beta}"']
\end{tikzcd}$$

Fig. 1. A visual representation of the property (P1) which concerns a morphic encoding of the pattern α via the image of a morphism $\sigma_{\alpha,\beta}$ in the sense that any morphism τ mapping a pattern β to $\sigma_{\alpha,\beta}(\alpha)$ necessarily induces a morphism φ_τ mapping β to α.

By considering the more general case that α is not necessarily equal to β, we do not introduce any substantial additional effort, but are able to take advantage of our construction again in Section 5 to easily prove some properties of terminal-free group pattern languages.

4.1. A morphic encoding

We begin by concentrating on the first property (P1) (cf. Fig. 1). Our morphism $\sigma_{\alpha,\beta}$ is a generalisation of a construction given by Jiang et al. [10] which satisfies the equivalent condition in a free monoid. We will see in the remainder of this section, our task is substantially more complicated due to the possible presence of contractions.

The following idea is central: for each variable $x \in \mathrm{var}(\alpha)$, the image $\sigma_{\alpha,\beta}(x)$ has a uniquely associated factor S_x which acts as an anchor, holding the place of x. This encodes the pre-image α implicitly in the image $\sigma_{\alpha,\beta}(\alpha)$, and we can recover α explicitly by replacing each occurrence of an anchor S_x with the variable x and erasing any remaining letters from Σ.

For any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ with $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ and set $S = \{S_x \mid x \in \mathrm{var}(\alpha)\}$ of anchors, we derive the morphism φ_τ in the same way. To ensure the replacement process is deterministic, we shall always consider sets of anchors which cannot overlap with one another. Formally, we define the following.

Definition 15. For any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ with $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ and set $S = \{S_x \mid x \in \mathrm{var}(\alpha)\}$ of anchors, let $\tau_S^{\mathrm{mod}} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ be the morphism such that for each $y \in \mathrm{var}(\beta)$, $\tau_S^{\mathrm{mod}}(y)$ is obtained from $\tau(y)$ by replacing each occurrence of each anchor S_x with the corresponding variable x. Furthermore, define $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ to be the morphism obtained by erasing all the letters from Σ in τ_S^{mod}.[5]

Let W be the (reduced) image $\sigma_{\alpha,\beta}(\alpha)$ and let W_S^{mod} be result of replacing each occurrence of an anchor $S_x \in S$ in W by the appropriate variable x. Then the following two properties are sufficient to ensure that $\varphi_{\tau,S}(\beta) = \alpha$ as desired.

(P3) W_S^{mod} has α as a subpattern, and
(P4) $\tau_S^{\mathrm{mod}}(\beta) = W_S^{\mathrm{mod}}$.

Similarly to the construction in Jiang et al. [10], we shall achieve properties (P3) and (P4) by building our morphism $\sigma_{\alpha,\beta}$ from a high number of *segments*: factors unique to $\sigma_{\alpha,\beta}(x)$ for a given variable x which provide possible choices for the anchor S_x. We use segments of the form $s_i = \mathtt{a}\mathtt{b}^i\mathtt{a}^{-i}\mathtt{b}$ due to the fact that it is impossible for any two occurrences (positive or negative) to overlap, meaning the result of replacing anchors with variables is uniquely defined.

One additional problem we have is that while in the free monoid it is trivial that any factor occurring in $\sigma_{\alpha,\beta}(x)$ will occur in $\sigma_{\alpha,\beta}(\alpha)$ if $x \in \alpha$, the same simple statement does not always hold in the free group context. For example, consider the morphism $\sigma : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ given by $\sigma(1) = \mathtt{ab}$ and $\sigma(2) = \mathtt{b}^{-1}$. Then $\sigma(1 \cdot 2) = \mathtt{abb}^{-1} = \mathtt{a}$, and hence the factors $\mathtt{b}$ in $\sigma(1)$ and $\mathtt{b}^{-1}$ in $\sigma(2)$ do not "survive" in the reduced image $\sigma(1 \cdot 2)$.

In order to guarantee that each segment s_i in $\sigma_{\alpha,\beta}$ does in fact survive in the reduced image $W = \sigma_{\alpha,\beta}(\alpha)$ we add contraction-blocking factors μ_i as prefixes and suffixes to each $\sigma_{\alpha,\beta}(x)$, $x \in \mathrm{var}(\alpha)$. In particular, we will use factors $\mathtt{a}\mathtt{b}^i\mathtt{a}$ as no one is a prefix or suffix of the other. This is sufficient to stop any contractions from occurring beyond these factors, and hence we guarantee that each segment s_i survives. We define these blocking factors, as well as segments, formally below for ease of reference.

Definition 16. For all $i \in \mathbb{N}$, let $\mu_i = \mathtt{a}\mathtt{b}^i\mathtt{a}$ and let $s_i = \mathtt{a}\mathtt{b}^i\mathtt{a}^{-i}\mathtt{b}$.

Remark 17. For any $i, j \in \mathbb{N}$ with $i \neq j$, we have $\mu_i\mu_j^{-1} = \mathtt{a}\mathtt{b}^{i-j}\mathtt{a}^{-1}$ (and likewise $\mu_i^{-1}\mu_j = \mathtt{a}^{-1}\mathtt{b}^{j-i}\mathtt{a}$) with $i - j \neq 0$.

The morphism $\sigma_{\alpha,\beta}$ will map each variable to the appropriate blocking factor – which must be unique to that variable – then the string of segments s_i which form our potential anchors, and finally a second blocking factor. In order to avoid any

[5] Note that we refine our notation from φ_τ to $\varphi_{\tau,S}$ to accommodate the fact that our construction relies on the choice of anchors.

confusion between parts of the blocking factors and the segments, we ensure that the smallest segments are longer than the largest blocking factors. Hence we get a class of morphisms $\sigma_{k,\Delta}$ as follows, where k is the number of distinct, unique segments per variable and Δ is the set of variables (and hence provides the minimum "length" of the segments).

Definition 18. Let $k \in \mathbb{N}$ and let $\Delta = \{y_1, y_2, \dots, y_m\}$ be a set of variables from $\mathbb{N}$. Let $\sigma_{k,\Delta} : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ be the morphism given by

$$\sigma_{k,\Delta}(y_i) = \mu_i \cdot s_{m+(i-1)k+1} \cdots s_{m+ik} \cdot \mu_i$$

for $1 \le i \le m$.

For example if $k = 3$ and $\Delta = \{1, 2, 3\}$, we have that $\sigma_{3,\Delta} : \mathcal{F}_\Delta \to \mathcal{F}_\Sigma$ is the morphism given by:

$$\sigma_{3,\Delta}(1) = \mathtt{a\ b\ a\ \ a\ b^4a^{-4}b\ a\ b^5a^{-5}b\ a\ b^6a^{-6}b\ \ a\ b\ a},$$
$$\sigma_{3,\Delta}(2) = \mathtt{a\ b^2a\ \ a\ b^7a^{-7}b\ a\ b^8a^{-8}b\ a\ b^9a^{-9}b\ \ a\ b^2a},$$
$$\sigma_{3,\Delta}(3) = \mathtt{a\ b^3a\ \ a\ b^{10}a^{-10}b\ a\ b^{11}a^{-11}b\ a\ b^{12}a^{-12}b\ \ a\ b^3a}.$$

Because the blocking factors μ_i severely restrict the manner in which any contractions may occur, it is straightforward to observe that, for a pattern $\alpha \in \mathcal{F}_\Delta$, the reduced image $\sigma_{k,\Delta}(\alpha)$ has the form

$$\mu_r^{p_1} U_1^{p_1} V_1 U_2^{p_2} V_2 \dots V_{n-1} U_n^{p_n} \mu_s^{p_n}$$

where $\alpha = x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$ and $r, s, x_i \in \mathbb{N}$, $p_i \in \{1, -1\}$ for $1 \le i \le n$, μ_r, μ_s are defined according to Definition 16, the factors U_i consist of k consecutive segments s_j which uniquely occur as factors of $\sigma_{k,\Delta}(x_i)$ and

$$V_i = \begin{cases} \mathtt{ab}^{q_i}\mathtt{aab}^{q_{i+1}}\mathtt{a} & \text{if } p_i = p_{i+1} = 1 \\ \mathtt{a}^{-1}\mathtt{b}^{-q_i}\mathtt{a}^{-2}\mathtt{b}^{-q_{i+1}}\mathtt{a}^{-1} & \text{if } p_i = p_{i+1} = -1 \\ \mathtt{ab}^{q_i - q_{i+1}}\mathtt{a}^{-1} & \text{if } p_i = 1, p_{i+1} = -1 \\ \mathtt{a}^{-1}\mathtt{b}^{q_{i+1} - q_i}\mathtt{a} & \text{if } p_i = -1, p_{i+1} = 1 \end{cases}$$

where $q_i, q_{i+1} \in \mathbb{N}$.

Consequently, each segment s_j occurs exactly once in each factor U_i for which s_j is a factor of $\sigma(x_i)$, and nowhere else. Hence we may draw the following conclusions.

Remark 19. For any $k \in \mathbb{N}$, $\Delta \subset \mathbb{N}$ and $\alpha \in \mathcal{F}_\mathbb{N}$ with $\mathrm{var}(\alpha) = \Delta$ such that $\alpha \ne \varepsilon$, we have that $\sigma_{k,\Delta}(\alpha) \ne \varepsilon$ and consequently, $\sigma_{k,\Delta}$ is injective.

Remark 20. Let $\alpha \in \mathcal{F}_\mathbb{N}$, let $\Delta = \mathrm{var}(\alpha)$, and let $k \ge 1$. For each $x \in \mathrm{var}(\alpha)$, let S_x be a segment s_i such that S_x is a factor of $\sigma_{k,\Delta}(x)$. Let $S = \{S_x \mid x \in \mathrm{var}(\alpha)\}$. Let W be the result of replacing each occurrence of S_x in $\sigma_{k,\Delta}(\alpha)$ with x. Then W has α as a subpattern.

Remark 20 allows us to conclude that property (P3) is satisfied whenever we take $\sigma_{\alpha,\beta}$ to be one of the morphisms $\sigma_{k,\mathrm{var}(\alpha)}$ and for any set of anchors $S = \{S_x \mid x \in \mathrm{var}(\alpha)\}$ such that S_x is a segment s_i occurring in $\sigma_{\alpha,\beta}(x)$. Now we turn our attention to the property (P4). We shall show that provided k is large enough, taking $\sigma_{\alpha,\beta}$ to be $\sigma_{k,\mathrm{var}(\alpha)}$ also guarantees the existence of at least one such set of anchors for which (P4) is also satisfied.

Essentially, we need to guarantee that replacing the anchors in a morphism τ (so, in the images of individual variables) before its application yields the same result as replacing the anchors directly in the reduced image, after its application. In the free monoid, this is straightforward: it is enough to guarantee that the image contains no occurrence of an anchor which crosses a boundary between the images of two variables as this is the only way an occurrence of an anchor can exist which is not directly produced by an occurrence in the image of an individual variable. However, in the free group setting we must also consider the possibility that occurrences of anchors are produced or removed due to contractions.

We provide the following criteria which, as we show in what follows, provides a sufficient condition (although in a negated form) for when a factor is a good candidate for being an anchor in the free group case.

Definition 21. Let $\beta \in \mathcal{F}_\mathbb{N}$ and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism. Let w be the unreduced image $\tau(\beta)$. Let $u \in \mathcal{F}_\Sigma$ be an unbordered word. Then u is *split by* τ *in* $\tau(\beta)$ if:

(i) there exists an unreduced occurrence of u in w, or
(ii) there exists an occurrence of u in w which partially overlaps with $\tau(x)$ for some $x \in \mathrm{var}(\beta)$, or
(iii) there exists an occurrence of u in w which partially overlaps with a maximal contraction.

Using Definition 21, we give the following criteria for a "good" set of anchor segments.

Definition 22. Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, let $\Delta = \mathrm{var}(\alpha)$ and let $k \in \mathbb{N}$. Let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\beta) = \sigma_{k,\Delta}(\alpha)$. We say a *set of anchor segments* for τ is a set $S = \{S_{x_1}, S_{x_2}, \ldots, S_{x_n}\}$ such that:

(i) $\{x_1, x_2, \ldots, x_n\} = \Delta$, and
(ii) $S_{x_i} = s_j$ for some $j \in \mathbb{N}$, $j > |\Delta|$, and
(iii) S_{x_i} is a factor of $\sigma_{k,\Delta}(x_i)$, and
(iv) S_{x_i} is not split by τ in $\tau(\beta)$.

For ease of exposition, we shall initially concentrate on the replacement of a single anchor segment S_x by the variable x. Recall from Section 2 that we use the notation $R[u \to v](w)$ to refer to the result of replacing all occurrences of u with v in the (possibly unreduced) word w. Hence, we want the following equation to hold:

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\tau)(\beta) \tag{1}$$

where the image $\tau(\beta)$ on the left hand side is reduced. As we shall see later, adapting Equation (1) to account for replacing all anchors and thus to (P4) is straightforward due to the fact that we shall consider only anchors which are segments s_i which cannot overlap, and moreover that performing the replacements cannot induce any new contractions.

The following three examples demonstrate why each of the three cases in Definition 21 should be avoided by the anchors S_x if we want the property (P4) to hold.

Example 23. Let $\beta = 1 \cdot 2$, and let $\tau : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) = \mathtt{ab}^{-1}$ and $\tau(2) = \mathtt{ba}$. Suppose that for $x = 1$ we choose $S_x = \mathtt{aa}$. Then since S_x does not occur in $\tau(1)$ or $\tau(2)$, we have $R[S_x \to x](\tau) = \tau$ and thus

$$R[S_x \to x](\tau)(\beta) = \tau(\beta) = \mathtt{ab}^{-1}\mathtt{ba} = \mathtt{aa}.$$

However, while the reduced word S_x does not occur directly in the unreduced image $\tau(\beta)$, an unreduced version of S_x does occur, which results in an occurrence of S_x in the reduced version of $\tau(\beta)$. Thus

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\mathtt{aa}) = x = 1.$$

Hence $R[S_x \to x](\tau)(\beta) \neq R[S_x \to x](\tau(\beta))$.

Example 24. Let $\beta = 1 \cdot 2$, and let $\tau : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) = \mathtt{aa}$ and $\tau(2) = \mathtt{bb}$. Suppose that for $x = 1$, we choose $S_x = \mathtt{ab}$. Then since S_x does not occur in $\tau(1)$ or $\tau(2)$, we have $R[S_x \to x](\tau) = \tau$ and thus

$$R[S_x \to x](\tau)(\beta) = \tau(\beta) = \mathtt{aabb}.$$

However, S_x *does* occur in $\tau(\beta)$, as a combination of the prefix of $\tau(1)$ and the suffix of $\tau(2)$. Therefore

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\mathtt{aabb}) = \mathtt{a}\,x\,\mathtt{b} = \mathtt{a} \cdot 1 \cdot \mathtt{b}.$$

Hence $R[S_x \to x](\tau)(\beta) \neq R[S_x \to x](\tau(\beta))$.

Example 25. Let $\beta = 1 \cdot 2$, and let $\tau : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_\Sigma$ be the morphism given by $\tau(1) = \mathtt{abb}$ and $\tau(2) = \mathtt{b}^{-1}\mathtt{a}$. Suppose that for $x = 1$, we choose $S_x = \mathtt{abb}$. Then $R[S_x \to x](\tau)$ is the morphism $\tau' : \mathcal{F}_{\{1,2\}} \to \mathcal{F}_{\Sigma \cup \{1\}}$ given by $\tau'(1) = 1$ and $\tau'(2) = \mathtt{b}^{-1}\mathtt{a}$. Hence

$$R[S_x \to x](\tau)(\beta) = 1 \cdot \mathtt{b}^{-1}\mathtt{a}.$$

However,

$$R[S_x \to x](\tau(\beta)) = R[S_x \to x](\mathtt{abbb}^{-1}\mathtt{a}) = R[S_x \to x](\mathtt{aba}) = \mathtt{aba}.$$

Hence $R[S_x \to x](\tau)(\beta) \neq R[S_x \to x](\tau(\beta))$.

Next, we shall prove that Definition 21 does indeed yield a condition for when a set of anchor segments results in Equation (1) and subsequently the property (P4) holding. We require the following lemmas regarding the combinatorics of replacements and how they interact with contractions and concatenations. The first is a direct observation which deals with concatenation and is therefore relevant to Condition (ii) of Definition 21.

Lemma 26. *Let $w, u, v \in \mathcal{F}_\Sigma$ such that u is unbordered, and suppose that $w = w_1 w_2 \dots w_n$ for non-empty words $w_i \in \mathcal{F}_\Sigma$, and suppose that no occurrence of u in w partially overlaps a factor w_i. Then we have that*

$$R[u \to v](w_1) R[u \to v](w_2) \dots R[u \to v](w_n) = R[u \to v](w).$$

Proof. The statement follows from the definitions: since every occurrence of u is contained entirely inside a factor w_i, replacing all the factors of u in each w_i with v is equivalent to replacing all the factors of u in w. □

Our second lemma addresses the contractions, and is therefore relevant to Conditions (i) and (iii).

Lemma 27. *Let $w, u, v \in \mathcal{F}_\Sigma$ be words such that u is unbordered and reduced, and w is unreduced. Suppose that $w = \varepsilon$. If no occurrence of u in w partially overlaps with a maximal contraction and every occurrence of u is reduced, then*

$$R[u \to v](w) = \varepsilon.$$

Proof. We first prove the following statement: Let $w_1, w_2 \in \mathcal{F}_\Sigma$ be such that $w = w_1 x w_2$ where x is a maximal primary contraction. If no occurrence of u in w partially overlaps with x, then

$$R[u \to v](w) = R[u \to v](w_1 w_2).$$

To verify this claim, note that because no further maximal contractions occur in x, there exist reduced words x_1, x_2 such that $x = x_1 x_2$ and $x_1 = x_2^{-1}$. It follows that $R[u \to v](x_1) = R[u \to v](x_2^{-1}) = R[u \to v](x_2)^{-1}$. Furthermore, since u is reduced, there cannot be an occurrence of u in x which is contained partly in x_1 and partly in x_2 as it would then contain a contraction. Therefore by Lemma 26 we have that

$$\begin{aligned} R[u \to v](x) &= R[u \to v](x_1) R[u \to v](x_2) \\ &= R[u \to v](x_2)^{-1} R[u \to v](x_2) \\ &= \varepsilon. \end{aligned}$$

Now, if u does not partially overlap with x as we have assumed, but an occurrence of u partially overlaps with either w_1 or w_2, then that occurrence must have x as a factor. This contradicts the assumption that every occurrence of u is reduced. Hence we can assume that u does not partially overlap with w_1, x or w_2 and hence by Lemma 26, we have:

$$R[u \to v](w) = R[u \to v](w_1)\ R[u \to v](x)\ R[u \to v](w_2)$$

and therefore:

$$R[u \to v](w) = R[u \to v](w_1) R[u \to v](w_2).$$

By the same argument there cannot be an occurrence of u in $w_1 w_2$ which partially overlaps either w_1 or w_2, since such an occurrence would imply an occurrence in $w_1 x w_2$ which is unreduced. So we can again apply Lemma 26 to get:

$$\begin{aligned} R[u \to v](w_1 w_2) &= R[u \to v](w_1) R[u \to v](w_2) \\ &= R[u \to v](w). \end{aligned}$$

Hence we have proven our claim. In order to see that the statement of the lemma follows, note that every occurrence of u in $w_1 w_2$ must be reduced and no occurrence of u can overlap a maximal contraction in $w_1 w_2$. Indeed, if an unreduced occurrence of u occurs in either w_1 or w_2, then it also occurs in w which is a contradiction. We have already seen that we cannot have an occurrence of u which partially overlaps with w_1 or w_2 in $w_1 w_2$, so all occurrences must be reduced. Similarly, any maximal contraction in $w_1 w_2$ will also correspond to a maximal contraction in w (possibly, but not necessarily also containing x). Therefore any occurrence of u partially overlapping a maximal contraction in $w_1 w_2$ results in an occurrence of u partially overlapping a maximal contraction in w which is again a contradiction.

To complete the proof of the lemma, it is enough to observe that if we continuously remove the primary maximal contractions x from w, we eventually reach ε, and thus the statement can be obtained through a straightforward induction. □

As claimed, using Lemmas 26 and 27, we are able to turn Definition 21 into a sufficient condition on τ and S_x such that Equation (1) is satisfied.

Lemma 28. *Let $\beta \in \mathcal{F}_\mathbb{N}$ and $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism. Let $u, v \in \mathcal{F}_\Sigma$ such that u is unbordered. If no occurrence of u is split by τ in $\tau(\beta)$, then*

$$R[u \to v](\tau)(\beta) = R[u \to v](\tau(\beta)).$$

Proof. Let w be the unreduced image $\tau(\beta)$. Suppose that u is not split by τ in $\tau(\beta)$ (and thus that none of Conditions (i), (ii) and (iii) of Definition 21 hold). By Lemma 26, if Condition (ii) does not hold, then $R[u \to v](\tau)(\beta) = R[u \to v](w)$.

Hence it remains to show that $R[u \to v](w) = R[u \to v](\tau(\beta))$. To do this, let $w = w_0 z_1 w_1 \dots z_m w_m$ where each z_i is a maximal contraction and $w_0 w_1 \dots w_m$ is the reduced image $\tau(\beta)$. If Condition (iii) of Definition 21 does not hold, then every occurrence of u is contained entirely within each w_i or z_i. Consequently, by Lemma 26,

$$R[u \to v](w) = R[u \to v](w_0)R[u \to v](z_1)R[u \to v](w_1)R[u \to v](z_2)\dots$$
$$\dots R[u \to v](z_m)R[u \to v](w_m).$$

Furthermore, since Conditions (i) and (iii) do not hold, by Lemma 27,

$$R[u \to v](z_i) = \varepsilon$$

for $1 \leq i \leq m$. Thus

$$R[u \to v](w) = R[u \to v](w_0)R[u \to v](w_1)\dots R[u \to v](w_m).$$

Finally we conclude that no occurrence of u can partially overlap any w_i in $w_0 w_1 \dots w_m$, otherwise that occurrence of u must also contain some z_j as a factor in $w_0 z_1 w_1 \dots z_m w_m$, and would therefore contradict our assumption that Condition (i) does not hold. Hence by Lemma 26

$$\begin{aligned} R[u \to v](w_0)R[u \to v](w_1)\dots R[u \to v](w_m) &= R[u \to v](w_0 w_1 \dots w_m) \\ &= R[u \to v](\tau(\beta)), \end{aligned}$$

and therefore

$$R[u \to v](\tau)(\beta) = R[u \to v](w) = R[u \to v](\tau(\beta))$$

as claimed. □

The following proposition confirms that the existence of a set of anchor segments as defined in Definition 22 for a morphism τ mapping β to $\sigma_{k,\Delta}(\alpha)$ is sufficient to guarantee that (P4) holds. For convenience in later proofs, we give a slightly more detailed statement regarding the morphism τ_S^{mod} from which the fact that $\varphi_{\tau,S}(\beta) = \alpha$ follows directly.

Proposition 29. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, $k \in \mathbb{N}$, $\Delta = \text{var}(\alpha)$ and let $\tau : \mathcal{F}_{\text{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\beta) = \sigma_{k,\Delta}(\alpha)$. Suppose there exists a set S of anchor segments for τ as per Definition 22. For each $x \in \text{var}(\beta)$, let w_x be the prefix of $\sigma_{k,\Delta}(x)$ and w'_x be the suffix of $\sigma_{k,\Delta}(x)$ such that $\sigma_{k,\Delta}(x) = w_x S_x w'_x$. Then*

$$\tau_S^{\text{mod}}(\beta) = W_S^{\text{mod}} = \left(w_{x_1} x_1 w'_{x_1}\right)^{p_1} \left(w_{x_2} x_2 w'_{x_2}\right)^{p_2} \cdots \left(w_{x_n} x_n w'_{x_n}\right)^{p_n}$$

where W_S^{mod} is the obtained from the reduced image $W = \sigma_{k,\Delta}(\alpha)$ by replacing each occurrence of an anchor $S_x \in S$ with the appropriate variable x, and $\alpha = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$, with $x_i \in \mathbb{N}$, $p_i \in \{1, -1\}$.

Proof. By definition, S_{x_1} is not split by τ in $\tau(\beta)$ and is unbordered, so by Lemma 28, we have that

$$R[S_{x_1} \to x_1](\tau)(\beta) = R[S_{x_1} \to x_1](\tau(\beta))$$

and since $\tau(\beta) = \sigma_{k,\Delta}(\alpha)$, this implies

$$R[S_{x_1} \to x_1](\tau)(\beta) = R[S_{x_1} \to x_1](\sigma_{k,\Delta}(\alpha)).$$

Let $\tau^{(1)} = R[S_{x_1} \to x_1](\tau)$. Note that because S_{x_1} and S_{x_2} occur independently (i.e., they do not overlap), and since no new contractions are introduced when replacing S_{x_1} by x_1 (since x_1 is a new symbol not already present), it follows from the fact that S_{x_2} is not split by τ in $\tau(\beta)$, that S_{x_2} is not split by $\tau^{(1)}$ in $\tau^{(1)}(\beta)$. In other words, replacing S_{x_1} with x_1 does not affect whether S_{x_2} satisfies any of the conditions of Definition 21. Hence, for $\tau^{(2)} = R[S_{x_2} \to x_2]\left(\tau^{(1)}\right)$, we have that

$$\tau^{(2)}(\beta) = R[S_{x_2} \to x_2]\left(\tau^{(1)}(\beta)\right) = R[S_{x_2} \to x_2]\Big(R[S_{x_1} \to x_1]\left(\sigma_{k,\Delta}(\alpha)\right)\Big).$$

By repeating the same logic we eventually have:

$$\tau^{(m)}(\beta) = R[S_{x_m} \to x_m]\Big(R[S_{x_{m-1}} \to x_{m-1}]\big(\dots R[S_{x_1} \to x_1](\sigma_{k,\Delta}(\alpha))\big)\Big)$$
$$= W_S^{\mathrm{mod}}.$$

Clearly $\tau^{(m)} = \tau_S^{\mathrm{mod}}$, so $\tau_S^{\mathrm{mod}}(\beta) = W_S^{\mathrm{mod}}$. It remains to consider the second equality. Note that by definition, we have that:

$$\sigma_{k,\Delta}(\alpha) = W = \left(w_{x_1} S_{x_1} w'_{x_1}\right)^{p_1} \left(w_{x_2} S_{x_2} w'_{x_2}\right)^{p_2} \dots \left(w_{x_n} S_{x_n} w'_{x_n}\right)^{p_n}$$

such that $\alpha = x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$, $x_i \in \mathbb{N}$, $p_i \in \{1, -1\}$ and $\sigma_{k,\Delta}(x_i) = w_{x_i} S_{x_i} w'_{x_i}$. Furthermore, by the construction of $\sigma_{k,\Delta}$ (see Remark 20), there are no other occurrences of each S_{x_i} in W. Hence the result of applying the replacements $R[S_{x_i} \to x_i]$ to the word W is:

$$W_S^{\mathrm{mod}} = \left(w_{x_1} x_1 w'_{x_1}\right)^{p_1} \left(w_{x_2} x_2 w'_{x_2}\right)^{p_2} \dots \left(w_{x_n} x_n w'_{x_n}\right)^{p_n}$$

and our statement holds. □

The next step is to prove that if we chose $\sigma_{\alpha,\beta}$ to be $\sigma_{k,\mathrm{var}(\alpha)}$ for large enough k, then we can guarantee that any morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ has a set of anchor segments satisfying Definition 22. As with the free monoid case, this centres around a counting argument: we shall show that for a given β, there is an upper bound on the number of segments s_i which may satisfy any of the three conditions of Definition 21.

The main observation we need is that the number of strict maximal contractions occurring in $\tau(\beta)$ is bounded by a function of $|\beta|$. We can then use this insight to show that the number of segments s_i which are split by τ in $\tau(\beta)$ is also bounded by a function of $|\beta|$.

Lemma 30. *Let $w_1, w_2, \dots, w_n \in \mathcal{F}_\Sigma$ be reduced words. Let w be the unreduced word $w_1 w_2 \cdots w_n$. Then there are at most $\frac{n(n-1)}{2}$ strictly maximal contractions in w.*

Proof. For the purposes of our proof, we will classify the strictly maximal contractions as follows: a primary strictly maximal contraction is *degree*-1. In general, a strictly maximal contraction u is *degree*-$k+1$ if it contains a degree-k strictly maximal contraction, and removing all degree-k strictly maximal contractions yields a primary contraction. For example, $\mathtt{a}\mathtt{a}^{-1}$ is degree-1 because it is primary, while $\mathtt{abaa}^{-1}\mathtt{bb}^{-2}\mathtt{a}^{-1}$ is degree-2 because removing the degree-1 strictly maximal contractions $\mathtt{aa}^{-1}$ and $\mathtt{bb}^{-1}$ yields the primary contraction $\mathtt{abb}^{-1}\mathtt{a}^{-1}$.

We will now count the maximum possible number of strictly maximal contractions of degree-m in w. We start with $m = 1$. Because each w_i is reduced, and because each primary contraction must contain a factor $\mathtt{aa}^{-1}$ for some $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$, we have at most $n-1$ primary strictly maximal contractions (and therefore strictly maximal contractions with degree-1) in w. Consider the word $w' = w'_1 w'_2 \dots w'_n$ obtained by removing all the primary strictly maximal contractions, where $w'_1, w'_2, \dots, w'_n$ are obtained by removing the corresponding parts of each primary strictly maximal contraction from $w_1, w_2, \dots w_n$ respectively. Note that this may result in w'_i being the empty word for some values i. Moreover, any factor of a reduced word is also reduced, so each w'_i is reduced for $1 \le i \le n$.

Now consider the primary strictly maximal contractions in w'. Note that each degree-2 strictly maximal contraction in w must contain at least one of these. By the same logic as before, there can be at most $n-1$. However, we claim that there can be at most $n-2$. This is clearly the case if $w'_i = \varepsilon$ for some i, $1 \le i \le n$ (since we then have $n-2$ or less transitions where a factor $\mathtt{aa}^{-1}$ may occur, $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$). We now show that if $w'_i \ne \varepsilon$ for all i, $1 \le i \le n$, then a much stronger statement holds, that there are *no* contractions. In particular, if $w'_i \ne \varepsilon$ for $1 \le i \le n$, then there exist $x_i, y_i, z_i \in \mathcal{F}_\Sigma$ for $1 \le i \le n$ such that $w_i = x_i y_i z_i$ (and hence the latter is reduced), $z_i x_{i+1}$ is a strictly maximal contraction in w for $1 \le i < n$, and $w'_i = y_i \ne \varepsilon$.

Now suppose to the contrary that we have a primary strictly maximal contraction in w'. Then, since any primary strictly maximal contraction contains a factor $\mathtt{aa}^{-1}$, $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$, and since each w'_i is reduced (and hence does not contain such a factor), we must have that for some k, $1 \le k < n$, $w'_k = y_k$ has a suffix $\mathtt{a}$ and $w'_{k+1} = y_{k+1}$ has a prefix $\mathtt{a}^{-1}$ for some $\mathtt{a} \in \Sigma \cup \Sigma^{-1}$. However, this contradicts the fact that the contraction $z_k x_{k+1}$ is a *maximal* contraction. Hence if there exist any primary strictly maximal contractions in w', at least one w'_i must be empty, and as we have already reasoned, there can be at most $n-2$ primary strictly maximal contractions in w'.

By repeating this argument, we see there are $n-p$ possible strictly maximal contractions of degree p until $p = n$ and we have a reduced word $w^{(p)}$. Thus we have at most $\sum\limits_{j=1}^{n-1} j = \frac{n(n-1)}{2}$ distinct strictly maximal contractions in w. □

Note that it follows directly from the definitions that if a segment s_i partially overlaps a maximal contraction, then it must partially overlap a strictly maximal contraction. We can therefore use Lemma 30, to infer that for any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ with $\tau(\beta) = \sigma_{k,\mathrm{var}(\alpha)}(\alpha)$, the number of segments s_i satisfying Condition (iii) of Definition 21 is bounded by a (fixed) function of $|\beta|$. By observing that there are at most $|\beta| - 1$ segments satisfying Conditions (i) and (ii) of the same definition, we can also bound the total number of segments which are split by τ in $\tau(\beta)$ by some function of $|\beta|$.

Hence, to guarantee the existence of anchor segments S_x satisfying Definition 22, we simply need to choose a value of k which is above this bound (and thus at least one segment occurring in each $\sigma_{k,\mathrm{var}(\alpha)}(x)$ cannot be split by τ in $\tau(\beta)$).

We will see from Proposition 32 that the exact number required is $k = (|\beta|+1)(|\beta|-1)+1$. Hence we define the morphism $\sigma_{\alpha,\beta}$ as follows.

Definition 31. For $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$, let $k = (|\beta|+1)(|\beta|-1)+1$. Then we define $\sigma_{\alpha,\beta}$ to be the morphism $\sigma_{k,\mathrm{var}(\alpha)}$ as per Definition 18.

The next proposition confirms that any morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ has at least one set of anchor segments satisfying Definition 22.

Proposition 32. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$ and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Then there exists a set S of anchor segments for τ.*

Proof. Let seg_x be the set of segments s_i such that s_i is a factor of $\sigma_{\alpha,\beta}(x)$. Then by definition,

$$|seg_x| = k = (|\beta|+1)(|\beta|-1)+1.$$

Moreover it is easily determined (e.g., from Remark 20 or the preceding discussion), that every segment $s \in seg_x$ occurs as a factor of $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. We have the following claim.

Claim 1. *Let $x \in \mathrm{var}(\alpha)$. Then there exists $s \in seg_x$ such that s is not split by τ in $\tau(\beta)$.*

Proof (Claim 1). We consider the maximum number of segments $s \in seg_x$ which may be split by τ in $\tau(\beta)$. We note that at most $|\beta|-1$ segments may satisfy Condition (ii) of Definition 21, since there are at most $|\beta|-1$ factors $\tau(x)\tau(y)$ of $\tau(\beta)$ with $x, y \in \mathbb{N} \cup \mathbb{N}^{-1}$. Similarly, any segment satisfying Condition (i) of Definition 21 contains a factor $\mathtt{a}\mathtt{a}^{-1}$, which must necessarily occur across the border of a factor $\tau(x)\tau(y)$ and so must also satisfy Condition (ii).

By Lemma 30, there are at most $\frac{|\beta|(|\beta|-1)}{2}$ different strictly maximal contractions in $\tau(\beta)$. Note that an occurrence of $s \in seg_x$ partially overlaps a maximal contraction if and only if it partially overlaps a strictly maximal contraction. By definition, it must occur partly inside and partly outside the contraction, so it crosses the edge of the contraction. Since there are two edges per strictly maximal contraction, and since no two occurrences of segments overlap, there can be at most two segments partially overlapping each maximal contraction, and hence at most $|\beta|(|\beta|-1)$ segments $s \in seg_x$ satisfying Condition (iii) of Definition 21. In total, we have at most

$$|\beta| - 1 + |\beta|(|\beta|-1) = (|\beta|+1)(|\beta|-1)$$

distinct segments $s \in seg_x$ which satisfy any of the conditions of Definition 21. Consequently, since $|seg_x| > (|\beta|+1)(|\beta|-1)$, there exists $S_x \in seg_x$ which doesn't satisfy any of the conditions of Definition 21, and hence is not split by τ in $\tau(\beta)$. □

By Claim 1, we can define a set S containing exactly one segment S_x from each seg_x, $x \in \mathrm{var}(\alpha)$ such that S_x is not split by τ in $\tau(\beta)$. It follows that S satisfies Conditions (i) and (iv) of Definition 22. By definition, each S_x is a segment occurring as a factor of $\sigma_{\alpha,\beta}(x)$, so S also satisfies Conditions (ii) and (iii). Thus S is a set of anchor segments for τ as required. □

Summarising our reasoning so far, we have the following theorem confirming that $\sigma_{\alpha,\beta}$ satisfies the property (P1).

Theorem 33. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. There exists a morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\Sigma}$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ if and only if there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$. In particular, there exists a set S of anchor segments for τ such that $\varphi_{\tau,S}(\beta) = \alpha$.*

Proof. The "if" statement is straightforward, since we can just take $\tau = \sigma_{\alpha,\beta} \circ \varphi$. The "only if" statement obtained as a straightforward consequence of Propositions 29 and 32. □

We also include the following remark which is necessary for our considerations later, in Section 5.

Remark 34. Our definition of $\sigma_{\alpha,\beta}$ relies only on the length of β and the set $\Delta = \mathrm{var}(\alpha)$: we choose $\sigma_{\alpha,\beta}$ based on $\sigma_{k,\Delta}$ where k is derived from $|\beta|$. All the results of this section also hold if, instead, we base $\sigma_{\alpha,\beta}$ on $\sigma_{k',\Delta}$ for any $k' \geq k$.

$$\begin{tikzcd} \beta \arrow[r, "\varphi_{\tau,S_1}"] \arrow[rr, bend left, "\sigma_{\alpha,\beta}\circ\varphi_{\tau,S}"] \arrow[rr, bend right, "\tau \neq \sigma_{\alpha,\beta}\circ\varphi_{\tau,S}"'] & \alpha \arrow[r, "\sigma_{\alpha,\beta}"] & \sigma_{\alpha,\beta}(\alpha) \end{tikzcd}$$

Fig. 2. For a given morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ we can construct a morphism $\varphi_{\tau,S}$ mapping β to α. In order to compare the ambiguity of τ and $\varphi_{\tau,S}$, we need to consider the possibility that $\tau \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$.

4.2. Comparing the ambiguity of τ and $\varphi_{\tau,S}$

Having fulfilled our aim in the previous section of constructing a morphism $\sigma_{\alpha,\beta}$ such that any morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ "encodes" an associated morphism $\varphi_{\tau,S}$ mapping β to α (cf. (P1)), we now turn our attention to the ambiguity of the morphisms τ. In particular, we wish to show that τ is as unambiguous as $\varphi_{\tau,S}$ (cf. (P2)).

In a rough sense, we wish to show that there are not "more" morphisms τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$, than morphisms $\varphi_{\tau,S}$ mapping β to α. In particular, we consider two cases based on whether or not $\tau = \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$ for every morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ and set S of anchor segments. The case that this is true is the simpler case since we get a direct correspondence between the morphisms τ and the morphisms $\varphi_{\tau,S}$.

Proposition 35. *Let $\alpha, \beta \in \mathcal{F}_\mathbb{N}$ and suppose that for every morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ and for every set S of anchor segments, $\tau = \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$. Then τ is ambiguous up to inner automorphism w.r.t. β if and only if $\varphi_{\tau,S}$ is ambiguous up to inner automorphism w.r.t. β. Moreover τ is ambiguous up to automorphism w.r.t. β if and only if $\varphi_{\tau,S}$ is ambiguous up to automorphism w.r.t. β.*

Proof. We shall prove the statement for ambiguity up to automorphism. The proof for ambiguity up to inner automorphism is a straightforward adaptation. Then for every morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ mapping β to $\sigma_{\alpha,\beta}(\alpha)$, there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ (namely $\varphi_{\tau,S}$) mapping β to α such that $\tau = \sigma_{\alpha,\beta} \circ \varphi$. Moreover, for every morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β to α we have that $\tau = \sigma_{\alpha,\beta} \circ \varphi$ is a morphism mapping β to $\sigma_{\alpha,\beta}(\alpha)$. Hence the set of all morphisms mapping β to $\sigma_{\alpha,\beta}(\alpha)$ is given by the set:

$$\mathcal{M} = \{\sigma_{\alpha,\beta} \circ \varphi \mid \varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)} \text{ is a morphism such that } \varphi(\beta) = \alpha\}.$$

Now let $\tau_1 = \sigma_{\alpha,\beta} \circ \varphi_1$ and $\tau_2 = \sigma_{\alpha,\beta} \circ \varphi_2$ be morphisms in $\mathcal{M}$ such that $\varphi_1, \varphi_2 : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ are morphisms mapping β to α. Note that, since $\sigma_{\alpha,\beta}$ is injective (cf. Remark 19), for every $x \in \mathrm{var}(\beta)$, $\sigma_{\alpha,\beta}(\varphi_1(x)) = \sigma_{\alpha,\beta}(\varphi_2(x))$ if and only if $\varphi_1(x) = \varphi_2(x)$. Consequently, $\tau_1 = \tau_2$ if and only if $\varphi_1 = \varphi_2$, and by the same reasoning, $\tau_1 = \tau_2 \circ \psi$ for some automorphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ if and only if $\varphi_1 = \varphi_2 \circ \psi$.

Hence, for $\tau = \sigma_{\alpha,\beta} \circ \varphi$, there exists a second morphism τ' such that $\tau(\beta) = \tau'(\beta) = \sigma_{\alpha,\beta}(\alpha)$ (i.e., $\tau' \in S$) and $\tau' \neq \tau \circ \psi$ for any automorphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\beta)}$ if and only if there exists $\varphi' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β to α such that $\varphi' \neq \varphi \circ \psi'$ for any automorphism $\psi' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\beta)}$. This is equivalent to saying that τ is ambiguous up to automorphism if and only if φ is, and our statement follows. $\square$

Now we consider the case that there exists some morphism τ mapping β to $\sigma_{\alpha,\beta}(\alpha)$ such that $\tau \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$ for some set S of anchor segments (cf. Fig. 2). In this case, we get the following technical statement which, as we shall see, is sufficient to construct a wide variety of morphisms φ mapping β to α.

Proposition 36. *Let $\alpha, \beta \in \mathcal{F}_\mathbb{N}$ and let $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. Suppose there exists a set S of anchor segments for τ such that $\tau \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$. Then there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that*

(i) $\psi(\beta) = \alpha$, *and*
(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ *for some* $x \in \mathrm{var}(\beta)$.

Proof. We begin by constructing our morphism ψ. To this end, for each $x \in \mathrm{var}(\alpha)$, let $S_x \in S$ be the anchor segment associated with x. Then from the definition of $\sigma_{\alpha,\beta}$, there exist (unique) $w_x, w'_x \in \mathcal{F}_\Sigma$ such that $\sigma_{\alpha,\beta}(x) = w_x S_x w'_x$ and $w_x S_x w'_x$ is reduced. Let τ_S^{mod} be defined according to Definition 15, and let $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ be the morphism given by $\rho(\mathtt{a}) = \mathtt{a}$, $\rho(\mathtt{b}) = \mathtt{b}$ and $\rho(x) = w_x^{-1} x {w'_x}^{-1}$ for all $x \in \mathrm{var}(\alpha)$. We define our morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ to be $\rho \circ \tau_S^{\mathrm{mod}}$. The following claim confirms that ψ satisfies Condition (i) of the proposition.

Claim 1. $\psi(\beta) = \alpha$.

Proof (Claim 1). Recall that $\psi = \rho \circ \tau_S^{\mathrm{mod}}$. By Proposition 29, we have

$$\tau_S^{\mathrm{mod}}(\beta) = \left(w_{x_1} x_1 w'_{x_1}\right)^{p_1} \left(w_{x_2} x_2 w'_{x_2}\right)^{p_2} \cdots \left(w_{x_n} x_n w'_{x_n}\right)^{p_n}$$

such that $\alpha = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $x_i \in \mathbb{N}$ and $p_i \in \{1, -1\}$, and $\sigma_{\alpha,\beta}(x_i) = w_{x_i} S_{x_i} w'_{x_i}$ such that $w_{x_i} S_{x_i} w'_{x_i}$ is reduced. Now, since $\rho(\mathtt{a}) = \mathtt{a}$ and $\rho(\mathtt{b}) = \mathtt{b}$, and each $w_{x_i}, w'_{x_i} \in \mathcal{F}_\Sigma$, we have $\rho(w_{x_i}) = w_{x_i}$ and $\rho(w'_{x_i}) = w'_{x_i}$. Thus

$$\begin{aligned}
\rho \circ \tau_S^{\mathrm{mod}}(\beta) &= \left(w_{x_1} \rho(x_1) w'_{x_1}\right)^{p_1} \left(w_{x_2} \rho(x_2) w'_{x_2}\right)^{p_2} \cdots \left(w_{x_n} \rho(x_n) w'_{x_n}\right)^{p_n} \\
&= \left(w_{x_1} w_{x_1}^{-1} x_1 {w'_{x_1}}^{-1} w'_{x_1}\right)^{p_1} \left(w_{x_2} w_{x_2}^{-1} x_2 {w'_{x_2}}^{-1} w'_{x_2}\right)^{p_2} \cdots \\
&\qquad \cdots \left(w_{x_n} w_{x_n}^{-1} x_n {w'_{x_n}}^{-1} w'_{x_n}\right)^{p_n} \\
&= x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \\
&= \alpha.
\end{aligned}$$

Therefore we have $\psi(\beta) = \rho \circ \tau_S^{\mathrm{mod}}(\beta) = \alpha$ as claimed. □

In order to address the second condition, we present the following claim.

Claim 2. *Suppose there exists $x \in \mathrm{var}(\beta)$ such that $\tau(x)$ is reduced and $\tau(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. Then $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$.*

Proof (Claim 2). In order to prove our claim, we must consider more closely the structure of the morphisms τ, τ_S^{mod} and ψ. Firstly, recall that τ_S^{mod} is obtained from τ by replacing the anchor segments S_x with their respective variables x. Thus, there exist $u_0, u_1, \ldots u_m \in \mathcal{F}_\Sigma$, $y_1, y_2, \ldots y_m \in \mathbb{N}$,[6] and $p_1, p_2, \ldots p_m \in \{1, -1\}$ such that

$$\tau(x) = u_0\, S_{y_1}^{p_1}\, u_1\, S_{y_2}^{p_2}\, u_2\, \ldots\, S_{y_m}^{p_m}\, u_m$$

and

$$\tau_S^{\mathrm{mod}}(x) = u_0\, {y_1}^{p_1}\, u_1\, {y_2}^{p_2}\, u_2 \ldots\, {y_m}^{p_m}\, u_m$$

where

$$y_1^{p_1}\, y_2^{p_2}\, \cdots\, y_m^{p_m} = \varphi_{\tau,S}(x).$$

Note that we may assume $\tau(x)$ is reduced as written above. Recall that $\psi = \rho \circ \tau_S^{\mathrm{mod}}$ where $\rho(\mathtt{a}) = \mathtt{a}$, $\rho(\mathtt{b}) = \mathtt{b}$, and $\rho(y) = w_y^{-1} y {w'_y}^{-1}$ such that $\sigma_{\alpha,\beta}(y) = w_y S_y w'_y$. Then since $u_i \in \mathcal{F}_\Sigma$ for $0 \le i \le m$, we have that $\rho(u_i) = u_i$ and hence:

$$\begin{aligned}
\psi(x) &= \rho \circ \tau_S^{\mathrm{mod}}(x) \\
&= \rho\left(u_0\, {y_1}^{p_1}\, u_1\, {y_2}^{p_2}\, u_2\, \ldots\, {y_m}^{p_m}\, u_m\right) \\
&= u_0\, \rho(y_1)^{p_1}\, u_1\, \ldots\, \rho(y_m)^{p_m}\, u_m \\
&= v_0\, {y_1}^{p_1}\, v_1\, {y_2}^{p_2}\, v_2\, \ldots\, {y_m}^{p_m}\, v_m,
\end{aligned}$$

where $v_0 = u_0 w_{y_1}^{-1}$ if $p_1 = 1$ and $v_0 = u_0 w'_{y_1}$ if $p_1 = -1$, likewise $v_m = {w'}_{y_m}^{-1} u_m$ if $p_m = 1$ and $v_m = w_{y_m} u_m$ if $p_m = -1$, and for $1 \le i < m$,

$$v_i = \begin{cases} {w'}_{y_i}^{-1}\, u_i\, w_{y_{i+1}}^{-1} & \text{if } p_i = p_{i+1} = 1, \\ w_{y_i}\, u_i\, w_{y_{i+1}}^{-1} & \text{if } p_i = -1, p_{i+1} = 1, \\ {w'}_{y_i}^{-1}\, u_i\, w'_{y_{i+1}} & \text{if } p_i = 1, p_{i+1} = -1, \\ w_{y_i}\, u_i\, w'_{y_{i+1}} & \text{if } p_i = p_{i+1} = -1. \end{cases}$$

Now, because for each variable $y \in \mathrm{var}(\alpha)$, $\sigma_{\alpha,\beta}(y) = w_y S_y w'_y$, we have

$$\sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x) = \left(w_{y_1} S_{y_1} w'_{y_1}\right)^{p_1} \left(w_{y_2} S_{y_2} w'_{y_2}\right)^{p_2} \ldots \left(w_{y_m} S_{y_m} w'_{y_m}\right)^{p_m}.$$

It is clear that if $v_i = \varepsilon$ for $0 \le i \le m$, then $\tau(x) = \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. Consequently there exists i such that $v_i \neq \varepsilon$. If v_i is not contracted in $\psi(x)$ then since $v_i \in \mathcal{F}_\Sigma$, we have that $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ and our claim holds.

[6] Note that in the case no anchor segments S_x occur in $\tau(x)$, we simply have $\tau(x) = u_0 = \tau_S^{\mathrm{mod}}(x) = \psi(x)$, and the statement is straightforward.

Suppose instead that v_i *is* contracted. Consider a contraction c occurring in $\psi(x)$ containing v_i. Since $v_i \neq \varepsilon$, c must include at least one variable $y \in \{y_1, y_2, \dots, y_m\}$. Since this y must also be contracted, c must necessarily contain a factor yuy^{-1} where $y = y_j^{p_j}$ for some j and $u = \varepsilon$ (u need not be reduced). Consider such a factor for which u is a short as possible. Then we may conclude that $u \in \mathcal{F}_\Sigma$ and hence that $y_{j+1}^{p_{j+1}} = y_j^{-p_j}$ and $u = u_j$. However this implies that there is a factor $S_{y_j}^{p_j} u_j S_{y_{j+1}}^{p_{j+1}}$ which is unreduced and equal to ε, which in turn implies that $\tau(x)$ is unreduced, a contradiction to our previous assumption. □

We are now ready to prove our statement. In particular, note that if there exists $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$ and such that for some $x \in \mathrm{var}(\beta)$, $\tau(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$ and $\tau(x)$ is *not* reduced, then for the morphism $\tau' : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau'(x)$ is the reduced version of $\tau(x)$, we still have $\tau'(\beta) = \tau(\alpha)$ and $\tau'(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. Hence we may assume w.l.o.g. that $\tau(x)$ is reduced for each $x \in \mathrm{var}(\beta)$. Therefore, if $\tau \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}$, then there exists $x \in \mathrm{var}(\beta)$ such that $\tau(x)$ is reduced and $\tau(x) \neq \sigma_{\alpha,\beta} \circ \varphi_{\tau,S}(x)$. By Claim (2), this implies that the morphism $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ satisfies Condition (ii) of the Proposition, and by Claim (1), ψ also satisfies Condition (i), so the statement holds. □

For a morphism ψ satisfying the two conditions of Proposition 36, the existence of a letter $\mathtt{a} \in \mathcal{F}_\Sigma$ which occurs in $\psi(x)$ for some $x \in \mathrm{var}(\beta)$ allows for the construction of many other morphisms ψ' mapping β to α. In particular, since $\mathtt{a}$ does not occur in α, we may replace each occurrence of $\mathtt{a}$ in ψ with any factor without disrupting the image α. Formally, ψ' can be obtained by composing ψ with a morphism $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\rho(y) = y$ for all $y \in \mathrm{var}(\alpha)$.

Example 37. Let $\beta = 1 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3$, let $\alpha = 1 \cdot 2^4 \cdot 3 \cdot 1 \cdot 2^2 \cdot 3$, and let $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ be the morphism given by $\psi(1) = 1 \cdot \mathtt{a}$, $\psi(2) = \mathtt{a}^{-1} \cdot 2^2 \cdot \mathtt{a}$ and $\psi(3) = \mathtt{a}^{-1} \cdot 3$. Then we have:

$$\begin{aligned}\psi(\beta) &= 1 \cdot \mathtt{a}\mathtt{a}^{-1} \cdot 2 \cdot 2 \cdot \mathtt{a}\mathtt{a}^{-1} \cdot 2 \cdot 2 \cdot \mathtt{a}\mathtt{a}^{-1} \cdot 3 \cdot 1 \cdot \mathtt{a}\mathtt{a}^{-1} \cdot 2 \cdot 2 \cdot \mathtt{a} \cdot 3\\ &= 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3\\ &= \alpha.\end{aligned}$$

Let $\gamma_{\mathtt{a}}$ be any pattern in $\mathcal{F}_{\mathrm{var}(\alpha)}$, and let $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the morphism such that $\rho(y) = y$ for all $y \in \mathrm{var}(\alpha)$, $\rho(\mathtt{a}) = \gamma_{\mathtt{a}}$. Then we have

$$\begin{aligned}\rho \circ \psi(\beta) &= 1\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}2 \cdot 2\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}2 \cdot 2\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}3 \cdot 1\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}2 \cdot 2\gamma_{\mathtt{a}} \cdot \gamma_{\mathtt{a}}^{-1}3\\ &= 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3\\ &= \alpha,\end{aligned}$$

while, for example, $\rho \circ \psi(1) = 1 \cdot \gamma_{\mathtt{a}}$. Hence each possible $\gamma_{\mathtt{a}}$ results in a different morphism $\rho \circ \psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ mapping β to α.

The ability to replace the letter $\mathtt{a}$ with any factor leads to a combinatorially rich set of morphisms mapping β to α. Intuitively, one might expect that this guarantees that such morphisms are ambiguous. While we are unable to give a proof of this in general for both versions of ambiguity, we are able to provide a proof for ambiguity up to inner automorphism in the case that $\alpha = \beta$ (and thus ψ fixes α).

Proposition 38. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. Suppose there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that:*

(i) $\psi(\alpha) = \alpha$, *and*
(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ *for some* $x \in \mathrm{var}(\alpha)$.

Then there exists a morphism $\psi' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an inner automorphism, and such that $\psi'(\alpha) = \alpha$.

Proof. For the purposes of this proof, we will say a word w is *enclosed* by a variable z if (the reduced version of) w has a prefix in $\{z, z^{-1}\}$ and a suffix in $\{z, z^{-1}\}$. Let $\alpha \in \mathcal{F}_{\mathbb{N}}$ and let $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ be a morphism satisfying Conditions (i) and (ii) of the proposition. In particular, we have a variable $x \in \mathrm{var}(\alpha)$ such that $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$. We will construct a morphism $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\rho(\alpha) = \alpha$ (and hence such that $\rho \circ \psi(\alpha) = \alpha$), and such that $\psi' = \rho \circ \psi$ is not an inner automorphism.

We define ρ as follows. W.l.o.g., suppose that $1, 2 \in \mathrm{var}(\alpha)$.[7] Let $\hat{\alpha}$ be a primitive root of α (recall this is unique up to inverse). Let $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$ be a word which is enclosed by 1, such that neither η nor η^{-1} are a factor of $\hat{\alpha}^n x \hat{\alpha}^{-n}$ for any $n \in \mathbb{Z}$.[8] Let $\rho : \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the morphism given by

$$\rho(y) = \begin{cases} 1^k \cdot 2 \cdot \eta \cdot 2 \cdot 1^k & \text{if } y = \mathtt{a}, \\ 1^{2k} \cdot 2 \cdot \eta \cdot 2 \cdot 1^{2k} & \text{if } y = \mathtt{b}, \\ y & \text{if } y \in \mathrm{var}(\alpha), \end{cases}$$

where $k = |\psi(x)| + 1$. Clearly, since $\mathtt{a}, \mathtt{b} \notin \mathrm{var}(\alpha)$, $\rho(\alpha) = \alpha$, and hence $\rho \circ \psi(\alpha) = \rho(\alpha) = \alpha$. Thus it remains to show that $\rho \circ \psi$ is not an inner automorphism.

Suppose initially that $\rho \circ \psi$ is an inner automorphism. Then there exists a primitive pattern $\tilde{\alpha} \in \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\rho \circ \psi$ is generated by $\tilde{\alpha}^n$ for some $n \in \mathbb{N}$ (i.e. $\rho \circ \psi(y) = \tilde{\alpha}^n y \tilde{\alpha}^{-n}$ for all $y \in \mathrm{var}(\alpha)$). Then $\rho \circ \psi(\alpha) = \tilde{\alpha}^n \alpha \tilde{\alpha}^{-n} = \alpha$. By Corollary 2, we know that α and $\tilde{\alpha}$ must share a primitive root, and since $\tilde{\alpha}$ is primitive, we must have that $\tilde{\alpha}$ is a primitive root of α so $\tilde{\alpha} \in \{\hat{\alpha}, \hat{\alpha}^{-1}\}$.

Hence, to show that $\rho \circ \psi$ is not an inner automorphism, it is sufficient to show that for some $y \in \mathrm{var}(\alpha)$, we have $\rho \circ \psi(y) \neq \hat{\alpha}^n y \hat{\alpha}^{-n}$ for any $n \in \mathbb{Z}$. We will choose $y = x$, and using the fact that $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$, prove that η or η^{-1} occurs as a factor of $\rho \circ \psi(x)$. It is trivial, of course, that η or η^{-1} occurs as a factor of the unreduced image $\rho \circ \psi(x)$, however, we must show that at least one occurrence of η is not partially or entirely contracted, but rather "survives" and is hence a factor of the reduced word $\rho \circ \psi(x)$. Therefore we must consider all possible contractions occurring in $\rho \circ \psi(x)$. Firstly, we split $\psi(x)$ into factors μ which contain only variables from $\mathbb{N}$, and factors u only containing letters from Σ. More formally, we note that there exist $\mu_0, \mu_1, \dots \mu_m \in \mathcal{F}_{\mathbb{N}}$, $u_1, u_2, \dots u_m \in \mathcal{F}_\Sigma$ such that

$$\psi(x) = \mu_0 \ u_1 \ \mu_1 \ u_2 \ \dots \ u_m \ \mu_m,$$

where each $u_i \neq \varepsilon$, $1 \leq i \leq m$, and each $\mu_i \neq \varepsilon$ for $1 \leq i \leq m-1$. Since $\psi(x)$ contains at least one letter from Σ, we have $m > 0$ (i.e., at least one u_i exists). Moreover, since $\rho(x) = x$ for all $x \in \mathbb{N}$, we have $\rho(\mu_i) = \mu_i$, and hence:

$$\rho \circ \psi(x) = \mu_0 \ \rho(u_1) \ \mu_1 \ \rho(u_2) \ \dots \ \rho(u_m) \ \mu_m. \tag{2}$$

It follows from the fact that $\psi(x)$ is reduced that each μ_i is reduced for $0 \leq i \leq m$. We now consider the factors $\rho(u_i)$ with the following claim:

Claim 1. *Let $u_i \in \mathcal{F}_\Sigma$. Then there exist $p_i, p_i' \in \{k, -k, 2k, -2k\}$ and $q_i, q_i' \in \{1, -1\}$ such that*

$$\rho(u_i) = 1^{p_i} \cdot 2^{q_i} \cdot w_i \cdot 2^{q_i'} \cdot 1^{p_i'}$$

where w_i is reduced, contains η or η^{-1} as a factor, and is enclosed by 1.

Proof (Claim 1). Let $u_i = \mathtt{a}_1^{q_1} \mathtt{a}_2^{q_2} \dots \mathtt{a}_\ell^{q_\ell}$ such that $\mathtt{a}_j \in \Sigma$ and $q_j \in \{1, -1\}$ for $1 \leq j \leq \ell$. We remark that since u_i is reduced, if $\mathtt{a}_j = \mathtt{a}_{j+1}$ then $q_j = q_{j+1}$. For each j, we have $\rho(\mathtt{a}_j^{q_j}) = \gamma_j \ \eta^{q_j} \ \gamma_j'$ where γ_j, γ_j' depend on $\mathtt{a}_j$ and q_j. We can therefore write the following:

$$\rho(u_i) = \overbrace{\gamma_1 \ \eta^{q_1} \ \gamma_1'}^{\rho(\mathtt{a}_1^{q_1})} \ \overbrace{\gamma_2 \ \eta^{q_2} \ \gamma_2'}^{\rho(\mathtt{a}_2^{q_2})} \ \overbrace{\gamma_3 \ \dots \ \gamma_{\ell-1}'}^{\rho(\mathtt{a}_3^{q_3} \dots \mathtt{a}_{\ell-1}^{q_{\ell-1}})} \ \overbrace{\gamma_\ell \ \eta^{q_\ell} \ \gamma_\ell'}^{\rho(\mathtt{a}_\ell^{q_\ell})}.$$

From the definition of ρ, γ_j' is comprised of either 2 or 2^{-1}, followed by a series of 1s or 1^{-1}s, while γ_{j+1} is comprised of a series of 1s or 1^{-1}s followed by 2 or 2^{-1}. More precisely, we have

$$\gamma_j' \ \gamma_{j+1} = 2^{q_j} \cdot 1^r \cdot 2^{q_{j+1}}$$

where we can infer from the definition of ρ that $r = 0$ if and only if $\mathtt{a}_j = \mathtt{a}_{j+1}$ and $q_j = -q_{j+1}$. However, this would contradict the fact that u_i is reduced, so we may assume $r \neq 0$. It follows that

$$\rho(u_i) = \gamma_1 \cdot \eta^{q_1} \cdot \overbrace{2^{q_1} \cdot 1^{r_1} \cdot 2^{q_2}}^{\gamma_1'\gamma_2} \cdot \eta^{q_2} \cdot \overbrace{2^{q_2} \cdot 1^{r_2} \cdot 2^{q_3}}^{\gamma_2'\gamma_3} \cdot \dots \cdot \overbrace{2^{q_{\ell-1}} 1^{r_{\ell-1}} \cdot 2^{q_\ell}}^{\gamma_{\ell-1}'\gamma_\ell} \cdot \eta^{q_\ell} \gamma_\ell'$$

7 The case that $|\mathrm{var}(\alpha)| = 1$ is trivial, since the only morphism fixing a unary pattern is the identity and thus the conditions (i) and (ii) of the proposition cannot be satisfied.

8 $\eta = 1^{2|\hat{\alpha}|}$ or $\eta = 1 \cdot 2^{2|\hat{\alpha}|} \cdot 1$ would suffice.

such that $r_j \neq 0$ for $1 \le j < \ell$. Recall that by definition, each η^{q_i} is reduced and is enclosed by 1. Hence the above word does not contain any contractions and is thus reduced. We can observe that our claim holds simply by taking

$$w_i = \eta^{q_1} \cdot 2^{q_1} \cdot 1^{r_1} \cdot 2^{q_2} \cdot \eta^{q_2} \cdot 2^{q_2} \cdot 1^{r_2} \cdot \ldots \cdot 1^{r_{\ell-1}} \cdot 2^{q_\ell} \cdot \eta^{q_\ell}$$

and noting that $\gamma_1 = 1^{p_i} \cdot 2^{q_i}$ and $\gamma'_\ell = 2^{q'_i} \cdot 1^{p'_i}$ for some $p_i, p'_i \in \{k, -k, 2k, -2k\}$ and $q_i, q'_i \in \{1, -1\}$. □

Now, by the application of Claim 1 to each $\rho(u_i)$ in (2), we can write

$$\rho \circ \psi(x) = \delta_0 \; w_1 \; \delta_1 \; w_2 \; \ldots w_m \; \delta_m \tag{3}$$

such that $\delta_0 = \mu_0 \cdot 1^{p_1} \cdot 2^{q_1}$, $\delta_m = 2^{q'_m} \cdot 1^{p'_m} \cdot \mu_m$ and for $1 \le i < m$,

$$\delta_i = 2^{q'_i} \cdot 1^{p'_i} \cdot \mu_i \cdot 1^{p_{i+1}} \cdot 2^{q_{i+1}}$$

where $p_j, p'_j q_j, q'_j$ and w_j are defined in accordance with Claim 1 for $1 \le j \le m$. In particular, each w_i contains a factor η or η^{-1}, is reduced and is enclosed by 1. We now claim that the reduced δ_i are non-empty and enclosed by 2.

Claim 2. *For each i, $1 \le i < m$, the (reduced) word δ_i is enclosed by 2.*

Proof (Claim 2). Firstly, suppose that μ_i does not consist only of 1s or 1^{-1}s. Then there exist $s_1, s_2 \in \mathbb{Z}$ such that $\mu_i = 1^{s_1} \cdot v \cdot 1^{s_2}$ where v is non-empty, reduced, and does not start or end with 1 or 1^{-1}. Hence

$$\begin{aligned}\delta_i &= 2^{q'_i} \cdot 1^{p'_i} \cdot 1^{s_1} \cdot v \cdot 1^{s_2} \cdot 1^{p_{i+1}} \cdot 2^{q_{i+1}} \\ &= 2^{q'_i} \cdot 1^{p'_i+s_1} \cdot v \cdot 1^{p_{i+1}+s_2} \cdot 2^{q_{i+1}}.\end{aligned}$$

Note that the claim holds provided $p'_i + s_1 \neq 0$ and $p_{i+1} + s_2 \neq 0$. To see that this is true, we simply recall that $|p'_i| \ge k$ and $|p_{i+1}| \ge k$, and since $k > |\psi(x)| > |\mu_i|$ we have $k > |s_1|$ and $k > |s_2|$.

Now suppose instead that μ_i *does* consist only of 1s or 1^{-1}s. Recall that by definition, $\mu_i \neq \varepsilon$. Hence there exists $s \in \mathbb{Z}\backslash\{0\}$ such that $\mu_i = 1^s$, so:

$$\begin{aligned}\delta_i &= 2^{q'_i} \cdot 1^{p'_i} \cdot 1^{s} \cdot 1^{p_{i+1}} \cdot 2^{q_{i+1}} \\ &= 2^{q'_i} \cdot 1^{p'_i+s+p_{i+1}} \cdot 2^{q_{i+1}}.\end{aligned}$$

Again, since $|p'_i| \ge k$ and $|p_{i+1}| \ge k$ and $k > |\psi(x)| > |\mu_i| > s$, we have $p'_i + s + p_{i+1} \neq 0$ and the claim follows. □

We now consider δ_0 and δ_m. Recall that $\delta_0 = \mu_0 1^{p_1} 2^{q_1}$. Since $|\mu_0| < k$ we have $\mu_0 = v1^s$ for some $s < k$ and such that v is reduced and does not end with 1 or 1^{-1}. It follows that the reduced word δ_0 equals $v1^{p_1+s}2^{q_1}$. Since $p_1 \ge k > s$, and since v does not end with 1 or 1^{-1}, there are no further possible contractions. A symmetrical argument can be made for δ_m. Hence, recalling (3), we have

$$\rho \circ \psi(x) = \delta_0 \; w_1 \; \delta_1 \; w_2 \; \ldots w_m \; \delta_m$$

such that, by Claim 2, each (reduced) δ_i is non empty and enclosed by 2, and by Claim 1, each w_i is reduced, contains η or η^{-1} as a factor and is enclosed by 1. Hence there are no contractions occurring outside the δ_i factors, and at least one factor η or η^{-1} survives in the reduced word $\rho \circ \psi(x)$, so $\rho \circ \psi(x) \neq \hat{\alpha}^n x \hat{\alpha}^{-n}$ for any $n \in \mathbb{Z}$. Hence, $\psi' = \rho \circ \psi$ is not an inner automorphism as required. □

We are now ready to prove our main result characterising the existence of injective morphisms which are unambiguous up to inner automorphism.

Theorem 39. *Let $\alpha \in \mathcal{F}_{\mathbb{N}}$. There exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to inner automorphism w.r.t. α if and only if every morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α is an inner automorphism.*

Proof. The "only if" direction is given by Proposition 9. Hence we consider the "if" direction. Suppose that α is only fixed by inner automorphisms (and thus the identity morphism is unambiguous up to inner automorphism w.r.t. α). Let $\sigma_{\alpha,\alpha} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be defined according to Definition 31. Note that $\sigma_{\alpha,\alpha}$ is injective. Let $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ be a morphism such that $\tau(\alpha) = \sigma_{\alpha,\alpha}(\alpha)$. Let S be a set of anchor segments for τ, and let $\varphi_{\tau,S} : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be defined according to Definition 15. Note that by Theorem 33, $\varphi_{\tau,S}(\alpha) = \alpha$, and hence $\varphi_{\tau,S}$ must be an inner automorphism. By Proposition 36, either $\tau = \sigma_{\alpha,\alpha} \circ \varphi_{\tau,S}$, or α is fixed by a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that, for some $x \in \mathrm{var}(\alpha)$, $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$. By Proposition 38, this implies that α is fixed by a morphism which is not an inner automorphism and that is a contradiction. Consequently, for any morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ with $\tau(\alpha) = \sigma_{\alpha,\alpha}(\alpha)$, we must have that $\tau = \sigma_{\alpha,\alpha} \circ \varphi_{\tau,S}$ for some inner automorphism $\varphi_{\tau,S}$. It follows that $\sigma_{\alpha,\alpha}$ is unambiguous up to inner automorphism w.r.t. α. $\square$

One way to prove the same result for the case of ambiguity up to automorphism would be to show that a stronger form of Proposition 38 holds, namely that if $\alpha = \beta$, and $\psi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ is a morphism satisfying the two conditions of Proposition 36, then we are able to produce a morphism ψ' which is not an automorphism such that $\psi'(\alpha) = \alpha$.

We expect this statement to indeed be true, due to the combinatorially rich set of morphisms fixing α induced by the existence of such a morphism ψ, as explained in Example 37. Nevertheless, the stronger statement seems to be considerably more complicated to prove, and thus we present it instead as a conjecture.

Conjecture 40. *Let $\alpha \in \mathcal{F}_\mathbb{N}$. If there exists a morphism $\psi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)\cup\Sigma}$ such that:*

(i) $\psi(\alpha) = \alpha$, and
(ii) $\Sigma \cap \mathrm{symb}(\psi(x)) \neq \emptyset$ for some $x \in \mathrm{var}(\alpha)$,

then there exists a morphism $\psi' : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ which is not an automorphism, and such that $\psi'(\alpha) = \alpha$.

Using the same reasoning as for the proof of Proposition 38, we are able to construct a morphism $\rho \circ \psi$ fixing α such that for any $\eta \in \mathcal{F}_\mathbb{N}$, there exists $x \in \mathrm{var}(\alpha)$, such that η occurs as a factor of $\rho \circ \psi(x)$. Hence we see that our conjecture holds for any pattern α for which there exists $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$ such that for every automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ with $\varphi(\alpha) = \alpha$, η does not occur as a factor of $\varphi(x)$ for any $x \in \mathrm{var}(\alpha)$. Hence we can reduce our conjecture to the following. Note that it follows from Theorem 11 and Proposition 9 that all injective morphisms are ambiguous with respect to morphically imprimitive patterns, so we only need to consider those which are morphically primitive.

Conjecture 41. *Given a morphically primitive pattern $\alpha \in \mathcal{F}_\mathbb{N}$, there exists $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$ such that, for every automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α, η does not occur as a factor of $\varphi(x)$ for any $x \in \mathrm{var}(\alpha)$.*

We provide the following two comments concerning Conjecture 41. Firstly, we note that for any such factor η there exists an automorphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ with η as a factor of $\varphi(x)$ for some $x \in \mathrm{var}(\alpha)$. For example, we may simply take the inner automorphism generated by η so that $\varphi(x) = \eta \cdot x \cdot \eta^{-1}$. It is worth pointing out however, that these inner automorphisms only fix the restricted set of patterns sharing a primitive root with η, and therefore do not all fix a single pattern α and are not sufficient to disprove the conjecture.

Secondly, we point out that there exist patterns α such that, for every $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$, there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α and such that η occurs as a factor of $\varphi(x)$ for some $x \in \mathrm{var}(\alpha)$. For example, let $\alpha = 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3$. For $\eta \in \mathcal{F}_{\mathrm{var}(\alpha)}$, let $\varphi_\eta : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ be the morphism given by $\varphi(1) = 1 \cdot \eta^{-1}$, $\varphi(2) = \eta \cdot 2 \cdot \eta^{-1}$ and $\varphi(3) = \eta \cdot 3$. Then

$$\begin{aligned}\varphi(\alpha) &= \overbrace{1 \cdot \eta^{-1}}^{\varphi(1)}\overbrace{\eta \cdot 2 \cdot \eta^{-1}}^{\varphi(2)}\overbrace{\eta \cdot 3}^{\varphi(3)}\;\overbrace{1 \cdot \eta^{-1}}^{\varphi(1)}\overbrace{\eta \cdot 2 \cdot \eta^{-1}}^{\varphi(2)}\overbrace{\eta \cdot 2 \cdot \eta^{-1}}^{\varphi(2)}\overbrace{\eta \cdot 3}^{\varphi(3)}\\ &= 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3 = \alpha.\end{aligned}$$

However, all such examples known to the authors can be shown to be morphically imprimitive, and thus do not possess an injective morphism which is unambiguous up to automorphism. Thus, while these comments provide an insight into the complexity of our open questions, they are not sufficient to disprove our conjecture. To the contrary, the structures involved seem to indicate a necessity that any pattern fixed by morphisms with “arbitrary” factors appearing in the images, must have some inherent ambiguous structure causing morphic imprimitivity, and hence appear to support the conjecture.

We conclude this section with the following theorem which deals with unambiguity up to automorphism, and also offers a characterisation of when a pattern in a free group possesses an injective morphism which is unambiguous up to automorphism subject to the correctness of Conjecture 40 (or Conjecture 41). Nevertheless, assuming the conjecture holds, we have a particularly interesting situation: not only are we able to recreate the same analogy to the free monoid as we have for ambiguity up to inner automorphism, but we are also able to generalise a second existing characterisation from the free monoid to the free group, implying that the weaker of the two forms of unambiguity has the closest relation to unambiguity in a free monoid.

Theorem 42. *Let $\alpha \in \mathcal{F}_\mathbb{N}$. If Conjecture 40 holds, then the following statements are equivalent:*

1. *There exists an injective morphism $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ which is unambiguous up to automorphism w.r.t. α.*
2. *The only morphisms $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ fixing α are automorphisms (i.e., α is a* test word*).*
3. *α is morphically primitive.*

Proof. The equivalence of Statements (2) and (3) is given in Theorem 11. Hence it remains to show the equivalence of Statements (1) and (2). This can be derived straight from the proof of Theorem 39, using on Conjecture 40 in the place of Proposition 38. $\square$

5. Application: properties of pattern languages

Finally, we take advantage of our construction from Section 4.1 to provide some simple proofs of properties of terminal-free pattern languages over a group alphabet. Firstly, we are able to characterise when two such languages satisfy a subset relation. It is unsurprising that our construction leads to this result, as it is a generalisation of the construction of Jiang et al. [10] whose purpose was exactly to prove the equivalent statement for pattern languages in a free monoid.

Theorem 43. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. Then $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$ if and only if there exists a morphism $\varphi : \mathcal{F}_{\mathbb{N}} \to \mathcal{F}_{\mathbb{N}}$ such that $\varphi(\beta) = \alpha$.*

Proof. Suppose firstly that there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$. By definition, for every $w \in L_\Sigma(\alpha)$, there exists $\sigma : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\sigma(\alpha) = w$. Clearly, $\sigma \circ \varphi(\beta) = w$, so for every $w \in L_\Sigma(\alpha)$, we have $w \in L_\Sigma(\beta)$ and thus $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$.

Now suppose that $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$. Then since $\sigma_{\alpha,\beta}(\alpha) \in L_\Sigma(\alpha)$ where $\sigma_{\alpha,\beta}$ is defined according to Definition 31, there exists a morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{\alpha,\beta}(\alpha)$. By Theorem 33, this implies the existence of $\varphi : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_{\mathrm{var}(\alpha)}$ such that $\varphi(\beta) = \alpha$. $\square$

Of course our characterisation of the inclusion problem for group pattern languages automatically provides a characterisation for the equivalence problem.

Corollary 44. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. Then $L_\Sigma(\alpha) = L_\Sigma(\beta)$ if and only if there exist morphisms $\varphi, \psi : \mathcal{F}_{\mathbb{N}} \to \mathcal{F}_{\mathbb{N}}$ such that $\varphi(\beta) = \alpha$ and $\psi(\alpha) = \beta$.*

Moreover, Day et al. [3] use the construction of Jiang et al. [10] to give a characterisation of when the union of two (monoid) pattern languages is again a (monoid) pattern language. By using the same technique, we are able to exploit our construction in Section 4.1 further to provide the same result for group pattern languages.

Theorem 45. *Let $\alpha, \beta \in \mathcal{F}_{\mathbb{N}}$. Then there exists $\gamma \in \mathcal{F}_{\mathbb{N}}$ satisfying $L_\Sigma(\alpha) \cup L_\Sigma(\beta) = L_\Sigma(\gamma)$ if and only if $L_\Sigma(\alpha) \subseteq L_\Sigma(\beta)$ and $L_\Sigma(\beta) = L_\Sigma(\gamma)$, or $L_\Sigma(\beta) \subseteq L_\Sigma(\alpha)$ and $L_\Sigma(\alpha) = L_\Sigma(\gamma)$.*

Proof. (Adapted from Day et al. [3].) The "if" direction is trivial. We consider the "only if" direction. Suppose that $L_\Sigma(\alpha) \cup L_\Sigma(\beta) = L_\Sigma(\gamma)$. Let $\sigma_{\gamma,\alpha}$ and $\sigma_{\gamma,\beta}$ be defined according to Definition 31. In particular, note that there exist $k_1, k_2 \in \mathbb{N}$ such that $\sigma_{\gamma,\alpha} = \sigma_{k_1,\mathrm{var}(\gamma)}$ (as defined in Definition 18) and $\sigma_{\gamma,\beta} = \sigma_{k_2,\mathrm{var}(\gamma)}$. Let $k = \max(k_1, k_2)$. Note that for the union relation to hold, there exists a morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$ such that $\tau(\alpha) = \sigma_{k,\mathrm{var}(\gamma)}(\gamma)$ or a morphism $\tau : \mathcal{F}_{\mathrm{var}(\beta)} \to \mathcal{F}_\Sigma$ such that $\tau(\beta) = \sigma_{k,\mathrm{var}(\gamma)}(\gamma)$. W.l.o.g. suppose that $\tau(\alpha) = \sigma_{k,\mathrm{var}(\gamma)}(\gamma)$ for some morphism $\tau : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_\Sigma$. By Theorem 33 and Remark 34, this implies that there exists a morphism $\varphi : \mathcal{F}_{\mathrm{var}(\alpha)} \to \mathcal{F}_{\mathrm{var}(\gamma)}$ such that $\varphi(\alpha) = \gamma$. It follows from Theorem 43 that $L_\Sigma(\gamma) \subseteq L_\Sigma(\alpha)$. It is clear that $L_\Sigma(\alpha) \subseteq L_\Sigma(\gamma)$ and $L_\Sigma(\beta) \subseteq L_\Sigma(\gamma)$, and hence by Corollary 44 $L_\Sigma(\alpha) = L_\Sigma(\gamma)$, and our statement holds. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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