Wrinkling of a compressed hyperelastic half-space with localized surface imperfections

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Abstract

We consider a variant of the classical Biot problem concerning the wrinkling of a compressed hyperelastic half-space. The traction-free surface is no longer flat but has a localized ridge or trench that is invariant in the x_1-direction along which the wrinkling pattern is assumed to be periodic. With the x_2-axis aligned with the depth direction, the localized imperfection is assumed to be slowly varying and localized in the x_3-direction, and an asymptotic analysis is conducted to assess the effect of the imperfection on the critical stretch for wrinkling. The imperfection introduces a length scale so that the critical stretch is now weakly dependent on wave number. It is shown that the imperfection increases the critical stretch (and hence reduces the critical strain) whether the imperfection is a ridge or trench, and the amount of increase is proportional to the square of the maximum gradient of the surface profile.

Keywords: Elastic half-space, wrinkling, bifurcation, nonlinear elasticity.

1. Introduction

Biot [1] was the first to examine the problem of possible surface wrinkling of a compressed hyperelastic half-space. The problem was further studied by Nowinski [2], Usmani and Beatty [3], Reddy [4, 5], Dowaikh and Ogden [6], Fu and Mielke [7], Destrade and Scott [8], Murphy and Destrade [9], and Chen et al. [10]. Some of these studies are in the context of surface waves in a pre-stressed hyperelastic half-space. For the case of a neo-Hookean half-space in a state of plane strain, the critical stretch was found to be 0.544. Since the wrinkling modes are non-dispersive due to lack of a natural length scale in the problem, a small-amplitude monochromatic mode will induce all higher harmonics at second order through nonlinear interactions. Based on this fact Ogden and Fu [11] attempted to determine the

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post-buckling solution through Fourier expansion and concluded that a convergent post-buckling solution (and hence a solution with enough regularity) cannot exist. Fu [12] then considered the deformation of a corrugated half-space and concluded that any post-buckling solution should probably contain static shocks. The corrugated half-space problem was also investigated by Cai and Hutchinson [13] with focus on interactions of a finite number of modes.

By bending a rectangular rubber block, Gent and Cho [14] demonstrated that creases, instead of periodic wrinkles, form on the compressed inner surface when the local stretch reaches 0.65, much earlier than what Biot predicted for periodic wrinkles. Subsequently, Gent and Cho's observation has been confirmed by numerical [15, 16], experimental [17, 18], and analytical studies [19–21].

Although Biot's buckling mode does not seem realizable in practice, it has nonetheless provided a major reference point in stability and bifurcation analysis of a variety of soft materials and structures. For instance, it often appears as the large wave number limit in a bifurcation analysis [22], and is closely associated with the complementing condition for the existence of a unique solution for boundary value problems in nonlinear elasticity [23]. Two variants of the Biot problem have received a lot of attention in recent years. The first is concerned with the buckling of a compressed half-space with material properties varying with depth; see Lee et al. [24], Wu et al. [25], Wu et al. [26], Diab and Kim [27], Yang and Chen [28], Chen et al. [29]. The second variant is concerned with the buckling of a coated half-space (or a film/substrate bilayer). For the latter there now exists a huge body of literature, driven by a variety of applications. We refer to the review articles by Yang et al. [30], Li et al. [31], Wang and Zhao [32], and Dimmock et al. [33] for a comprehensive list of the literature and discussion of applications from different perspectives.

In this paper, we propose and study another variant of the Biot problem by taking into account a geometrical imperfection on the free surface. The imperfection takes the form of a localized ridge or trench that varies slowly in the direction perpendicular to the direction of periodic wrinkling; see Fig. 1. Our aim is to assess how such an imperfection affects the critical stretch for periodic wrinkling. The other extreme variant of the Biot problem is concerned with the case when the imperfection is fast-varying such that it is wedge-like. The latter problem has recently been studied by Lestringant et al. [34].

The current problem has a counterpart in the dynamical setting of topography-guided surface waves; see Samuel et al. [35] and Fu et al. [36]. As the half-space is compressed, the surface wave speed will change with respect to the stretch. When the stretch is such that the surface wave speed vanishes, the surface wave mode becomes the wrinkling mode that is studied in the current paper.

The rest of this paper is divided into six sections as follows. After problem formulation in Section 2, the next three sections are concerned with the asymptotic solutions at leading-, second-, and third-orders. The leading-order problem recovers Biot's classical problem, the second-order problem is mainly concerned with the determination of the anti-plane displacement component, and it is at the third order that we derive an eigenvalue problem that determines the leading-order correction to the critical stretch due to surface imperfections. The eigenvalue problem is solved numerically and asymptotically in Section 6, and a summary and further discussions are presented in the concluding section.

2. Problem formulation

We first summarize the incremental equations for a general homogeneous elastic body composed of a non-heat-conducting incompressible elastic material. Such a material is assumed to possess an initial unstressed configuration B_0. A purely homogeneous static deformation is imposed upon B_0 to produce a finitely stressed equilibrium configuration denoted by B_e. A problem of major interest in continuum mechanics is whether such a configuration is the only one possible. One way to answer this question is to superimpose a small amplitude perturbation on B_e and then solve the resulting incremental boundary value problem. It is now well-known that the linearized incremental equilibrium equations and incompressibility condition may be written in the form [37]

$$\chi_{ij,j} = 0, \quad u_{i,i} = 0, \tag{2.1}$$

where χ_{ij} is the incremental stress tensor and u_i the incremental displacement superposed on B_e. Throughout this paper, we employ the summation convention whereby Latin subscripts range between 1 and 3 whereas Greek subscripts range between 1 and 2, and we use a comma to denote differentiation with respect to the coordinates in B_e.

The incremental stress components χ_{ij} are given by

$$\chi_{ij} = \mathcal{A}_{jilk} u_{k,l} + \bar{p} u_{j,i} - p^* \delta_{ji}, \tag{2.2}$$

where the instantaneous elastic moduli $\mathcal{A}_{jilk}$ are defined by [38]

$$\mathcal{A}_{jilk} = \bar{J}^{-1} \bar{F}_{jA} \bar{F}_{lB} \left. \frac{\partial^2 W}{\partial F_{iA} \partial F_{kB}} \right|_{F=\bar{F}}, \tag{2.3}$$

and $\bar{p}$ and p^* are the Lagrangian multipliers associated with the primary finite deformation and the incremental deformation, respectively. In the above definition, W is the strain-energy function per unit volume in the reference configuration, which is a function of the deformation gradient F, the $\bar{F}$ is the value of F associated with the primary deformation, and $\bar{J} = \det \bar{F} \equiv 1$. For a neo-Hookean material, we have

$$W = \frac{1}{2}\mu(\mathrm{tr}B - 3), \quad \mathcal{A}_{jilk} = \mu\delta_{ik}\bar{B}_{jl}, \tag{2.4}$$

where B is the left Cauchy-Green strain tensor $(= FF^T)$, $\bar{B} = \bar{F}\bar{F}^T$, and μ is the ground-state shear modulus.

We now specialize the above equations to a hyperelastic half-space that is defined by

$$-h(\varepsilon x_3) \leq x_2 < \infty, \quad -\infty < x_1, x_3 < \infty \tag{2.5}$$

in the finitely deformed configuration B_e, where h is a continuously differentiable even function to be specified and ε is a small positive parameter so that h is a slowly-varying function of x_3. In writing down (2.5), we have assumed that x_i and h have all been scaled by a length scale L. In the next section, this L will be taken to be the inverse of the wave number of the wrinkling mode. Therefore the term "slowly varying"above means that h varies over a length scale much larger than the wavelength of the wrinkling mode. Equivalently, for a surface profile of arbitrary variation, our analysis will be valid in the large wave number limit. We further assume that h is localized in the sense that $h \to 0$ as $x_3 \to \pm\infty$ so that h represents a localized ridge (if $h > 0$) or a trench (if $h < 0$) that maintains its shape in the x_1-direction; see Fig. 1.

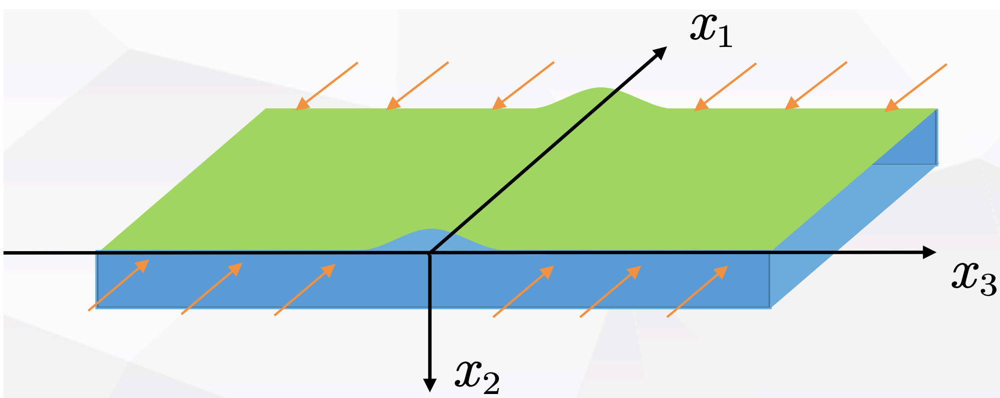


Figure 1: Half-space with localized surface imperfections.

The finite deformation from B_0 to B_e is assumed to correspond to a uni-axial compression with deformation gradient given by $\bar{F} = \mathrm{diag}\{\lambda, \lambda^{-1}, 1\}$. Strictly speaking, this is not an exact solution since it cannot satisfy the traction-free boundary condition at the (curved) free surface exactly; a shear stress of order ε would be required to maintain such a homogeneous deformation, but we assume that its effect can be neglected. Our task is then to find the critical value of λ at which a static inhomogeneous solution can bifurcate from this primary homogeneous solution. The particular inhomogeneous solution that we are looking for is periodic in the x_1-direction and is localized in the other two directions. When h is identically zero, this reduces to the classical Biot problem. Our aim is to determine how a surface imperfection assumed above affects the critical value for wrinkling. It is worth noting that in contrast to the original Biot problem, the current problem has two length scales: the height of the localized ridge/trench and the length scale over which it is varying. As a result, the critical stretch will be a function of the wave number. This dispersive nature of our problem will further be discussed when numerical results are presented.

Thus, our objective is to solve the incremental equations (2.1) subject to the boundary condition

$$\boldsymbol{\chi}\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \ x_2 = -h(\varepsilon x_3), \tag{2.6}$$

and the decay condition

$$\boldsymbol{u} \to \boldsymbol{0} \quad \text{as } \ x_2 \to \infty, \tag{2.7}$$

where $\boldsymbol{n}$ is any vector normal to the free surface (it does not need to be normalized since (2.6) is still valid if the left-hand side is multiplied by any constant). We take the convenient choice

$$\boldsymbol{n} = \{0, 1, \varepsilon h'\}^T, \tag{2.8}$$

where h' denotes $h'(s)$, the derivative of $h(s)$, evaluated at $s = \varepsilon x_3$. As a result, with the use of $(2.1)_2$ and (2.2), the equilibrium equations $(2.1)_1$ and boundary condition (2.6) may be written as

$$\mathcal{A}_{\alpha i\beta k}u_{k,\alpha\beta} + (\mathcal{A}_{\alpha i3k} + \mathcal{A}_{3i\alpha k})u_{k,3\alpha} + \mathcal{A}_{3i3k}u_{k,33} - p^*_{,i} = 0, \quad -h(\varepsilon x_3) < x_2 < \infty, \tag{2.9}$$

$$\begin{aligned}\mathcal{A}_{2i\alpha k}u_{k,\alpha} + \varepsilon h'(\varepsilon x_3)\mathcal{A}_{3i\alpha k}u_{k,\alpha} + \mathcal{A}_{2i3k}u_{k,3} + \varepsilon h'(\varepsilon x_3)\mathcal{A}_{3i3k}u_{k,3} + \bar{p}u_{2,i} - p^*\delta_{2i}\\ +\varepsilon h'(\varepsilon x_3)(\bar{p}u_{3,i} - p^*\delta_{3i}) = 0, \quad \text{on } x_2 = -h(\varepsilon x_3).\end{aligned} \tag{2.10}$$

We recall our summation convention that Greek letters only range between 1 and 2. A property concerning the elastic moduli $\mathcal{A}_{jilk}$ that we use repeatedly is that each such modulus is equal to zero whenever the subscript 1, 2 or 3 in it appears an odd number of times; for instance $\mathcal{A}_{1112} = \mathcal{A}_{1332} = 0$ etc. Therefore, if the subscript i in (2.9) is equal to 1 or 2, then the subscript k in the first term may be replaced by a Greek subscript since $\mathcal{A}_{\alpha 1\beta k}$ and $\mathcal{A}_{\alpha 2\beta k}$ with $k = 3$ must necessarily be zero.

To facilitate application of the boundary conditions, we employ the variable substitution $x_i \to x'_i$ where

$$x'_1 = x_1, \quad x'_2 = x_2 + h(\varepsilon x_3), \quad x'_3 = \varepsilon x_3, \tag{2.11}$$

so that the free surface corresponds to $x_2' = 0$. We have assumed that the dependence of the incremental solution on x_3 is also through εx_3, which means that it is also slowly varying in the x_3-direction. We have

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1'}, \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2'}, \quad \frac{\partial}{\partial x_3} = \varepsilon \frac{\partial}{\partial x_3'} + \varepsilon h'(x_3') \frac{\partial}{\partial x_2'}.$$

In terms of the new variables, the incremental equilibrium equations and traction-free boundary conditions become

$$\mathcal{A}_{\alpha i\beta k} u_{k,\alpha\beta} - p^*_{,i} + \varepsilon(\mathcal{A}_{\alpha i3k} + \mathcal{A}_{3i\alpha k})(u_{k,3\alpha} + h' u_{k,2\alpha})$$

$$+\varepsilon^2 \mathcal{A}_{3i3k}(u_{k,33} + h'' u_{k,2} + 2h' u_{k,23} + h'^2 u_{k,22}) = 0, \quad 0 < x_2 < \infty, \tag{2.12}$$

$$\mathcal{A}_{2i\alpha k} u_{k,\alpha} + \varepsilon h' \mathcal{A}_{3i\alpha k} u_{k,\alpha} + \varepsilon \mathcal{A}_{2i3k}(u_{k,3} + h' u_{k,2}) + \varepsilon^2 h' \mathcal{A}_{3i3k}(u_{k,3} + h' u_{k,2})$$

$$+\bar{p} u_{2,i} - p^* \delta_{2i} + \varepsilon h'(\bar{p} u_{3,i} - p^* \delta_{3i}) = 0, \quad \text{on} \;\; x_2 = 0, \tag{2.13}$$

where here and hereafter the primes on x_i' are dropped.

We anticipate that due to the geometrical imperfection the critical stretch should be expanded as

$$\lambda = \lambda_{\rm cr} + \varepsilon^2 \hat{\lambda}, \tag{2.14}$$

where $\lambda_{\rm cr}$ is the critical stretch in the Biot problem and $\hat{\lambda}$ is the leading-order correction due to the surface imperfection. The order of the correction term is determined by the fact that the effects of h and this term should both operate at the third order of successive approximations.

Corresponding to the above expansion, the elastic moduli and $\bar{p}$ must also be expanded:

$$\mathcal{A}_{jilk} = \mathcal{A}^{(0)}_{jilk} + \varepsilon^2 \hat{\lambda} \mathcal{A}^{(1)}_{jilk} + \cdots, \quad \bar{p} = \bar{p}_0 + \varepsilon^2 \hat{\lambda} \bar{p}_1 + \cdots, \tag{2.15}$$

where

$$\mathcal{A}^{(0)}_{jilk} = \mathcal{A}_{jilk}\Big|_{\lambda=\lambda_{\rm cr}}, \quad \mathcal{A}^{(1)}_{jilk} = \frac{\partial \mathcal{A}_{jilk}}{\partial \lambda}\Big|_{\lambda=\lambda_{\rm cr}}, \quad \bar{p}_0 = \bar{p}\Big|_{\lambda=\lambda_{\rm cr}}, \quad \bar{p}_1 = \frac{\partial \bar{p}}{\partial \lambda}\Big|_{\lambda=\lambda_{\rm cr}}.$$

We now look for an asymptotic solution of the form

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots, \quad p^* = p^{*(0)} + \varepsilon p^{*(1)} + \varepsilon^2 p^{*(2)} + \cdots, \tag{2.16}$$

where all the functions on the right-hand sides are to be determined at successive orders.

On substituting the above asymptotic solution into (2.12) and (2.13), and then equating the coefficients of like powers of ε, we obtain the following three sets of boundary value problems.

Leading order:

$$\begin{aligned}\mathcal{A}^{(0)}_{\alpha\gamma\beta k}u^{(0)}_{k,\alpha\beta}-p^{*(0)}_{,\gamma}=&0,\quad \text{in } B_e, &(2.17)\\ \mathcal{A}^{(0)}_{2\gamma\alpha k}u^{(0)}_{k,\alpha}+\bar{p}_0u^{(0)}_{2,\gamma}-p^{*(0)}\delta_{2\gamma}=&0,\quad \text{on } x_2=0, &(2.18)\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{\alpha 3\beta k}u^{(0)}_{k,\alpha\beta}=&0,\quad \text{in } B_e, &(2.19)\\ \mathcal{A}^{(0)}_{2i\alpha k}u^{(0)}_{k,\alpha}=&0,\quad \text{on } x_2=0. &(2.20)\end{aligned}$$

Second order:

$$\begin{aligned}\mathcal{A}^{(0)}_{\alpha\gamma\beta k}u^{(1)}_{k,\alpha\beta}-p^{*(1)}_{,\gamma}=&-(\mathcal{A}^{(0)}_{\alpha\gamma 3k}+\mathcal{A}^{(0)}_{3\gamma\alpha k})(u^{(0)}_{k,3\alpha}+h'u^{(0)}_{k,2\alpha}),\quad \text{in } B_e, &(2.21)\\ \mathcal{A}^{(0)}_{2\gamma\alpha k}u^{(1)}_{k,\alpha}+\bar{p}_0u^{(1)}_{2,\gamma}-p^{*(1)}\delta_{2\gamma}=&-h'(\mathcal{A}^{(0)}_{3\gamma\alpha k}u^{(0)}_{k,\alpha}+\mathcal{A}^{(0)}_{2\gamma 3k}u^{(0)}_{k,2}+\bar{p}_0u^{(0)}_{3,\gamma}-p^{*(0)}\delta_{3\gamma})\\ &-\mathcal{A}^{(0)}_{2\gamma 3k}u^{(0)}_{k,3},\quad \text{on } x_2=0, &(2.22)\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{\alpha 3\beta k}u^{(1)}_{k,\alpha\beta}-(p^{*(0)}_{,3}+h'p^{*(0)}_{,2})=&-(\mathcal{A}^{(0)}_{\alpha 33k}+\mathcal{A}^{(0)}_{33\alpha k})(u^{(0)}_{k,3\alpha}+h'u^{(0)}_{k,2\alpha}),\quad \text{in } B_e, &(2.23)\\ \mathcal{A}^{(0)}_{23\alpha k}u^{(1)}_{k,\alpha}+\bar{p}_0(u^{(0)}_{2,3}+h'u^{(0)}_{2,2})=&-h'(\mathcal{A}^{(0)}_{33\alpha k}u^{(0)}_{k,\alpha}+\mathcal{A}^{(0)}_{233k}u^{(0)}_{k,2}-p^{*(0)})\\ &-\mathcal{A}^{(0)}_{233k}u^{(0)}_{k,3},\quad \text{on } x_2=0. &(2.24)\end{aligned}$$

Third order:

$$\begin{aligned}\mathcal{A}^{(0)}_{\alpha\gamma\beta k}u^{(2)}_{k,\alpha\beta}-p^{*(2)}_{,\gamma}=&-\hat{\lambda}A^{(1)}_{\alpha\gamma\beta k}u^{(0)}_{k,\alpha\beta}-(\mathcal{A}^{(0)}_{\alpha\gamma 3j}+\mathcal{A}^{(0)}_{3\gamma\alpha j})(u^{(1)}_{j,3\alpha}+h'u^{(1)}_{j,2\alpha})\\ &-\mathcal{A}^{(0)}_{3\gamma 3k}(u^{(0)}_{k,33}+h''u^{(0)}_{k,2}+2h'u^{(0)}_{k,23}+h'^2u^{(0)}_{k,22}),\quad \text{in } B_e, &(2.25)\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{2\gamma\alpha k}u^{(2)}_{k,\alpha}+\bar{p}_0u^{(2)}_{2,\gamma}-p^{*(2)}\delta_{2\gamma}=&-\hat{\lambda}A^{(1)}_{2\gamma\alpha k}u^{(0)}_{k,\alpha}-h'(\mathcal{A}^{(0)}_{3\gamma\alpha j}u^{(1)}_{j,\alpha}+\mathcal{A}^{(0)}_{2\gamma 3j}u^{(1)}_{j,2}\\ &+\bar{p}_0u^{(1)}_{3,\gamma}-p^{*(1)}\delta_{3\gamma})-\mathcal{A}^{(0)}_{2\gamma 3j}u^{(1)}_{j,3}\\ &-h'\mathcal{A}^{(0)}_{3\gamma 3k}(u^{(0)}_{k,3}+h'u^{(0)}_{k,2})-\hat{\lambda}\bar{p}_1u^{(0)}_{2,\gamma},\quad \text{on } x_2=0. &(2.26)\end{aligned}$$

Note that in writing down the equilibrium equations and boundary conditions at each order, we have taken i to be γ (1 or 2) and 3, separately, and at third order the equations corresponding to $i=3$ have not been written out since they are not required in subsequent analysis. It is seen immediately that due to the symmetry properties of the elastic moduli, stated after equation (2.10), the problem for $u^{(k)}_\alpha$ is decoupled from the problem for $u^{(k)}_3$ at each order ($k=1,2,3$).

The above boundary value problems are to be solved in conjunction with the incompressibility equations

$$u^{(0)}_{1,1}+u^{(0)}_{2,2}=0, \tag{2.27}$$

$$u_{1,1}^{(1)} + u_{2,2}^{(1)} = -u_{3,3}^{(0)} - h' u_{3,2}^{(0)}, \tag{2.28}$$

$$u_{1,1}^{(2)} + u_{2,2}^{(2)} = -u_{3,3}^{(1)} - h' u_{3,2}^{(1)}, \tag{2.29}$$

obtained from the original incompressibility condition $(2.1)_2$.

3. Leading-order problem

It can be seen that the subscript k in (2.19) and (2.20) must necessarily be equal to 3, and as a result the problem for $u_3^{(0)}$ is decoupled from the problem for $u_\gamma^{(0)}$. We take $u_3^{(0)} = 0$ since our focus is on the connection with the classical Biot problem. It is also seen from (2.17) that $\mathcal{A}_{\alpha\gamma\beta k}^{(0)}$ is only non-zero if k is equal to 1 or 2. Thus, we may replace k by a Greek subscript and obtain

$$\mathcal{A}_{\alpha\gamma\beta\delta}^{(0)} u_{\delta,\alpha\beta}^{(0)} - p_{,\gamma}^{*(0)} = 0. \tag{3.1}$$

The $p^{*(0)}$ can be eliminated by cross-differentiating the two equations corresponding to $\gamma = 1, 2$, which yields the single equation

$$\mathcal{A}_{\alpha 1\beta\delta}^{(0)} u_{\delta,2\alpha\beta}^{(0)} - \mathcal{A}_{\alpha 2\beta\delta}^{(0)} u_{\delta,1\alpha\beta}^{(0)} = 0. \tag{3.2}$$

The $p^{*(0)}$ in the boundary condition (2.18) with $\gamma = 2$ can be eliminated by differentiating this equation with respect to x_1 and then eliminating $p_{,1}^{*(0)}$ with the use of (3.1). As a result, the two boundary conditions become

$$\mathcal{A}_{21\alpha\delta}^{(0)} u_{\delta,\alpha}^{(0)} + \bar{p}_0 u_{2,1}^{(0)} = 0, \quad \text{on } \; x_2 = 0, \tag{3.3}$$

$$\mathcal{A}_{22\alpha\delta}^{(0)} u_{\delta,1\alpha}^{(0)} - \mathcal{A}_{\alpha 1\beta\delta}^{(0)} u_{\delta,\alpha\beta}^{(0)} + \bar{p}_0 u_{2,12}^{(0)} = 0, \quad \text{on } \; x_2 = 0. \tag{3.4}$$

The incompressibility condition (2.27) can be satisfied by introducing a "stream" function ϕ such that

$$u_1^{(0)} = \phi_{,2}, \qquad u_2^{(0)} = -\phi_{,1}. \tag{3.5}$$

Equations (3.2)$-$(3.4) then define a boundary value problem for a single function ϕ.

We look for a solution of the form

$$\phi = f(x_3) z(x_2) \mathrm{e}^{\mathrm{i}x_1} + \text{c.c}, \tag{3.6}$$

where f and z are scalar functions to be determined, and c.c. denotes the complex conjugate of the preceding term. Note that the wave number in the above expression is unity; this is because we have used the inverse of the original wavenumber to non-dimensionalize x_1. The actual wave number will be restored when numerical results are discussed.

On substituting (3.6) into (3.2)−(3.4), we obtain the following reduced boundary value problem for z:

$$\mathcal{L}[z] = 0, \quad \text{for } 0 < x_2 < \infty, \tag{3.7}$$

$$\mathcal{B}_1[z] = 0, \quad \mathcal{B}_2[z] = 0, \quad \text{on } x_2 = 0, \tag{3.8}$$

where the three differential operators $\mathcal{L}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by

$$\mathcal{L}[z] = \gamma z^{(4)} - 2\beta z'' + \alpha z, \tag{3.9}$$

$$\mathcal{B}_1[z] = \gamma(z'' + z), \tag{3.10}$$

$$\mathcal{B}_2[z] = \gamma z^{(3)} - (2\beta + \gamma)z', \tag{3.11}$$

together with

$$\alpha = \mathcal{A}^{(0)}_{1212}, \quad 2\beta = \mathcal{A}^{(0)}_{1111} + \mathcal{A}^{(0)}_{2222} - 2\mathcal{A}^{(0)}_{1122} - 2\mathcal{A}^{(0)}_{1221}, \quad \gamma = \mathcal{A}^{(0)}_{2121}. \tag{3.12}$$

In obtaining these expressions we have made use of the fact that $\mathcal{A}^{(0)}_{2121} - \mathcal{A}^{(0)}_{2112} - \bar{p}_0$ is equal to the principal stress in the x_2-direction which in this case is identically zero; see, e.g. Dowaikh and Ogden [6].

Through integration by parts, it can be shown that the above operators have the property that for any two sufficiently differentiable functions $g_1(x_2)$ and $g_2(x_2)$,

$$\int_0^\infty (g_1\mathcal{L}[g_2] - g_2\mathcal{L}[g_1])dx_2 = \{g_1'\mathcal{B}_1[g_2] - g_1\mathcal{B}_2[g_2] - g_2'\mathcal{B}_1[g_1] + g_2\mathcal{B}_2[g_1]\}\big|_{x_2=0}. \tag{3.13}$$

In particular, if $g_1 = z$, the above identity then reduces to

$$\int_0^\infty z\mathcal{L}[g_2]dx_2 = \{z'\mathcal{B}_1[g_2] - z\mathcal{B}_2[g_2]\}\big|_{x_2=0}. \tag{3.14}$$

This reduced identity will be used in Section 5 to derive the amplitude equation for the unknown function $f(x_3)$.

The boundary value problem (3.7) and (3.8) can be solved by elementary methods. It has a non-trivial decaying solution only if the following bifurcation condition is satisfied:

$$\alpha\gamma + 2(\beta + \gamma)\sqrt{\alpha\gamma} - \gamma^2 = 0. \tag{3.15}$$

See, e.g., Dowaikh and Ogden [6]. When the material is neo-Hookean, it gives the critical stretch $\lambda_{\rm cr} = 0.544$ first obtained by Biot [1].

4. Second-order problem

With $u_3^{(0)}$ identically zero, the boundary value problem for $u_\alpha^{(1)}$ is the same as that for $u_\alpha^{(0)}$. Its solution takes a form similar to (3.5) and (3.6), but this solution is not required in subsequent analysis since $u_\alpha^{(1)}$ will not appear in any of the equations from now on. Thus, we shall focus on the following boundary value problem for the anti-plane component $u_3^{(1)}$:

$$\mathcal{A}^{(0)}_{1313}u^{(1)}_{3,11} + \mathcal{A}^{(0)}_{2323}u^{(1)}_{3,22} = p^{*(0)}_{,3} + h'p^{*(0)}_{,2} - (\mathcal{A}^{(0)}_{\alpha 33\gamma} + \mathcal{A}^{(0)}_{33\alpha\gamma})(u^{(0)}_{\gamma,3\alpha} + h'u^{(0)}_{\gamma,2\alpha}), \quad \text{in } B_e, \tag{4.1}$$

$$\mathcal{A}^{(0)}_{2323}u^{(1)}_{3,2} = -\bar{p}_0(u^{(0)}_{2,3} + h'u^{(0)}_{2,2}) - h'(\mathcal{A}^{(0)}_{33\alpha\delta}u^{(0)}_{\delta,\alpha} + \mathcal{A}^{(0)}_{2332}u^{(0)}_{2,2} - p^{*(0)}) - \mathcal{A}^{(0)}_{2332}u^{(0)}_{2,3}, \quad \text{on } x_2 = 0. \tag{4.2}$$

The $p^{*(0)}$ in the above equations can be eliminated by differentiating each equation with respect to x_1 and then eliminating $p^{*(0)}_{,1}$ with the use of (3.1). We then obtain

$$\begin{aligned}\mathcal{A}^{(0)}_{1313}u^{(1)}_{3,111} + \mathcal{A}^{(0)}_{2323}u^{(1)}_{3,122} =& -(\mathcal{A}^{(0)}_{\alpha 33\gamma} + \mathcal{A}^{(0)}_{33\alpha\gamma})(u^{(0)}_{\gamma,13\alpha} + h'u^{(0)}_{\gamma,12\alpha}) + \mathcal{A}^{(0)}_{\alpha 1\beta\delta}u^{(0)}_{\delta,\alpha\beta 3}\\ &+ h'\mathcal{A}^{(0)}_{\alpha 2\beta\delta}u^{(0)}_{\delta,\alpha\beta 1}, \quad \text{in } B_e,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{2323}u^{(1)}_{3,12} =& -\bar{p}_0(u^{(0)}_{2,13} + h'u^{(0)}_{2,12}) + h'\mathcal{A}^{(0)}_{\alpha 1\beta\delta}u^{(0)}_{\delta,\alpha\beta} - \mathcal{A}^{(0)}_{2332}u^{(0)}_{2,13}\\ &- h'(\mathcal{A}^{(0)}_{33\alpha\delta}u^{(0)}_{\delta,1\alpha} + \mathcal{A}^{(0)}_{2332}u^{(0)}_{2,12}), \quad \text{on } x_2 = 0.\end{aligned} \tag{4.4}$$

The form of the right-hand sides of (4.3) and (4.4) suggests that $u_3^{(1)}$ should take the form

$$u_3^{(1)} = \mathrm{i}(f'w_1 + fh'w_2)e^{\mathrm{i}x_1} + \text{c.c}, \tag{4.5}$$

where the factor i is inserted so that w_1 and w_2 are both real functions. On substituting this expression into (4.3) and (4.4) and then equating the coefficients of f and f', we obtain

$$\begin{aligned}\mathcal{A}^{(0)}_{2323}w_1'' - \mathcal{A}^{(0)}_{1313}w_1 =& (\mathcal{A}^{(0)}_{1111} - \mathcal{A}^{(0)}_{1122} - \mathcal{A}^{(0)}_{1221} - \mathcal{A}^{(0)}_{1331} + \mathcal{A}^{(0)}_{2332} - \mathcal{A}^{(0)}_{1133} + \mathcal{A}^{(0)}_{2233})z'\\ &- \mathcal{A}^{(0)}_{2121}z''', \quad \text{in } B_e,\end{aligned} \tag{4.6}$$

$$w_1' = z \quad \text{on } x_2 = 0, \tag{4.7}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{2323}w_2'' - \mathcal{A}^{(0)}_{1313}w_2 =& (\mathcal{A}^{(0)}_{1122} + \mathcal{A}^{(0)}_{1221} + \mathcal{A}^{(0)}_{2332} + \mathcal{A}^{(0)}_{2233} - \mathcal{A}^{(0)}_{2222} - \mathcal{A}^{(0)}_{1133} - \mathcal{A}^{(0)}_{1331})z''\\ &+ \mathcal{A}^{(0)}_{1212}z, \quad \text{in } B_e,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{2323}w_2' =& -(\mathcal{A}^{(0)}_{1221} - \mathcal{A}^{(0)}_{1111} + \mathcal{A}^{(0)}_{1122} - \mathcal{A}^{(0)}_{2323} + \mathcal{A}^{(0)}_{1133} - \mathcal{A}^{(0)}_{2233})z',\\ &- \mathcal{A}^{(0)}_{2121}z''', \quad \text{on } x_2 = 0.\end{aligned} \tag{4.9}$$

The above equations are to be solved subject to the additional decay conditions $w_1 \to 0$ and $w_2 \to 0$ as $x_2 \to \infty$. A unique solution for w_1 and w_2 can be found by elementary methods.

5. Third-order problem

In view of the symmetry properties of $\mathcal{A}^{(0)}_{jilk}$, we can replace the subscript k in (2.25) and (2.26) by a Greek letter and j by 3 to obtain

$$\begin{aligned}\mathcal{A}^{(0)}_{\alpha\gamma\beta\delta}u^{(2)}_{\delta,\alpha\beta} - p^{*(2)}_{,\gamma} = &- \hat{\lambda}A^{(1)}_{\alpha\gamma\beta\delta}u^{(0)}_{\delta,\alpha\beta} - (\mathcal{A}^{(0)}_{\alpha\gamma 33} + \mathcal{A}^{(0)}_{3\gamma\alpha 3})(u^{(1)}_{3,3\alpha} + h'u^{(1)}_{3,2\alpha}) \\ &- \mathcal{A}^{(0)}_{3\gamma 3\delta}(u^{(0)}_{\delta,33} + h''u^{(0)}_{\delta,2} + 2h'u^{(0)}_{\delta,23} + h'^2u^{(0)}_{\delta,22}), \quad \text{in } B_e \qquad (5.1)\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{(0)}_{2\gamma\alpha\delta}u^{(2)}_{\delta,\alpha} + \bar{p}_0u^{(2)}_{2,\gamma} - p^{*(2)}\delta_{2\gamma} = &- \hat{\lambda}A^{(1)}_{2\gamma\alpha\delta}u^{(0)}_{\delta,\alpha} - h'(\mathcal{A}^{(0)}_{3\gamma\alpha 3}u^{(1)}_{3,\alpha} + \mathcal{A}^{(0)}_{2\gamma 33}u^{(1)}_{3,2} \\ &+ \bar{p}_0u^{(1)}_{3,\gamma} - p^{*(1)}\delta_{3\gamma}) - \mathcal{A}^{(0)}_{2\gamma 33}u^{(1)}_{3,3} \\ &- h'\mathcal{A}^{(0)}_{3\gamma 3\delta}(u^{(0)}_{\delta,3} + h'u^{(0)}_{\delta,2}) - \hat{\lambda}\bar{p}_1u^{(0)}_{2,\gamma}, \quad \text{on } x_2 = 0 \qquad (5.2)\end{aligned}$$

The incremental pressure $p^{*(2)}$ can be eliminated in the same manner as how $p^{*(0)}$ was eliminated from the leading-order problem. We then look for a solution of the form

$$u^{(2)}_1 = U(x_2, x_3)e^{\mathrm{i}x_1} + c.c \quad u^{(2)}_2 = -\mathrm{i}V(x_2, x_3)e^{\mathrm{i}x_1} + c.c. \qquad (5.3)$$

On substituting these expressions into the incompressibility condition (2.29), we obtain

$$U = \frac{\partial V}{\partial x_2} - \xi_3, \qquad (5.4)$$

where

$$\xi_3 = f''w_1 + f'h'(w_1' + w_2) + f(h''w_2 + h'^2w_2'). \qquad (5.5)$$

When U is eliminated in favour of V with the use of (5.4), the boundary value problem (5.1) and (5.2) reduces to the following boundary value problem for V:

$$\mathcal{L}[V] = \xi_1' - \xi_2 - (A^{(0)}_{1111} - A^{(0)}_{1221} - A^{(0)}_{2211})\xi_3' + A^{(0)}_{2121}\xi_3''', \ \ 0 < x_2 < \infty, \qquad (5.6)$$

$$\mathcal{B}_1[V] = \zeta_1 + A^{(0)}_{2121}\xi_3', \quad x_2 = 0, \qquad (5.7)$$

$$\mathcal{B}_2[V] = \xi_1 - \zeta_2 - A^{(0)}_{1111}\xi_3 + A^{(0)}_{1122}\xi_3 + A^{(0)}_{2121}\xi_3'', \quad x_2 = 0, \qquad (5.8)$$

where the primes on ξ_1 and ξ_3 signify differentiation with respect to x_2, and

$$
\begin{aligned}
\xi_1 =& \hat{\lambda} A^{(1)}_{1111} f z' - \hat{\lambda}(A^{(1)}_{1122} + A^{(1)}_{2112}) f z' - \hat{\lambda} A^{(1)}_{2121} f z''', \\
&+ (A^{(0)}_{1133} + A^{(0)}_{3113})[(f'' w_1 + f' h' w_2 + f h'' w_2) + h'(f' w_1' + f h' w_2')], \\
&- A^{(0)}_{3131}(f'' z' + h'' f z'' + 2h' f' z'' + h'^2 f z'''), \\
\xi_2 =& \hat{\lambda} A^{(1)}_{1212} f z + \hat{\lambda}(A^{(1)}_{1221} + A^{(1)}_{2211}) f z'' - \hat{\lambda} A^{(1)}_{2222} f z'' \\
&+ (A^{(0)}_{2233} + A^{(0)}_{3223})[(f'' w_1' + f' h' w_2' + f h'' w_2') + h'(f' w_1'' + f h' w_2'')] \\
&- A^{(0)}_{3232}(f'' z + h'' f z' + 2h' f' z' + h'^2 f z''), \\
\zeta_1 =& - \hat{\lambda} A^{(1)}_{2112} f z - \hat{\lambda} A^{(1)}_{2121} f z'' + h'(A^{(0)}_{3113} + \bar{p}_0)(f' w_1 + h' f w_2) \\
&- h' A^{(0)}_{3131}(f' z' + h' f z'') - \hat{\lambda} \bar{p}_1 f z, \\
\zeta_2 =& \hat{\lambda} A^{(0)}_{1122} f z' - \hat{\lambda} A^{(0)}_{2222} f z' + h'(A^{(0)}_{3223} + A^{(0)}_{2233} + \bar{p}_0)(f' w_1' + f h' w_2') \\
&+ A^{(0)}_{2233}(f'' w_1 + f' h' w_2 + f h'' w_2) - h' A^{(0)}_{3232}(f' z + h' f z') - \hat{\lambda} \bar{p}_1 f z'.
\end{aligned}
$$

Since the left-hand sides of (5.6)−(5.8) involve the same operators as those in the leading order problem, this boundary value problem has a solution only if the right-hand sides satisfy a solvability condition. This condition may be obtained by replacing g_2 by V in the identity (3.14). After simplification, we obtain

$$
c_4 f''(x_3) + c_3 h' f'(x_3) + (c_2 h'^2 + c_1 h'' + c_0 \hat{\lambda}) f(x_3) = 0, \tag{5.9}
$$

where the coefficients are all real and are given by

$$
\begin{aligned}
c_4 =& \int_0^\infty z[(-A^{(0)}_{1133} - A^{(0)}_{3113} + A^{(0)}_{2233} + A^{(0)}_{3223} + A^{(0)}_{1111} - A^{(0)}_{1221} - A^{(0)}_{1122}) w_1' \\
&+ (A^{(0)}_{3131} z'' - A^{(0)}_{3232} z) - A^{(0)}_{2121} w_1'''] dx_2 + A^{(0)}_{2121} z'(0) w_1'(0) - A^{(0)}_{2121} z(0) w_1''(0) \\
&+ A^{(0)}_{3131} z(0) z'(0) + A^{(0)}_{2233} z(0) w_1(0) + (A^{(0)}_{1111} - A^{(0)}_{1122} - A^{(0)}_{1133} - A^{(0)}_{1331}) z(0) w_1(0),
\end{aligned}
$$

$$
\begin{aligned}
c_3 =& \int_0^\infty z[(-A^{(0)}_{1133} - A^{(0)}_{3113} + A^{(0)}_{2233} + A^{(0)}_{3223} + A^{(0)}_{1111} - A^{(0)}_{1221} - A^{(0)}_{1122})(w_1'' + w_2') \\
&+ 2(A^{(0)}_{3131} z''' - A^{(0)}_{3232} z') - A^{(0)}_{2121}(w_1'''' + w_2''')] dx_2 + (A^{(0)}_{3113} + \bar{p}_0) z'(0) w_1(0) \\
&- A^{(0)}_{3131} z'^2(0) + A^{(0)}_{2121} z'(0)(w_1''(0) + w_2'(0)) - (A^{(0)}_{1133} + A^{(0)}_{3113}) z(0)(w_1'(0) + w_2(0)) \\
&+ 2A^{(0)}_{3131} z(0) z''(0) + (A^{(0)}_{3223} + A^{(0)}_{2233} + \bar{p}_0) z(0) w_1'(0) + A^{(0)}_{2233} z(0) w_2(0) \\
&- A^{(0)}_{3232} z^2(0) + (A^{(0)}_{1111} - A^{(0)}_{2211}) z(0)(w_1'(0) + w_2(0)) - A^{(0)}_{2121} z(0)(w_1'''(0) + w_2''(0)),
\end{aligned}
$$

$$
\begin{aligned}
c_2 = \int_0^\infty & z[(-A^{(0)}_{1133} - A^{(0)}_{3113} + A^{(0)}_{2233} + A^{(0)}_{3223} + A^{(0)}_{1111} - A^{(0)}_{1221} - A^{(0)}_{2211})w_2'' \\
& + (A^{(0)}_{3131} z'''' - A^{(0)}_{3232} z'') - A^{(0)}_{2121} w_2''''] dx_2 + (A^{(0)}_{3113} + \bar{p}_0) z'(0) w_2(0) \\
& - A^{(0)}_{3131} z'(0) z''(0) + A^{(0)}_{2121} z'(0) w_2''(0) - A^{(0)}_{2121} z(0) w_2'''(0) \\
& + A^{(0)}_{3131} z(0) z'''(0) - A^{(0)}_{3232} z(0) z'(0) \\
& + (A^{(0)}_{1111} - A^{(0)}_{1122} + A^{(0)}_{3223} + A^{(0)}_{2233} + \bar{p}_0 - A^{(0)}_{1133} - A^{(0)}_{3113}) z(0) w_2'(0),
\end{aligned}
$$

$$
\begin{aligned}
c_1 = \int_0^\infty & z[(-A^{(0)}_{1133} - A^{(0)}_{3113} + A^{(0)}_{2233} + A^{(0)}_{3223} + A^{(0)}_{1111} - A^{(0)}_{1221} - A^{(0)}_{2211})w_2' \\
& + (A^{(0)}_{3131} z''' - A^{(0)}_{3232} z') - A^{(0)}_{2121} w_2'''] dx_2 + A^{(0)}_{2121} z'(0) w_2'(0) + A^{(0)}_{3131} z(0) z''(0) - A^{(0)}_{2121} z(0) w_2''(0) \\
& - (A^{(0)}_{1133} + A^{(0)}_{3113} + A^{(0)}_{1122} - A^{(0)}_{2233} - A^{(0)}_{1111}) z(0) w_2(0),
\end{aligned}
$$

$$
\begin{aligned}
c_0 = \int_0^\infty & z[A^{(1)}_{2121} z'''' + (2A^{(1)}_{1122} + A^{(1)}_{2112} - A^{(1)}_{1111} + A^{(1)}_{1221} - A^{(1)}_{2222}) z'' + A^{(1)}_{1212} z] dx_2 \\
& - z(0)[(A^{(1)}_{1111} - 2A^{(1)}_{1122} + A^{(1)}_{2222} + 2\bar{p}_1) z'(0) - A^{(1)}_{2121} z'''(0)] - A^{(1)}_{2121} z'(0) z''(0).
\end{aligned}
$$

For each specified surface profile $h(x_3)$, equation (5.9) is to be solved subject to the decay conditions $f(x_3) \to 0$ as $x_3 \to \pm\infty$. This is an eigenvalue problem for $\hat{\lambda}$ which will be solved in the next section.

6. Numerical and asymptotic results

We have evaluated the coefficients for a variety of materials, including neo-Hookean, Gent, Mooney-Rivlin, and Ogden models. It is found that c_3 is identically zero for all the material models considered although we have not been able to prove this result analytically. We thus rewrite (5.9) as

$$
f''(x_3) + (d_2 h'^2 + d_1 h'' + d_0 \hat{\lambda}) f(x_3) = 0, \tag{6.10}
$$

where

$$
(d_0, d_1, d_2) = (c_0, c_1, c_2)/c_4.
$$

Recall that all the coordinates and parameters/functions that have the dimension of length have been scaled by $1/k$, where k is the original wave number. Denoting the unscaled coordinates, imperfection profile, and amplitude function by x_i^*, h^*, and f^*, respectively, we then have

$$
x_3 = k x_3^*, \quad h(x_3) = k h^*(x_3^*), \quad f(x_3) = k f^*(x_3^*),
$$

and

$$h'(x_3) = \frac{dh^*}{dx_3^*}, \quad h''(x_3) = \frac{1}{k}\frac{d^2h^*}{dx_3^{*2}}, \quad f''(x_3) = \frac{1}{k}\frac{d^2f^*}{dx_3^{*2}}.$$

The dimensional form of the amplitude equation (6.10) is then given by

$$f^{*\prime\prime}(x_3^*) + (k^2 d_2 h^{*\prime 2} + k d_1 h^{*\prime\prime} + k^2 d_0 \hat{\lambda}) f^*(x_3^*) = 0, \tag{6.11}$$

where a prime now signifies differentiation with respect to x_3^*. Note, however, that x_3^* is not the original dimensional coordinate in the anti-plane direction, but that coordinate multiplied by ε. Equation (6.11) shows that the current problem is dispersive: the correction $\hat{\lambda}$ to the critical stretch is dependent on the wave number even though the leading order term is not.

To facilitate interpretation of numerical results, it is convenient to scale x_3^*, h^* and f^* by the maximum value of $|h^*(x_3^*)|$, h_0 say. Thus we write

$$x_3^* = h_0 \hat{x}_3, \quad h^*(x_3^*) = h_0 \hat{h}(\hat{x}_3), \quad f^*(x_3^*) = h_0 \hat{f}(\hat{x}_3).$$

On substituting these expressions into (6.11) and then dropping the hats, we obtain

$$f''(x_3) + k^2 h_0^2 \left\{ d_2 h'^2 + \frac{d_1}{kh_0} h'' + d_0 \hat{\lambda} \right\} f(x_3) = 0. \tag{6.12}$$

As a consistency check, if h_0 is taken to be $1/k$, this recovers (6.10). Recalling equation (2.14), we see that the correction to the critical stretch due to the surface imperfection is a function of kh_0 and ε.

Equation (6.12) subject to the decay conditions $f(x_3) \to 0$ as $x_3 \to \pm\infty$ is an eigenvalue problem that has a non-trivial solution only for special values of $\hat{\lambda}$. This eigenvalue problem needs to be solved numerically in general. However, asymptotic solutions can be obtained when kh_0 is either large or small.

When kh_0 is sufficiently small, it can be shown that the eigenvalue problem has at most one eigenvalue given by

$$\sqrt{-d_0\hat{\lambda}} = \frac{1}{2}(d_2 + d_1^2) kh_0 \int_{-\infty}^{\infty} h'^2 dx_3 + O((kh_0)^2). \tag{6.13}$$

See Simon [39], Klaus [40], and also Fu et al. [36]. Thus, this single eigenvalue can exist only if

$$d_2 + d_1^2 > 0. \tag{6.14}$$

In Table 1 we have shown the values of the coefficients for some commonly used material models.

In view of the fact that $d_0 < 0$ and the product under the radical sign in (6.13) must be positive, we may conclude that localized solutions exist only if $\hat{\lambda} > 0$, that is localized surface

Table 1: Coefficients of the amplitude equation

	d_0	d_1	d_2
neo-Hookean	-3.087	0	0.436
Gent ($J_m = 200$)	-3.043	0.006	0.443
Gent ($J_m = 97$)	-2.995	0.013	0.450
Gent ($J_m = 30$)	-2.764	0.046	0.487
Gent ($J_m = 10$)	-1.652	0.194	0.661
Mooney-Rivlin ($\mu_2/\mu_1 = 0.1$)	-3.360	0.077	0.365
Ogden	-3.811	0.039	0.250

imperfection will increase the critical stretch, thus making the half-space easier to wrinkle. It is seen that d_1 is identically zero for the neo-Hookean material model. This means that when this model is used, a surface ridge or trench has the same effect on the critical stretch, but this symmetry is lost when the other material models are used. However, according to the asymptotic expansion (6.13) the sign of h does not affect the leading order term.

On the other hand, when kh_0 is large, a WKB analysis can be conducted to find the eigenvalues; see Bender and Orszag [41]. The eigenfunctions would localize near the points where the term $d_2 h'^2$ attains a maximum or minimum. Since $h(x_3)$ has been assumed to be an even function and to decay to zero as $|x_3| \to \infty$, we have $h'(0) = 0$ and so there must exist at least two such maximum points, symmetrically located on the two sides of the origin. For the examples that we shall consider later, there exist exactly two such maximum points. Focus on the positive one, which we denote by x_0. It is known that for a minimum of $d_0\hat{\lambda}$ (and hence a maximum of $\hat{\lambda}$) this point should be a second-order turning point and the associated eigenfunctions exist in a thin layer of order $(kh_0)^{-1/2}$ around x_0. The asymptotic solution takes the form

$$f(x_3) = f_0(\xi) + (kh_0)^{-1/2} f_1(\xi) + (kh_0)^{-1} f_2(\xi) + \cdots , \tag{6.15}$$

$$\hat{\lambda} = \hat{\lambda}_0 + (kh_0)^{-1}\hat{\lambda}_1 + (kh_0)^{-2}\hat{\lambda}_2 + \cdots , \tag{6.16}$$

where the boundary layer variable ξ is defined by

$$\xi = (kh_0)^{\frac{1}{2}}(x_3 - x_0), \tag{6.17}$$

and the constants $\hat{\lambda}_i$ (i=0, 1, ...) are to be determined. On substituting (6.15)$-$(6.17) into (6.12), and then equating the coefficients of like powers of kh_0, we find that the leading order problem can be satisfied only if

$$\hat{\lambda}_0 = -\frac{d_2}{d_0} h'^2(x_0). \tag{6.18}$$

The second-order problem requires that $h''(x_0)=0$, which we have assumed already. Finally, at third order, we obtain

$$f_2''(\xi)+(d_0\hat{\lambda}_1+d_2h_0'h_0'''\xi^2)f_2(\xi)=0, \tag{6.19}$$

where $h_0'=h'(x_0),\ h_0'''=h'''(x_0)$. This equation can be reduced to the Weber's equation

$$g''(s)-(\frac{1}{4}s^2-\frac{d_0}{a^2}\hat{\lambda}_1)g(s)=0, \tag{6.20}$$

with the substitutions $\xi=s/a,\ f_2(\xi)=g(s)$, where

$$a=(-4d_2h_0'h_0''')^{\frac{1}{4}}.$$

Equation (6.20) has localized solutions only if

$$\frac{d_0}{a^2}\hat{\lambda}_1=m+\frac{1}{2}, \quad (m=0,1,2,...), \tag{6.21}$$

and the associated solutions are given by

$$g(s)=\mathrm{e}^{-\frac{1}{4}s^2}\mathrm{He}_m(\frac{s}{\sqrt{2}}), \quad \text{or} \quad s\mathrm{e}^{-\frac{1}{4}s^2}\mathrm{He}_m(\frac{s}{\sqrt{2}}), \tag{6.22}$$

where He_m are the Hermite polynomials. The two solutions given by (6.22) are symmetric and anti-symmetric modes, respectively. Thus, we obtain the following two-term expansion for $\hat{\lambda}$:

$$\hat{\lambda}=-\frac{d_2}{d_0}h'^2(x_0)+\frac{\sqrt{-d_2h_0'h_0'''}}{d_0kh_0}(2m+1), \quad (m=0,1,2,...). \tag{6.23}$$

For each m, there exist both a symmetric and an anti-symmetric mode given by (6.22), and there are infinite pairs of such modes. These predictions will shortly be verified by our numerical results.

The amplitude equation (6.12) is now solved numerically following the procedure outlined in Fu et al. [36]. In the limit $x_3\to\pm\infty$, equation (6.12) can be approximated by

$$f''(x_3)+(kh_0)^2d_0\hat{\lambda}f(x_3)=0. \tag{6.24}$$

It is clear that $f(x_3)$ will have the required decay behaviour as $x_3\to\pm\infty$ only if $d_0\hat{\lambda}<0$, which is consistent with the asymptotic expression (6.13). We then have

$$f'(x_3)\pm kh_0\sqrt{-d_0\hat{\lambda}}\,f(x_3)\to 0, \quad \text{as} \quad x_3\to\pm\infty. \tag{6.25}$$

Since $h(x_3)$ has been assumed to be an even function of x_3, $f(-x_3)$ is a solution of (6.12) whenever $f(x_3)$ is a solution. Thus, the eigen solutions of (6.12) are either even or odd. For the even (symmetric) modes, we may impose, without loss of generality, the conditions

$$f(0)=1, \quad f'(0)=0, \tag{6.26}$$

and the decay behaviour through

$$e(\hat{\lambda}) \equiv f'(L) + kh_0\sqrt{-d_0\hat{\lambda}}\, f(L) = 0, \tag{6.27}$$

where L is a sufficiently large positive constant and the first equation in (6.27) defines the error function $e(\hat{\lambda})$. For each fixed $\hat{\lambda}$, the $e(\hat{\lambda})$ can be evaluated after integrating (6.12) subject to the initial conditions (6.26). We first plot $e(\hat{\lambda})$ against $\hat{\lambda}$ to show the approximate locations of any zeros and then use the Newton-Raphson method to find the exact values of the zeros. All of our symbolic manipulations and numerical integrations are carried out with the aid of Mathematica [42].

For the odd (anti-symmetric) modes, the initial conditions (6.26) are replaced by

$$f(0) = 0, \quad f'(0) = 1, \tag{6.28}$$

and we integrate (6.12) subject to the initial conditions (6.28) and iterate on $\hat{\lambda}$ in order to satisfy the decay condition (6.27). To avoid having to adjust L for different values of $\hat{\lambda}$, we solve the eigenvalue problem in terms of a scaled variable $\tilde{x}_3$ defined by $\tilde{x}_3 = kh_0\sqrt{-d_0\hat{\lambda}}\, x_3$. Then it is found sufficient to choose L to be between 15 and 25.

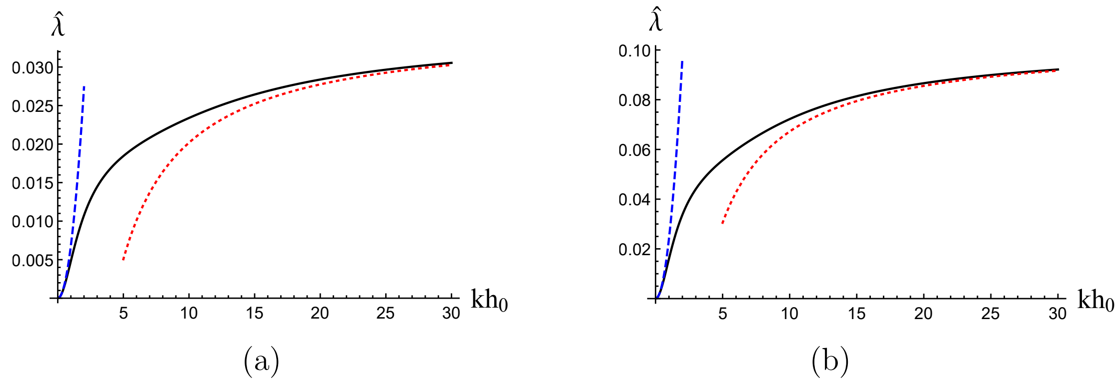
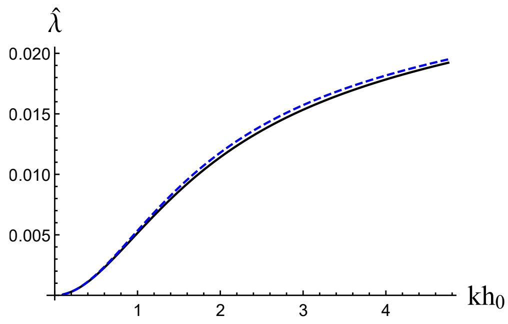


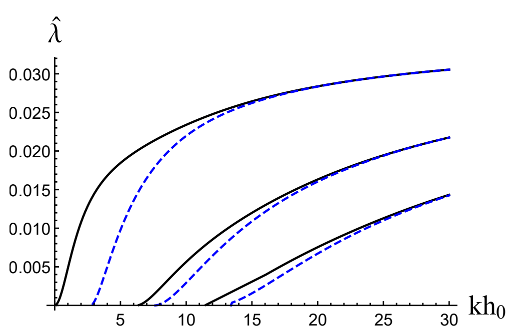
Figure 2: Variation of $\hat{\lambda}$ with respect to kh_0 when (a) $h(x_3) = \text{sech}(x_3)$, and (b) $h(x_3) = \mathrm{e}^{-x_3^2}$ and the material is neo-Hookean. Solid line: numerical results; dashed line (blue): asymptotic results given by the leading term in (6.13); dotted line (red): asymptotic results given by (6.23).

As an illustrative example, consider the case when the topography is described by the 'Gaussian bump' $h(x_3) = e^{-x_3^2}$ and the less localized bump $h(x_3) = \text{sech}(x_3)$. Fig. 2 shows the variation of $\hat{\lambda}$ with respect to kh_0 for the first mode when the material is neo-Hookean. There is excellent agreement between the asymptotic results (large or small kh_0) and the numerical results. For instance, the leading-order asymptotic result for small kh_0 is capable of approximating the exact result with a relative error less than 5% for kh_0 up to 0.3. It is also observed $\hat{\lambda}$ is larger for the more localized Gaussian bump.



(a) (b)

Figure 3: Effect of changing the sign of h: variation of $\hat{\lambda}$ with respect to kh_0 when $h(x_3) = \text{sech}(x_3)$ (solid line) or $h(x_3) = -\text{sech}(x_3)$ (dashed line). (a) Gent material model with $J_m = 97$; and (b) Gent material model with $J_m = 10$.



(a) (b)

Figure 4: Existence of higher symmetric (solid lines) and anti-symmetric (dashed lines) modes when the material is neo-Hookean. (a) $h(x_3) = \text{sech}(x_3)$; (b) $h(x_3) = \mathrm{e}^{-x_3^2}$.

To show the effect of a trench-like surface topography, we consider the Gent material model with $J_m = 97$ and show in Fig. 3 the effect of changing the sign of h. It is seen that changing the sign has a negligible effect for small kh_0, as indicated by the asymptotic result (6.13), but for the larger values of kh_0, the effect is noticeable. However, the effect is only significant for small enough values of J_m. Also, the trench-like surface topography corresponds to a larger value of $\hat{\lambda}$, and so is slightly easier to wrinkle.

It is found that the current eigenvalue problem has an infinite number of symmetric and anti-symmetric modes. Fig. 4 shows the dependence of $\hat{\lambda}$ on kh_0 for the first three symmetric and anti-symmetric modes. It can be seen that the symmetric (solid curves) and anti-symmetric modes (dashed curves) alternately emerge as kh_0 increases and each pair converges to the same curve in the large kh_0 limit. For each fixed kh_0, the first symmetric mode has the largest value of $\hat{\lambda}$ and is therefore the critical mode.

Finally, in Fig.5 we have shown the eigenfunctions corresponding to the first four modes for a typical case. It is seen that the m-th mode has $m-1$ zeros in the interval, as in standard Sturm-Liouville eigenvalue problems, and the eigenfunctions for the symmetric mode always exhibit a "trench"at $x_3 = 0$. As predicted by the asymptotic analysis, for large kh the first two eigenfunctions are localized near the two second-order turning points at $x_3 = \pm x_0$, which explains the trench behaviour around $x_3 = 0$.

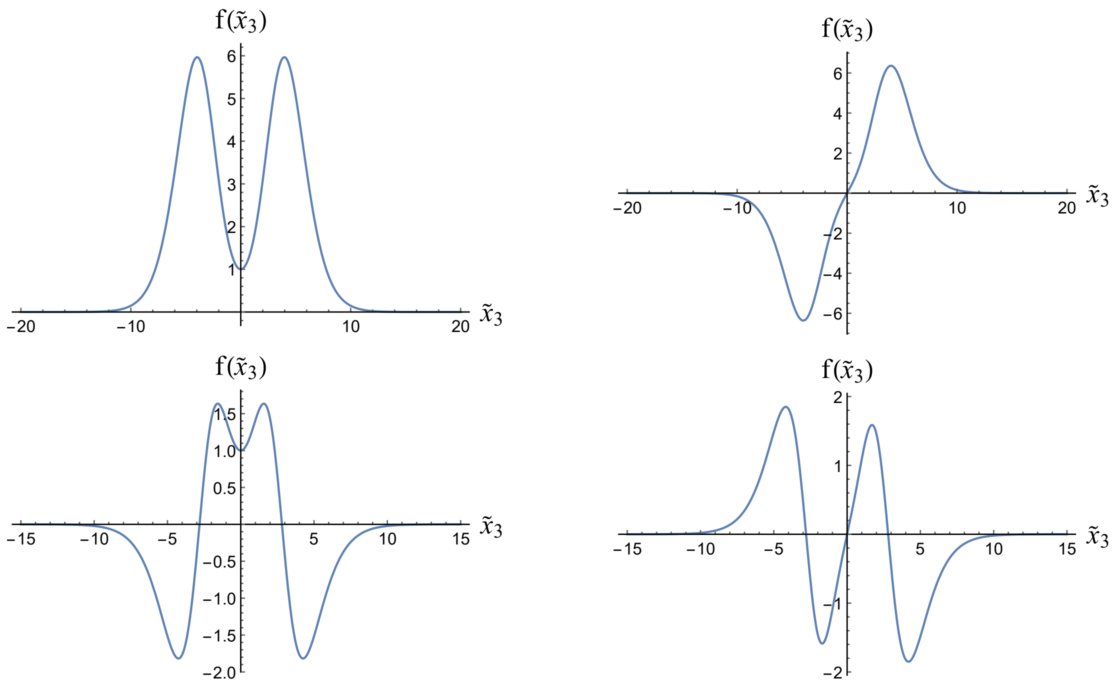


Figure 5: Eigenfunctions for the first four modes corresponding to $kh_0 = 15$ in Fig. 4(a).

7. Conclusion

In this paper we have studied the bifurcation condition for wrinkling of a compressed hyperelastic half-space with geometrical surface imperfections. This can be viewed as a variant of the classical Biot problem in that the traction-free surface is flat except for a straight, infinite length, ridge or trench, the profile of which is invariant in the x_1-direction. For an arbitrary surface profile, the determination of the wrinkling condition would be a fully numerical problem, but under the assumption that the topography is slowly varying and localized in the x_3-direction, an asymptotic analysis becomes possible. The necessary small parameter ε characterizes the ratio of the wavelength of the wrinkling modes to the lengthscale over which the surface imperfections vary; the maximum height h_0 of the surface imperfection is assumed to be of the same order as the wavelength. Thus, our results are only relevant when kh_0 is of order one or larger although the case of small kh_0 (for the first mode) was considered to validate our numerical scheme.

The main result is that the critical stretch has the asymptotic expansion $\lambda = \lambda_{\rm cr} + \varepsilon^2 \hat{\lambda}$, where $\lambda_{\rm cr}$ is the classical Biot value and $\hat{\lambda}$ is determined by solving an eigenvalue problem consisting of (6.11) and the associated decay conditions. The $\hat{\lambda}$ depends on kh_0 and there is an infinite number of wrinkling modes. The maximum of $\hat{\lambda}$ is obtained in the limit $kh_0 \to \infty$, and hence from (6.23) the maximum of λ is given by

$$\lambda_{\max} = \lambda_{\rm cr} - \frac{d_2}{d_0} \left[\varepsilon h'(x_0)\right]^2 . \tag{7.29}$$

Note that the factor $\varepsilon h'(x_0)$ in the second term is simply the maximum gradient of the surface profile in terms of the original dimensional variables. The above expression may be taken to be the critical stretch for an incompressible half-space with surface imperfections to wrinkle. Since d_2/d_0 is negative, the above formula shows that a geometrical surface imperfection would increase the critical stretch by an amount that is proportional to the square of the maximum gradient of the surface profile.

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