PROCEEDINGS A

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Research

Article submitted to journal

Subject Areas:
mechanics, applied mathematics

Keywords:
nonlocal, elasticity, plate, asymptotic, boundary layer, dynamics

Author for correspondence:
J. Kaplunov
e-mail: j.kaplunov@keele.ac.uk

A nonlocal asymptotic theory for thin elastic plates

R. Chebakov[1], J. Kaplunov[2], and G. A. Rogerson[3]

[1,2,3]School of Computing and Mathematics, Keele University, Keele, ST5 5BG, UK

The 3D dynamic nonlocal elasticity equations for a thin plate are subject to asymptotic analysis assuming the plate thickness to be much greater than a typical microscale size. The integral constitutive relations, incorporating the variation of an exponential nonlocal kernel across the thickness, are adopted. Long-wave low-frequency approximations are derived for both bending and extensional motions. Boundary layers specific for nonlocal behaviour are revealed near the plate faces. It is established that the effect of the boundary layers leads to first-order corrections to the bending and extensional stiffness in the classical 2D plate equations.

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1. Introduction

Nonlocal elasticity is a basic framework for continuous modelling of micro- and nanoplates, finding numerous applications in energy storage, chemical and biological sensors, renewable energy devices, field emission devices, micro- and nano-electromechanical systems, etc., see Kiani (2011a), Kiani (2011b), Shen (2011), Alibeigloo (2011), Ashoori et al. (2016) and references therein. The general nonlocal theory is presented, for example, in the book by Eringen (2002), see also more recent papers by Arash and Wang (2012), Schwartz et al. (2012), Benvenuti and Simone (2013), Abdollahi and Boroomand (2013), Abdollahi and Boroomand (2014), and Salehipour et al. (2015). In addition, here we mention publications by Owen and Paroni (2000) and Owen and Paroni (2015) on the so-called 'structured deformations'.

The original nonlocal integral formulations can sometimes be reduced to differential ones, taking the form of singularly perturbed equations in linear elasticity with a second order perturbation in the microscale parameter, e.g., see Eringen (1983) and Peerlings et al. (2001). However, such a transformation is not generally possible for tackling boundary value problems (Fernandez-Saez et al., 2016). Indeed, near the boundaries of a non-locally elastic solid the effect of boundary layers becomes essential. In particular, asymptotic analysis of a half-space in Chebakov et al. (2016) reveals a first-order nonlocal correction to the boundary conditions on a free surface. The importance of the boundary layers in nonlocal elasticity was also emphasised earlier in Bazant et al. (2010) and Abdollahi and Boroomand (2014).

The existing 2D nonlocal models for thin elastic plates are usually based on the above mentioned differential constitutive relations, e.g., see Lu et al. (2007), Duan and Wang (2007), Aghababaei and Reddy (2009), Pradhan and Phadikar (2009), Malekzadeh et al. (2011), Xu et al. (2014), Thai et al. (2014), Jung and Han (2014), and Mousavi et al. (2017). In these models, 3D $\rightarrow$ 2D reduction is carried out using ad-hoc assumptions neglecting the variation of nonlocal properties across the thickness. As a result, the nonlocal corrections appear only at second-order in the microscale parameter related to the longitudinal length-scale which considerably exceeds the thickness. To our best knowledge, the consideration in Sajadi et al. (2017), taking into account the variation of several nonlocal integral kernels through the thickness, is the only exception. In the cited paper, the nonlocal bending moments and shear forces are calculated starting from the traditional engineering hypotheses underlying the classical theory for plate bending and extension.

In this paper we take into consideration the variation of nonlocal properties across the plate thickness stating from the full integral constitutive relations in Eringen (1983). The long-wave low-frequency approximations of the 3D dynamic equations in nonlocal elasticity are derived for plate bending and extension. The asymptotic approach using direct integration through the thickness is adapted, e.g., see Goldenveizer et al. (1993), Kaplunov et al. (1998), Kaplunov et al. (2000), and Kaplunov et al. (2006). For the sake of simplicity, we specify a single small parameter, which is equal both to the ratio of the thickness to a characteristic longitudinal lengthscale and that of the thickness to a microscale size. Also, we restrict ourselves to a commonly used exponential nonlocal kernel, see Eringen (1983).

It is established that the scaling characteristic of the local theory for plate bending and extension (Goldenveizer et al. (1993) and Kaplunov et al. (1998)) also appears to be relevant within the present context. At the same time, the nonlocal stresses contain specific boundary layer components adjacent to the plate faces, along with counterparts of usual local stresses demonstrating a polynomial variation across the thickness. The boundary layer and polynomial stress components are related via boundary conditions. It is remarkable that such boundary layer components of strains and displacements arise only at higher orders. In passing we note that similar boundary layers were earlier observed in 1D problem for a nonlocal beam (Pisano and Fuschi, 2003).

As might be expected, the leading-order approximations of 3D equations in nonlocal elasticity are identical to the conventional 'local' set up. The nonlocal effect due to the boundary layers comes at the next order. In terms of the chosen small parameter, it is four times greater than that in the aforementioned nonlocal differential formulations for elastic plates.

The nonlocal equations of motion obtained in the paper are presented in the form of the associated local ones with slightly modified bending and extensional stiffness. These stiffnesses involve nonlocal first order corrections to their local analogues.

2. Statement of the problem

Consider an elastic plate of thickness $2h$ and microscale size a with traction-free faces, see Figure 1. The 3D equations of motion in nonlocal elasticity can be written as

$$s_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2}\,, \tag{2.1}$$

with $s_{mn}, m, n = 1, 2, 3$ nonlocal stresses, u_n displacement vector, ρ volume density, and t time. The nonlocal stresses are expressed though their local counterparts σ_{mn} as, e.g., see Eringen (1983)

$$s_{mn}(\mathbf{x}) = \int\limits_V K\left(|\mathbf{x}' - \mathbf{x}|, a\right) \sigma_{mn}(\mathbf{x}')\, dv(\mathbf{x}')\,, \tag{2.2}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is a reference point, V the domain occupied by the plate, $K(\mathbf{x}, a)$ is a nonlocal kernel normalised by

$$\int\limits_{V_\infty} K\left(|\mathbf{x}'|, a\right)\, dv(\mathbf{x}') = 1\,. \tag{2.3}$$

In what follows, we define the nonlocal kernel as

$$K(|\mathbf{x}|, a) = \frac{1}{\pi^{3/2} a^3} \exp\left[-\frac{\mathbf{x}\cdot\mathbf{x}}{a^2}\right]. \tag{2.4}$$

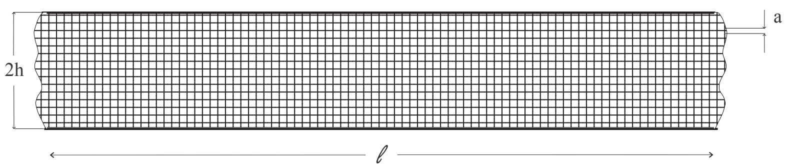

Figure 1. A thin plate treated in the framework of nonlocal elasticity.

The constitutive relations for an isotropic material are given by

$$\sigma_{mn} = \lambda e_{ll} \delta_{mn} + 2\mu e_{mn}\,, \tag{2.5}$$

with

$$e_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)\,, \tag{2.6}$$

where e_{mn} are linear elastic strains, δ_{mn} the Kronecker's delta, and λ and μ are the Lamé constants.

The boundary conditions on the traction-free faces $x_3 = \pm h$ are

$$s_{3n} = 0\,, \tag{2.7}$$

where $n = 1, 2, 3$.

Let us assume that the plate half-thickness h is much greater than the microscale parameter a and at the same time is much smaller than a typical macroscale size ℓ, i.e., $a \ll h \ll \ell$. For the sake of simplicity, we specify a single small parameter given by

$$\eta = \frac{h}{\ell} = \frac{a}{h} \ll 1\,. \tag{2.8}$$

We also assume that a characteristic time scale T is much greater than the time the shear wave travels the distance between the plate faces, i.e., $T \gg h/c_2$. It is known from the asymptotic plate theory (Goldenveizer et al. (1993) and Kaplunov et al. (1998)) that

$$T = \eta^{-2}\frac{h}{c_2} \tag{2.9}$$

for low-frequency bending motion and

$$T = \eta^{-1}\frac{h}{c_2} \tag{2.10}$$

for low-frequency extensional motion.

For a plate of thickness $2h$, ($-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, and $-h \le x_3 \le h$), (2.2) becomes

$$s_{mn}(\mathbf{x}) = \frac{1}{\pi^{3/2}a^3}\int_{-h}^{h} dx_3' \int_{-\infty}^{\infty} dx_1' \int_{-\infty}^{\infty} dx_2' \, \exp\left[-\frac{(\mathbf{x}'-\mathbf{x})^2}{a^2}\right]\sigma_{mn}(\mathbf{x}')\,. \tag{2.11}$$

On expanding the stresses σ_{mn} in Taylor series in x_1 and x_2 about the reference point $\mathbf{x}' = (x_1, x_2, x_3')$ and neglecting $O(\eta^4)$ terms, we obtain

$$s_{mn}(\mathbf{x}) = \frac{1}{a\sqrt{\pi}}\int_{-h}^{h} \exp\left[-\frac{(x_3'-x_3)^2}{a^2}\right]\sigma_{mn}(x_1,x_2,x_3')dx_3'\,. \tag{2.12}$$

Below we adapt for nonlocal plate bending and extension the asymptotic approach previously developed within the classical framework, e.g., see Goldenveizer et al. (1993), Kaplunov et al. (1998), and Kaplunov et al. (2006) and references therein.

3. Plate bending

Let us specify dimensionless variables by

$$x_i = \ell\xi_i\,, \quad x_3 = h\zeta_p = a\zeta_q\,, \quad \text{and} \quad t = T\tau\,, \tag{3.1}$$

with the time scale T given by (2.9). Thus, nonlocal asymptotic analysis operates with two different transverse dimensionless variables ξ_p and ξ_q related to the plate half-thickness and microscale parameter. As a consequence, the nonlocal stresses s_{mn} are separated below into two components $p_{mn}(\zeta_p)$ and $q_{mn}(\zeta_q)$, demonstrating slow and fast variation across the thickness, respectively, $m, n = 1, 2, 3$. In this case the scaling is nevertheless similar to that in the asymptotic theory for plate bending and given by

$$u_i = \eta\ell v_i\,,\; u_3 = \ell v_3\,, \tag{3.2}$$

$$e_{ii} = \eta\varepsilon_{ii}\,,\; e_{ij} = \eta\varepsilon_{ij}\,,\; e_{3i} = \eta^2\varepsilon_{3i}\,,\; e_{33} = \eta\varepsilon_{33}\,, \tag{3.3}$$

and

$$\begin{aligned} s_{ii} &= \eta\mu(p_{ii}+q_{ii})\,,\; s_{ij} = \eta\mu(p_{ij}+q_{ij})\,,\\ s_{3i} &= \eta^2\mu(p_{3i}+\eta q_{3i})\,,\; s_{33} = \eta^3\mu(p_{33}+\eta^2 q_{33})\,, \end{aligned} \tag{3.4}$$

where $i \neq j = 1, 2$ and Einstein's summation convention is not employed; here the dimensionless quantities v, ε, and p and q are assumed to be of the same asymptotic order.

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On inserting (3.4) into (2.12) and (3.3) into (2.5), we obtain

$$
\begin{aligned}
p_{ij}+q_{ij} &= \frac{2}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\,[-(\zeta_q'-\zeta_q)^2]\varepsilon_{ij}d\zeta_q' \,, \\
p_{ii}+q_{ii} &= \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\,[-(\zeta_q'-\zeta_q)^2]\left(\kappa^{-2}\varepsilon_{ii}+(\kappa^{-2}-2)(\varepsilon_{jj}+\varepsilon_{33})\right)d\zeta_q' \,, \\
p_{3i}+\eta q_{3i} &= \frac{2}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\,[-(\zeta_q'-\zeta_q)^2]\varepsilon_{3i}d\zeta_q' \,, \\
\eta^2(p_{33}+\eta^2 q_{33}) &= \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\,[-(\zeta_q'-\zeta_q)^2]\left(\kappa^{-2}\varepsilon_{33}+(\kappa^{-2}-2)(\varepsilon_{ii}+\varepsilon_{jj})\right)d\zeta_q' \,,
\end{aligned}
\tag{3.5}
$$

with $\kappa=\frac{c_2}{c_1}$. We also have from (2.6) that

$$
\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) \,, \\
\varepsilon_{ii} &= \frac{\partial v_i}{\partial \xi_i} \,, \\
\eta^2\varepsilon_{3i} &= \frac{1}{2}\left(\frac{\partial v_3}{\partial \xi_i}+\frac{\partial v_i}{\partial \zeta_p}\right) \,, \\
\eta^2\varepsilon_{33} &= \frac{\partial v_3}{\partial \zeta_p} \,.
\end{aligned}
\tag{3.6}
$$

In addition, the nonlocal equations of motion (2.1) and the boundary conditions (2.7) at $\zeta_p=\pm 1(\zeta_q=\pm\eta^{-1})$ become

$$
\begin{aligned}
\frac{\partial p_{3i}}{\partial \zeta_p}+\frac{\partial q_{3i}}{\partial \zeta_q} &= -\frac{\partial (p_{ii}+q_{ii})}{\partial \xi_i}-\frac{\partial (p_{ij}+q_{ij})}{\partial \xi_j}+\eta^2\frac{\partial^2 v_i}{\partial \tau^2} \,, \\
\frac{\partial p_{33}}{\partial \zeta_p}+\eta\frac{\partial q_{33}}{\partial \zeta_q} &= -\frac{\partial (p_{3i}+\eta q_{3i})}{\partial \xi_i}-\frac{\partial (p_{3j}+\eta q_{3j})}{\partial \xi_j}+\frac{\partial^2 v_3}{\partial \tau^2} \,,
\end{aligned}
\tag{3.7}
$$

and

$$
p_{3i}+\eta q_{3i}=0 \,,\; p_{33}+\eta^2 q_{33}=0 \,. \tag{3.8}
$$

Now we expand all the dimensionless quantities in (3.2)-(3.4) in asymptotic series as

$$
\begin{pmatrix} v_n \\ p_{mn} \\ q_{mn} \\ \varepsilon_{mn} \end{pmatrix} = \begin{pmatrix} v_n^{(0)} \\ p_{mn}^{(0)} \\ q_{mn}^{(0)} \\ \varepsilon_{mn}^{(0)} \end{pmatrix} + \eta \begin{pmatrix} v_n^{(1)} \\ p_{mn}^{(1)} \\ q_{mn}^{(1)} \\ \varepsilon_{mn}^{(1)} \end{pmatrix} + \ldots \,. \tag{3.9}
$$

Substituting these expansions into equations (3.5)$_{1,2,4}$, (3.6), and (3.7), we have at leading order

$$\begin{aligned}
&\frac{\partial v_3^{(0)}}{\partial \zeta_p} = 0\,,\\
&\frac{\partial v_i^{(0)}}{\partial \zeta_p} = -\frac{\partial v_3^{(0)}}{\partial \xi_i}\,,\\
&\varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^{(0)}}{\partial \xi_j} + \frac{\partial v_j^{(0)}}{\partial \xi_i}\right)\,,\\
&\varepsilon_{ii}^{(0)} = \frac{\partial v_i^{(0)}}{\partial \xi_i}\,,\\
&\varepsilon_{33}^{(0)} = -(1-2\kappa^2)(\varepsilon_{ii}^{(0)} + \varepsilon_{jj}^{(0)})\,,\\
&p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right]\varepsilon_{ij}^{(0)}\, d\zeta_q'\,,\\
&p_{ii}^{(0)} + q_{ii}^{(0)} = \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right]\left(\kappa^{-2}\varepsilon_{ii}^{(0)} + (\kappa^{-2}-2)(\varepsilon_{jj}^{(0)} + \varepsilon_{33}^{(0)})\right) d\zeta_q'\,,\\
&\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(0)} + q_{ii}^{(0)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(0)} + q_{ij}^{(0)})}{\partial \xi_j}\,,\\
&\frac{\partial p_{33}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{3i}^{(0)}}{\partial \xi_i} - \frac{\partial p_{3j}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_3^{(0)}}{\partial \tau^2}\,,
\end{aligned} \tag{3.10}$$

and also, at $\zeta_p = \pm 1$ ($\zeta_q = \pm\eta^{-1}$),

$$p_{3i}^{(0)} = 0\,,\ p_{33}^{(0)} = 0\,. \tag{3.11}$$

First, on integrating the equations (3.10)$_{1,2}$ with respect to ζ_p and using the obtained expressions in (3.10)$_{3-5}$, we arrive at

$$\begin{aligned}
&v_3^{(0)} = w_3^{(0)}, \quad v_i^{(0)} = -\zeta_p \frac{\partial w_3^{(0)}}{\partial \xi_i}\,,\\
&\varepsilon_{ij}^{(0)} = -\zeta_p \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j}, \quad \varepsilon_{ii}^{(0)} = -\zeta_p \frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2}, \quad \varepsilon_{33}^{(0)} = (1-2\kappa^2)\zeta_p \Delta_\xi w_3^{(0)}\,,
\end{aligned} \tag{3.12}$$

where $w_3^{(0)} = w_3^{(0)}(\xi_i, \xi_j, \tau)$ and $\Delta_\xi = \frac{\partial^2}{\partial \xi_i^2} + \frac{\partial^2}{\partial \xi_j^2}$. We remark that the variation of displacements and strains across the thickness, predicted by (3.12), is similar to that in the asymptotic theory of plate bending as above.

Equation (3.10)$_6$ may now be expressed as

$$p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2\eta}{\sqrt{\pi}}\frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j}\int_{-\eta^{-1}}^{\eta^{-1}} \zeta_q' \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q'\,, \tag{3.13}$$

which can be transformed to

$$p_{ij}^{(0)} + q_{ij}^{(0)} = -\frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j}\zeta_p \left\{2 - \operatorname{erfc}(\eta^{-1} - \zeta_q) - \operatorname{erfc}(\eta^{-1} + \zeta_q)\right\} \tag{3.14}$$

by integrating by parts and omitting $O(\eta)$-terms. Thus,

$$\begin{aligned}
&p_{ij}^{(0)}(\zeta_p) = -2\zeta_p \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j}\,,\\
&q_{ij}^{(0)}(\zeta_q) = Q_1(\zeta_q)\frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j}\,,
\end{aligned} \tag{3.15}$$

where

$$Q_1(\zeta_q) = \zeta_q \eta \left(\operatorname{erfc}(\eta^{-1} - \zeta_q) + \operatorname{erfc}(\eta^{-1} + \zeta_q)\right)\,, \tag{3.16}$$

with $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int\limits_x^\infty e^{-t^2} dt$. Here and below all the p stress components having polynomial variations across the thickness are virtually the same as those in plate theory, while the q stress components correspond to boundary layers with width the microscale size a and localised near the plate faces.

Next, equation (3.10)$_7$ becomes

$$p_{ii}^{(0)} + q_{ii}^{(0)} = -\frac{\eta}{\sqrt{\pi}} \left(4(1-\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + 2(1-2\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) \int_{-\eta^{-1}}^{\eta^{-1}} \zeta_q' \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \,. \tag{3.17}$$

Similarly to the derivation above, we have

$$\begin{aligned} p_{ii}^{(0)}(\zeta_p) &= -\zeta_p \left(4(1-\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + 2(1-2\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) , \\ q_{ii}^{(0)}(\zeta_q) &= \frac{1}{2} Q_1 \left(4(1-\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + 2(1-2\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) . \end{aligned} \tag{3.18}$$

In what follows, we treat separately the p and q parts of the equation (3.10)$_8$, having

$$\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(0)}}{\partial \xi_j} \tag{3.19}$$

and

$$\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial q_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial q_{ij}^{(0)}}{\partial \xi_j} \,. \tag{3.20}$$

Integration of equation (3.19) with respect to ζ_p yields

$$p_{3i}^{(0)} = 2\zeta_p^2 (1-\kappa^2) \Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} + C(\xi_1, \xi_2, \tau) \,, \tag{3.21}$$

where the arbitrary function C can be found using the related boundary condition (3.11)$_1$, thus

$$p_{3i}^{(0)} = 2(\zeta_p^2 - 1)(1-\kappa^2) \Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} + C(\xi_1, \xi_2, \tau) \,. \tag{3.22}$$

Next, we obtain from (3.10)$_9$

$$p_{33}^{(0)} = \zeta_p \left(2\left(1 - \frac{\zeta_p^2}{3}\right)(1-\kappa^2) \Delta_\xi^2 w_3^{(0)} + \frac{\partial^2 w_3^{(0)}}{\partial \tau^2} \right) , \tag{3.23}$$

and finally, satisfying the boundary condition (3.11)$_2$, we arrive at the Kirchhoff equation in the 2D theory for plate bending

$$\frac{4}{3}(1-\kappa^2) \Delta_\xi^2 w_3^{(0)} + \frac{\partial^2 w_3^{(0)}}{\partial \tau^2} = 0 \,. \tag{3.24}$$

In order to incorporate the nonlocal phenomena of interest, we also need to determine $q_{3i}^{(0)}$ from equation (3.20). Substituting $q_{ij}^{(0)}$ from (3.15) and $q_{ii}^{(0)}$ from (3.18) into this equation, we obtain

$$\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -2(1-\kappa^2) Q_1 \Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} \,, \tag{3.25}$$

after neglecting asymptotically small terms resulting in

$$q_{3i}^{(0)}(\zeta_q) = 2(1-\kappa^2) R_1(\zeta_q) \Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} \,. \tag{3.26}$$

It is convenient to introduce the following notation

$$R_k(\zeta_q) = (\eta^{-1} - \zeta_q)\,\mathrm{erfc}\,(\eta^{-1} - \zeta_q) - (-1)^k(\eta^{-1} + \zeta_q)\,\mathrm{erfc}\,(\eta^{-1} + \zeta_q) \\ -\frac{1}{\sqrt{\pi}}\left(\exp\left[-\left(\eta^{-1} - \zeta_q\right)^2\right] - (-1)^k \exp\left[-\left(\eta^{-1} + \zeta_q\right)^2\right]\right), \tag{3.27}$$

with $k = 1$ and $k = 2$ corresponding to plate bending and plate extension, respectively, with the latter one considered in the next section.

At first order we obtain the same relations as (3.10) to within the suffix substitution $(0) \to (1)$, except equation $(3.10)_{10}$, which can be splitted into two parts as

$$\frac{\partial p_{33}^{(1)}}{\partial \zeta_p} = -\frac{\partial p_{3i}^{(1)}}{\partial \xi_i} - \frac{\partial p_{3j}^{(1)}}{\partial \xi_j} + \frac{\partial^2 v_3^{(1)}}{\partial \tau^2} \tag{3.28}$$

and

$$\frac{\partial q_{33}^{(0)}}{\partial \zeta_q} = -\frac{\partial q_{3i}^{(0)}}{\partial \xi_i} - \frac{\partial q_{3j}^{(0)}}{\partial \xi_j}. \tag{3.29}$$

The related boundary conditions on the faces $\zeta_p = \pm 1$ ($\zeta_q = \pm\eta^{-1}$) become

$$p_{3i}^{(1)} = -q_{3i}^{(0)}, \; p_{33}^{(1)} = 0. \tag{3.30}$$

As a result, all the formulae for the quantities $v_i^{(1)}, v_3^{(1)}, \varepsilon_{ij}^{(1)}, \varepsilon_{ii}^{(1)}, \varepsilon_{33}^{(1)}, p_{ij}^{(1)}, q_{ij}^{(1)}, p_{ii}^{(1)}, q_{ii}^{(1)}$, and $p_{3i}^{(1)}$ can be obtained from the associated formulae of the leading order approximation, see (3.12), (3.15), (3.18), and (3.19), by the same suffix substitution $(0) \to (1)$. In particular, substitution of the counterpart of the formula (3.22), i.e.,

$$p_{3i}^{(1)} = \zeta_p^2 2(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(1)}}{\partial \xi_i} + C(\xi_1, \xi_2, \tau), \tag{3.31}$$

into the boundary condition $(3.30)_1$ leads to

$$C = 2(1-\kappa^2)\Delta_\xi \left[\frac{1}{\sqrt{\pi}}\frac{\partial w_3^{(0)}}{\partial \xi_i} - \frac{\partial w_3^{(1)}}{\partial \xi_i}\right], \tag{3.32}$$

thus

$$p_{3i}^{(1)} = 2(1-\kappa^2)\Delta_\xi \left[\frac{\partial w_3^{(1)}}{\partial \xi_i}(\zeta_p^2 - 1) + \frac{1}{\sqrt{\pi}}\frac{\partial w_3^{(0)}}{\partial \xi_i}\right]. \tag{3.33}$$

Then, we deduce from $(3.28)_{10}$, using the (3.33), that

$$p_{33}^{(1)} = \zeta_p\left(2(1-\kappa^2)\left[(1 - \frac{\zeta_p^2}{3})\Delta_\xi^2 w_3^{(1)} + \frac{1}{\sqrt{\pi}}\Delta_\xi^2 w_3^{(0)}\right] + \frac{\partial^2 w_3^{(1)}}{\partial \tau^2}\right). \tag{3.34}$$

Finally, satisfying the boundary condition $(3.30)_2$ we arrive at a PDE for the first order correction $w_3^{(1)}$, given by

$$\frac{4}{3}(1-\kappa^2)\left[\Delta_\xi^2 w_3^{(1)} - \frac{3}{2\sqrt{\pi}}\Delta_\xi^2 w_3^{(0)}\right] + \frac{\partial^2 w_3^{(1)}}{\partial \tau^2} = 0. \tag{3.35}$$

Let us now multiply (3.35) by η and add the result to the leading order equation (3.24). We then have the sought for 2D nonlocal equation for plate bending. In the original variables it may be written as

$$D'\Delta^2 u_3 + 2\rho h u_{3,tt} = 0\,, \tag{3.36}$$

where $u_3 = \ell(w_3^{(0)} + \eta w_3^{(1)})$ and the nonlocal bending stiffness D' is given by

$$D' = D\left(1 - \frac{3a}{2h\sqrt{\pi}}\right)\,, \tag{3.37}$$

with $D = \frac{8\mu h^3(1-\kappa^2)}{3}$ denoting the bending stiffness in the classical Kirchhoff theory for plate bending. In terms of Young modulus E and Poisson ratio ν, $D = \frac{2Eh^3}{3(1-\nu^2)}$.

It is worth noting that the nonlocal bending stiffness D' in (3.37) to within higher order terms in η, mainly exponentially small ones, coincides with that obtained in Sajadi et al. (2017) using classical Kirchhoff hypotheses in thin plate theory [1].

4. Plate extension

Now we operate with the dimensionless variables (3.1) with the time scale (2.10) characteristic of low-frequency extensional motion. In this case the scaling, similar to that in the above mentioned asymptotic plate theory is given by

$$u_i = \ell v_i\,,\ u_3 = \eta\ell v_3\,, \tag{4.1}$$

$$e_{ii} = \varepsilon_{ii}\,,\ e_{ij} = \varepsilon_{ij}\,,\ e_{33} = \varepsilon_{33}\,, \tag{4.2}$$

and

$$\begin{aligned} s_{ii} &= \mu(p_{ii} + q_{ii})\,,\\ s_{ij} &= \mu(p_{ij} + q_{ij})\,,\\ s_{3i} &= \eta\mu(p_{3i} + \eta q_{3i})\,,\\ s_{33} &= \eta^2\mu(p_{33} + \eta^2 q_{33})\,. \end{aligned} \tag{4.3}$$

In addition, we define

$$\gamma_{i3} = \frac{\partial u_i}{\partial x_3} = \eta g_{i3}\,, \tag{4.4}$$

where g_{i3} is assumed to be a quantity of order unity.

Similarly to the previous section, we insert the formulae (3.1), (2.10), and (4.1)-(4.3) into the relations in Section 2 and (4.4), yielding

$$\begin{aligned} p_{ij} + q_{ij} &= \frac{2}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right]\varepsilon_{ij}\, d\zeta_q'\,,\\ p_{ii} + q_{ii} &= \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right]\left(\kappa^{-2}\varepsilon_{ii} + (\kappa^{-2} - 2)(\varepsilon_{jj} + \varepsilon_{33})\right) d\zeta_q'\,,\\ \eta^2(p_{33} + \eta^2 q_{33}) &= \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right]\left(\kappa^{-2}\varepsilon_{33} + (\kappa^{-2} - 2)(\varepsilon_{ii} + \varepsilon_{jj})\right) d\zeta_q'\,, \end{aligned} \tag{4.5}$$

with

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_i}\right)\,,\\ \varepsilon_{ii} &= \frac{\partial v_i}{\partial \xi_i}\,,\\ \varepsilon_{33} &= \frac{\partial v_3}{\partial \zeta_p}\,. \end{aligned} \tag{4.6}$$

We also obtain

$$\frac{\partial v_i}{\partial \zeta_p} = \eta^2 g_{i3}\,. \tag{4.7}$$

[1]The expression "$(3\eta - 2\eta^3)$" in the formulae (22) and (26) in Sajadi et al. (2017) is seemingly in error and should be "$(3\eta^{-1} - 2\eta^{-3})$".

The equations of motion and boundary conditions at $\zeta_p = \pm 1$ ($\zeta_q = \pm\eta^{-1}$) take the forms

$$\begin{aligned}\frac{\partial p_{3i}}{\partial \zeta_p} + \frac{\partial q_{3i}}{\partial \zeta_q} &= -\frac{\partial (p_{ii}+q_{ii})}{\partial \xi_i} - \frac{\partial (p_{ij}+q_{ij})}{\partial \xi_j} + \frac{\partial^2 v_i}{\partial \tau^2}\,,\\ \frac{\partial p_{33}}{\partial \zeta_p} + \eta\frac{\partial q_{33}}{\partial \zeta_q} &= -\frac{\partial (p_{3i}+\eta q_{3i})}{\partial \xi_i} - \frac{\partial (p_{3j}+\eta q_{3j})}{\partial \xi_j} + \frac{\partial^2 v_3}{\partial \tau^2}\,,\end{aligned} \tag{4.8}$$

and

$$p_{3i} + \eta q_{3i} = 0\,,\; p_{33} + \eta^2 q_{33} = 0\,. \tag{4.9}$$

The strain ε_{33} in $(4.5)_2$, to within the error $O(\eta^2)$, can be expressed from $(4.5)_3$, resulting in

$$\begin{aligned}p_{ii} + q_{ii} = \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q'-\zeta_q)^2\right]\Big(4(1-\kappa^2)\varepsilon_{ii} + 2(1-2\kappa^2)\varepsilon_{jj}\Big)\,d\zeta_q'\\ +\eta^2(1-2\kappa^2)(p_{33}+\eta^2 q_{33})\,.\end{aligned} \tag{4.10}$$

Let us substitute the expansions (3.9) into the relations above. Thus, we readily obtain

$$\begin{aligned}&\frac{\partial v_i^{(0)}}{\partial \zeta_p} = 0\,,\\ &\varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^{(0)}}{\partial \xi_j} + \frac{\partial v_j^{(0)}}{\partial \xi_i}\right),\\ &\varepsilon_{ii}^{(0)} = \frac{\partial v_i^{(0)}}{\partial \xi_i}\,,\\ &p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q'-\zeta_q)^2\right]\varepsilon_{ij}^{(0)}\,d\zeta_q'\,,\\ &p_{ii}^{(0)} + q_{ii}^{(0)} = \frac{1}{\sqrt{\pi}}\int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q'-\zeta_q)^2\right]\Big(4(1-\kappa^2)\varepsilon_{ii}^{(0)} + 2(1-2\kappa^2)\varepsilon_{jj}^{(0)}\Big)\,d\zeta_q'\,,\\ &\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_i^{(0)}}{\partial \tau^2}\,,\\ &\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial q_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial q_{ij}^{(0)}}{\partial \xi_j}\,,\\ &\frac{\partial p_{33}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{3i}^{(0)}}{\partial \xi_i} - \frac{\partial p_{3j}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_3^{(0)}}{\partial \tau^2}\,,\end{aligned} \tag{4.11}$$

and at $\zeta_p = \pm 1$ ($\zeta_q = \pm\eta^{-1}$)

$$p_{3i}^{(0)} = 0\,,\; p_{33}^{(0)} = 0\,. \tag{4.12}$$

First, we deduce from $(4.11)_{1-3}$

$$\begin{aligned}&v_i^{(0)} = w_i^{(0)}\,,\\ &\varepsilon_{ii}^{(0)} = \frac{\partial w_i^{(0)}}{\partial \xi_i},\quad \varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i}\right),\end{aligned} \tag{4.13}$$

where $w_i^{(0)} = w_i^{(0)}(\xi_i, \xi_j, \tau)$. Then, integration in $(4.11)_{4,5}$ yields

$$\begin{aligned}&p_{ij}^{(0)} = \frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i}\,,\\ &q_{ij}^{(0)} = -\frac{1}{2}Q_2\left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i}\right),\end{aligned} \tag{4.14}$$

and

$$p_{ii}^{(0)} = 4(1-\kappa^2)\frac{\partial w_i^{(0)}}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j^{(0)}}{\partial \xi_j}\,,$$
$$q_{ii}^{(0)} = -\frac{1}{2}Q_2\left(4(1-\kappa^2)\frac{\partial w_i^{(0)}}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j^{(0)}}{\partial \xi_j}\right)\,, \tag{4.15}$$

where

$$Q_2 = (\zeta_q \eta)^{-1} Q_1\,, \tag{4.16}$$

with Q_1 given by (3.16).

Next, we integrate equation $(4.11)_6$ with respect to ζ_p and satisfy the boundary condition $(4.12)_1$, arriving at

$$4(1-\kappa^2)\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_i^{(0)}}{\partial \xi_j^2} + (3-4\kappa^2)\frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 w_i^{(0)}}{\partial \tau^2} = 0\,. \tag{4.17}$$

Finally, integration of $(4.11)_7$ with respect to ζ_q, and neglecting asymptotically small terms, gives

$$q_{3i}^{(0)} = -\frac{1}{2}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j}\right)\right] R_2\,, \tag{4.18}$$

with R_2 defined by (3.27) at $k = 2$.

At first order, we have for $p_{3i}^{(1)}$ and $w_i^{(1)}$ the same equation as (4.17). It has to be considered with the boundary conditions

$$p_{3i}^{(1)} = -q_{3i}^{(0)} \tag{4.19}$$

at $\zeta_p = \pm 1$ ($\zeta_q = \pm\eta^{-1}$). The solution of the formulated boundary value problem is

$$\left[\Delta_\xi w_i^{(1)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(1)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(1)}}{\partial \xi_i \partial \xi_j}\right) - \frac{\partial^2 w_i^{(1)}}{\partial \tau^2}\right]$$
$$-\frac{1}{2\sqrt{\pi}}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j}\right)\right] = 0\,. \tag{4.20}$$

Combining the equations (4.20) and (4.17), we have in the original variables

$$\frac{A'}{2}\left((1-\nu)\Delta \mathbf{u} + (1+\nu)\,\mathrm{grad\,div}\,\mathbf{u}\right) - 2\rho h \mathbf{u}_{tt} = 0\,, \tag{4.21}$$

with $\mathbf{u} = \ell(w_1^{(0)} + \eta w_1^{(1)},\ w_2^{(0)} + \eta w_2^{(1)})$ and

$$A' = A\left(1 - \frac{a}{2h\sqrt{\pi}}\right)\,, \tag{4.22}$$

where $A = \frac{2Eh}{1-\nu^2}$ is the extensional stiffness in the classical theory for plate extension. The nonlocal stiffness A' also coincides to within higher order term in η with that in Sajadi et al. (2017).

5. Numerical illustrations

First, we compute the functions Q_k and R_k, see (3.16), (3.27), and (4.16), characterising the boundary layer components of the nonlocal stresses s_{ii}, s_{ij}, and s_{3i} ($i \neq j = 1, 2$) for plate bending ($k = 1$) and extension ($k = 2$). As might be expected, the functions R_1 and Q_2 are even, whereas Q_1 and R_2 are odd in the thickness co-ordinate.

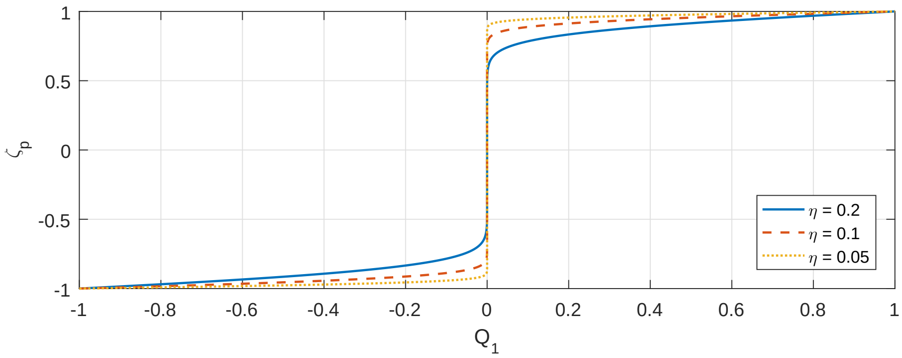

Figure 2. The boundary layer component of the nonlocal stresses s_{ii} and s_{ij} $(i \neq j = 1, 2)$ for plate bending.

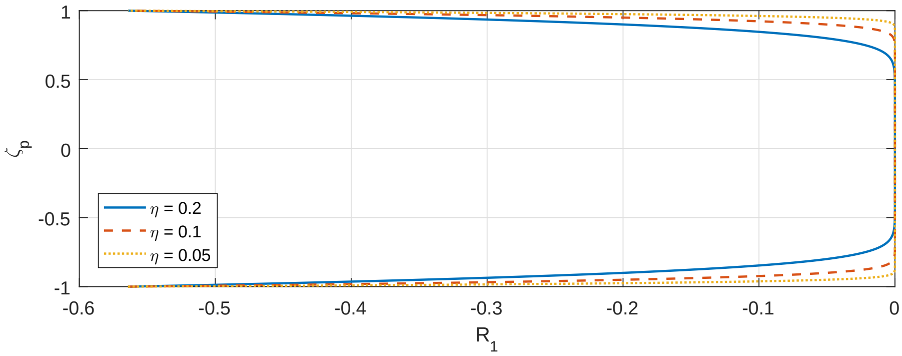

Figure 3. The boundary layer component of the nonlocal stresses s_{3i} $(i = 1, 2)$ for plate bending.

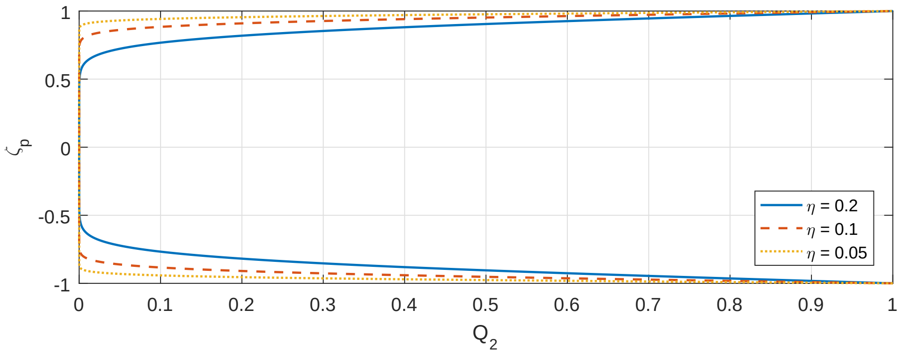

Figure 4. The boundary layer component of the nonlocal stresses s_{ii} and s_{ij} $(i \neq j = 1, 2)$ for plate extension.

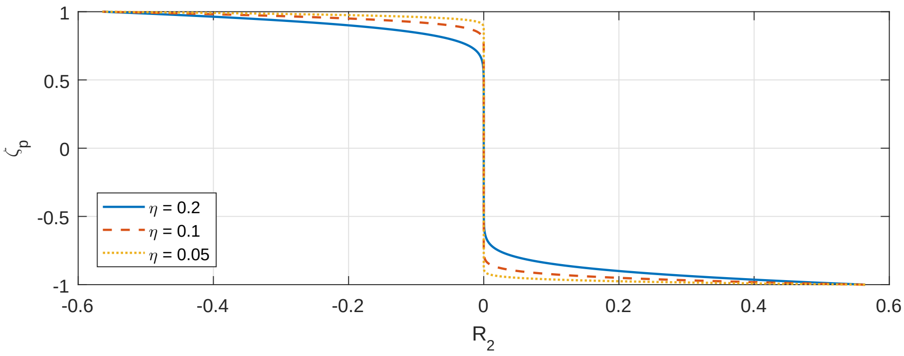

Figure 5. The boundary layer component of the nonlocal stresses s_{3i} $(i = 1, 2)$ for plate extension.

The graphs in Figures 2 - 5 are plotted for $\eta = 0.2, 0.1$ and 0.05, where we recall that $\eta = a/h$. All of the plots show localisation over narrow $O(\eta)$ zones of plate faces, demonstrating a monotonic exponential decay typical for various boundary layer phenomena. The curves in Figures 2 and 5 as well as in Figures 3 and 4 have a pretty similar shape just taking slightly different numerical values.

We also present dispersion curves for a plane harmonic wave propagating with frequency ω and wave number k. In this case the dispersion relation associated with the derived equations of motion (3.36) and (4.21) are

$$\frac{4}{3}(1-\kappa^2)K^4\left(1-\frac{3}{2}\frac{\eta}{\sqrt{\pi}}\right)-\varOmega^2=0 \tag{5.1}$$

and

$$4(1-\kappa^2)K^2\left(1-\frac{\eta}{2\sqrt{\pi}}\right)-\varOmega^2=0\,, \tag{5.2}$$

where $\varOmega = \frac{\omega h}{c_2}$, $K = kh$, and $\kappa = \sqrt{\frac{1-2\nu}{2-2\nu}}$. At $\eta = 0$ these formulae coincide with those in the classical theories of plate bending and extension.

Numerical data are presented in Figures 6 - 7 for $\eta = 0$, 0.1 and $\nu = 0.3$, where we also plot the curves corresponding to the fundamental Rayleigh-Lamb antisymmetric (Figure 6) and symmetric (Figure 7) modes calculated from the transcendental relations

$$\gamma^4\frac{\sinh\alpha}{\alpha}\cosh\beta-\beta^2K^2\cosh\alpha\frac{\sinh\beta}{\beta}=0 \tag{5.3}$$

and

$$\gamma^4\cosh\alpha\frac{\sinh\beta}{\beta}-\alpha^2K^2\frac{\sinh\alpha}{\alpha}\cosh\beta=0\,, \tag{5.4}$$

where $\alpha^2 = K^2 - \kappa^2\varOmega^2$, $\beta^2 = K^2 - \varOmega^2$, and $\gamma = K^2 - \frac{\varOmega^2}{2}$ (Kaplunov et al., 1998).

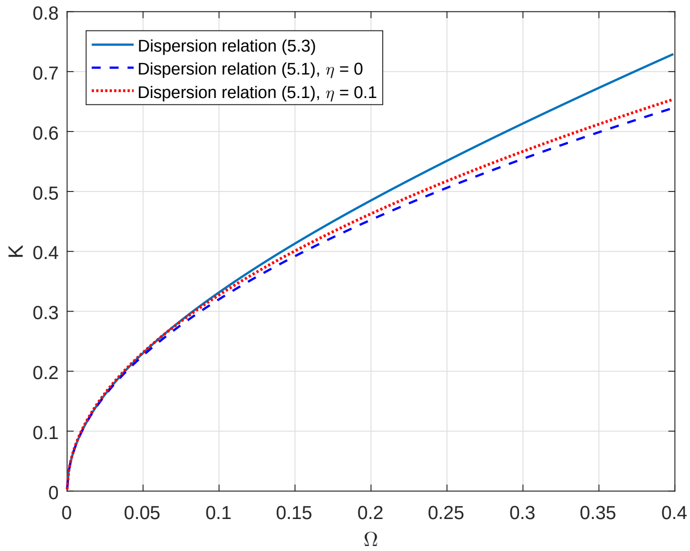

Figure 6. Dispersion of bending wave.

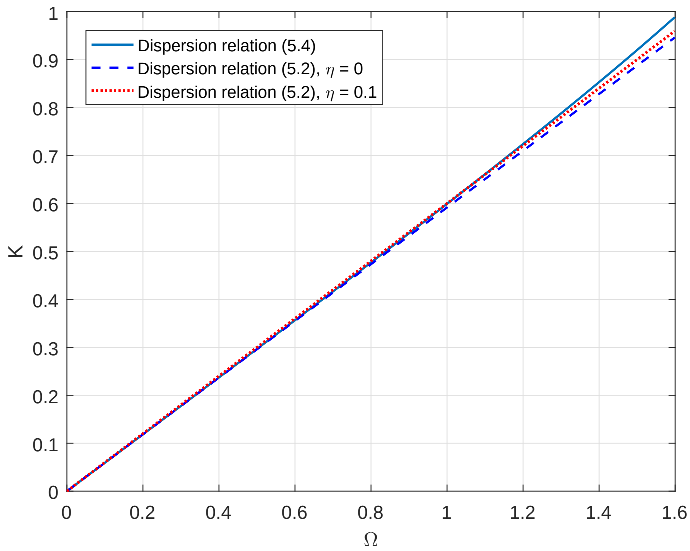

Figure 7. Dispersion of extensional wave.

These figures confirm that the nonlocal corrections to the classical plate theories are meaningful only at relatively low frequencies. It is also interesting to note that the curves corresponding to the nonlocal plate theory ($\eta = 0.1$ in Figures 6 and 7) intersect with those calculated from the Rayleigh-Lamb dispersion equations.

6. Concluding remarks

The main outcome of the presented study is that the effect of the boundary layers, arising due to nonlocal interactions, is just a modifying of the bending and extensional stiffness in the classical equations of plate motion. In this case the nonlocal stiffness D' and A' are defined by formulae (3.37) and (4.22), respectively, while the boundary layers are expressed in terms of the complementary error function, given by explicit relations (3.16), (3.27), and (4.16).

The range of validity of the equations (3.36) and (4.21) is not really restricted to the considered set up of a single small parameter, in which $h/\ell \sim a/h \ll 1$. In fact, these equations are also applicable at $h^2/\ell^2 \ll a/h \ll 1$, whereas at $a/h \sim h^2/\ell^2$, the terms of $O(h^2/l^2)$ typical for the asymptotic versions of Timoshenko-Reissner theories have to be retained, e.g., see Goldenveizer et al. (1993) and Elishakoff et al. (2015).

The proposed methodology has potential to be extended to thin elastic shells and beams taking into account anisotropy and more general nonlocal kernels. Another important area of further development is concerned with asymptotic justification of the nonlocal constitutive relations (2.2) in Eringen (1983) near plate faces, e.g., by homogenising the associated discrete lattice structure, e.g., see Eringen and Kim (1977) and also Picu (2002). The 1D procedure sketched by Eringen and Kim (1977) has potential to be extended to the 3D case; in doing so, we should apparently assume the so-called 'an effective cut-off length' (Eringen, 1983), (Picu, 2002) much greater than the microscale and much smaller that the plate thickness.

It also follows from Eringen and Kim (1977) that the discrete models involving nonlocal interactions adequately incorporate near surface phenomena, supporting boundary layers near plate faces similar to the considered kernel normalised over 3D domain by formula (2.3). Of course, such normalisation does not allow the classical limit near plate faces, which in apparently might not be dictated by physics of the studied nonlocal problem. At the same time, calculation of the alternative kernels arising from normalisation over the plate thickness (Sajadi et al., 2017) requires extra boundary conditions which have to be properly justified.

Ethics. This work does not involve any aspects such as collection of human data or such.

Data Accessibility. This work does not have any experimental data.

Authors' Contributions. All authors developed the asymptotic approach and wrote the paper. All authors gave final approval for publication.

Competing Interests. We have no competing interests.

Funding. Ph.D. studies of R. Chebakov were partly supported by Keele University. The support is gratefully acknowledged.

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