ELASTIC WAVES IN PERIODICALLY HETEROGENEOUS TWO-DIMENSIONAL MEDIA: LOCALLY PERIODIC AND ANTI-PERIODIC MODES

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Abstract

Propagation of anti-plane waves through a discrete square lattice and through a continuous fibrous medium is studied. In the long-wave limit, for periodically heterogeneous structures the solution can be periodic or anti-periodic across the unit cell. It is shown that combining periodicity and anti-periodicity conditions in different directions of the translational symmetry allows one to detect different types of modes that do not arise in the purely periodic case. Such modes may be interpreted as counterparts of non-classical waves appearing in phenomenological theories. Dispersion diagrams of the discrete square lattice are evaluated in a closed analytical from. Dispersion properties of the fibrous medium are determined using Floquet-Bloch theory and Fourier series approximations. Influence of a viscous damping is taken into account.

Key words: wave propagation; heterogeneous media; phononic bands; dispersion; Floquet-Bloch waves; gradient elasticity; Biot's theory.

1. Introduction

Wave propagation in heterogeneous materials and structures is characterised by a number of significant phenomena, which can never be observed in homogeneous solids. In recent decades, intensive studies have been devoted to photonic and phononic band gaps (Sigalas and Economou, 1993; Kushwaha et al., 1994; Nicorovici et al., 1995; Movchan et al., 2002), negative refraction (Pendry, 2000; Grbic and Eleftheriades, 2004), dynamic anisotropy and waves focusing (Ayzenberg-Stepanenko and Slepyan, 2008; Colquitt et al., 2012), acoustic diodes (Liang et al., 2009), acoustically invisible cloaks (Milton et al., 2006; Norris and Shuvalov, 2011; Colquitt et al., 2013), waves localisation in structures with defects (Craster et al., 2010b; Colquitt et al., 2011;

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Andrianov et al., 2014). All these remarkable effects and properties have a great practical importance for a large variety of applications in physics, engineering, biomechanics, and many other areas. Extended survey of the recent progress in the field can be found in the review papers by Maldovan (2013) and Hussein et al. (2014).

Theoretical modelling of waves in periodically heterogeneous media can be performed using Floquet-Bloch theory. Named after the original theorems of Floquet (1883) and Bloch (1928), it has been documented in the classical book by Brillouin (2003) and later utilized by many authors. According to this approach, a solution is represented as an effective wave multiplied by a modulation function; such a modulation aims to describe the influence of the microstructure. Spatial periodicity of the medium implies similar periodicity conditions for the modulation function. This yields an eigenvalue problem, which allows to determine dispersion relations between the frequency and the wave number. In order to evaluate numerical results, unknown fields in a heterogeneous medium are usually approximated by some series expansions. The most popular implementations are based on Fourier series (Sigalas and Economou, 1993; Kushwaha et al., 1994; Vasseur et al., 1994) and on Rayleigh multipole expansions (Nicorovici et al., 1995; Poulton et al., 2000; Movchan et al., 2002).

Another approach is provided by a two-scale asymptotic homogenisation procedure. Supposing the wave length to be considerably larger than the characteristic size of the microstructure, it becomes possible to separate macro- and micro-scale components of the solution. The latter are evaluated from a recurrent sequence of cell problems. Next, application of the homogenising operator over the unit cell domain allows one to obtain homogenised (so called effective) differential equations governing the macroscopic response of the material. The method of asymptotic homogenisation was originally developed for quasi-static problems (see, for example, a review by Kalamkarov et al. (2009) and references therein). Later, taking into account more terms of two-scale asymptotic expansions gave a possibility to derive new higher-order macroscopic equations that are applicable for modelling dynamic problems and can predict the dispersion properties of the medium (Boutin and Auriault, 1993; Fish and Chen, 2001; Andrianov et al., 2008; Soubestre and Boutin, 2012; Auriault and Boutin, 2012).

More recently, the two-scale asymptotic procedure was successfully generalised to a high-frequency domain. Choosing the leading order approximation corresponding to resonant frequencies of the unit cell (i.e., to standing waves excited at the band gaps thresholds) made it possible to describe a dynamic response of heterogeneous materials even when the wave length is comparable to the length-scale of the microstructure. The original idea has been proposed by Daya and Potier-Ferry (2001) and was further considerably developed by Craster et al. (2010a, 2010b),

Nolde et al. (2011), Antonakakis et al. (2014) and Colquitt et al. (2015). The high-frequency homogenisation technique is closely related to the effective mass framework established in the solid state physics (see, for example, Kittel, 2005).

The approaches described above are based on rigorous theoretical origins in the sense that the obtained macroscopic models encapsulate information about the microstructure and all the coefficients of these models can be determined basing on the geometry of the unit cell and properties of the components. An alternative way to describe the dynamic behaviour of heterogeneous media is provided by gradient continuum theories. According to them, the influence of the microstructure is modelled phenomenologically by allowing the medium some additional internal degrees of freedom. As a result, it becomes possible to derive macroscopic equations capable to predict scattering of the wave field by the microstructure. However, the coefficients of such equations are usually not known a priori and are expected to be determined experimentally. The origins of the gradient elasticity can be traced back to Cauchy (1851), who suggested to use higher-order spatial derivatives in continuum equations for discrete lattices, and to Voigt (1887), who included molecular rotations into the lattice models of crystals. Gradient theories for elastic continua were introduced by Cosserat (1909) and Le Roux (1911). Later on, various models were developed and specified by Aero and Kuvshinskii (1961), Toupin (1962), Kröner (1963), Mindlin (1964), Kunin (1966, 1982, 1983), Herrmann and Achenbach (1968), Levin (1971), and many others. Eringen (1983) has shown that gradient models can be also derived from integral non-local theories. Transformation of non-local problems into gradient ones were recently presented by Borino and Polizzotto (2014). A detailed review of the gradient elasticity can be found in the paper by Askes and Aifantis (2011).

In a general case, assigning a medium by additional degrees of freedom can induce new forms of motion. That is why gradient theories may predict some new kinds of waves, which do not appear in the classical theory of elasticity. As examples, we can note waves associated with micro-rotations in Cosserat continuum and so called "slow" or "second" wave appearing in Biot's theory. Biot (1956a, 1956b, 1962) has introduced a phenomenological theory to describe propagation of elastic waves in porous media. In the framework of Biot's model, two types of longitudinal waves can arise. The first wave corresponds to the case when solid and fluid components move in-phase on the micro level, which is consistent with the classical theory. The second wave requires a relative shift between the components, so they can move out-of-phase. Existence of the second Biot's mode was questioned many times. Experimental confirmations were reported by Plona (1980), Lakes et al. (1983), Hosokawa and Otani (1997).

The aim of the paper is to bridge a gap between rigorous and phenomenological approaches. We would like to show that theoretical models are able to predict some additional types of waves, which may be interpreted as counterparts of non-classical waves appearing in gradient theories. Our analysis is based on so called "envelope continualisation" approach, which is originated back to the papers by Il'iushina (1969) and Kovalev and Kosevich (1975). In physics it is usually referred to as the "semi-discrete approximation" (see, for example, Remoissenet (1986) and Dauxois (1991)).

As an illustrative example, let us consider wave propagation through a discrete lattice of identical particles connected by elastic springs. The classical solution is obtained using the irreducible unit cell that contains a single particle. According to this model, in the long-wave limit the neighbouring particles move nearly in-phase. For continuous heterogeneous media it implies that the wave field is locally periodic across the unit cell. The shape of the wave is schematically sketched in Fig. 1a. As the wave length tends to infinity, the frequency vanishes and no vibrations occur, so the motion is simply a rigid body translation.

However, there exist another possibility to describe the dynamic behaviour of the lattice. Let us now consider a unit cell containing two particles. If in the long-wave limit the neighbouring particles move out-of-phase, then the solution is represented as a combination of the envelope wave propagating on the macro level and the carrier wave that corresponds to sawtooth vibrations of the lattice (Fig. 1b). In continuous media such types of waves can be predicted by imposing anti-periodicity conditions at the opposite sides of the unit cell. At the long-wave limit, the motion turns to a standing wave with zero group velocity and a non-zero frequency.

As we can see, the choice of the reference unit cell can principally influence the solution. Rallu et al. (2017) studied different waves arising in one-dimensional lattices and reticulated beams with double and triple periodic macro cells. Vasiliev et al. (2010) have shown that considering a macro cell consisting of several translational cells increases the number of independent fields and brings additional degrees of freedom. Thus, multi-field models of different orders can be developed.

In this paper, we specialise the analysis to the waves which are periodic and anti-periodic with respect to the only one primitive unit cell. Our motivation is that such waves can present the main theoretical importance for the development of homogenised models. In the long-wave case they describe the low-frequency and the high-frequency limiting states of vibrations, which correspond accordingly to the continuous and to the semi-discrete approximation. Increasing the number of particles in the macro cell would provide us with additional models valid in a middle-frequency range, but it does not determine new forms of motion at the long-wave limit. The middle-frequency approximations can also be developed matching the low- and high-frequency solutions with the help of two-point Padé approximants (see a recent contribution by Colquitt et al. (2018)).
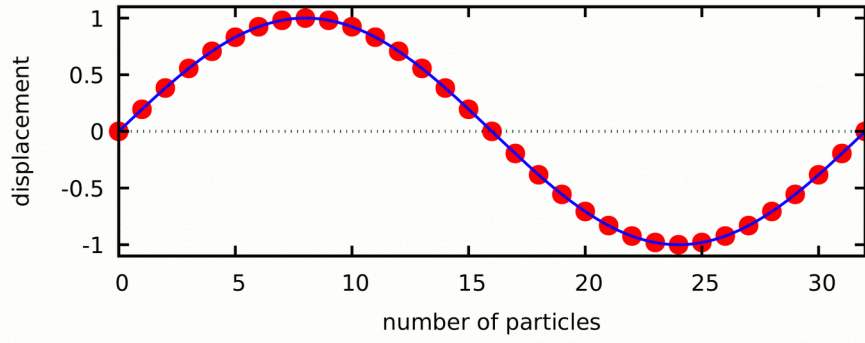
a) periodic mode
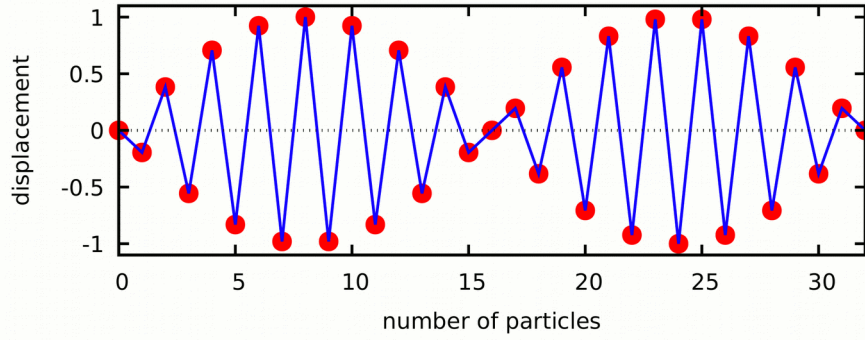
b) anti-periodic mode

Fig. 1. Shape of the waves in the one-dimensional monatomic lattice.

It is important to note that propagation of high-frequency long waves in heterogeneous structures can represent a fundamental analogy with the waves in thin-walled elastic waveguides (Craster et al., 2014; Kaplunov et al., 2017). The local solution across the unit cell corresponds to the solution within the transverse cross section of a thin rod, plate or shell. A theoretical framework for the asymptotic theories of long wave motion in thin-walled structures was developed by Rogerson et al. (2006, 2007, 2009), Lutianov and Rogerson (2010), Mukhomodyarov and Rogerson (2012).

The paper is organised as follows. In Section 2 we consider anti-plane waves in a two-dimensional lattice. The discrete model allows us to derive explicit results for the dispersion properties in a closed analytical form. The influence of a viscous damping is studied and continuous approximations in the long-wave case are presented. In Section 3 we generalise the developed solutions to a continuous medium. Propagation of anti-plane shear waves through a square array of cylindrical inclusions embedded into an isotopic matrix is considered. Dispersion relations are

obtained using Floquet-Bloch theory accompanied by periodic and anti-periodic boundary conditions at the edges of the unit cell. Numerical results are evaluated using a Fourier series approximation. The developed solutions are verified by some comparisons with data of other authors. Section 4 is devoted to the conclusions.

2. Two-dimensional lattice

We start with a problem of propagation of anti-plane waves through an infinite periodic square lattice consisting of identical particles connected by elastic springs. Discrete lattice-type models are widely used to describe vibrations in crystals (Kittel, 2005); in foams (Cantat et al., 2013); in cellular structures and bone tissues (Gibson et al., 2010). Some novel applications include modelling of polymer molecules, atomic lattices (e.g., graphene), and nanocrystalline materials (Friesecke, James, 2000; Potapov et al., 2009). Discrete models can also appear in engineering, e.g., for simulating lightweight truss structures with attached masses. Waves in complex lattices and cracked media were studied by Slepyan (2002). For a detailed review of the subject we refer to Andrianov et al. (2010).

In contrast to continuous media, discrete lattices often allow evaluation of exact dispersion relations in a closed analytical form, which is particularly convenient for further analysis. That is why they can be considered as benchmark models to highlight the main effects of the wave propagation in heterogeneous structures.

A single-particle unit cell of the lattice under consideration is displayed at Fig. 2. The mass of the particle is m. Every particle is supposed to interact with its eight neighbours. The interaction forces are linear and represent an analogy to anti-plane shear in a continuous medium. We assume that the interaction forces can depend on the distance between particles, so the springs have different stiffness, namely, c and c_0 in the orthogonal and in the diagonal directions respectively.

The classical solution can be developed as follows. The equation of motion of a typical particle is given by

$$m\frac{d^2 u_{n_1,n_2}}{dt^2} = c\left(u_{n_1+1,n_2} + u_{n_1-1,n_2} + u_{n_1,n_2+1} + u_{n_1,n_2-1} - 4u_{n_1,n_2}\right) + \\ c_0\left(u_{n_1+1,n_2+1} + u_{n_1-1,n_2-1} + u_{n_1+1,n_2-1} + u_{n_1-1,n_2+1} - 4u_{n_1,n_2}\right),$$

(1)

where u is the displacement in the direction transverse to the $x_1 x_2$ plane and subscripts n_1, n_2 denote the number of the particle, $n_1, n_2 = 0, \pm 1, \pm 2, \ldots$.
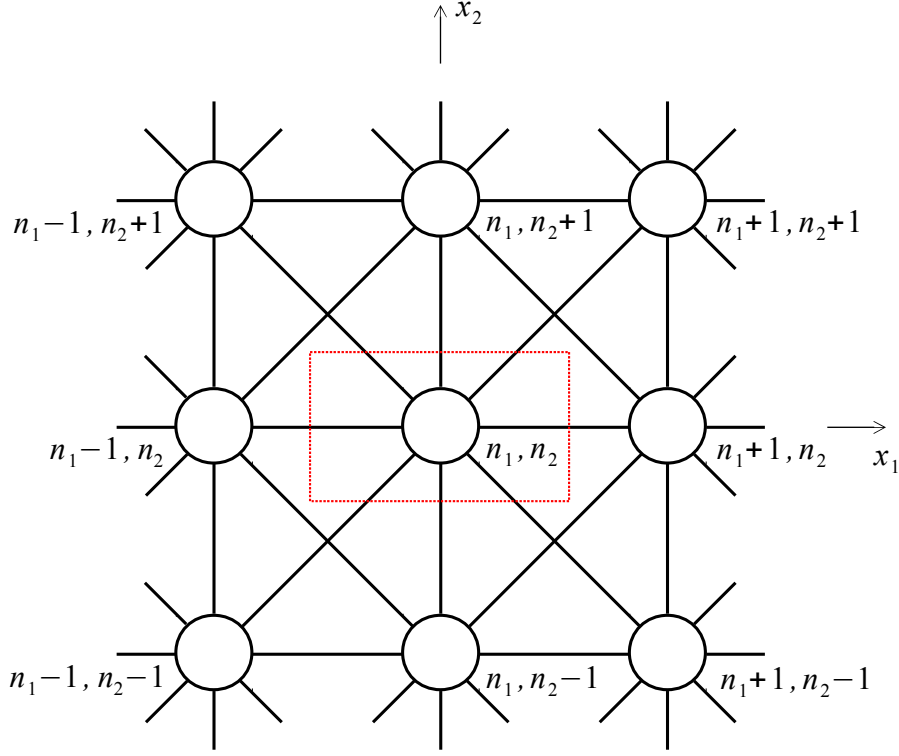
Fig. 2. Square lattice with a single-particle unit cell.

Let us consider a harmonic wave propagating with amplitude A, frequency ω and non-dimensional wave vector $\mathbf{k}$ having components k_1, k_2

$$u_{n_1, n_2} = A \exp(i \omega t) \exp\left[-i\left(k_1 n_1 + k_2 n_2\right)\right] . \tag{2}$$

Substituting (2) into (1), we obtain the dispersion relation

$$\bar{\omega}^2 = 4\left[\sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right)\right] + \frac{4 c_0}{c}\left[\sin^2\left(\frac{k_1 + k_2}{2}\right) + \sin^2\left(\frac{k_1 - k_2}{2}\right)\right], \tag{3}$$

where $\bar{\omega}$ is the non-dimensional frequency, $\bar{\omega} = \omega\left(m/c\right)^{1/2}$.

The monatomic lattice under consideration has only one pass band. Equation (3) describes a conventional acoustic mode, whose frequency vanishes as the wave length tends to infinity. This long-wave limit corresponds to non-dispersive propagation that happens in homogeneous media. However, in heterogeneous structures there can appear additional types of modes, which do not

match the classical long-wave zero-frequency limit. To show this, let us consider a unit cell that includes four particles (Fig. 3).
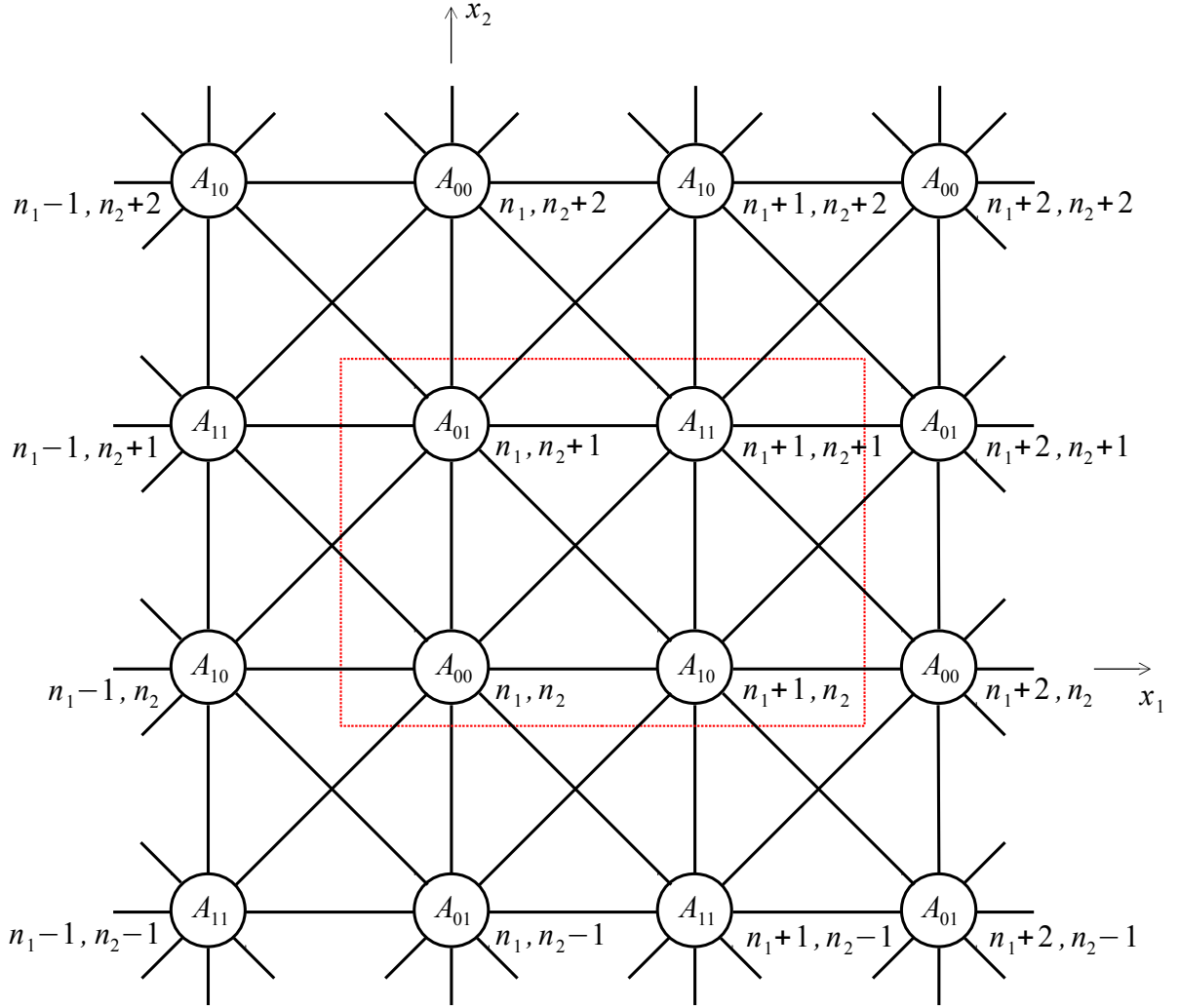

Fig. 3. Lattice with a four-particles unit cell. Different amplitudes of the particles are indicated.

We use different notations for the displacements of odd and even particles, i.e.

$$
u_{n_1,n_2} =
\begin{cases}
u_{n_1,n_2}^{(00)} & \text{for } n_1, n_2 = \pm 0, \pm 2, \pm 4, \ldots; \\
u_{n_1,n_2}^{(01)} & \text{for } n_1 = \pm 0, \pm 2, \pm 4, \ldots, n_2 = \pm 1, \pm 3, \pm 5, \ldots; \\
u_{n_1,n_2}^{(10)} & \text{for } n_1 = \pm 1, \pm 3, \pm 5, \ldots, n_2 = \pm 0, \pm 2, \pm 4, \ldots; \\
u_{n_1,n_2}^{(11)} & \text{for } n_1, n_2 = \pm 1, \pm 3, \pm 5, \ldots .
\end{cases}
$$

The equations of motion for the particles in the unit cell take the form:

$$m \frac{d^2 u^{(00)}_{n_1, n_2}}{dt^2} = c\left(u^{(10)}_{n_1+1, n_2} + u^{(10)}_{n_1-1, n_2} + u^{(01)}_{n_1, n_2+1} + u^{(01)}_{n_1, n_2-1} - 4 u^{(00)}_{n_1, n_2}\right) + $$
$$c_0\left(u^{(11)}_{n_1+1, n_2+1} + u^{(11)}_{n_1-1, n_2-1} + u^{(11)}_{n_1+1, n_2-1} + u^{(11)}_{n_1-1, n_2+1} - 4 u^{(00)}_{n_1, n_2}\right),$$

$$m \frac{d^2 u^{(01)}_{n_1, n_2+1}}{dt^2} = c\left(u^{(11)}_{n_1+1, n_2+1} + u^{(11)}_{n_1-1, n_2+1} + u^{(00)}_{n_1, n_2+2} + u^{(00)}_{n_1, n_2} - 4 u^{(01)}_{n_1, n_2+1}\right) + $$
$$c_0\left(u^{(10)}_{n_1+1, n_2+2} + u^{(10)}_{n_1-1, n_2} + u^{(10)}_{n_1+1, n_2} + u^{(10)}_{n_1-1, n_2+2} - 4 u^{(01)}_{n_1, n_2}\right),$$

$$m \frac{d^2 u^{(10)}_{n_1+1, n_2}}{dt^2} = c\left(u^{(00)}_{n_1+2, n_2} + u^{(00)}_{n_1, n_2} + u^{(11)}_{n_1+1, n_2+1} + u^{(11)}_{n_1+1, n_2-1} - 4 u^{(10)}_{n_1+1, n_2}\right) + $$
$$c_0\left(u^{(01)}_{n_1+2, n_2+1} + u^{(01)}_{n_1, n_2-1} + u^{(01)}_{n_1+2, n_2-1} + u^{(01)}_{n_1, n_2+1} - 4 u^{(10)}_{n_1+1, n_2}\right),$$

$$m \frac{d^2 u^{(11)}_{n_1+1, n_2+1}}{dt^2} = c\left(u^{(01)}_{n_1+2, n_2+1} + u^{(01)}_{n_1, n_2+1} + u^{(10)}_{n_1+1, n_2+2} + u^{(10)}_{n_1+1, n_2} - 4 u^{(11)}_{n_1+1, n_2+1}\right) + $$
$$c_0\left(u^{(00)}_{n_1+2, n_2+2} + u^{(00)}_{n_1, n_2} + u^{(00)}_{n_1+2, n_2} + u^{(00)}_{n_1, n_2+2} - 4 u^{(11)}_{n_1+1, n_2+1}\right). \tag{4}$$

Let us suppose that the particles in the unit cell can vibrate with different amplitudes A_{00}, A_{01}, A_{10}, A_{11}, i.e.

$$u^{(j_1 j_2)}_{n_1, n_2} = A_{j_1 j_2} \exp(i\omega t) \exp[-i(k_1 n_1 + k_2 n_2)], \quad j_1, j_2 = 0,1. \tag{5}$$

Substituting (5) into (4), we come to a system of equations for the unknown amplitudes. In the matrix form it reads

$$[\mathbf{M}(\mathbf{k}) + \bar{\omega}^2 \mathbf{I}_4] \cdot \mathbf{A} = 0. \tag{6}$$

Here $\mathbf{A} = [A_{00}, A_{01}, A_{10}, A_{11}]^T$; $\mathbf{I}_4$ is the 4th order identity matrix; $\mathbf{M}$ is the symmetric matrix

$$\mathbf{M} = \begin{vmatrix} -4\left(1+\dfrac{c_0}{c}\right) & 2\cos(k_2) & 2\cos(k_1) & \dfrac{2 c_0}{c}[\cos(k_1+k_2)+\cos(k_1-k_2)] \\ \dots & -4\left(1+\dfrac{c_0}{c}\right) & \dfrac{2 c_0}{c}[\cos(k_1+k_2)+\cos(k_1-k_2)] & 2\cos(k_1) \\ \dots & \dots & -4\left(1+\dfrac{c_0}{c}\right) & 2\cos(k_2) \\ \dots & \dots & \dots & -4\left(1+\dfrac{c_0}{c}\right) \end{vmatrix}.$$

Equation (6) determines the eigenvalue problem that allows us to evaluate the dispersion relation. Depending on the ratio between the amplitudes of the particles, four different types of waves can be distinguished.

1. $A_{00}=A_{01}=A_{10}=A_{11}$. In this case the particles in the unit cell move in-phase. In the long-wave limit as $k_1, k_2 \rightarrow 0$ one obtains a solution, which is periodic across the unit cell in both the x_1 and x_2 directions. We refer to this type of wave as a periodic mode and will denote it by the abbreviation P1P2. This is the conventional acoustic mode that appears in the classical theory. The dispersion relation is given by equation (3).

2. $A_{00}=A_{01}=-A_{10}=-A_{11}$. The neighbouring particles move out-of-phase in the direction x_1 and in-phase in the direction x_2. In the long-wave limit the solution is locally anti-periodic by x_1 and periodic by x_2. This mode is denoted as A1P2. The dispersion relation reads

$$\bar{\omega}^2 = 4\left[\cos^2\left(\frac{k_1}{2}\right)+\sin^2\left(\frac{k_2}{2}\right)\right]+\frac{4c_0}{c}\left[\cos^2\left(\frac{k_1+k_2}{2}\right)+\cos^2\left(\frac{k_1-k_2}{2}\right)\right]. \tag{7}$$

3. $A_{00}=-A_{01}=A_{10}=-A_{11}$. The wave is periodic by x_1 and anti-periodic by x_2 (P1A2 mode). The solution is identical to case 2 with the interchanging $k_1 \Leftrightarrow k_2$:

$$\bar{\omega}^2 = 4\left[\sin^2\left(\frac{k_1}{2}\right)+\cos^2\left(\frac{k_2}{2}\right)\right]+\frac{4c_0}{c}\left[\cos^2\left(\frac{k_1+k_2}{2}\right)+\cos^2\left(\frac{k_1-k_2}{2}\right)\right]. \tag{8}$$

4. $A_{00}=-A_{01}=-A_{10}=A_{11}$. In this case the solution is anti-periodic in both the x_1 and x_2 directions (A1A2 mode). The dispersion relation is

$$\bar{\omega}^2 = 4\left[\cos^2\left(\frac{k_1}{2}\right)+\cos^2\left(\frac{k_2}{2}\right)\right]+\frac{4c_0}{c}\left[\sin^2\left(\frac{k_1+k_2}{2}\right)+\sin^2\left(\frac{k_1-k_2}{2}\right)\right]. \tag{9}$$

From equations (6)–(9), it is easy to see that the anti-periodicity of the solution in the direction x_1 (or x_2) results in a shift of the corresponding component of the wave vector to half of its period: $k_1 \Leftrightarrow k_1+\pi$ (or $k_2 \Leftrightarrow k_2+\pi$).

From the physical point of view, the developed solutions can be interpreted as follows. The original difference equation (1) accounts for the displacement of individual particles. We have introduced new equations (4), which describe a motion of the centre of masses of the unit cell and motions around this centre. Then, the periodic modes represent a displacement of the centre of masses. The anti-periodic modes describe oscillations around a stationary centre of masses.

Let us consider two numerical examples. In the first case, we suppose that the interaction forces are inversely proportional to the distance between the particles, then $c_0 = c/\sqrt{2}$. The dispersion curves are presented at Fig. 4. The right part of the diagram displays the solution for the x_1 direction of the wave propagation ($k_1 = k$, $k_2 = 0$) and the left part corresponds to the diagonal direction ($k_1 = k_2 = k/\sqrt{2}$). When the frequency $\bar{\omega}$ exceeds the cut-off threshold, the wave number k becomes complex. Then the signal decays exponentially with the attenuation coefficient equal to the imaginary part of the wave number.
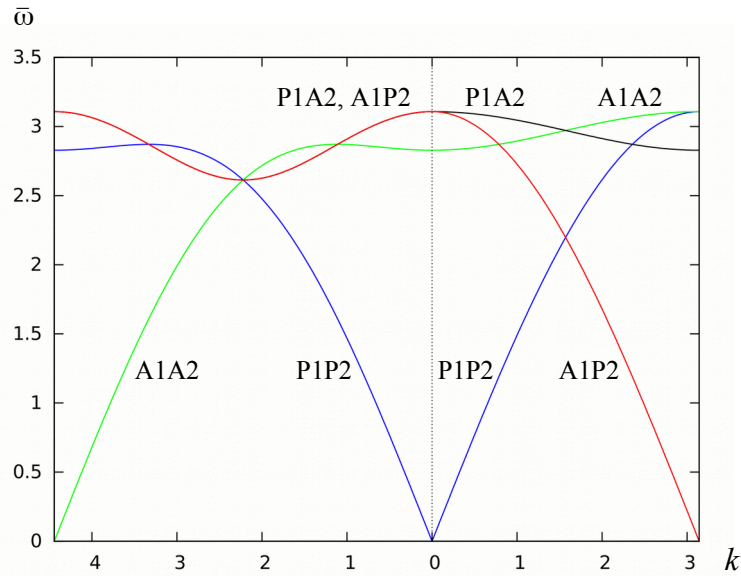

Fig. 4. Dispersion curves of the square lattice with $c_0 = c/\sqrt{2}$.

Blue – P1P2 mode; red – A1P2 mode; black – P1A2 mode; green – A1A2 mode.

In the long-wave limit the frequency of the periodic mode P1P2 vanishes, $\bar{\omega} \to 0$ as $k \to 0$, so no vibrations occur. Alternatively, all the anti-periodic modes tend to standing waves with zero group velocities and non-zero frequencies. This regime can be considered as "hidden" or "trapped" modes, when no energy is transmitted on the macro scale, but on the micro scale the lattice exhibits sawtooth oscillations.

Let us analyse waves propagating in the direction x_1 (the right part of the diagram at Fig. 4). The modes P1P2 and A1P2 turn to each other by the interchanging $k \Leftrightarrow k + \pi$. Both these modes describe the same displacement field, however, for the mode P1P2 the wave number is associated with the carrier wave and for the mode A1P2 – with the envelope wave.

Meanwhile, the modes P1A2 and A1A2 describe a particularly different type of motion when the wave is anti-periodic in the direction transverse to the direction of propagation. This represents a clear analogue with the propagation of high-frequency long waves in thin-walled elastic waveguides (see, for example, Craster et al. (2014) and Kaplunov et al. (2017)). For wave propagating in the direction x_1, the local solution across the unit cell in the direction x_2 corresponds to the solution within the transverse cross section of a thin rod, plate or shell. We note that these types of waves arise specifically in multi-dimensional problems and they cannot be described by one-dimensional models.

Analysis of the dispersion surfaces provides information about the directions of energy propagation within the lattice. Numerical results are presented in the electronic supplementary material (Section S1). Fig. S1 displays the dispersion surfaces covering the entire region of the wave numbers $-\pi \leq k_1, k_2 \leq \pi$. For the modes P1P2 and A1A2, in the long-wave limit the lattice exhibits isotropic response, so the isofrequency curves are almost circular near the origin. However, when the wave length decreases and approaches the size of the microstructure, the dynamic properties of the lattice become strongly anisotropic. This effect is caused particularly by the heterogeneity of the medium.

Energy transmission is defined by the group velocity vector $\mathbf{v}_g = \partial \bar{\omega} / \partial \mathbf{k}$. Thus, one can easily see that the directions of the energy flow at a given frequency are identified by the normals to the corresponding isofrequency curve. When the isofrequency curves are nearly circular (e.g., in the low frequency case for P1P2 and A1A2 modes) energy can propagate in all directions almost uniformly. As the frequency increases, the energy flows become focused in certain specific directions. This effect results in a strong "beaming" of the wave field. It allows to design phononic crystals and metamaterials with wave focusing capabilities, which is remarkably important for many engineering applications.

In the second example, let us consider a "degenerate" lattice, when the stiffness of the diagonal springs is zero: $c_0 = 0$. The dispersion curves are displayed at Fig. 5. We may notice that the "degeneracy" leads to some changes in symmetry of the problem. As a result, localised flat-band modes (P1A2 and A1P2) appear in the diagonal direction of propagation. They transmit no energy at any value of the wave number. In Section 3 we will observe a similar phenomenon in a

heterogeneous medium with nearly touching cylindrical inclusions. The dispersion surfaces of the "degenerative" lattice are presented in the electronic supplement, Fig. S2.
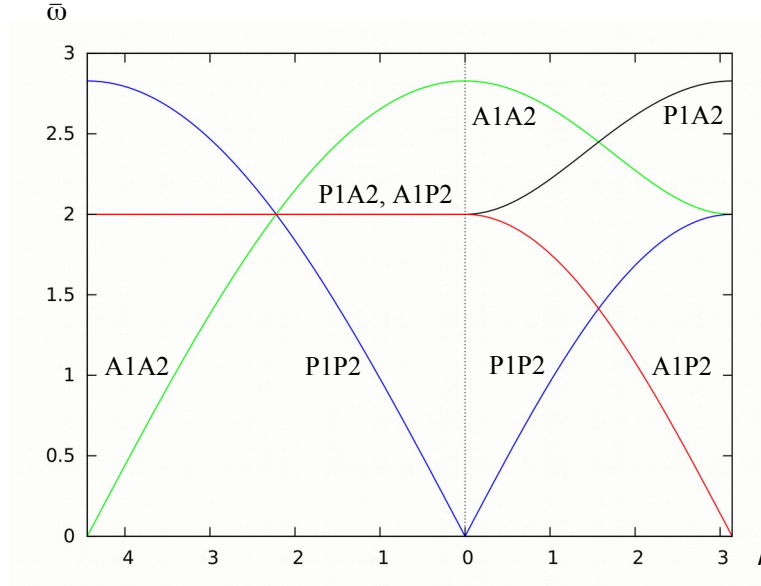

Fig. 5. Dispersion curves of the "degenerative" lattice with $c_0 = 0$.
Blue – P1P2 mode, red – A1P2 mode, black – P1A2 mode, green – A1A2 mode.

The propagation of anti-periodic modes can be essentially influenced by dissipation effects, which arise in real structures due to the viscosity of materials. In order to simulate the viscoelastic behaviour of the lattice, we apply the classical Kelvin-Voigt model (see, for example, Flügge (1975), Christensen (1982)). The developed solution and the obtained numerical results are given in the electronic supplement, Section S2. We may conclude that the presence of viscosity provides a "mode selection mechanism". Anti-periodic modes can be observed on a relatively short timescale, while on a long timescale periodic modes become dominant.

Continuous approximations of the discrete lattice are considered in the electronic supplement, Section S3. We present macroscopic dynamic equations applicable for different periodic and anti-periodic modes in the long-wave case. These equations may be used for solving macroscopic boundary value problems subject to appropriate boundary and initial conditions.

3. Fibrous heterogeneous medium

We aim to generalise the results presented above to a continuous medium. Let us consider the propagation of anti-plane shear waves in an infinite heterogeneous solid consisting of a matrix Ω_1

and a periodic square array of cylindrical inclusions Ω_2 (Fig. 6). This model may correspond to unidirectional fibre-reinforced composite materials. It may also be applicable to describe the dynamics of two-dimensional porous media. The waves propagate in the plane $x_1 x_2$ and the governing equation reads

$$\nabla_x[\mu(\mathbf{x})\nabla_x u]=\rho(\mathbf{x})\frac{\partial^2 u}{\partial t^2}\,,\tag{10}$$

where $\mathbf{x}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$; $\nabla_x=(\partial/\partial x_1)\mathbf{e}_1+(\partial/\partial x_2)\mathbf{e}_2$; $\mathbf{e}_1$ and $\mathbf{e}_2$ are the Cartesian unit vectors; u is the displacement in the direction transverse to the plane $x_1 x_2$; μ and ρ characterise physical properties of the medium, namely, the shear modulus and the mass density. Due to the heterogeneity, the coefficients μ and ρ are represented by functions of co-ordinates: $\mu(\mathbf{x})=\mu_s$, $\rho(\mathbf{x})=\rho_s$ for $\mathbf{x}\in\Omega_s$; μ_s and ρ_s are the properties of the components, $s=1,2$.
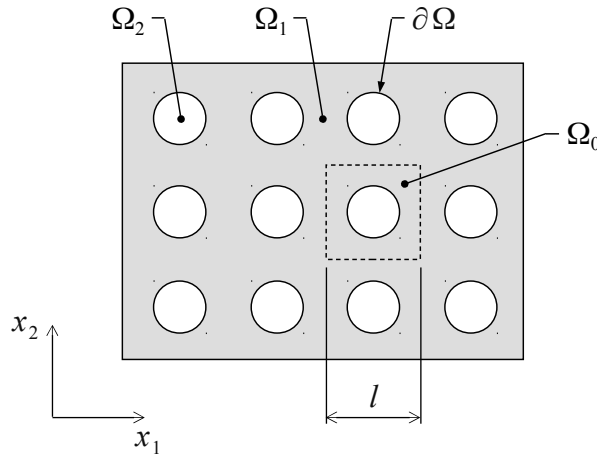

Fig. 6. Fibrous heterogeneous medium.

Functions $\mu(\mathbf{x})$, $\rho(\mathbf{x})$ are piecewise continuous in the unit cell domain Ω_0. Therefore, a function u that satisfies equation (10) should be treated as a weak (generalised) solution (see, for example, Lebedev and Vorovich (2002)). The governing model (10) can be equivalently represented by two equations with constant coefficients (namely, for the matrix and for the inclusions domains) accompanied by conditions of perfect bonding at the interface $\partial\Omega$. However, for the further evaluations we prefer to use the form (10).

Let us note that equation (10) can have different physical interpretations. Here we shall discuss this with respect to the elastic anti-plane shear waves. However, it can also describe in-plane propagation of electromagnetic and optical waves through an array of dielectric cylinders.

The dispersion properties can be evaluated using Floquet-Bloch theory (Brillouin, 2003). According to this approach, the solution of the partial differential equation (10) with periodic coefficients $\mu(\mathbf{x}), \rho(\mathbf{x})$ can be written as follows

$$u = F(\mathbf{x})\exp[-i(K_1 x_1 + K_2 x_2)]\exp(i\omega t), \tag{11}$$

where $F(\mathbf{x})$ is a function of the same period as $\mu(\mathbf{x}), \rho(\mathbf{x})$ and the term $\exp[-i(K_1 x_1 + K_2 x_2)]$ describes a phase shift across the single period. Considering the propagation of a harmonic wave, the variables K_1, K_2 can be interpreted as the components of the wave vector.

In the classical theory, $F(\mathbf{x})$ is considered to be periodic across the unit cell. In order to predict periodic as well as anti-periodic modes, we introduce a new set of boundary conditions

for P1P2 mode $\qquad F(x_1, x_2) = F(x_1+l, x_2), \quad F(x_1, x_2) = F(x_1, x_2+l);$ $\qquad(12)$

for A1P2 mode $\qquad F(x_1, x_2) = -F(x_1+l, x_2), \quad F(x_1, x_2) = F(x_1, x_2+l);$

for P1A2 mode $\qquad F(x_1, x_2) = F(x_1+l, x_2), \quad F(x_1, x_2) = -F(x_1, x_2+l);$

for A1A2 mode $\qquad F(x_1, x_2) = -F(x_1+l, x_2), \quad F(x_1, x_2) = -F(x_1, x_2+l);$

where l is the size of the unit cell.

A solution of the input problem can be developed using series expansions. Among the most popular methods, we note Rayleigh multipole expansions (see, for example, Nicorovici et al. (1995), Poulton et al. (2000), Movchan et al. (2002)) and Fourier series (so called plane-wave expansions, Sigalas and Economou (1993), Kushwaha et al. (1994), Vasseur et al. (1994)). The main differences between these two approaches are as follows. Rayleigh series are based on Bessel functions. They meet strictly the constitutive equation (10) and the bonding conditions at the fibre-matrix interface, whereas conditions (12) imposed at the outer boundary of the unit cell are satisfied approximately. The Fourier series incorporate the periodic and anti-periodic expressions (12) exactly, but the local fields at the interface are evaluated approximately. Generally, both approaches give explicit representations of the displacement field in terms of infinite series and, following a proper truncation, lead to the same results for the effective properties of the medium.

Here we employ Fourier series expansions. Let us represent

for P1P2 mode $\qquad F(\mathbf{x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} F_{n_1 n_2} \exp\left[i\frac{2\pi}{l}(n_1 x_1 + n_2 x_2)\right],$ $\qquad(13)$

for A1P2 mode $\qquad F(\mathbf{x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} F_{n_1 n_2} \exp\left\{ i \frac{2\pi}{l} [(n_1 + \frac{1}{2}) x_1 + n_2 x_2] \right\},$

for P1A2 mode $\qquad F(\mathbf{x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} F_{n_1 n_2} \exp\left\{ i \frac{2\pi}{l} [n_1 x_1 + (n_2 + \frac{1}{2}) x_2] \right\},$

for A1A2 mode $\qquad F(\mathbf{x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} F_{n_1 n_2} \exp\left\{ i \frac{2\pi}{l} [(n_1 + \frac{1}{2}) x_1 + (n_2 + \frac{1}{2}) x_2] \right\}.$

Properties of the medium are given by the expansions

$$\mu(\mathbf{x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \mu_{n_1 n_2} \exp\left[i \frac{2\pi}{l} (n_1 x_1 + n_2 x_2) \right], \qquad (14)$$

$$\rho(\mathbf{x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \rho_{n_1 n_2} \exp\left[i \frac{2\pi}{l} (n_1 x_1 + n_2 x_2) \right],$$

with coefficients $\mu_{n_1 n_2}$, $\rho_{n_1 n_2}$ evaluated as follows

$$\mu_{n_1 n_2} = \frac{1}{S_0} \iint_{\Omega_0} \mu(\mathbf{x}) \exp\left[-i \frac{2\pi}{l} (n_1 x_1 + n_2 x_2) \right] dS,$$

$$\rho_{n_1 n_2} = \frac{1}{S_0} \iint_{\Omega_0} \rho(\mathbf{x}) \exp\left[-i \frac{2\pi}{l} (n_1 x_1 + n_2 x_2) \right] dS,$$

where $\iint_{\Omega_0} (\cdot) dS$ is the integral over a specific unit cell Ω_0; S_0 is the area of the unit cell, $S_0 = l^2$.

Substituting ansatz (11) and expansions (13), (14) into the wave equation (10) and collecting the terms $\exp[i(2\pi/l)(m_1 x_1 + m_2 x_2)]$, $m_1, m_2 = 0, \pm 1, \pm 2, \ldots$, we come to infinite systems of linear algebraic equations for the unknown amplitudes $F_{n_1 n_2}$:

for P1P2 mode

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} F_{n_1 n_2} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \left\{ \mu_{j_1 j_2} [(k_1 + 2\pi n_1)(k_1 + 2\pi m_1) + \right.$$
$$\left. (k_2 + 2\pi n_2)(k_2 + 2\pi m_2)] - (\bar{\omega} v_1)^2 \rho_{j_1 j_2} \right\} = 0, \qquad (15)$$

for A1P2 mode

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}F_{n_1n_2}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\left\{\mu_{j_1j_2}\big[(k_1+\pi+2\pi n_1)(k_1+\pi+2\pi m_1)+\right.$$
$$\left.(k_2+2\pi n_2)(k_2+2\pi m_2)\big]-(\bar{\omega}v_1)^2\rho_{j_1j_2}\right\}=0,$$

for P1A2 mode

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}F_{n_1n_2}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\left\{\mu_{j_1j_2}\big[(k_1+2\pi n_1)(k_1+2\pi m_1)+\right.$$
$$\left.(k_2+\pi+2\pi n_2)(k_2+\pi+2\pi m_2)\big]-(\bar{\omega}v_1)^2\rho_{j_1j_2}\right\}=0,$$

for A1A2 mode

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}F_{n_1n_2}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\left\{\mu_{j_1j_2}\big[(k_1+\pi+2\pi n_1)(k_1+\pi+2\pi m_1)+\right.$$
$$\left.(k_2+\pi+2\pi n_2)(k_2+\pi+2\pi m_2)\big]-(\bar{\omega}v_1)^2\rho_{j_1j_2}\right\}=0,$$

where $j_1=m_1-n_1$, $j_2=m_2-n_2$; $k_1=K_1l$, $k_2=K_2l$; $\bar{\omega}$ is the non-dimensional frequency, $\bar{\omega}=\omega l/v_1$; v_1 is the velocity of the wave propagation through the matrix, $v_1=\sqrt{\mu_1/\rho_1}$.

Systems (15) has a non-trivial solution if and only if the determinant of the matrix of the coefficients is zero. Equating the determinant to zero, we derive a dispersion relation between $\bar{\omega}$ and k_1, k_2.

In the numerical examples presented below the dispersion relations are calculated approximately in *Maple* by truncation of the infinite systems (15). From the physical point of view such a truncation means cutting off higher frequencies. Increasing the number of the equations kept improves the accuracy of the solution in the high-frequency domain. Convergence of the results depends also on the volume fraction of the inclusions and on the contrast between the properties of the components. Expansions (13), (14) can lack convergence in the case of nearly touching and perfectly rigid (or perfectly soft) inclusions, when rapid oscillations of the physical fields occur in the narrow gaps between the fibres. This computational difficulty is not associated with Fourier series only, but it is rather typical for many approaches based on series expansions, e.g., for Rayleigh multipole method (Perrins et al., 1979) and for the boundary shape perturbation method (Andrianov et al., 2005, 2018). Li (1996, 1997) has proposed a new formulation of the Fourier analysis for piecewise-continuous structures, which improved significantly the convergence of the procedure.

In order to verify the developed approach, let us consider numerical results for the quasi-static effective shear modulus $\tilde{\mu}$. It can be evaluated from the dispersion relation for P1P2 mode in the long-wave limit as follows

$$\widetilde{\mu} = \widetilde{\rho} \lim_{K \to 0} (d\,\omega/d\,K)^2,$$

where $\widetilde{\rho}$ is the effective mass density, $\widetilde{\rho} = \rho_1(1-f) + \rho_2 f$; f is the volume fraction of the inclusions, $0 \leq f \leq \pi/4 = 0.785\ldots$.

If we truncate system (15), and let $-n_{max} \leq n_1, n_2 \leq n_{max}$, the number of the equations kept is $(2n_{max}+1)^2$. In Table 1 we compare the developed solution for $\widetilde{\mu}$ with the convergent proved data presented by Perrins et al. (1979). Calculations are made for the case of porous inclusions, $\mu_2 = 0$, which requires the highest computational costs. We observe that results obtained for $n_{max} = 2$ and $n_{max} = 3$ are very close, which confirms a fast practical convergence of the procedure. Keeping $n_{max} = 3$ provides a good accuracy even for large values of the inclusions volume fraction. At $n_{max} = 3$, the 49 equations of system (15) are taken into account. This order of approximation is close to Perrins et al. (1979), who considered 40 equations for $f = 0.77$. Let us also note that in the case of perfectly rigid fibres, $\mu_2 \to \infty$, the effective modulus can be easily determined using the data from Table 1 with the help of Keller's theorem (Keller, 1964): $\widetilde{\mu}(\mu_2/\mu_1) = 1/\widetilde{\mu}(\mu_1/\mu_2)$.

Table 1. Effective shear modulus $\widetilde{\mu}/\mu_1$ of the fibrous medium with porous inclusions.

Inclusions volume fraction f	Fourier series solution			Perrins et al. (1979)
	$n_{max} = 1$	$n_{max} = 2$	$n_{max} = 3$	
0.1	0.867	0.833	0.821	0.818
0.2	0.705	0.670	0.667	0.667
0.3	0.560	0.539	0.538	0.538
0.4	0.437	0.426	0.425	0.425
0.5	0.333	0.325	0.325	0.325
0.6	0.239	0.231	0.230	0.230
0.7	0.147	0.136	0.135	0.135
0.74	0.109	0.094	0.091	0.091
0.76	0.089	0.071	0.067	0.065

The implemented procedure may bring a residual error in the solution on micro level. The Fourier series are continuous and differentiable throughout the unit cell domain Ω_0. Therefore, the evaluated displacements and stresses are also continuous across the interface $\partial\Omega$, which corresponds to the perfect bonding between the matrix and fibres. But the truncation of the infinite

expansions (13), (14) does not allow us to derive the exact values of the local fields. Evaluation of the effective properties implies a homogenisation, which can partially compensate for the inaccuracies of the local solution. This is evidenced by the results shown in Table 1. In this paper we are interested primarily in the effective behaviour of the medium; analysis of the local fields is not within the scope of our study.

The continuous heterogeneous solid exhibits an infinite sequence of pass and stop bands. Here we present the results for the lowest branches of the dispersion curves, since the low-frequency domain is in many cases the most important for the engineering practice. The calculations are performed at $n_{max}=3$ and the evaluated graphs are almost indistinguishable from the case $n_{max}=2$.

Fig. 7 displays the dispersion diagrams for a high-contrast material with porous inclusions, $\mu_2=0$. We can observe that for $f=0.7$ the pattern of the dispersion curves is qualitatively similar to the case of the lattice having no diagonal interactions between the particles (see Fig. 5). In particular, localised nearly flat-band modes appear in the diagonal direction of propagation. This can be explained in such a way. In the case of densely packed pores the geometry of the structure can be approximately described by the square lattice model: nearly rhombus areas of the matrix correspond to the particles, while narrow matrix strips separating neighbouring pores play a role of the massless springs. The analogy between dispersion diagrams of continuous and discrete structures gives a possibility to tune relatively simple lattice-type models in such a way as to describe the dynamic properties of more complicated materials. Carta and Brun (2012), Brito-Santana et al. (2015) developed lattice approximations for periodic laminates. Lattice and frame models of two-dimensional heterogeneous media were considered by Movchan et al. (2002), Martinsson and Movchan (2003).

In Fig. 8, the dispersion diagrams are presented for a low-contrast material with $\mu_2/\mu_1=0.5$, $\rho_1=\rho_2$. The results are similar to the high-contrast case with the main exception that now the flat-band modes do not exist. We may conclude that if the fibres are not too soft and the both components are noticeably involved in the deformation, the lattice-type models can not provide a reasonable approximation.

Let us note that for a porous material with a relatively low volume fraction f of the inclusions the dispersion curves for P1P2 and A1A2 modes propagating in the diagonal direction are nearly straight (see Figs. 7*a*, 8*a*). In this case, the wave propagation is non-dispersive within the whole pass band except a narrow and visually non-distinguishable frequency region in the vicinity of the cut-off threshold. As f increases, the structure of the material becomes more heterogeneous and the dispersion effect increases as it can be observed at Figs. 7*b*, 8*b*.
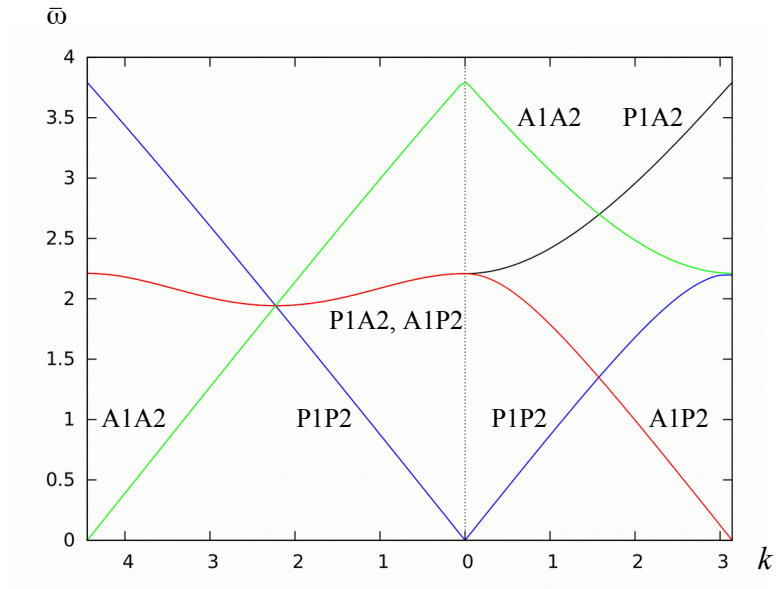
a) $f = 0.3$
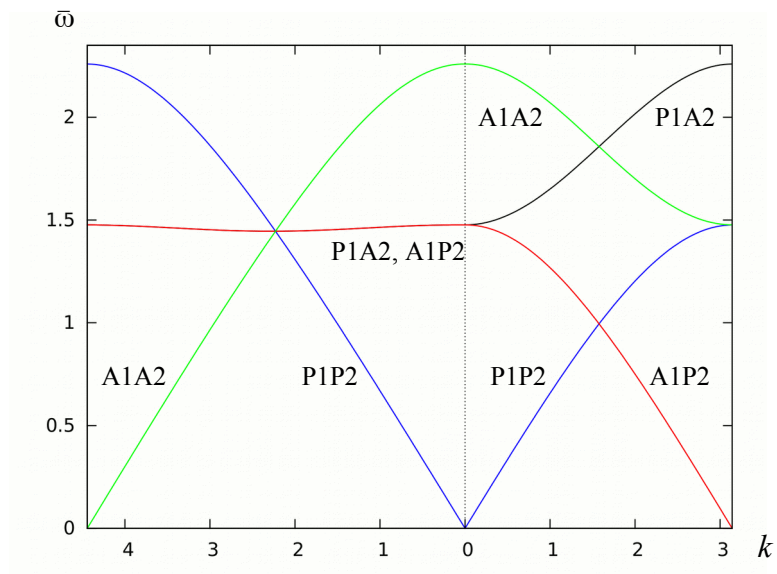
b) $f = 0.7$

Fig. 7. Low-frequency dispersion curves of the high-contrast fibrous medium; $\mu_2 = 0$.

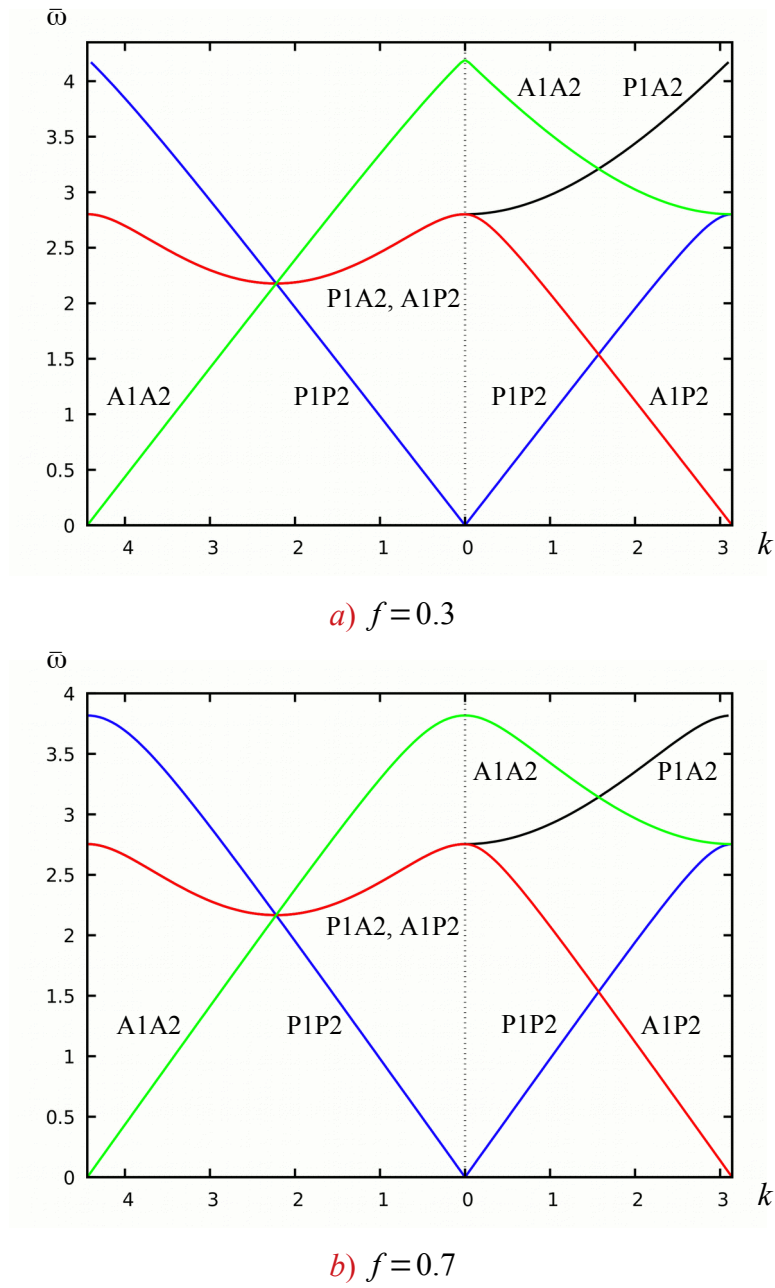Blue – P1P2 mode, red – A1P2 mode, black – P1A2 mode, green – A1A2 mode.

a) $f = 0.3$

b) $f = 0.7$

Fig. 8. Low-frequency dispersion curves of the low-contrast fibrous medium; $\mu_2/\mu_1 = 0.5$, $\rho_1 = \rho_2$.
Blue – P1P2 mode, red – A1P2 mode, black – P1A2 mode, green – A1A2 mode.

4. Conclusions

We considered propagation of elastic anti-plane waves through two-dimensional periodically heterogeneous structures. Dispersion properties of the discrete square lattice are determined in a closed analytical from. Solution for the continuous fibrous medium is developed using Floquet-Bloch theory and Fourier series approximations. The practical convergence of the procedure is

verified by comparing the results obtained for the quasi-static effective modulus with available data from the literature.

The aim of the paper is to highlight that imposing different boundary conditions at the opposite sides of the unit cell gives a possibility to describe propagation of various types of modes. If in the long-wave limit the wave field is spatially periodic across the unit cell, then classical modes arise. This is consistent with a continuous approximation of the discrete lattice. However, if the solution is anti-periodic within the unit cell, we can predict new types of modes. In lattice models, this corresponds to the "envelope continualisation" limit. For two-dimensional structures, combining periodic and anti-periodic conditions in different directions of the translational symmetry reveals a number of modes having different dispersive properties and velocities of propagation. We have shown that anti-periodic modes can be interpreted as theoretical counterparts of non-classical waves that appear in phenomenological approaches, such as gradient and Biot's theories.

It is also shown that within the acoustic band, viscous damping results in a faster attenuation of anti-periodic modes. Therefore, in real materials and structures anti-periodic modes should be detected on a relatively short timescale, while on a long timescale periodic modes are expected to dominate.

For the sake of clarity, we have considered the anti-plane problems only. Let us note that in the case of in-plane deformation longitudinal and shear waves are coupled and propagate together. This brings much more admissible combinations of periodic and anti-periodic conditions for different components of the displacement field. Therefore, the problems of in-plane wave propagation should demonstrate a considerably richer variety of non-classical modes.

References

Aero E., Kuvshinskii E., 1961. Fundamental equations of the theory of elastic media with rotationally interacting particles. Soviet Phys. – Solid State 2, 1272–1281.

Andrianov I.V., Awrejcewicz J., Danishevskyy V., 2018. Asymptotical Mechanics of Composites. Modelling Composites without FEM. Springer, Berlin Heidelberg, New York.

Andrianov I.V., Awrejcewicz J., Weichert D., 2010. Improved continuous models for discrete media. Math. Prob. Eng. 2010, 986242.

Andrianov I.V., Bolshakov V.I., Danishevskyy V.V., Weichert D., 2008. Higher-order asymptotic homogenization and wave propagation in periodic composite materials. Proc. R. Soc. A 464, 1181–1201.

Andrianov I.V., Danishevskyy V.V., Guillet A., Pareige P., 2005. Effective properties and micro-mechanical response of filamentary composite wires under longitudinal shear. Eur. J. Mech. A Solids 24, 195–206.

Andrianov I.V., Danishevskyy V.V., Kushnierov Ie.A., 2014. Spatial localization of linear elastic waves in composite materials with defects. ZAMM 94, 1001–1010.

Antonakakis T., Craster R.V., Guenneau S., 2014. Homogenisation for elastic photonic crystals and dynamic anisotropy. J. Mech. Phys. Solids 71, 84–96.

Askes H., Aifantis E.C., 2011. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. Int. J. Solids Struct. 48, 1962–1990.

Auriault J.-L., Boutin C., 2012. Long wavelength inner-resonance cut-off frequencies in elastic composite materials. Int. J. Solids Struct. 49, 3269–3281.

Ayzenberg-Stepanenko M.V., Slepyan L.I., 2008. Resonant-frequency primitive waveforms and star waves in lattices. J. Sound Vib. 313 ,812–821.

Biot M.A., 1956a. Theory of propagation of elastic waves in a fluid-saturated porous solid I. Low-frequency range. J. Acoust. Soc. Am. 28, 168–178.

Biot M.A., 1956b. Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range. J. Acoust. Soc. Am. 29, 179–191.

Biot M.A., 1962. Generalized theory of acoustic propagation in porous dissipative media. J. Acoust. Soc. Am. 34, 1254–1264.

Bloch F., 1928. Über die Quantenmechanik der Elektronen in Kristallgittern. Z. Physik 52, 555–600.

Borino G., Polizzotto C., 2014. A method to transform a nonlocal model into a gradient one within elasticity and plasticity. Eur. J. Mech. A/Solids 46, 30–41.

Boutin C., Auriault J.L., 1993. Rayleigh scattering in elastic composite materials. Int. J. Eng. Sci. 31, 1669–1689.

Brillouin L., 2003. Wave Propagation in Periodic Structures: Electric Filters and Crystal Lattices, 2nd ed. Dover, Mineola, New York.

Brito-Santana H., Wang Yue-Sheng, Rodríguez-Ramos R., Bravo-Castillero J., Guinovart-Díaz R., Tita V., 2015. A dispersive nonlocal model for shear wave propagation in laminated composites with periodic structures. Eur. J. Mech. A/Solids 49, 35–48.

Brun M., Guenneau S., Movchan A. B., 2009. Achieving control of in-plane elastic waves. Appl. Phys. Lett. 94, 061903.

Cantat I., Cohen-Addad S., Elias F., Graner F., Höhler R., Pitois O., 2013. Foams: Structure and Dynamics. Oxford University Press, Oxford.

Cauchy A.L., 1851. Note sur l'équilibré et les mouvements vibratoires des corps solides. C. R. Acad. Paris. 32, 323–326.

Carta G., Brun M., 2012. A dispersive homogenization model based on lattice approximation for the prediction of wave motion in laminates. J. Appl. Mech. 79, 021019.

Colquitt D.J., Craster R.V., Makwana M., 2015. High frequency homogenisation for elastic lattices. Q. J. Mech. Appl. Math. 68, 203–230.

Colquitt D.J., Danishevskyy V.V., Kaplunov J.D., 2018. Composite dynamic models for periodically heterogeneous media. Math. Mech. Solids, to appear.

Colquitt D.J., Jones I.S., Movchan N.V., Movchan A.B., Brun M., McPhedran R.C., 2013 Making waves round a structured cloak: lattices, negative refraction and fringes. Proc. R. Soc. A 469, 20130218.

Colquitt D.J., Jones I.S., Movchan N.V., Movchan A.B., McPhedran R.C., 2012. Dynamic anisotropy and localization in elastic lattice systems. Wave Random Complex 22, 143–159.

Colquitt D.J., Jones I.S., Movchan N.V., Movchan A.B., 2011. Dispersion and localization of elastic waves in materials with microstructure. Proc. R. Soc. A 467, 2874–2895.

Cosserat E., Cosserat F., 1909. Théorie des Corps Déformables. Paris, Hermann et fils.

Craster R.V., Joseph L.M., Kaplunov J., 2014. Long-wave asymptotic theories: The connection between functionally graded waveguides and periodic media. Wave Motion 51, 581–588.

Craster R.V., Kaplunov J., Pichugin A.V., 2010a. High frequency homogenization for periodic media. Proc. R. Soc. Lond. A 466, 2341–2362.

Craster R.V., Kaplunov J., Postnova J., 2010b. High-frequency asymptotics, homogenisation and localisation for lattices. Quart. J. Mech. Appl. Math. 63, 497–519.

Christensen R.M., 1982. Theory of Viscoelasticity. An Introduction, 2nd ed. Academic Press, New York.

Dauxois T., 1991. Dynamics of breather modes in a nonlinear "helicoidal" model of DNA. Phys. Lett. A l59, 390–395.

Daya E.M., Potier-Ferry M., 2001. Vibrations of long repetitive structures by a double scale asymptotic method, Struct. Eng. Mech. 12, 215–230.

Eringen A., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. 54, 4703–4710.

Fish J, Chen W., 2001. Higher-order homogenization of initial/boundary-value problem. J. Eng. Mech. 127, 1223–1230.

Floquet G., 1883. Sur les équations différentielles linéaires à coefficients périodiques. Ann. École Norm. Sup. 12, 47–88.

Flügge W., 1975. Viscoelasticity. New York, Springer.

Friesecke G., James R.D., 2000. A scheme for the passage from atomic to continuum theory for thin films, nanotubes and nanorods. J. Mech. Phys. Solids 48, 1519–1540.

Gibson L.J., Ashby M.F., Harley B.A., 2010. Cellular Materials in Nature and Medicine. Cambridge University Press, Cambridge.

Grbic A., Eleftheriades G.V., 2004. Overcoming the diffraction limit with a planar left-handed transmission-line lens. Phys. Rev. Lett. 92, 117403.

Herrmann G., Achenbach J.D., 1968. Application of theories of generalized Cosserat continua to the dynamics of composite materials. In: E. Kröner (Ed.) Mechanics of Generalized Continua. Berlin, Heidelberg, New York, Springer, 69–79.

Hosokawa A., Otani T., 1997. Ultrasonic wave propagation in bovine cancellous bone. J. Acoust. Soc. Am. 101, 1–5.

Hussein M.I., Leamy M.J., Ruzzene M., 2014. Dynamics of phononic materials and structures: historical origins, recent progress, and future outlook. Appl. Mech. Rev. 66, 040802.

Il'iushina E.A., 1969. On a model of continuous medium, taking into account the microstructure. J. Appl. Math. Mech. 33, 896–902.

Kalamkarov A.L., Andrianov I.V., Danishevskyy V.V., 2009. Asymptotic homogenization of composite materials and structures. Appl. Mech. Rev. 62, 030802.

Kaplunov J.D., Prikazchikov D.A., Sergushova O., 2017. Lowest vibration modes of strongly inhomogeneous elastic structures. In: Mechanics for Materials and Technologies (eds. Altenbach H., Goldstein R.V., Murashkin E.), Springer.

Keller J.B., 1964. A theorem on the conductivity of a composite medium. J. Math. Phys. 5, 548–549.

Kittel C., 2005. Introduction to Solid State Physics, 8th ed. Wiley, New York.

Kovalev A.S., Kosevich A.M., 1975. Self-localization of vibrations in a one-dimensional anharmonic chain. Soviet Physics JETP 40, 891–896.

Kröner E., 1963. On the physical reality of torque stresses in continuum mechanics. Int. J. Eng. Sci. 1, 261–278.

Kunin I.A., 1966. Theory of elasticity with spatial dispersion. One-dimensional complex structure. J. Appl. Math. Mech. 30, 1025–1034.

Kunin I.A., 1982. Elastic Media with Microstructure. I. One-Dimensional Models. Berlin, New York, Springer.

Kunin I.A., 1983. Elastic Media with Microstructure. II. Three-Dimensional Models. Berlin, New York, Springer.

Kushwaha M.S., Halevi P., Martinez G., Dobrzynski L., Djafari-Rouhani B., 1994. Theory of acoustic band structure of periodic elastic composites. Phys. Rev. B 49, 2313–2322.

Lakes R.S., Yoon H.S., Katz J.L., 1983. Slow compressional wave propagation in wet human cortical bone. Science 220, 513–515.

Le Roux J., 1911. Étude géométrique de la torsion et de la flexion dans la déformation. Ann. Scient. de l'École Normale Sup, 3e série, tome 28, 523–579.

Lebedev L.P., Vorovich I.I., 2002. Functional Analysis in Mechanics. New York, Springer.

Levin V., 1971. The relation between mathematical expectations of stress and strain tensors in elastic microheterogeneous media. J. Appl. Math. Mech. 35, 694–701.

Li L., 1996. Use of Fourier series in the analysis of discontinuous periodic structures. J. Opt. Soc. Am. A 13, 1870–1876.

Li L., 1997. New formulation of the Fourier modal method for crossed surface-relief gratings. J. Opt. Soc. Am. A 14, 2758–2767.

Liang B., Yuan B., Cheng J.C., 2009. Acoustic diode: rectification of acoustic energy flux in one-dimensional systems. Phys. Rev. Lett. 103, 104301.

Lutianov M., Rogerson G.A., 2010. Long wave motion in layered elastic media. Int. J. Eng. Sci. 48, 1856–1871.

Madeo A., 2015. Generalised Continuum Mechanics and Engineering Applications. London, ISTE Press.

Maldovan M., 2013. Sound and heat revolution in phononics. Nature 503, 209–217.

Martinsson P.G., Movchan A.B., 2003. Vibrations of lattice structures and phononic band gaps. Q. J. Mech. Appl. Math. 56, 45–64.

Milton G.W., Briane M., Willis J.R., 2006. On cloaking for elasticity and physical equations with a transformation invariant form. New J. Phys. 8, 248.

Mindlin R.D., 1964. Microstructure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78.

Movchan A.B., Movchan N.V., Poulton C.G., 2002. Asymptotic Models of Fields in Dilute and Densely Packed Composites. London, Imperial College Press.

Mukhomodyarov R.R., Rogerson G.A., 2012. Long-wave dispersion phenomena in a layer subject to elastically restrained boundary conditions. ZAMP 63, 171–188.

Nicorovici N.A., McPhedran R.C., Botten L.C., 1995. Photonic band gaps for arrays of perfectly conducting cylinders. Phys. Rev. E 52, 1135–1145.

Nolde E., Craster R.V., Kaplunov J., 2011. High frequency homogenization for structural mechanics. J. Mech. Phys. Solids 59, 651–671.

Norris A.N., Shuvalov A.L., 2011. Elastic cloaking theory. Wave Motion 48, 525–538.

Pendry J.B., 2000. Negative refraction makes a perfect lens. Phys. Rev. Lett. 85, 3966–3969.

Perrins W.T., McKenzie D.R., McPhedran R.C., 1979. Transport properties of regular arrays of cylinders. Proc. R. Soc. Lond. A 369, 207–225.

Plona T.J., 1980. Observation of a second bulk compressive wave in a porous medium at ultrasonic frequencies. Appl. Phys. Lett. 36, 259–261.

Potapov A.I., Pavlov I.S., Lisina S.A., 2009. Acoustic identification of nanocrystalline media. J. Sound Vib. 322, 564–580.

Poulton C.G., Movchan A.B., McPhedran R.C., Nicorovici N.A., Antipov Y.A., 2000. Eigenvalue problems for doubly periodic structures and phononic band gaps. Proc. R. Soc. Lond. A 457, 2561–2568.

Rallu A., Hans S., Boutin C., 2017. High frequency frequency reduced reduced models models for repetitive beams. Procedia Engineering 199, 1501–1506.

Remoissenet M., 1986. Low-amplitude breather and envelope solitons in quasi-one-dimensional physical models. Phys. Rev. B 33, 2386–2392.

Rogerson G.A., Kaplunov J.D., Nolde E.V., 2006. An asymptotic analysis of initial-value problems for thin elastic plates. Proc Royal Soc. A 462, 2541–2561.

Rogerson G.A., Prikazchikova L.A., 2009. Generalisations of long wave theories for pre-stressed compressible elastic plates. Int. J. Non-Linear Mech. 44, 520–529.

Rogerson G.A., Sandiford K.J., Prikazchikova L.A., 2007. Abnormal long wave dispersion phenomena in a slightly compressible elastic plate with non-classical boundary conditions. Int. J. Non-Linear Mech. 42, 298–309.

Sigalas M.M., Economou E.N., 1993. Band structure of elastic waves in two dimensional systems. Solid State Commun. 86, 141–143.

Slepyan L.I., 2002. Models and Phenomena in Fracture Mechanics. Berlin, New York, Springer.

Soubestre J., Boutin C., 2012. Non-local dynamic behaviour of linear fibre reinforced materials. Mech. Mater. 55, 16–32.

Toupin R.A., 1962. Perfectly elastic materials with couple stresses. Arch. Ration. Mech. An. 11, 385–414.

Vasiliev A.A., Dmitriev S.V., Miroshnichenko A.E., 2010. Multi-field approach in mechanics of structural solids. Int. J. Solids Struct. 47, 510–525.

Vasseur J.O., Djafari-Rouhani B., Dobrzynski L., Kushwaha M.S., Halevi P., 1994. Complete acoustic band gaps in periodic fibre reinforced composite materials: the carbon/epoxy composite and some metallic systems. J. Phys. Condens. Matter 6, 8759–8770.

Voigt W., 1887. Theoretische Studien über die Elastizitätsverhältnisse der Kristalle. Abh. Ges. Wissen. Göttingen 34, 3–100.