UDK 539.3

Elastic surface waves induced by internal sources

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Abstract

The paper is focused on surface wave field induced by an internal time-harmonic point source, embedded in an elastic half-space. Using the superposition principle, first the disturbances caused by embedded source in an unbounded half-space are analyzed. The problem is then reformulated in terms of the discrepant stresses on the surface of a homogeneous half-space. The consideration relies on the hyperbolic-elliptic asymptotic model for surface elastic waves, neglecting the contribution of the bulk waves. Explicit results for surface wave contribution are obtained, including the arising frequency-dependent factor.

Key words: hyperbolic-elliptic model, internal sources, Rayleigh waves.

1. Introduction

Surface waves induced by internal sources possess several important applications, including seismic engineering [8], as well as structural vibrations caused by underground dynamics, see e.g. [5]. Among the contributions addressing the effect of embedded sources, we mention the reciprocity based approach [16-18], as well as related analytic treatments of scattering surface waves from defects, see e.g. [1,6, 15, 20], and also numerical analysis of cracks located in an elastic media [12, 13]. The conventional formulation of the problem involves consideration of both bulk and surface waves. However, in the near-surface vicinity only surface wave contribution can be of interest, motivating a special theory.

Such a theory for Rayleigh and Rayleigh-type waves extracting the contribution of the studied wave to the overall dynamic response has been presented recently, see [9,10] and references therein. The main advantage of the proposed approach is reduction of the vector formulation in conventional elasticity to a scalar boundary value problem for the Laplace equation, with the boundary condition on the surface in the form of a hyperbolic equation containing prescribed loading in the right-hand side. The main range of applicability of the described approximate theory to dynamic problems of elasticity include the far-field zone, as well as the case of near-resonant excitation of surface waves, e.g. moving load problems, see e.g. [11].

The current paper aims at extension of the methodology in [10] to Rayleigh wave field induced by an embedded source. First, a brief description of the hyperbolic-elliptic model for the Rayleigh wave is presented. Then, radiation of longitudinal waves from the time-harmonic point internal source in an unbounded media is considered. Then, using the superposition principle, the problem is reformulated to that for a homogeneous half-space subject to appropriate boundary conditions on the surface. This allows direct implementation of the aforementioned asymptotic formulation for surface waves. The closed form solution in terms of elementary functions is obtained and illustrated numerically, showing the effect of depth of the source and frequency.

2. A specialized formulation for the Rayleigh wave field

Here we describe the hyperbolic-elliptic model for the Rayleigh wave field induced by surface stresses on a linearly elastic, isotropic half-space $0 \le x_2 < \infty$ within the framework of plane strain assumption for which the displacement $u_3 = 0$ and both displacements u_1 and u_2 are independent of x_3, for more detail see [10]. The original formulation of the boundary value problem involves the equations of motion

$$\Delta\varphi - c_1^{-2}\frac{\partial^2\varphi}{\partial t^2} = 0, \qquad \Delta\psi - c_2^{-2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{1}$$

for the scalar Lamé elastic potentials φ and ψ, subject to prescribed stresses on the surface $x_2 = 0$ i.e.

$$\sigma_{12} = P_1(x_1, t), \qquad \sigma_{22} = P_2(x_1, t). \tag{2}$$

In above Δ denotes a 2D Laplacian in Cartesian coordinates x_1 and x_2, $c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and $c_2 = \sqrt{\dfrac{\mu}{\rho}}$ are the longitudinal and transverse wave speeds, respectively, with λ and μ standing for the Lamé elastic parameters, and ρ being the volume mass density.

The asymptotic hyperbolic-elliptic formulation for the Rayleigh wave field contains pseudo-static elliptic equations for the potentials

$$\frac{\partial^2 \varphi}{\partial x_2^2} + \alpha_R^2 \frac{\partial^2 \varphi}{\partial x_1^2} = 0, \qquad \frac{\partial^2 \psi}{\partial x_2^2} + \beta_R^2 \frac{\partial^2 \psi}{\partial x_1^2} = 0, \tag{3}$$

where

$$\alpha_R = \sqrt{1 - \frac{c_R^2}{c_1^2}}, \qquad \beta_R = \sqrt{1 - \frac{c_R^2}{c_2^2}},$$

with c_R denoting the Rayleigh wave speed, being a unique solution of

$$4\alpha_R \beta_R - (1 + \beta_R^2)^2 = 0.$$

The potentials are related

$$\psi(x_1, \beta_R x_2, t) = \vartheta \varphi^*(x_1, \beta_R x_2, t), \tag{4}$$

as shown by Chadwick in [3], and earlier by Sobolev [19], where the asterisk denotes a harmonic conjugate, and

$$\vartheta = \frac{2\alpha_R}{1 + \beta_R^2}.$$

The boundary condition on the surface is provided by a hyperbolic equation

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{1}{c_R^2} \frac{\partial^2 \varphi}{\partial t^2} = A(P_2 + \vartheta^{-1} P_1^*), \tag{5}$$

where the asterisk in the right-hand side may be interpreted in a sense of the Hilbert transform, with

$$A_R = \frac{1+\beta_R^2}{2\mu B_R}, \qquad B_R = \frac{\alpha_R}{\beta_R}(1-\beta_R^2) + \frac{\beta_R}{\alpha_R}(1-\alpha_R^2) - 1 + \beta_R^4 .$$

The displacement components are conventionally expressed as

$$u_1 = \frac{\partial \varphi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \qquad u_2 = \frac{\partial \varphi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}, \tag{6}$$

which in view of (3) become

$$\begin{aligned} u_1(x_1, x_2, t) &= \frac{\partial \varphi(x_1, \alpha_R x_2, t)}{\partial x_1} - \frac{1+\beta_R^2}{2} \frac{\partial \varphi(x_1, \beta_R x_2, t)}{\partial x_1} \\ u_2(x_1, x_2, t) &= \frac{\partial \varphi(x_1, \alpha_R x_2, t)}{\partial x_2} - \frac{2}{1+\beta_R^2} \frac{\partial \varphi(x_1, \beta_R x_2, t)}{\partial x_2} \end{aligned} \tag{7}$$

Thus, the solution may be expressed in terms of a single plane harmonic functions, say elastic potential φ, being a solution of the elliptic equation $(3)_1$, subject to a hyperbolic equation (5) on the boundary $x_2 = 0$. The second potential is then given by (4), with the displacement field expressed as (7).

3. Internal point source

Let us now consider the Rayleigh wave field emerging due to action of an embedded time-harmonic point source. Suppose that the source is located at the origin, at a given depth a from the surface $x_2 = -a$, see Fig.1.

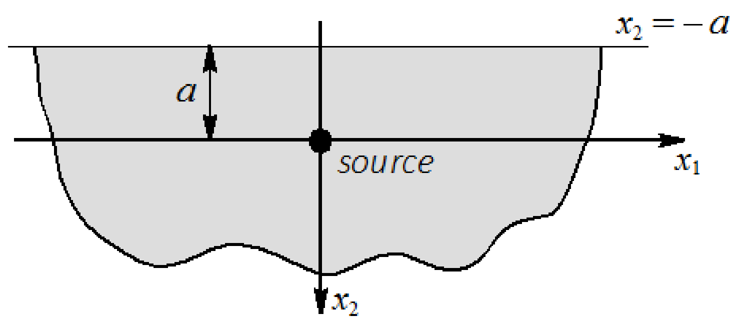

Figure 1. Schematic location of an internal source.

Applying the superposition principle, first, consider a problem of radiation from a time-harmonic point source in a 2D unbounded medium. For the sake of definiteness, in this paper we focus on the longitudinal potential φ. The solution is then given by the Green function

$$\varphi = \frac{i}{4k^2} H_0^{(1)}(kr) e^{-i\omega t}, \tag{8}$$

where is ω frequency, $k = \omega / c_1$ is wave number, $r = \sqrt{x_1^2 + x_2^2}$ is polar radius, and $H_0^{(1)}$ is the Hankel function of the first kind, see e.g. [2]. Note that in what follows the factor $e^{-i\omega t}$ is omitted for the sake of brevity. Now, assuming $\psi = 0$, the appropriate stress components can be obtained in the form

$$s_{12} = \frac{i\mu}{2} \frac{x_1 x_2}{x_1^2 + x_2^2} H_2^{(1)}\left(k\sqrt{x_1^2 + x_2^2}\right), \tag{9}$$

$$s_{22} = \frac{i}{4}\left[\frac{2\mu}{k} \frac{x_2^2 - x_1^2}{\left(x_1^2 + x_2^2\right)^{3/2}} H_1^{(1)}\left(k\sqrt{x_1^2 + x_2^2}\right) - \frac{\lambda x_1^2 + (\lambda + 2\mu) x_2^2}{x_1^2 + x_2^2} H_0^{(1)}\left(k\sqrt{x_1^2 + x_2^2}\right)\right] \tag{10}$$

Hence, in view of superposition principle, we can now formulate a non-homogeneous problem for a homogeneous half-plane (without a source) as

$$\sigma_{12} = -s_{12}\,|_{x_2 = -a}, \qquad \sigma_{22} = -s_{22}\,|_{x_2 = -a}. \tag{11}$$

Let us introduce the dimensionless scaling

$$\xi_1 = \frac{x_1}{a}, \qquad \xi_2 = \frac{x_2}{a}, \qquad k_1 = ka = \frac{\omega a}{c_1}.$$

Then, the loading terms in the right-hand side in conditions (2) take the form

$$P_1 = \frac{i\mu}{2} \frac{\xi_1}{\xi_1^2 + 1} H_2^{(1)}\left(k_1\sqrt{\xi_1^2 + 1}\right), \tag{12}$$

$$P_2 = \frac{i\mu}{2}\left[\frac{1}{k_1} \frac{\xi_1^2 - 1}{\left(\xi_1^2 + 1\right)^{3/2}} H_1^{(1)}\left(k_1\sqrt{\xi_1^2 + 1}\right) + \left(\frac{\lambda}{2\mu} + \frac{1}{\xi_1^2 + 1}\right) H_0^{(1)}\left(k_1\sqrt{\xi_1^2 + 1}\right)\right]. \tag{13}$$

In view of the time-harmonic regime, the hyperbolic equation (5) transforms to

$$\frac{d^2\varphi}{d\xi_1^2}+k_R^2\varphi=a^2A_R\left(P_2+\vartheta^{-1}P_1^*\right), \qquad (14)$$

with P_1^* being the Hilbert transform of P_1 given by (12), P_2 defined by (13), and the wave number $k_R=\dfrac{\omega a}{c_R}$.

We apply a Fourier transform in ξ_1 to the 2D counterpart of the equation (14) and $(3)_1$ over the interior, having

$$\varphi^F\left(\xi_1,0,t\right)=\frac{-a^2A_R\left(P_2+\vartheta^{-1}P_1^*\right)^F}{k^2-k_R^2}, \qquad (15)$$

$$\frac{d^2\varphi^F}{d\xi_2^2}-k^2\alpha_R^2\varphi^F=0, \qquad (16)$$

where φ^F denotes the transformed potential and k is Fourier transform parameter.

Then, solving the equation (16) subject to the boundary condition (15), we obtain a decaying solution in the form

$$\varphi^F\left(\xi_1,\xi_2,t\right)=\frac{-a^2A_R\left(P_2+\vartheta^{-1}P_1^*\right)^F}{k^2-k_R^2}e^{-\alpha_R|k|\xi_2}.$$

The Rayleigh wave contribution can be extracted from the latter by taking residues at the poles $k=\pm k_R$, which gives

$$\varphi\left(\xi_1,\xi_2,t\right)=-i\frac{a^2\left(1+\beta_R^2\right)}{4k_RB_R}\left(I_2(\omega)+\vartheta^{-1}I_1(\omega)\right)e^{k_R\left(i|\xi_1|-\alpha_R\xi_2\right)}, \qquad (17)$$

where integrals $I_1(\omega)$ and $I_2(\omega)$ allow table evaluation, see e.g. [6],

$$I_1(\omega)=\frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{\xi}{\xi^2+1}H_2^{(1)}\left(k_1\sqrt{\xi^2+1}\right)\frac{1}{\eta-\xi}d\xi\cos(k_R\eta)d\eta=-\frac{k_R}{k_1^2}e^{-\sqrt{k_R^2-k_1^2}} \qquad (18)$$

and

$$I_2(\omega) = i\int_0^\infty \left[\frac{1}{k_1} \frac{\xi^2 - 1}{\left(\xi^2+1\right)^{3/2}} H_1^{(1)}\left(k_1\sqrt{\xi^2+1}\right) + \left(\frac{\lambda}{2\mu} + \frac{1}{\xi^2+1}\right) H_0^{(1)}\left(k_1\sqrt{\xi^2+1}\right)\right]\cos(k_R\xi)d\xi = -\left(\frac{c_1}{c_R}\right)^2 \frac{\left(1-\frac{1}{2}\left(\frac{c_R}{c_2}\right)^2\right)}{\sqrt{k_R^2-k_1^2}} e^{-\sqrt{k_R^2-k_1^2}} \quad (19)$$

Therefore, the potential (17) may now be rewritten with (18) and (19) as

$$\varphi\left(\xi_1, \xi_2, t\right) = iA(a,\omega)e^{k_R\left(i|\xi_1| - \alpha_R \xi_2\right)}, \quad (20)$$

where an amplitude is expressed as

$$A(a,\omega) = \frac{a^2 \beta_R}{B_R k_1^2} e^{-\sqrt{k_R^2 - k_1^2}}. \quad (21)$$

It is worth noting that the function (20) is not smooth at $\xi_1 = 0$, due to the presence of $|\xi_1|$. Moreover, this has been noticed previously for the Rayleigh pole contribution in the associated plane problem in elasticity, see [4].

On employing the relations $(3)_2$ and (4), the related transverse potential ψ is obtained as

$$\psi\left(\xi_1, \xi_2, t\right) = sign(\xi_1)\vartheta A(a,\omega)e^{k_R\left(i|\xi_1| - \beta_R \xi_2\right)}. \quad (22)$$

The expression (21) readily confirms the expectation that the amplitude decreases with increase in depth of the source. The dependence of the dimensionless expression $A(a,\omega)/a^2$ on the dimensionless wave number k_1 is presented in Fig.2 for several values of the Poisson coefficient (ν=0.2, ν=0.3 and ν=0.4). It is clear from this illustration that the amplitude is a decaying function of the wavenumber.

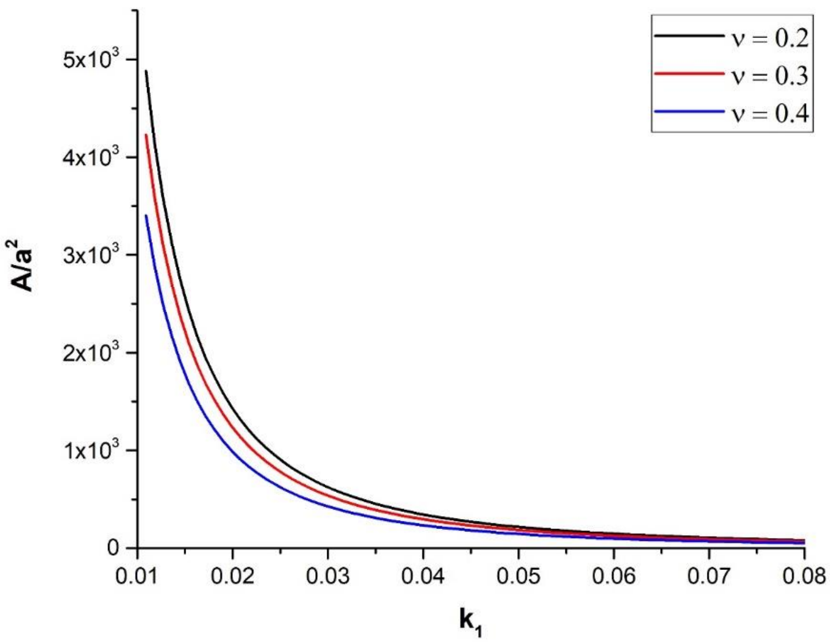

Figure 2 The dependence of the dimensionless amplitude on dimensionless wave number k_1 *at different values of the Poisson coefficient* ν.

Concluding remarks

In this paper, the effect of an embedded point time-harmonic source on propagation of surface waves in an elastic half-space has been studied. The superposition principle allowed reduction of the formulation of the problem to a simpler one for a homogeneous half-space, subject to discrepant surface stresses. This in turn enabled the application of the hyperbolic-elliptic formulation for the Rayleigh wave, providing elegant explicit expressions for the Lamé potentials (20), (22).

The approach could be extended to interfacial waves, using the asymptotic theories exposed in [10]. The effects of anisotropy can also be incorporated, see e.g. [14]. Finally, similar ideas could be developed for transient and moving sources, as well as wave fields with non-homogeneous initial conditions.

Acknowledgements. The research was supported by the Russian Foundation for Basic Research (RFBR) (Project No. 18-58-41001) and by the Ministry of Education and Science of the Republic of Kazakhstan, Grant IRN AP05132743. AAC acknowledges support through the Erasmus + KA107 framework. Finally, fruitful discussions with Prof J. Kaplunov are gratefully acknowledged.

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