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Low-frequency vibrations of coated elastic structures

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Submitted in partial fulfilment of the requirements of the degree of

Doctor of Philosophy

Keele University

School of Computing and Mathematics

December 2019

Declaration

I certify that this thesis submitted for the degree of Doctor of Philosophy is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Disseminations

Journal Papers:

- A Alzaidi, J Kaplunov, and LA Prikazchikova. Elastic bending wave on the edge of a semi-infinite plate reinforced by a strip plate. Mathematics and Mechanics of Solids, 24(10):3319–3330, 2019.

- A Alzaidi, J Kaplunov, and LA Prikazchikova. The edge bending wave on a plate reinforced by a beam (L). The Journal of the Acoustical Society of America, 146(2):1061-1064, 2019.

Presentations

- A flexural wave on a coated edge of a thin elastic plate, the British Applied Mathematics Colloquium, the School of Mathematics and Statistics, St Andrews University, March 2018.

- Dynamics of strongly inhomogeneous multi-component engineering structures,3nd International Scientific Conference on the Development of Industrial, Faculty of Industrial Engineering, Slovenia, April 2018.

- The bending wave on a coated edge of a thin elastic plate, Elasticity day, The university of Manchester, May 2018.

- The bending wave on a reinforced edge of a thin elastic plate, 7th International Conference and Exhibition on Mechanical and Aerospace, San Francisco, USA, September 2019.

Acknowledgements

First and foremost, I would like to thank God for achieving this work. I would like to express my very great appreciation to my supervisory team, including Prof. Julius Kaplunov, Dr Danila Prikazchikov and Dr Ludmila Prikazchikova, for all the invaluable comments, inspiration, guidance, and advice, which help me throughout the challenging and exciting three years to develop my research tools and techniques. I am grateful to have worked with them. Thank you all for support gave me in completing this project.

I would also like to acknowledge the help provided by Saudi Arabia embassy in London and Taif university , which gave me the opportunity to continue to produce this work.

I shall not forget to thank the light of my path and the most precious in my life my father, mother, wife, son, brothers and sisters for all the patience, prayers and understanding throughout the period of this research.

Finally, I do direct my sincere thanks to those who gave me a hand, and gave me an advice.

Abstract

The thesis deals with 1D and 2D scalar equations governing the dynamic behaviour of coated elastic structures. Low-frequency vibration of a composite rod, beam, rectangular plate and circular plate are studied. The main focus is on physical effects that occur in composite elastic structures with a thin coating. We start with two auxiliary 1D problems for two-component rods and beams.

Then elastic waves localised near the edge of a semi-infinite plate reinforced by a strip plate are considered within the framework of the 2D classical Kirchhoff theory for plate bending. The boundary value problem for the strip plate is subject to an asymptotic analysis assuming that a typical wave length is much greater the strip thickness. As a result, effective conditions along the interface, corresponding to a plate reinforced by a beam with a narrow rectangular cross-section, are established. They support an approximate dispersion relation perturbed from that for the homogeneous plate with a free edge. The accuracy of the approximate dispersion relation is tested by comparison with the numerical data obtained from the 'exact' matrix relation for a composite plate. The effect of the problem parameters on the localisation rate is studied.

In addition, edge bending waves on a thin isotropic semi-infinite plate reinforced by a beam are considered within the framework of the classical plate and beam theories. The boundary conditions at the plate edge incorporate both dynamic bending and twisting of the beam. A dispersion relation is derived along with its long-wave approximation. The effect of the problem parameters on the cut-off frequencies of the

wave in question is studied asymptotically. The obtained results are compared with calculations for the case when the reinforcement takes the form of a plate strip. Finally, a circular plate reinforced by a thin annular strip of the same thickness is considered. Asymptotic treatment of a strip circular plate with a free outer edge and its inner edge subject to prescribed deflection and rotation is presented. The effective boundary conditions are derived, and approximate dispersion relation is deduced.

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Chapter 1

Introduction

Low-frequency mechanical vibrations of composite elastic structures have been the subject of extensive studies, see e.g. the classical textbooks [51],[85], and also [142] for a recent account. In the last few decades, composite elastic structures have attracted significant interest of scientists due to the appearance of new applications connected to the development of multi-layered structures with high contrast in the geometrical and mechanical properties. Multi-layered composite structures with high-contrast material parameters possess many industrial applications. For instance, in aircraft and aerospace engineering, multi-layered structures are widely used, see e.g. [21],[22],[97]. Other obvious applications include solar panels and laminated glass [15],[100]. We also mention a related sub-area of acoustic metamaterials, see [32] and [132]. In addition, there are promising applications of coated structures, related to rapidly developing fields in modern engineering and technology, in particular, associated with structural mechanics and biomedical sciences, see e.g. [23],[28],[54],[86],[106],[109],[121], [135].

Composite elastic structures are produced by combining two or more materials. These may often have high-contrast properties, say, in stiffness, density and geometrical parameters. The main reason for the popularity of layered structures is that by putting two or more materials together one may result in a structure with unique properties which are different from each of the individual materials' properties. Thus, the desirable properties of multi-layered structures, for example, increasing stiffness and at the same time reducing the overall weight the of structure, can be obtained by choosing an appropriate combination of materials. This provides a motivation for investigation of the dynamics of such composite and layered elastic structures.

Propagation of waves in elastic sandwich plates are still among the popular research directions of elasticity. Many types of sandwich plates have been the subject of interest for a long period due to their wide implementation in civil and aircraft engineering. The first analytical investigation of bending and buckling in sandwich plates was seemingly made by Reissner in [117]. He considered a plate consisting of a core layer with two facing membranes both identical, where his analysis relied on assumption that the face-parallel stresses in the core and the face stresses over the thickness of the membranes are negligible. Then, Reissner's problem was modified, and the governing equations for the sandwich plates with orthotropic cores were related to the bi-harmonic equations from classical plate theory, see [30].

Later, numerous papers studying vibrations in sandwich plates have appeared, using mainly numerical computations, however, asymptotic methods have been employed as well, for a review of these achievements see [26],[53],[103]. Among the relatively recent contributions on the subject we mention [7],[15],[20],[27],[67],[94] dealing with analysis of the dispersion phenomenon in sandwich structures. It is known that the asymptotic structures in a single-layered plate for bending, extension, thickness stretch resonance and thickness shear resonance phenomenon are preserved within the multi-layer problem. A step forward in study of wave propagation in layered structures has been made through a recently developed multiparametric analysis, incorporating high-contrast properties, allowing unexpected low cut-off frequencies, and, as a result, requiring special two-mode long-wave low frequency theories for the bending of sandwich plates, see [67], resulting on the simpler considerations for elastic rods [68],[72].

The presence of a thin layer in composite elastic structures, including coated ones, stimulates the use of asymptotic methods in order to rely on a small geometric parameter, typically the ratio between the thickness of the layer and a typical length, which emerges naturally in the analysis. Asymptotic methods have also been very popular in statics and low-frequency dynamics of thin plates [4],[43],[47],[116] and shells [13],[14],[48],[80]. A number of contributions, applying asymptotic methods in more general dynamic problems considering long-wave high-frequency [49],[52],[60] and short-wave high-frequency [118] regions. In addition, the method of direct asymptotic integration of the equations in elasticity [50],[58],[59], were also applied in more general dynamic problems with high-frequency approximations considered for both long- and short-wave limits. We also mention here papers [5],[6],[10],[11],[12],[40] developing asymptotic approaches to contact problems in layered structures, pre-stressed and anisotropic materials investigated in [2],[25],[59],[63],[83], [102],[111],[112],[119], and bodies with clamped faces studied in [57],[62]. We also note the deep parallels between long-wave asymptotic theories for functionally graded waveguides and periodic structures observed in [33].

One of the popular asymptotic methods in modelling the effect of a thin coating is to derive the so-called "effective boundary conditions", imposed on the interface between the coating and substrate. Over the last few decades, a number of studies of effective boundary conditions have been presented. Originally, Tiersten in [131] was the first to derive such conditions using adhoc considerations originating from the classical theory of plate extension. Three decades later the problem was revisited in [24] and suggested that the results of [131] are not asymptotically consistent. A perturbation scheme in [34], accounting for the influence of the coating, revealed that the extra terms in [24] are in fact of a higher order, and also justified at leading order the consistency of the original effective boundary conditions in [131]. It can be seen that the boundary conditions in [24] were also discussed after the publication of the critical comments in [34], e.g. see [46],[95],[141] along with [110],[143]. Among numerous publications on the subject, we mention [18],[19],[101],[140]. Recently, the refined effective boundary conditions were proposed in [74]. The effective boundary conditions illustrating the effect of an isotropic elastic layer are established in [137], anisotropic elastic layer in [149],[150], and in particular orthotropic elastic layer in [136],[139].

The effective boundary conditions provide an approximate formulation for studying surface wave propagation in coated elastic solids, see e.g. [34],[66],[138], and also a recent achievement [73], allowing surface waves in case of a coated half-space with a clamped surface. One of the novel results in this thesis is related to consideration of other types of *localised* waves in coated solids, i.e. bending edge waves.

Localised elastic waves have a long and interesting history. It began with the famous paper [115] by Lord Rayleigh, describing the waves propagating along the surface of an elastic half-space and decaying away into the interior. Then, after the discovery of the Rayleigh wave in an elastic half-space, edge waves in semi-infinite elastic plates were considered. Generally, edge waves occurring in elastic structures, can be divided into two main parts, namely, into flexural and extensional edge waves. The extensional edge waves are longitudinal ones propagating along the edge of a material, see e.g. [113] and references therein. We also mention the fundamental contribution [43] which derived the approximate boundary conditions at the free edge of the Kirchhoff plate. Konenkov [79] was the first to demonstrate the existence of flexural edge waves in a semi-infinite isotropic thin elastic plate, see [104] discussing the interesting history of discovery of flexural edge waves. It indicated that Konenkov discovered these waves within the framework of Kirchhoff plate theory. Unfortunately, his result was not widely known in the western scientific circles limited by the scarcity of Soviet literature available at that time. After 14 years, the bending edge waves were rediscovered independently by Sinha in [126] and Thurston and McKenna in [130]. As found recently, there was also an earlier underlying work [55] within the framework of the stability of elastic plates, for more details and a more recent review of achievements see [84]. A related problem of edge resonance has been studied in [45],[107],[120],[123],[133],[152].

In the following years, the edge bending wave on an elastic plate has received much attention, taking into consideration effects of anisotropy, contact with elastic foundations, and three dimensional dynamic phenomena. We also mention [44] and [93] dealing with edge waves propagating along the edge of unsymmetrical plates. Among considerations of edge waves within 3D formulation of elasticity, we cite [44],[69],[81] and [151]. Another recent approach is related to development of asymptotic parabolic-elliptic models for edge bending waves [64],[65],[70]. The latter consists of a parabolic beam-like equation along the edge complemented with a 'pseudo-static' elliptic equation describing decay over the interior. They enable one extract the edge wave contribution from the overall dynamic response and appear to be in line with a general physical idea of edge wave phenomena. Its generalisation to the wave on a stiffened edge of 'plate-beam' structures appears to be of interest.

Stiffened plates are important components of civil, aerospace and naval structures

[42],[77],[98],[105],[125]. In spite of numerous contributions analysing their dynamic behaviour, e.g. see [31],[89],[90],[92],[99],[114],[144],[147], the bending wave localised near a reinforced plate edge has not yet been investigated. Such a wave has been studied in a great detail for a homogeneous plate with a traction free edge, beginning with the paper [79], see also the review articles [84] and [104] along with more recent publications [61],[71] dealing with an elastically supported plate.

In contrast to the non-dispersive Rayleigh wave on an elastic half-space described by a hyperbolic-elliptic formulations, the edge bending wave on a plate demonstrates dispersion governed by a specialised parabolic-elliptic model [65], [70]. Another distinct feature for the bending edge wave is their remarkably low decay rate, degenerating at zero Poisson ratio, see the references above. It might be expected that an edge reinforcement would control the localisation of the edge wave. In many cases the reinforcement apparently can be modelled by a plate strip or a beam attached to the edge governed by 1D or 2D equations, respectively.

Static and dynamic behaviour of stiffened plates was intensively studied in numerous publications within the framework of the classical bending theories for plates and beams also taking into consideration beam torsion, see e.g. [37],[38],[91],[108],[122]. At the same time, to the best of the authors' knowledge, edge waves in stiffened plates have only been analysed in two papers [17],[96], dealing with a semi-infinite strip with simply supported sides. Bending vibrations of an elastic strip were earlier investigated in various setups, e.g. see [78]. We also mention the recent contributions

[8] and [9] treating a semi-infinite plate reinforced by a beam or flexural strip along the edge, more details of which will be presented in Chapters 3 and 4.

A further extension of our results is related to circular plates, originating from Airy in [3]. One of the first studies of flexural vibrations of axially symmetric circular disks was by Deresiewicz and Mindlin [35] where they have used the classical thin plate theory as well as Mindlin plate theory to obtain mode shapes for free circular disks. The free vibration of axisymmetric orthotropic non-uniform circular discs with shear deformation has been studied in [128] using Chebyshev collocation technique and Mindlin plate theory. A number of papers dealing with circular plates, include in particular [29],[41] studying vibrations of plates with clamped edges, [56],[127],[153] analysing vibrations within 3D framework, also accounting for the effects of nonlinearity [129],[134], variable thickness [124],[145], edge supports [16],[88], as well as studying vibrations in circular plates within the framework of Mindlin theory [87],[146]. The waves localised near the edge of a circular disk were studied by Destrade and Fu in [36].

The present study is concerned with analysis of the propagation of flexural edge waves in case of the edge stiffened by thinly coated plate. The particular focus of this work is on physical effects which occur in composite coated elastic structures. First, two auxiliary problems for a composite rod and composite beam will be considered in Chapter 2. The continuity conditions are assumed between the components. The analysis is carried out for the case of one end being fixed, and another subject to external loading. Starting from the equations of motion for harmonic waves, the expressions for displacements for the left and right components are determined. The asymptotic analysis is carried out by using the asymptotic integration method to obtain the effective stress in rod and moment and shear force in the beam on the interface between the components. In other words, the effective boundary conditions are derived, replacing the effect of the loading through the geometrically small components. In addition, the exact solution is also expanded in Taylor series and compared with the asymptotic results in order to have additional verification of the solution.

In Chapter 3 we restrict ourselves to a semi-infinite plate perfectly bonded with a narrow strip plate of the same thickness, within the framework of the Kirchoff theory. We develop an asymptotic approach based on the derivation of effective conditions along the structure's interface, similarly, in a sense, to the developments for a coated elastic half-space, e.g. see [34], justifying at leading order the widely known effective conditions in [131] established using adhoc arguments.

The main part of this Chapter 3 is concerned with asymptotic treatment of a flexural strip with a free upper edge and its lower edge subject to prescribed deflection and rotation. The long-wave limit is analysed assuming that a typical wavelength is much greater than the strip width. In contrast to a coated half-space, the simplest effective conditions for a reinforced plate follow only from a fourth order asymptotic expansion in small width for deflection, since the shear force at the lower edge of interest is proportional to the fourth order deflection derivative. The consistency of the derived effective conditions is tested by comparison with the leading-order behaviour of the exact space and time-harmonic solution of the original problem for a flexural strip. As might be expected, the derived effective boundary conditions may be re-written in terms of a beam attached to the edge of the plate. The last formulation was previously used for static and dynamic analysis of reinforced plates, e.g. see [38],[39],[91],[108].

The proposed conditions are then adapted for obtaining an approximate dispersion relation for the sought for edge wave. An explicit correction, expressing the effect of the reinforcement, readily comes from the shortened relation and seems to be useful for a better qualitative insight into the influence of the density and stiffness of the plate strip material on edge wave propagation. Approximate results are displayed along with the numerical data calculated from the full dispersion relation for a composite plate taking the form of the determinant of a six-order matrix. The influence of the density and stiffness of the plate strip material on the edge wave propagation is discussed. In addition, we extend our work in two cases which are a clamped and mixed upper edge, with its lower edge subject to prescribed deflection and rotation.

Chapter 4 is concerned with bending vibrations localised along the edge of a semi-infinite plate, stiffened by a beam. A dispersion relation is derived together with its long-wave asymptotic approximations. At the leading order the latter coincides with the dispersion relation for the plate bending wave on a free edge [79]. Next order solution reveals the influence of stiffening on the edge wave localisation. Using the results of Chapter 3, a comparison of the dispersion relation for a plate reinforced by a beam with a narrow rectangular cross-section and that for a plate reinforced by a flexural strip is performed, justifying the adapted 'plate-beam' formulation. As might be expected the theory for a plate stiffened by a narrow beam is only valid over the long-wave region, and as we move outside of it, the distinction between the plate and beam reinforcement results becomes more pronounced.

The effect of material and geometric parameters on edge wave localisation is also investigated. A special focus is on the asymptotic evaluation of the cut-offs of the studied edge wave which have been earlier discovered in [17],[96]. The possible situation when the cut-offs are located outside the range of validity of the adapted classical structural theories is addressed.

Finally, in Chapter 5 we extend the previous results to a finite circular plate perfectly bonded with a narrow annular strip plate of the same thickness. We focus on asymptotic treatment of a thin annular plate strip with a free outer edge and its inner edge subject to prescribed deflection and rotation. Following a usual procedure described in Chapter 3, the effective boundary conditions are derived, along with the approximate dispersion relation. Then we conclude in Chapter 6.

Chapter 2

Harmonic vibrations of a composite beam and rod

This chapter describes harmonic vibrations of a composite beam and rod. In Section 2.1, we review harmonic vibrations of a composite rod. We obtain the exact solution for a two component rod in dimensionless variables and investigate it asymptotically. We also tested the asymptotic results obtained. In Section 2.2, we extend our work to investigation of vibrations in a composite beam. At the right end of a composite beam, two cases of the boundary conditions are imposed. In the first one we assume no transverse shear force at the right end while in the second case we consider no moment in the same end. To do this we introduced appropriate scaling for frequencies together with corresponding dimensionless spatial variables and obtained the exact solution for a two component beam. Next, the asymptotic analysis is carried out by using the asymptotic integration method to obtain the effective moment and shear force in the beam on the interface between the components. Then, the validation of the asymptotic results are obtained and comparison of asymptotic solution and exact solution is presented.

2.1 Harmonic axial vibrations of a composite rod

2.1.1 Problem statement

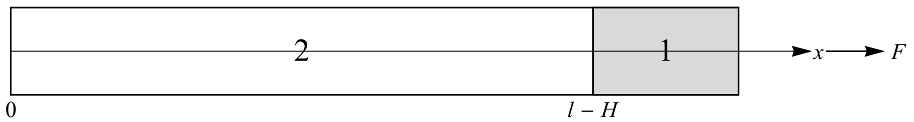

FIGURE 2.1: A composite rod

For the 1D analysis of laminated structures we start with rather basic problem considering a linear elastic two-component rod of finite length. Both components are assumed to be isotropic. Let the x axis be taken to lie along the rod with the length of the components $(l-H)$ and H with a force F applied at the right end of the rod. (see Figure 2.1).

Hereinafter the index 1 will be used to denote problem parameters and variables corresponding to the right component, whereas the index 2 will denote the same for the left component.

The equations of motion can be written in the form [82]

$$\frac{d^2u_j}{dx^2} + \frac{\omega^2}{c_j^2}u_j = 0, \qquad j = 1,2 \tag{2.1}$$

where u_j are the longitudinal displacements, $c_j = \sqrt{\frac{E_j}{\rho_j}}$ are the wave speeds, E_j are the Young's moduli, ρ_j are the material densities for the relevant component of the rod and ω is frequency.

The boundary conditions are taken in the form

$$u_2 = 0 \;\; at \;\; x = 0,$$
$$E_1 \frac{du_1}{dx} = F \;\; at \;\; x = l. \tag{2.2}$$

Traction and displacement continuity at the interface between the components is given by

$$u_1 = u_2 \quad at \quad x = l - H,$$

and

$$E_1 \frac{du_1}{dx} = E_2 \frac{du_2}{dx} \quad at \quad x = l - H. \tag{2.3}$$

The general solution of the linear ordinary differential equations with constant coefficient (2.1) is given by

$$u_j = A^{(j)} \cos \frac{\omega}{c_j} x + B^{(j)} \sin \frac{\omega}{c_j} x, \quad j = 1, 2 \tag{2.4}$$

where $A^{(j)}$ and $B^{(j)}$ are arbitrary constants.

Substituting (2.4) into the boundary conditions (2.2) and continuity relations (2.3) leads to $A^{(2)} = 0$ and the system of three simulate equations

$$\begin{pmatrix} -s\sin(s) & s\cos(s) & 0 \\ \cos(s_1) & \sin(s_1) & -\sin(s_2) \\ \frac{-E_1}{c_1}\sin(s_1) & \frac{E_1}{c_1}\cos(s_1) & \frac{-E_2}{c_2}\cos(s_2) \end{pmatrix} \begin{pmatrix} A^{(1)} \\ B^{(1)} \\ B^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{Fl}{E_1} \\ 0 \\ 0 \end{pmatrix}, \tag{2.5}$$

where $s = \dfrac{\omega l}{c_1}$, $s_1 = \dfrac{\omega(l-H)}{c_1}$, $s_2 = \dfrac{\omega(l-H)}{c_2}$, which possesses non-trivial solutions. Using Cramer's rule, we get the exact solution as

$$u_1 = \frac{c_1 F\left(E_1 c_2 \sin(s_2)\cos\left(\frac{\omega x}{c_1} - s_1\right) + E_2 c_1 \cos(s_2)\sin\left(\frac{\omega x}{c_1} - s_1\right)\right)}{E_1\omega\left(E_1 c_2 \sin\left(s - s_1\right)\sin(-s_2) + E_2 c_1 \cos\left(s - s_1\right)\cos(-s_2)\right)}, \tag{2.6}$$

$$u_2 = \frac{c_1 c_2 F \sin\left(\frac{x\omega}{c_2}\right)}{E_1 c_2 \omega \sin\left(s - s_1\right)\sin(-s_2) + E_2 c_1 \omega \cos\left(s - s_1\right)\cos(-s_2)}, \tag{2.7}$$

where $A^{(j)}$ and $B^{(j)}$ are presented in Appendix A.1.

2.1.2 Dimensionless equations

In order to investigate exact solutions asymptotically, we convert all variables into dimensionless form. We introduce the following dimensionless variables

$\xi_1 = (\frac{x}{l} - 1)\frac{1}{\varepsilon} + 1, \quad \xi_2 = \frac{x}{l-H}, \quad \Omega = \frac{\omega l}{c_1}$ and $\varepsilon = \frac{H}{l} \ll 1$ is assumed to be small.

Now we can rewrite the exact solutions (2.6) and (2.7) in dimensionless form as

$$u_1 = \frac{E_1 Fl \sin(c\Omega(\varepsilon-1))\cos(\xi_1\Omega\varepsilon) - E_2 cFl\cos(c\Omega(\varepsilon-1))\sin(\xi_1\Omega\varepsilon)}{E_1^2\Omega\sin(\Omega\varepsilon)\sin(c\Omega(1-\varepsilon)) - E_1E_2c\Omega\cos(\Omega\varepsilon)\cos(c\Omega(\varepsilon-1))}, \tag{2.8}$$

$$u_2 = \frac{Fl\sin(c\xi_2\Omega(1-\varepsilon))}{E_2c\Omega\cos(\Omega\varepsilon)\cos(c\Omega(\varepsilon-1)) - E_1\Omega\sin(\Omega\varepsilon)\sin(c\Omega(1-\varepsilon))}, \tag{2.9}$$

where $c = \frac{c_1}{c_2}$.

In order to confirm our result we are setting $x = l - H$ which implies $\xi_1 = 0$ and $\xi_2 = 1$ when $\varepsilon \to 0$ into (2.8) and (2.9), we obtain

$$u_1 = u_2 = \frac{Fl}{E_2 c\Omega}\tan(c\Omega),$$

which is an additional verification of the solution.

2.1.3 Asymptotic analysis of a composite rod

In this section, we apply an asymptotic approach to obtain an approximate solution for a two-component rod. In particular, we restrict our attention to perturbation scheme for u_1. To this aim, we use dimensionless variables in the above section to rewrite the equation of motion (2.1) as

$$\frac{d^2u_1}{d\xi_1^2} + \varepsilon^2\Omega^2 u_1 = 0, \tag{2.10}$$

subject to

$$\begin{aligned} &u_1 = u_H, \ \ at \ \ \xi_1 = 0, \\ &\frac{du_1}{d\xi_1} = \varepsilon\frac{lF}{E_1} \ \ at \ \ \xi_1 = 1, \end{aligned} \tag{2.11}$$

where function $u_H = \dfrac{Fl}{E_2 c\Omega}\tan(c\Omega)$ is a given displacement on the interface.

A deflection u_1 can be expanded into an asymptotic series in terms of ε as

$$u_1 = u_1^{(0)} + u_1^{(1)}\varepsilon + u_1^{(2)}\varepsilon^2 + u_1^{(3)}\varepsilon^3 + u_1^{(4)}\varepsilon^4 + \dots \tag{2.12}$$

Substituting expansion (2.12) into the boundary value problem (2.10)-(2.11), we arrive at the problem formulated for various asymptotic orders $n = 0, 1, 2, \dots$, namely

$$\frac{d^2u_1^{(n)}}{d\xi_1^2} + \Omega^2 u_1^{(n-2)} = 0, \tag{2.13}$$

subject to

$$
\begin{aligned}
&u_1^{(n)} = u_H^{(n)}, && \xi_1 = 0, \\
&\frac{du_1^{(n)}}{d\xi_1} = \frac{l^{(n)}F^{(n)}}{E_1^{(n)}} && \text{at} \quad \xi_1 = 1,
\end{aligned}
\tag{2.14}
$$

where quantities with the negative superscript are set to be equal to zero. The only non-zero components $u_H^{(n)}$ and $\frac{l^{(n)}F^{(n)}}{E_1^{(n)}}$ are $u_H^{(0)} = u_H$ and $\frac{l^{(1)}F^{(1)}}{E_1^{(1)}} = \frac{lF}{E_1}$, respectively.

Substituting subsequently $n = 0, 1, 2, 3$ and 4 into (2.13)-(2.14) we obtain corrections for a displacement u_1 in the form

$$
\begin{aligned}
u_1^{(0)} &= u_H, \\
u_1^{(1)} &= \frac{lF}{E_1}\xi_1, \\
u_1^{(2)} &= \Omega^2 u_H\Big(-\frac{1}{2}\xi_1^2 + \xi_1\Big), \\
u_1^{(3)} &= \frac{lF}{E_1}\Omega^2\Big(-\frac{1}{6}\xi_1^3 + \frac{1}{2}\xi_1\Big), \\
u_1^{(4)} &= \Omega^4 u_H\Big(\frac{1}{24}\xi_1^4 - \frac{1}{6}\xi_1^3 + \frac{1}{3}\xi_1\Big).
\end{aligned}
\tag{2.15}
$$

Finally, using expansion (2.15) together with the following relation

$$
\sigma_1 = \frac{E_1}{H}\frac{du_1}{d\xi_1}
\tag{2.16}
$$

to obtain stress on the interface at $\xi_1 = 0$ in the form

$$
\sigma_1 = F + \frac{\Omega^2 E_1}{l}u_H\varepsilon + \frac{1}{2}F\Omega^2\varepsilon^2 + \frac{1}{3}\frac{E_1\Omega^4}{l}u_H\varepsilon^3 + \ldots..
\tag{2.17}
$$

Thus, the original problem may be reduced to consideration of the left component only with the effective force (2.17).

2.1.4 Testing of asymptotic results

In order to validate the asymptotic results obtained in the previous section, consider a problem for the right component over the domain $l - H \leqslant x \leqslant l$. We take the equation of motion (2.10) subject to boundary conditions (2.11). The solution of the formulated problem is then sought for in the form (2.4) for $j = 1$, and we finally arrive at a set of two linear algebraic equations which can be written in a matrix form as

$$\begin{pmatrix} -\sin(\Omega) & \cos(\Omega) \\ \cos((1-\varepsilon)\Omega) & \sin((1-\varepsilon)\Omega) \end{pmatrix} \begin{pmatrix} A^{(1)} \\ B^{(1)} \end{pmatrix} = \begin{pmatrix} \dfrac{Fl}{E_1\Omega} \\ u_H \end{pmatrix}. \tag{2.18}$$

The sought for constants $A^{(1)}$ and $B^{(1)}$ are presented in Appendix A.2.

Next, we rewrite the solution (2.4) for u_1 in terms of dimensionless variables and expand it into Taylor series about $\varepsilon = 0$ arriving at the asymptotic expansion

$$\begin{aligned} u_1 \quad &= u_H + \frac{Fl\xi_1}{E_1}\varepsilon - \frac{1}{2}\left(\left(\xi_1 - 2\right)\xi_1\Omega^2 u_H\right)\varepsilon^2 - \frac{\left(Fl\xi_1\left(\xi_1^2 - 3\right)\Omega^2\right)}{6E_1}\varepsilon^3 \\ &+ \frac{1}{24}\xi_1\left(\left(\xi_1 - 4\right)\xi_1^2 + 8\right)\Omega^4 u_H\varepsilon^4 + O\left(\varepsilon^5\right). \end{aligned} \tag{2.19}$$

Clearly, the remainder also includes higher powers of Ω. It can be easily confirmed that formula (2.19) coincides with asymptotic solution (2.12) which is an additional verification of the asymptotic solution.

Let us now test the stress on the interface at $\xi_1 = 0$. Substituting (2.19) into (2.16), we obtain (2.17) as expected.

Now, we would like to find the asymptotic solution for the left component. We can rewrite the equation of motion (2.1) in the following form

$$\frac{d^2u_2}{d\xi_2^2} + (1-\varepsilon)^2c^2\Omega^2u_2 = 0, \tag{2.20}$$

subject to

$$\begin{aligned} &u_2 = 0, \;\; at \;\; \xi_2 = 0, \\ &\frac{du_2}{d\xi_2} = \frac{Fl}{E_2}(1-\varepsilon) + \frac{Fl\Omega^2}{2E_2}\varepsilon^2(1-\varepsilon) + \frac{\Omega^2E_1}{E_2}\varepsilon(1-\varepsilon)(1+\frac{1}{3}\Omega^2\varepsilon^2)u_2 \;\; at \;\; \xi_2 = 1. \end{aligned} \tag{2.21}$$

We also rewrite the general solution (2.4) in dimensionless variables for $j = 2$ as

$$u_2 = A^{(2)}\cos(\Omega c(1-\varepsilon)\xi_2) + B^{(2)}\sin(\Omega c(1-\varepsilon)\xi_2). \tag{2.22}$$

Substituting (2.22) into the boundary conditions (2.21) leads to $A^{(2)} = 0$ and $B^{(2)} = \frac{1}{m_1}\Big(\frac{Fl}{E_2}(1-\varepsilon)(1+\frac{\Omega^2\varepsilon^2}{2})\Big)$,

where

$m_1 = \Omega c(1-\varepsilon)\cos((1-\varepsilon)c\Omega) - \frac{\Omega^2E_1}{E_2}\varepsilon(1-\varepsilon)(1+\frac{1}{3}\Omega^2\varepsilon^2)\sin((1-\varepsilon)c\Omega)$.

Then we get the asymptotic solution for the left component as

$$u_2 = -\frac{3Fl\,(\Omega^2\varepsilon^2+2)\sin(c\xi_2\Omega(1-\varepsilon))}{2\Omega\,(E_1\Omega\varepsilon\,(\Omega^2\varepsilon^2+3)\sin(c\Omega(1-\varepsilon)) - 3cE_2\cos(c\Omega(\varepsilon-1)))}, \tag{2.23}$$

where we assume that Ω is not a resonant frequency, so that the denominator is non-zero. Now, we introduce new dimensionless variable

$$\tilde{u}_2 = \frac{u_2}{l}\frac{E_2}{F}.$$

Thus, we can rewrite the scaled displacement $\tilde{u}_2$, following from the exact solution (2.9) and the asymptotic solution (2.23) for the left component as

$$\tilde{u}_2 = \frac{\sin\left(c\xi_2\Omega(1-\varepsilon)\right)}{c\Omega\cos(\Omega\varepsilon)\cos(c\Omega(\varepsilon-1)) - E\Omega\sin(\Omega\varepsilon)\sin(c\Omega(1-\varepsilon))}, \tag{2.24}$$

$$\tilde{u}_2 = -\frac{3\left(\Omega^2\varepsilon^2+2\right)\sin\left(c\xi_2\Omega(1-\varepsilon)\right)}{2\Omega\left(E\Omega\varepsilon\left(\Omega^2\varepsilon^2+3\right)\sin(c\Omega(1-\varepsilon)) - 3c\cos(c\Omega(\varepsilon-1))\right)} + O\left(\varepsilon\right), \tag{2.25}$$

where $E = \dfrac{E_1}{E_2}$. Figures 2.2-2.7 demonstrate the exact solution of the left component $\tilde{u}_2$ (2.24), compared with the asymptotic solution (2.25) for the same component for the values $E = 1$, $c = 1$ and several values of Ω and ε. Clearly, with large values of the frequency Ω the oscillations are becoming more dense. Note that the substantial difference between approximate and exact solutions in Figure 2.7 is possibly due to the higher order powers of Ω in the reminder of (2.19). Also, for reasonably small values of ε, e.g. Figures 2.5, 2.6 the asymptotic solutions is providing a good approximation. In order to illustrate it further, we present the following Figures 2.8-2.12 showing the maximum error over $0 \le \xi_2 \le 1$ between the exact solution (2.24) and the asymptotic solution (2.25) with respect to Ω and ε. All Figures 2.8-2.12 demonstrate that the maximum error is monotonically increasing for increasing frequency.

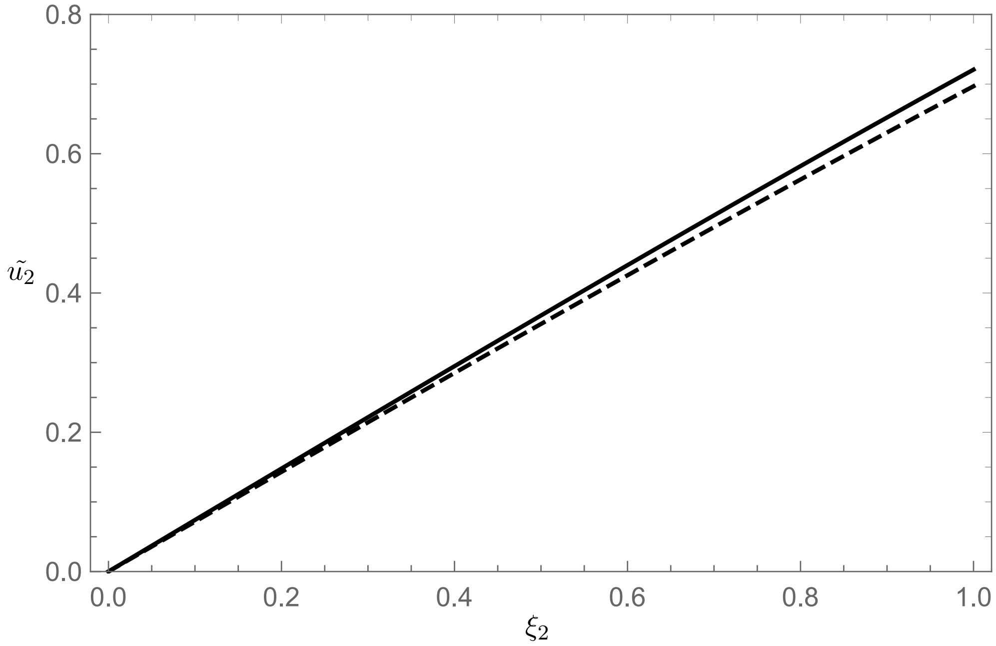

FIGURE 2.2: Comparison of asymptotic solution (2.25) (dashed line) and exact solution (2.24) (solid line) for $\Omega = 1$, $\varepsilon = 0.6$.

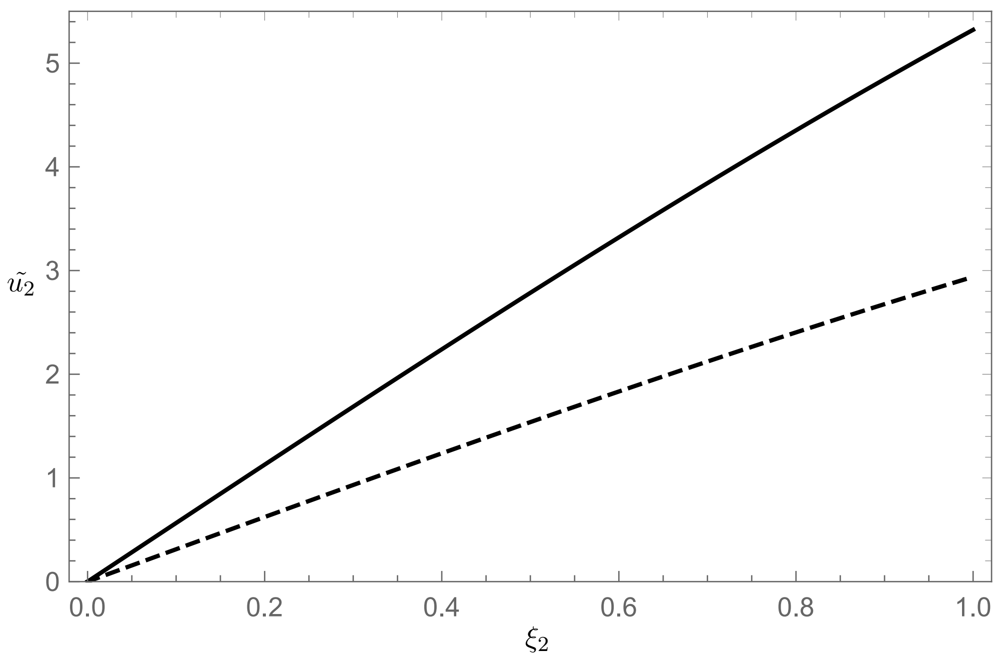

FIGURE 2.3: Comparison of asymptotic solution (2.25) (dashed line) and exact solution (2.24) (solid line) for $\Omega = 1.5$, $\varepsilon = 0.6$.

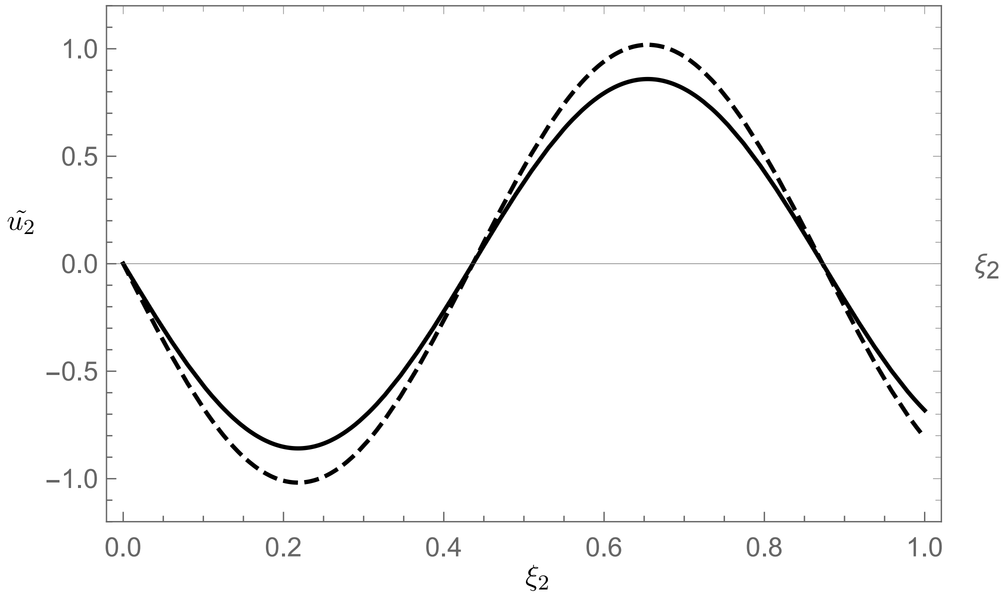

FIGURE 2.4: Comparison of asymptotic solution (2.25) (dashed line) and exact solution (2.24) (solid line) for $\Omega = 8$, $\varepsilon = 0.1$.

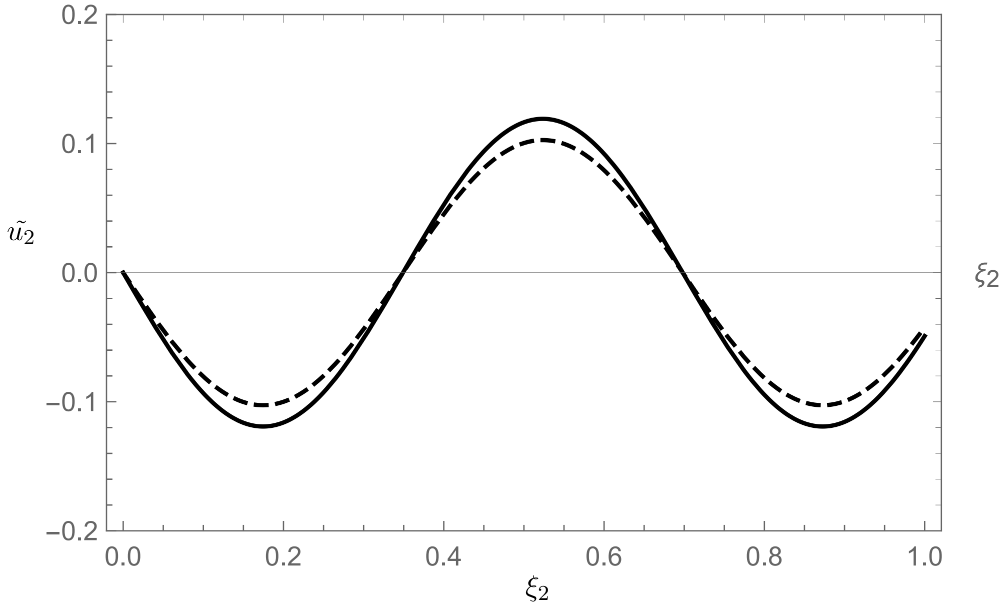

FIGURE 2.5: Comparison of asymptotic solution (2.25) (dashed line) and exact solution (2.24) (solid line) for $\Omega = 10$, $\varepsilon = 0.1$.

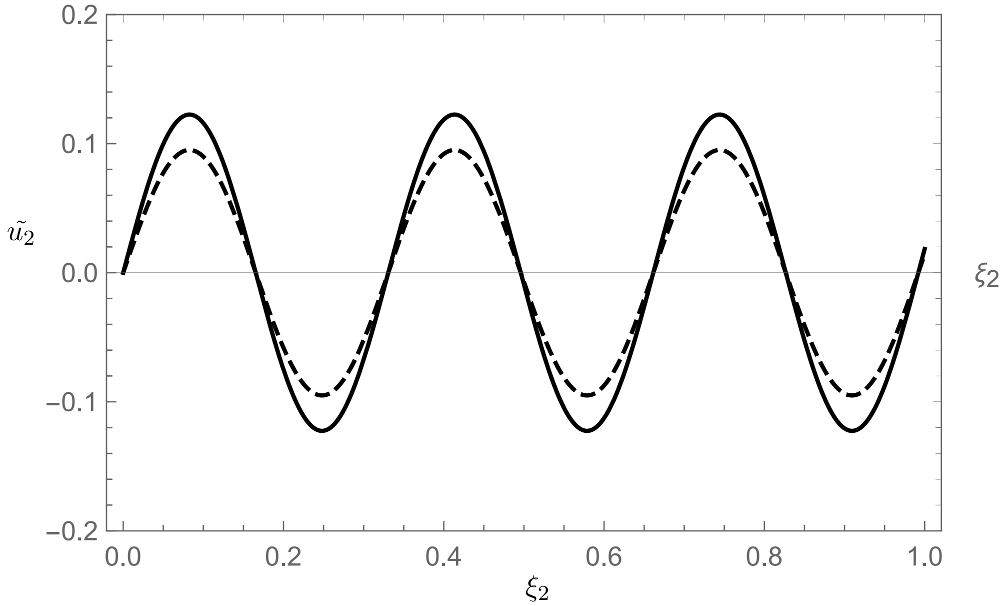

FIGURE 2.6: Comparison of asymptotic solution (2.25) (dashed line) and exact solution (2.24) (solid line) for $\Omega = 20$, $\varepsilon = 0.05$.

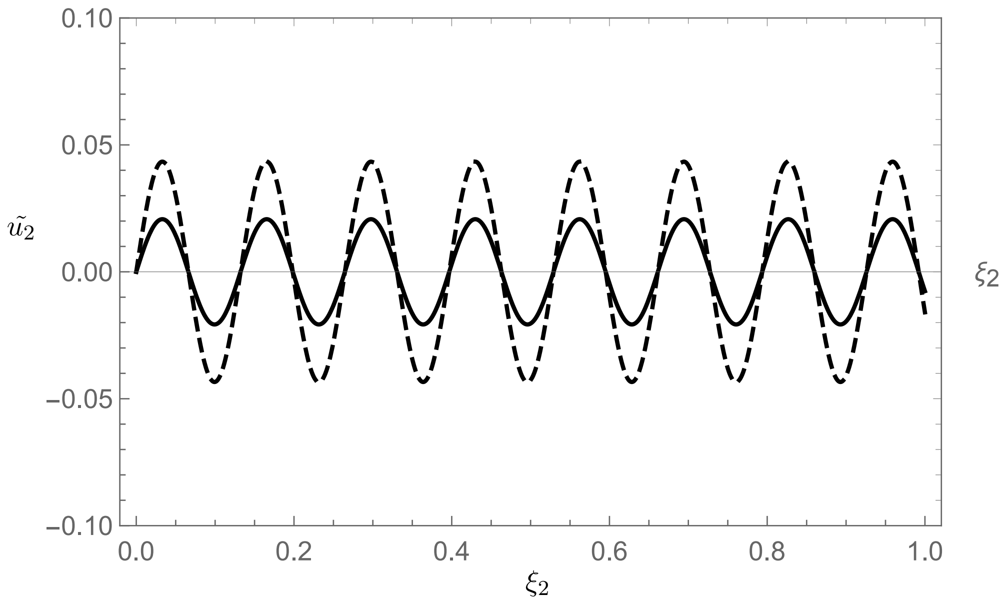

FIGURE 2.7: Comparison of asymptotic solution (2.25) (dashed line) and exact solution (2.24) (solid line) for $\Omega = 50$, $\varepsilon = 0.05$.

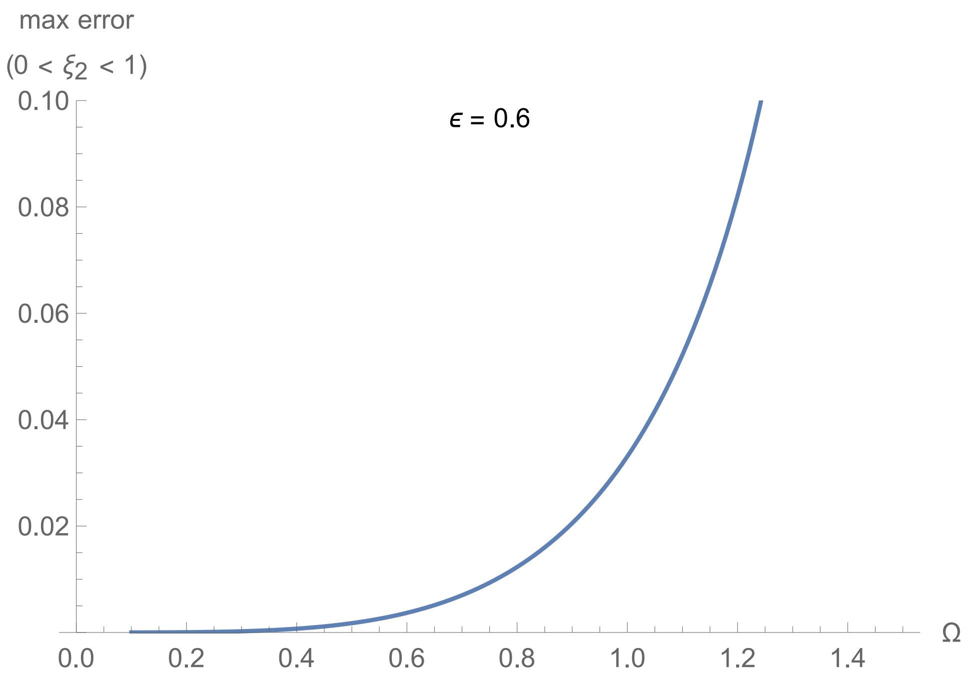

FIGURE 2.8: The maximum error between asymptotic solution (2.25) and exact solution (2.24) for $\varepsilon = 0.6$.

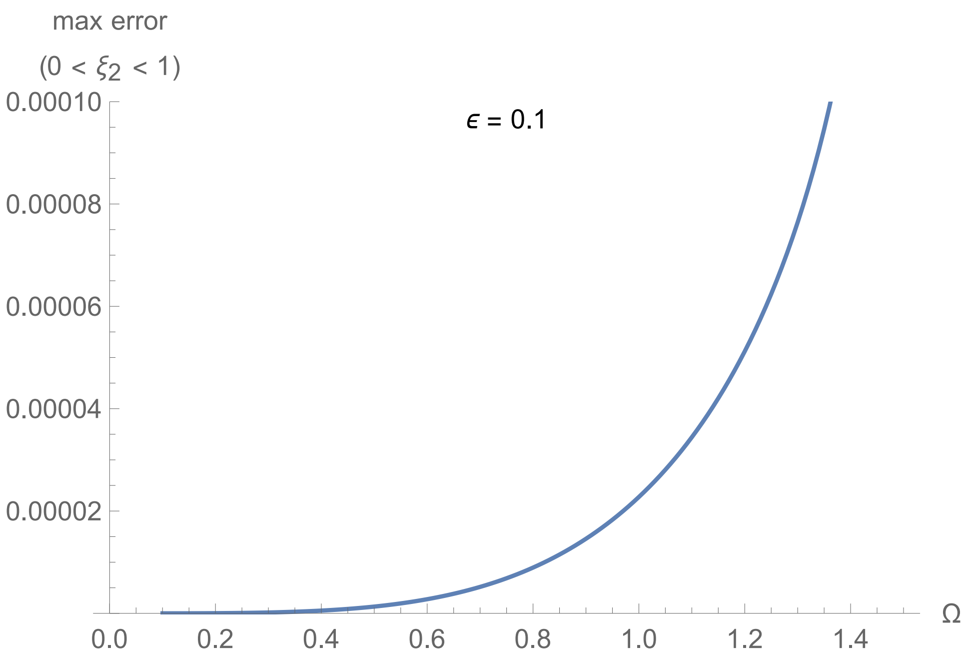

FIGURE 2.9: The maximum error between asymptotic solution (2.25) and exact solution (2.24) for $\varepsilon = 0.1$.

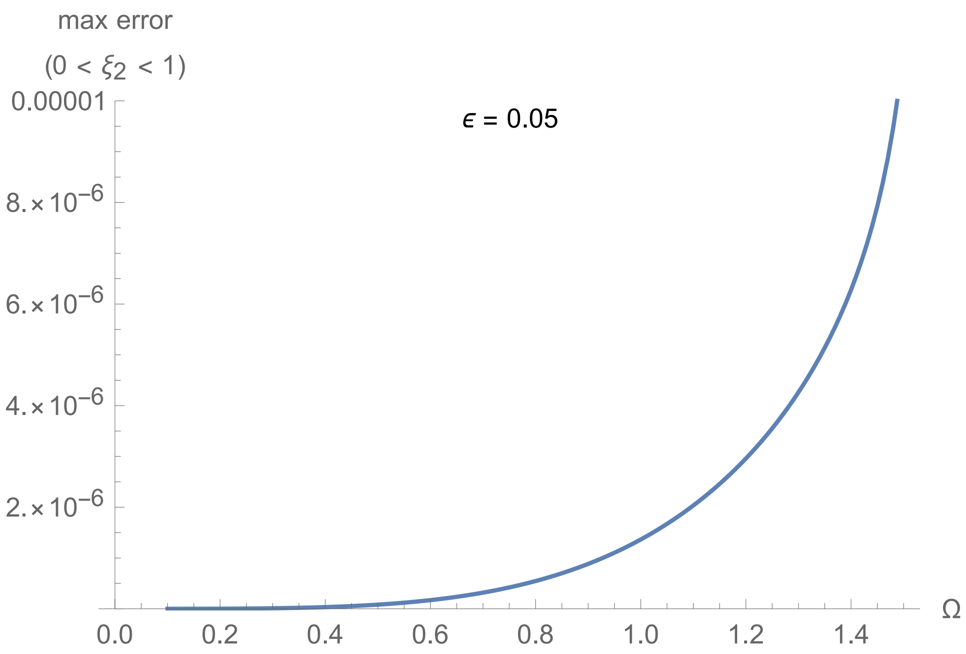

FIGURE 2.10: The maximum error between asymptotic solution (2.25) and exact solution (2.24) for $\varepsilon = 0.05$.

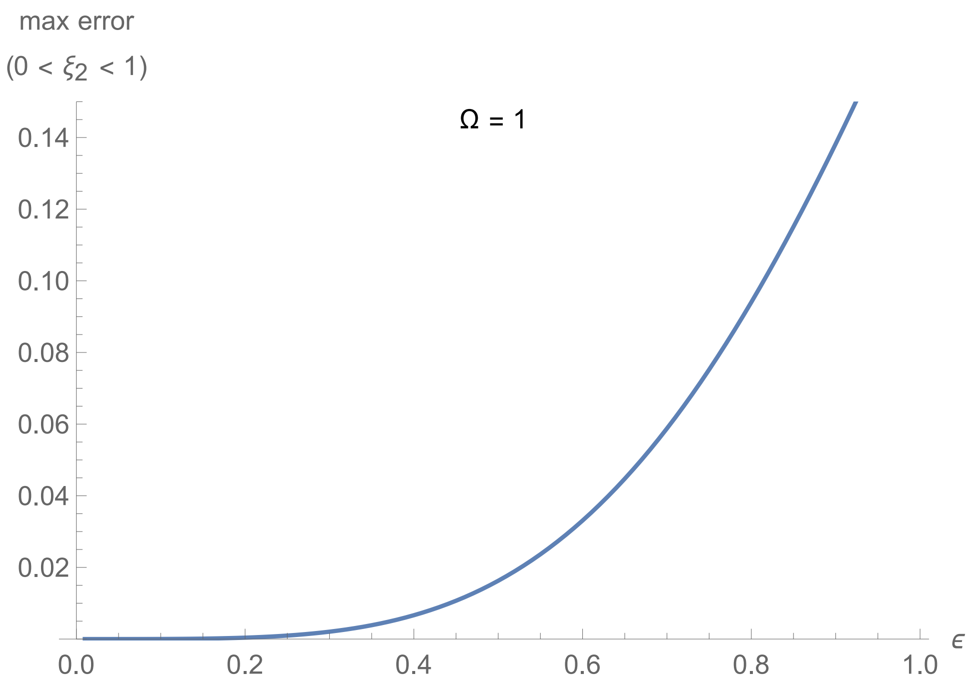

FIGURE 2.11: The maximum error between asymptotic solution (2.25) and exact solution (2.24) for $\Omega = 1$.

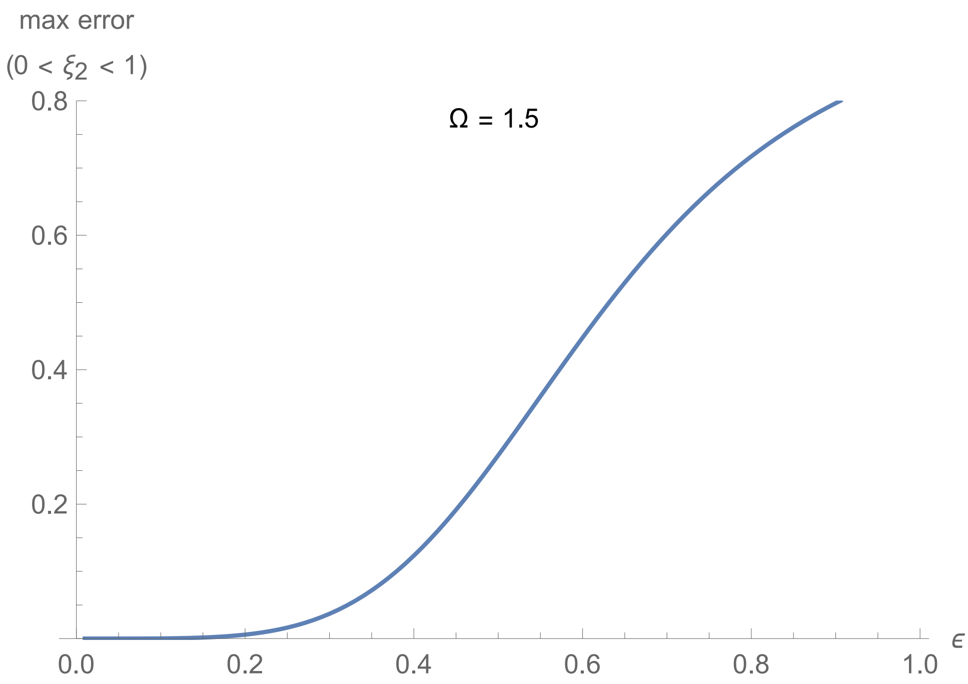

FIGURE 2.12: The maximum error between asymptotic solution (2.25) and exact solution (2.24) for $\Omega = 1.5$.

2.2 Harmonic vibrations of a composite beam

2.2.1 Formulation of the problem

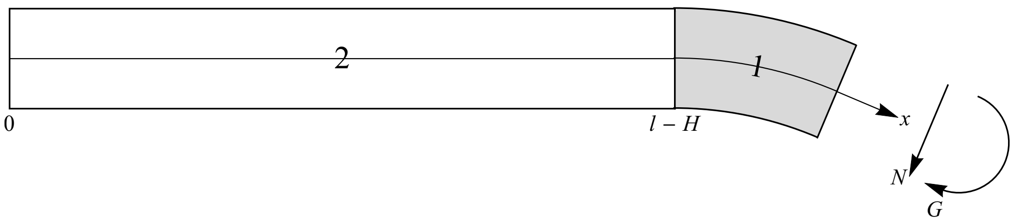

FIGURE 2.13: A composite beam

In this section we extend our work from a rod to a beam. We consider a linear elastic two-component beam of finite length with the components labelled of 1 and 2. These two-components are characterised with the same geometric small parameter $\varepsilon \ll 1$ as for a rod. Let the x axis be taken to lie along the beam with the length of the components $(l - H)$ and H and a moment G applied, along with the modified shear force N at the right end. (see Figure 2.13). In view of the linearity of the problem, below we consider two cases separately. In the first case the excitation is purely moment-type, i.e. $N = 0$, $G \neq 0$. The second case is associated with excitation due to the modified shear force only, thus $G = 0$, $N \neq 0$.

Hereinafter the index 1 will be used to denote the quantities corresponding to the right component, whereas the index 2 will denote the same for the left component. The equation of motion can be written in the form

$$D_j \frac{\partial^4 w_j}{\partial x^4} - 2\rho_j h \omega^2 w_j = 0, \qquad j = 1, 2 \tag{2.26}$$

where w_j are the transverse displacements, $D_j = \dfrac{2E_j h^3}{3(1-\nu_j^2)}$ are bending stiffness, E_j are the Young's moduli, ρ_j are the material densities for relevant components of the beam and ω is frequency.

The boundary conditions at the clamped left end of a beam are taken in the form

$$w_2 = 0, \quad \frac{\partial w_2}{\partial x} = 0, \quad at \;\; x = 0. \tag{2.27}$$

At the right end of a beam two cases of the boundary conditions are imposed . We consider two cases. In the first one we assume no transverse shear force at the end $(N = 0)$,

$$D_1 \frac{\partial^2 w_1}{\partial x^2} = G, \quad \frac{\partial^3 w_1}{\partial x^3} = 0, \quad at \;\; x = l, \tag{2.28}$$

and in the second case we assume no moment at the end $(G = 0)$,

$$\frac{\partial^2 w_1}{\partial x^2} = 0, \quad D_1 \frac{\partial^3 w_1}{\partial x^3} = N, \quad at \;\; x = l. \tag{2.29}$$

Traction and displacement continuity relations at the interface between the components are given by

$$\begin{aligned} & w_1 = w_2, \quad \frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x}, \quad D_1 \frac{\partial^2 w_1}{\partial x^2} = D_2 \frac{\partial^2 w_2}{\partial x^2}, \\ & D_1 \frac{\partial^3 w_1}{\partial x^3} = D_2 \frac{\partial^3 w_2}{\partial x^3} \quad at \;\; x = (l - H). \end{aligned} \tag{2.30}$$

The general solution of linear ordinary differential equations (2.26) is given by

$$w_j = \alpha_1^{(j)} \sinh(\beta_j x) + \alpha_2^{(j)} \cosh(\beta_j x) + \alpha_3^{(j)} \cos(\beta_j x) + \alpha_4^{(j)} \sin(\beta_j x), \quad j = 1, 2 \tag{2.31}$$

where $\alpha_i^{(j)}$, $i = 1 - 4$ are arbitrary constants and $\beta_j = \left(\dfrac{2\rho_j h\omega^2}{D_j}\right)^{\frac{1}{4}}$.

2.2.2 First case (excitation by bending moment)

Substituting general solution (2.31) into the boundary conditions (2.27), (2.28) and continuity relations (2.30) leads to the relations $\alpha_3^{(2)} = -\alpha_2^{(2)}$, $\alpha_4^{(2)} = -\alpha_1^{(2)}$, which simplifies the original 8×8 system to a set of six linear algebraic equations which can be written in a matrix form as

$$\mathrm{Q^b} \cdot \alpha = \mathrm{U^b}, \tag{2.32}$$

where $\alpha = \left(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_4^{(1)}, \alpha_1^{(2)}, \alpha_2^{(2)}\right)^T$, $\mathrm{U^b} = \left(0, \dfrac{G}{D_1\beta_1^2}, 0, 0, 0, 0\right)^T$ are vectors and $\mathrm{Q^b}$ is a 6×6 matrix with the non-zero components given by

$$Q_{11}^b = \cosh(\beta_1 l), \;\; Q_{12}^b = \sinh(\beta_1 l), \;\; Q_{13}^b = \sin(\beta_1 l), \;\; Q_{14}^b = -\cos(\beta_1 l),$$

$$Q_{21}^b = \sinh(\beta_1 l), \;\; Q_{22}^b = \cosh(\beta_1 l), \;\; Q_{23}^b = -\cos(\beta_1 l), \;\; Q_{24}^b = -\sin(\beta_1 l),$$

$$Q_{31}^b = \sinh(\beta_1(l-H)), \;\; Q_{32}^b = \cosh(\beta_1(l-H)),$$

$$Q_{33}^b = \cos(\beta_1(l-H)), \;\; Q_{34}^b = \sin(\beta_1(l-H)),$$

$$Q_{35}^b = \sin(\beta_2(l-H)) - \sinh(\beta_2(l-H)),$$

$$Q_{36}^b = \cos(\beta_2(l-H)) - \cosh(\beta_2(l-H)),$$

$$Q_{41}^b = \beta_1 \cosh(\beta_1(l-H)),\ \ Q_{42}^b = \beta_1 \sinh(\beta_1(l-H)),$$

$$Q_{43}^b = -\beta_1 \sin(\beta_1(l-H)),\ \ Q_{44}^b = \beta_1 \cos(\beta_1(l-H)),$$

$$Q_{45}^b = \beta_2(\cos(\beta_2(l-H)) - \cosh(\beta_2(l-H))),$$

$$Q_{46}^b = -\beta_2(\sin(\beta_2(l-H)) + \sinh(\beta_2(l-H))),$$

$$Q_{51}^b = D_1\beta_1^2 \sinh(\beta_1(l-H)),\ \ Q_{52}^b = D_1\beta_1^2 \cosh(\beta_1(l-H)),$$

$$Q_{53}^b = -D_1\beta_1^2 \cos(\beta_1(l-H)),\ \ Q_{54}^b = -D_1\beta_1^2 \sin(\beta_1(l-H)),$$

$$Q_{55}^b = -D_2\beta_2^2(\sin(\beta_2(l-H)) + \sinh(\beta_2(l-H)),$$

$$Q_{56}^b = -D_2\beta_2^2(\cos(\beta_2(l-H)) + \cosh(\beta_2(l-H))),$$

$$Q_{61}^b = D_1\beta_1^3 \cosh(\beta_1(l-H)),\ \ Q_{62}^b = D_1\beta_1^3 \sinh(\beta_1(l-H)),$$

$$Q_{63}^b = D_1\beta_1^3 \sin(\beta_1(l-H)),\ \ Q_{64}^b = -D_1\beta_1^3 \cos(\beta_1(l-H)),$$

$$Q_{65}^b = -D_2\beta_2^3(\cos(\beta_2(l-H)) + \cosh(\beta_2(l-H))),$$

$$Q_{66}^b = D_2\beta_2^3(\sin(\beta_2(l-H)) - \sinh(\beta_2(l-H))). \tag{2.33}$$

The sought for constants $\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \alpha_4^{(1)}, \alpha_1^{(2)}, \alpha_2^{(2)}$ are presented in Appendix B.1. Using Cramer's rule to solve (2.32), we get the exact solution as

$$\begin{aligned}
w_1 \quad &= \Bigg(- \left(\frac{1}{2} - \frac{i}{2}\right) G\Bigg(- \Bigg((1+i)\cos\left((l-x)\beta_1\right) + i\cos\left(((1+i)H - il + ix)\beta_1\right) \\
&- \cos\left(((1+i)H - l + x)\beta_1\right) - (1+i)\cosh\left((l-x)\beta_1\right) - i\cosh\Bigg(((1+i)H - il \\
&+ix)\beta_1\Bigg) + \cosh\left(((1+i)H - l + x)\beta_1\right)\Bigg)\Bigg(\cos\left((l-H)\beta_2\right)\cosh\left((l-H)\beta_2\right) \\
&-1\Bigg) D_1^2\beta_1^4 + 2D_1D_2\beta_2\Bigg(i\Bigg(\cosh\left((H-l+x)\beta_1\right)\sin\left(H\beta_1\right) - \cos\left((H-l+x)\beta_1\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \sinh(H\beta_1)\Big)(\sin((1+i)(l-H)\beta_2) - \sinh((1+i)(l-H)\beta_2))\,\beta_1^2 \\
&-\Big(\cos(((1+i)H - il + ix)\beta_1) + i\cos(((1+i)H - l + x)\beta_1) \\
&+\cosh\Big(((1+i)H - il + ix)\beta_1\Big) + i\cosh(((1+i)H - l + x)\beta_1)\Big)\sin((l-H)\beta_2) \\
&\times \sinh((l-H)\beta_2)\,\beta_2\beta_1 - \Big(\cosh(H\beta_1)\sin((H-l+x)\beta_1) + \cos(H\beta_1) \\
&\times \sinh((H-l+x)\beta_1)\Big)\Big(\sin((1+i)(l-H)\beta_2) + \sinh((1+i)(l-H)\beta_2)\Big)\beta_2^2\Big)\beta_1 \\
&+\Big((1+i)\cos((l-x)\beta_1) - i\cos(((1+i)H - il + ix)\beta_1) \\
&+\cos(((1+i)H - l + x)\beta_1) - (1+i)\cosh((l-x)\beta_1) \\
&+i\cosh(((1+i)H - il + ix)\beta_1) - \cosh(((1+i)H - l + x)\beta_1)\Big)\Big(\cos((l-H)\beta_2) \\
&\times \cosh((l-H)\beta_2) + 1\Big)D_2^2\beta_2^4\Big)\Big)\Big(D_1\beta_1^2\Big(2(\cos(H\beta_1)\cosh(H\beta_1) - 1) \\
&\times (\cos((H-l)\beta_2)\cosh((H-l)\beta_2) - 1)\,D_1^2\beta_1^4 \\
&+D_1D_2\beta_2\Big((\sin((1+i)H\beta_1) + \sinh((1+i)H\beta_1)) \\
&(\sin((1+i)(H-l)\beta_2) - \sinh((1+i)(H-l)\beta_2))\,\beta_1^2 \\
&-4\sin(H\beta_1)\sin((H-l)\beta_2)\sinh(H\beta_1)\sinh((H-l)\beta_2)\,\beta_2\beta_1 \\
&+(\sin((1+i)H\beta_1) - \sinh((1+i)H\beta_1)) \\
&(\sin((1+i)(H-l)\beta_2) + \sinh((1+i)(H-l)\beta_2))\,\beta_2^2\Big)\beta_1 + 2\Big(\cos(H\beta_1) \\
&\times \cosh(H\beta_1) + 1\Big)(\cos((H-l)\beta_2)\cosh((H-l)\beta_2) + 1)\,D_2^2\beta_2^4\Big)\Big)^{-1},
\end{aligned}
\tag{2.34}
$$

and

$$
\begin{aligned}
w_2 \quad &= \Big(G\Big(2\left(\sinh\left(\beta_2 x\right)-\sin\left(\beta_2 x\right)\right)\Big(\beta_1^2 D_1\Big(\beta_1\left(\sin\left(\beta_1 H\right)+\sinh\left(\beta_1 H\right)\right)\\
&\times\Big(\cos\left(\beta_2(H-l)\right)-\cosh\left(\beta_2(H-l)\right)\Big)-\beta_2\left(\cos\left(\beta_1 H\right)-\cosh\left(\beta_1 H\right)\right)\\
&\times\left(\sin\left(\beta_2(H-l)\right)+\sinh\left(\beta_2(H-l)\right)\right)\Big)+\beta_2^2 D_2\Big(\beta_1\left(\sin\left(\beta_1 H\right)-\sinh\left(\beta_1 H\right)\right)\\
&\times\left(\cos\left(\beta_2(H-l)\right)+\cosh\left(\beta_2(H-l)\right)\right)+\beta_2\left(\cos\left(\beta_1 H\right)+\cosh\left(\beta_1 H\right)\right)\\
&\times\left(\sinh\left(\beta_2(H-l)\right)-\sin\left(\beta_2(H-l)\right)\right)\Big)\Big)-\left(\cosh\left(\beta_2 x\right)-\cos\left(\beta_2 x\right)\right)\\
&\times\Big(2\beta_1^2 D_1\Big(\beta_1\Big(\sin\left(\beta_1 H\right)+\sinh\left(\beta_1 H\right)\Big)\left(\sinh\left(\beta_2(H-l)\right)-\sin\left(\beta_2(H-l)\right)\right)\\
&-\beta_2\left(\cos\left(\beta_1 H\right)-\cosh\left(\beta_1 H\right)\right)\left(\cos\left(\beta_2(H-l)\right)-\cosh\left(\beta_2(H-l)\right)\right)\Big)\\
&-2\beta_2^2 D_2\Big(\beta_1\left(\sin\left(\beta_1 H\right)-\sinh\left(\beta_1 H\right)\right)\left(\sin\left(\beta_2(H-l)\right)+\sinh\left(\beta_2(H-l)\right)\right)\\
&+\beta_2\left(\cos\left(\beta_1 H\right)+\cosh\left(\beta_1 H\right)\right)\left(\cos\left(\beta_2(H-l)\right)+\cosh\left(\beta_2(H-l)\right)\right)\Big)\Big)\Big)\Big)\\
&\times\Big(4\beta_2\Big(\frac{1}{2}\beta_2\beta_1 D_1 D_2\Big(\beta_1^2\left(\sin\left((1+i)\beta_1 H\right)+\sinh\left((1+i)\beta_1 H\right)\right)\\
&\times\Big(\sin\left((1+i)\beta_2(H-l)\right)-\sinh\left((1+i)\beta_2(H-l)\right)\Big)\\
&-4\beta_2\beta_1\sin\left(\beta_1 H\right)\sinh\left(\beta_1 H\right)\sin\left(\beta_2(H-l)\right)\sinh\left(\beta_2(H-l)\right)\\
&+\beta_2^2\left(\sin\left((1+i)\beta_1 H\right)-\sinh\left((1+i)\beta_1 H\right)\right)\Big(\sin\left((1+i)\beta_2(H-l)\right)\\
&+\sinh\left((1+i)\beta_2(H-l)\right)\Big)\Big)+\beta_1^4 D_1^2\left(\cos\left(\beta_1 H\right)\cosh\left(\beta_1 H\right)-1\right)\\
&\times\left(\cos\left(\beta_2(H-l)\right)\cosh\left(\beta_2(H-l)\right)-1\right)+\beta_2^4 D_2^2\left(\cos\left(\beta_1 H\right)\cosh\left(\beta_1 H\right)+1\right)\\
&\times\left(\cos\left(\beta_2(H-l)\right)\cosh\left(\beta_2(H-l)\right)+1\right)\Big)\Big)^{-1}.
\end{aligned}
\tag{2.35}
$$

2.2.2.1 Dimensionless equations

We convert all variables into dimensionless form in order to investigate the exact solution asymptotically. We introduce the following dimensionless variables and problem parameters

$$\Omega^4 = \frac{2\rho_1 h\omega^2 l^4}{D_1}, \quad \xi_1 = (\frac{x}{l} - 1)\frac{1}{\varepsilon} + 1, \quad \xi_2 = \frac{x}{l-H} \text{ and } \varepsilon = \frac{H}{l} \ll 1. \tag{2.36}$$

Now we can rewrite the exact solutions (2.34) and (2.35) in the dimensionless form as

$$\begin{aligned}
w_1 \quad &= \Bigg(- \left(\frac{1}{2} + \frac{i}{2}\right) Gl^2 \Big(\upsilon^4 (\cos(\upsilon(\epsilon - 1)\rho\Omega) \cosh(\upsilon(\epsilon - 1)\rho\Omega) + 1) \\
&\times \Big(\cos\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) + \cosh(\epsilon\Omega) \left(\cos\left(\epsilon\Omega\xi_1\right) - \cosh\left(\epsilon\Omega\xi_1\right)\right) - \cos(\epsilon\Omega) \cosh\left(\epsilon\Omega\xi_1\right) \\
&- \sin\left(\epsilon\Omega\xi_1\right) \sinh(\epsilon\Omega) + \left(\sinh(\epsilon\Omega) - \sin(\epsilon\Omega)\right) \sinh\left(\epsilon\Omega\xi_1\right) \Big) D_2^2 \rho^4 + 2\upsilon \Big(\sinh(\epsilon\Omega) \\
&\times \Big(\cosh(\upsilon(\epsilon - 1)\rho\Omega) \sin(\upsilon(\epsilon - 1)\rho\Omega) \Big(\upsilon^2 \cos(\epsilon\Omega) \cosh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) \rho^2 \\
&+ \cos\left(\epsilon\Omega\xi_1\right) - \sin(\epsilon\Omega) \sinh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) \Big) \Big) + \sinh(\upsilon(\epsilon - 1)\rho\Omega) \\
&\times \Big(- \cos(\upsilon(\epsilon - 1)\rho\Omega) \cos\left(\epsilon\Omega\xi_1\right) + \upsilon\rho \Big(\cosh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) (\upsilon\rho \cos(\epsilon\Omega) \\
&\times \cos(\upsilon(\epsilon - 1)\rho\Omega) + \sin(\epsilon\Omega) \sin(\upsilon(\epsilon - 1)\rho\Omega) - \sin(\upsilon(\epsilon - 1)\rho\Omega) \sin\left(\epsilon\Omega\xi_1\right) \Big) \\
&+ (\cos(\upsilon(\epsilon - 1)\rho\Omega) \sin(\epsilon\Omega) - \upsilon\rho \cos(\epsilon\Omega) \sin(\upsilon(\epsilon - 1)\rho\Omega)) \sinh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) \Big) \Big) \\
&+ \cosh(\epsilon\Omega) \Big(\cosh(\upsilon(\epsilon - 1)\rho\Omega) \sin(\upsilon(\epsilon - 1)\rho\Omega) \Big(\upsilon^2 \rho^2 \Big(\sin\left(\epsilon\Omega\xi_1\right) \\
&+ \cos(\epsilon\Omega) \sinh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) \Big) - \cosh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) \sin(\epsilon\Omega) \Big) \\
&+ \sinh(\upsilon(\epsilon - 1)\rho\Omega) \Big(\cosh\left(\epsilon\Omega\left(\xi_1 - 1\right)\right) (\cos(\upsilon(\epsilon - 1)\rho\Omega) \sin(\epsilon\Omega) \\
&- \upsilon\rho \cos(\epsilon\Omega) \sin(\upsilon(\epsilon - 1)\rho\Omega)) + \upsilon\rho \Big(- \cos\left(\epsilon\Omega\xi_1\right) \sin(\upsilon(\epsilon - 1)\rho\Omega)
\end{aligned}$$

$$
\begin{aligned}
&+\upsilon\rho\cos(\upsilon(\epsilon-1)\rho\Omega)\sin\left(\epsilon\Omega\xi_1\right)+\left(\upsilon\rho\cos(\epsilon\Omega)\cos(\upsilon(\epsilon-1)\rho\Omega)\right.\\
&\left.\left.\left.\left.+\sin(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega))\sinh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\right)\right)\right)\right)D_1D_2\rho-(\cos(\upsilon(\epsilon-1)\rho\Omega)\\
&\times\cosh(\upsilon(\epsilon-1)\rho\Omega)-1)\Big(\cos\left(\epsilon\Omega\left(\xi_1-1\right)\right)+\cos(\epsilon\Omega)\cosh\left(\epsilon\Omega\xi_1\right)\\
&-\cosh(\epsilon\Omega)\left(\cos\left(\epsilon\Omega\xi_1\right)+\cosh\left(\epsilon\Omega\xi_1\right)\right)+\sin\left(\epsilon\Omega\xi_1\right)\sinh(\epsilon\Omega)\\
&+(\sin(\epsilon\Omega)+\sinh(\epsilon\Omega))\sinh\left(\epsilon\Omega\xi_1\right)\Big)D_1^2\Big)\Big)\Big(\Omega^2\upsilon_1\Big((1+i)\upsilon^4\\
&\times(\cos(\epsilon\Omega)\cosh(\epsilon\Omega)+1)(\cos(\upsilon(\epsilon-1)\rho\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)+1)D_2^2\rho^4\\
&+\upsilon\Big(\cosh(\epsilon\Omega)\sin(\epsilon\Omega)\Big(\left(\upsilon^2\rho^2+i\right)\sin((1+i)\upsilon(\epsilon-1)\rho\Omega)\\
&+\left(\upsilon^2\rho^2-i\right)\sinh((1+i)\upsilon(\epsilon-1)\rho\Omega)\Big)\\
&-(1+i)\sinh(\epsilon\Omega)\Big(\left(\upsilon^2\rho^2-1\right)\cos(\epsilon\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\\
&+\Big(\left(\upsilon^2\rho^2+1\right)\cos(\epsilon\Omega)\cos(\upsilon(\epsilon-1)\rho\Omega)\\
&+2\upsilon\rho\sin(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\Big)\sinh(\upsilon(\epsilon-1)\rho\Omega)\Big)\Big)D_1D_2\rho+(1+i)(\cos(\epsilon\Omega)\\
&\times\cosh(\epsilon\Omega)-1)(\cos(\upsilon(\epsilon-1)\rho\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)-1)D_1^2\Big)\Big)^{-1},
\end{aligned}
\tag{2.37}
$$

and

$$
\begin{aligned}
w_2 \quad &= \Big(Gl^2\Big(\left(\sinh\left(\upsilon\xi_2\rho\Omega(1-\epsilon)\right)-\sin\left(\upsilon\xi_2\rho\Omega(1-\epsilon)\right)\right)\Big(\upsilon^3D_2\rho^3(\cos(\Omega\epsilon)\\
&+\cosh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon))-\sinh(\upsilon\rho\Omega(1-\epsilon)))+\upsilon^2D_2\rho^2(\sin(\Omega\epsilon)\\
&-\sinh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1))+\cosh(\upsilon\rho\Omega(\epsilon-1)))\\
&+\upsilon D_1\rho(\cos(\Omega\epsilon)-\cosh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon))+\sinh(\upsilon\rho\Omega(1-\epsilon)))\\
&+D_1(\sin(\Omega\epsilon)+\sinh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1))-\cosh(\upsilon\rho\Omega(\epsilon-1)))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\cosh\left(\upsilon\xi_2\rho\Omega(\epsilon-1)\right)-\cos\left(\upsilon\xi_2\rho\Omega(\epsilon-1)\right)\right)\Big(-\upsilon^3D_2\rho^3(\cos(\Omega\epsilon)\\
&+\cosh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1))+\cosh(\upsilon\rho\Omega(\epsilon-1)))+\upsilon^2D_2\rho^2(\sin(\Omega\epsilon)\\
&-\sinh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon))+\sinh(\upsilon\rho\Omega(1-\epsilon)))\\
&+D_1(\sin(\Omega\epsilon)+\sinh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon))-\sinh(\upsilon\rho\Omega(1-\epsilon)))\\
&-\upsilon D_1\rho(\cos(\Omega\epsilon)-\cosh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1))\\
&-\cosh(\upsilon\rho\Omega(\epsilon-1)))\Big)\Big)\Big(2\upsilon\rho\Omega^2\Big(\upsilon^4D_2^2\rho^4(\cos(\Omega\epsilon)\cosh(\Omega\epsilon)+1)(\cos(\upsilon\rho\Omega(\epsilon-1))\\
&\times\cosh(\upsilon\rho\Omega(\epsilon-1))+1)+\upsilon D_1D_2\rho\Big(-\sinh(\Omega\epsilon)\Big(\sinh(\upsilon\rho\Omega(\epsilon-1))\Big(\left(\upsilon^2\rho^2+1\right)\\
&\times\cos(\Omega\epsilon)\cos(\upsilon\rho\Omega(\epsilon-1))+2\upsilon\rho\sin(\Omega\epsilon)\sin(\upsilon\rho\Omega(\epsilon-1))\Big)\\
&+\left(\upsilon^2\rho^2-1\right)\cos(\Omega\epsilon)\sin(\upsilon\rho\Omega(\epsilon-1))\cosh(\upsilon\rho\Omega(\epsilon-1))\Big)\\
&+\left(\frac{1}{2}+\frac{i}{2}\right)\sin(\Omega\epsilon)\cosh(\Omega\epsilon)\Big(\left(1-i\upsilon^2\rho^2\right)\sin((1+i)\upsilon\rho\Omega(\epsilon-1))+\left(-1-i\upsilon^2\rho^2\right)\\
&\times\sinh((1+i)\upsilon\rho\Omega(\epsilon-1))\Big)\Big)+D_1^2(\cos(\Omega\epsilon)\cosh(\Omega\epsilon)-1)(\cos(\upsilon\rho\Omega(\epsilon-1))\\
&\times\cosh(\upsilon\rho\Omega(\epsilon-1))-1)\Big)\Big)^{-1},
\end{aligned}
\tag{2.38}
$$

where $\upsilon=\left(\frac{D_1}{D_2}\right)^{\frac{1}{4}}$ and $\rho=\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{4}}$.

Now, we test our result by setting $x=l-H$, implies that $\xi_1=0$ and $\xi_2=1$, into (2.37) and (2.38) for $\varepsilon\to0$, we obtain

$$
w_1=w_2=\frac{Gl^2\sin(\upsilon\rho\Omega)\sinh(\upsilon\rho\Omega)}{\upsilon^2D_2\rho^2\Omega^2(\cos(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega)+1)},
$$

which is an additional verification of the solutions.

2.2.2.2 Asymptotic analysis a composite beam

In this section, we apply an asymptotic approach to obtain an approximate solution for linear elastic two-component beam. We restrict our attention to perturbation scheme applied to w_1. To this end, we use dimensionless variables (2.36) to rewrite the equation of motion (2.26) to get

$$\frac{\partial^4 w_1}{\partial \xi_1^4} - \varepsilon^4 \Omega^4 w_1 = 0, \tag{2.39}$$

subject to

$$\begin{aligned} &w_1 = w_H, \qquad (1-\varepsilon)\frac{\partial w_1}{\partial \xi_1} = \varepsilon w_{H\xi_1}, \quad \text{at} \quad \xi_1 = 0, \\ &D_1 \frac{\partial^2 w_1}{\partial \xi_1^2} = \varepsilon^2 G l^2, \qquad \frac{\partial^3 w_1}{\partial \xi_1^3} = 0, \quad \text{at} \quad \xi_1 = 1, \end{aligned} \tag{2.40}$$

where functions

$$w_H = \frac{Gl^2 \sin(\upsilon\rho\Omega)\sinh(\upsilon\rho\Omega)}{\upsilon^2 D_2 \rho^2 \Omega^2 (\cos(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega)+1)}$$

and

$$w_{H\xi_1} = \frac{Gl^2 \cos(\upsilon\rho\Omega)\sinh(\upsilon\rho\Omega) + Gl^2 \sin(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega)}{\upsilon D_2 \rho\Omega + \upsilon D_2 \rho\Omega \cos(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega)}$$

are given on the interface.

Deflection w_1 can be expanded into an asymptotic series in terms of ε as

$$w_1 = w_1^{(0)} + w_1^{(1)}\varepsilon + w_1^{(2)}\varepsilon^2 + w_1^{(3)}\varepsilon^3 + w_1^{(4)}\varepsilon^4 + \dots \tag{2.41}$$

Substituting expansion (2.41) into the boundary value problem (2.39)-(2.40), we arrive at the problem formulated at the various asymptotic orders $n = 0, 1, 2, \ldots,$ namely

$$\frac{\partial^4 w_1^{(n)}}{\partial \xi_1^4} - \Omega^4 w_1^{(n-4)} = 0, \tag{2.42}$$

subject to

$$\begin{aligned}
&w_1^{(n)} = w_H^{(n)}, \qquad \xi_1 = 0,\\
&\frac{\partial w_1^{(n)}}{\partial \xi_1} - \frac{\partial w_1^{(n-1)}}{\partial \xi_1} = w_{H\xi_1}^{(n)} \quad \text{at} \quad \xi_1 = 0,\\
&D_1 \frac{\partial^2 w_1^{(n)}}{\partial \xi_1^2} = G^{(n)} l^{(n)} \quad \text{at} \quad \xi_1 = 1,\\
&\frac{\partial^3 w_1^{(n)}}{\partial \xi_1^3} = 0 \quad \text{at} \quad \xi_1 = 1,
\end{aligned} \tag{2.43}$$

where quantities with the negative superscript are set to be equal to zero. The only non-zero components $w_H^{(n)}$, $w_{H\xi_1}^{(n)}$ and $G^{(n)} l^{(n)}$ are $w_H^{(0)} = w_H$, $w_{H\xi_1}^{(1)} = w_{H\xi_1}$ and $G^{(2)} l^{(2)} = G l^2$, respectively.

Substituting subsequently $n = 0, 1, 2, 3$ and 4 into (2.42)-(2.43) we obtain

$$\begin{aligned}
&w_1^{(0)} = w_H,\\
&w_1^{(1)} = w_{H\xi_1} \xi_1,\\
&w_1^{(2)} = w_{H\xi_1} \xi_1 + \frac{1}{2} \frac{G l^2}{D_1} \xi_1^2,\\
&w_1^{(3)} = w_{H\xi_1} \xi_1,\\
&w_1^{(4)} = w_{H\xi_1} \xi_1 + \frac{1}{4} \Omega^4 w_H \xi_1^2 - \frac{1}{6} \Omega^4 w_H \xi_1^3 + \frac{1}{24} \Omega^4 w_H \xi_1^4.
\end{aligned} \tag{2.44}$$

Finally, using expansion (2.44) together with the following relations

$$M_1 = \frac{D_1}{H^2}\frac{\partial^2 w_1}{\partial \xi_1^2}, \tag{2.45}$$

$$N_1 = \frac{D_1}{H^3}\frac{\partial^3 w_1}{\partial \xi_1^3}, \tag{2.46}$$

to obtain moment and shear force on the interface at $\xi_1 = 0$ in the form

$$M_1 = G + \frac{D_1 \Omega^4}{2l^2} w_H \varepsilon^2 + \dots, \tag{2.47}$$

$$N_1 = -\frac{D_1 \Omega^4}{l^3} w_H \varepsilon + \dots \tag{2.48}$$

These formulae above present the expansion of the moment and shear force for ε on the interface.

2.2.2.3 Testing of asymptotic formulae

In order to validate the asymptotic results obtained in the previous section, consider the right component over the domain $l - H \leqslant x \leqslant l$. We take equation of motion (2.39) subject to boundary conditions (2.40). The solution of the formulated problem is then sought for in the form (2.31) for $j = 1$, and we finally arrive at a set of four linear algebraic equations which can be written in a matrix form as

$$\bar{\mathrm{Q}^{\mathrm{b}}} \cdot \bar{\alpha} = \bar{\mathrm{U}^{\mathrm{b}}}, \tag{2.49}$$

where $\bar{\alpha} = \left(\bar{\alpha}_1^{(1)}, \bar{\alpha}_2^{(1)}, \bar{\alpha}_3^{(1)}, \bar{\alpha}_4^{(1)}\right)^T$, $\bar{\mathrm{U}}^{\mathrm{b}} = (0, Gl^2, w_H, w_{H\xi_1})^T$ are vectors and $\bar{\mathrm{Q}}^{\mathrm{b}}$ is a 4×4 matrix with its non-zero components given by

$$
\begin{aligned}
&\bar{Q^b}_{11} = \cosh(\Omega), \;\; \bar{Q^b}_{12} = \sinh(\Omega), \\
&\bar{Q^b}_{13} = \sin(\Omega), \;\; \bar{Q^b}_{14} = -\cos(\Omega), \\
&\bar{Q^b}_{21} = \Omega^2 D_1 \sinh(\Omega), \;\; \bar{Q^b}_{22} = \Omega^2 D_1 \cosh(\Omega), \\
&\bar{Q^b}_{23} = -\Omega^2 D_1 \cos(\Omega), \;\; \bar{Q^b}_{24} = -\Omega^2 D_1 \sin(\Omega), \\
&\bar{Q^b}_{31} = \sinh((1-\varepsilon)\Omega), \;\; \bar{Q^b}_{32} = \cosh((1-\varepsilon)\Omega), \\
&\bar{Q^b}_{33} = \cos((1-\varepsilon)\Omega), \;\; \bar{Q^b}_{34} = \sin((1-\varepsilon)\Omega), \\
&\bar{Q^b}_{41} = (1-\varepsilon)\Omega \cosh((1-\varepsilon)\Omega), \;\; \bar{Q^b}_{42} = (1-\varepsilon)\Omega \sinh((1-\varepsilon)\Omega), \\
&\bar{Q^b}_{43} = -(1-\varepsilon)\Omega \sin((1-\varepsilon)\Omega), \;\; \bar{Q^b}_{44} = (1-\varepsilon)\Omega \cos((1-\varepsilon)\Omega). \qquad (2.50)
\end{aligned}
$$

The sought for constants $\bar{\alpha}_i^{(1)}$, $i = 1, 2, 3$ and 4 are presented in Appendix B.2. Next, we rewrite solution (2.31) for w_1 in terms of dimensionless variables and expand it into Taylor series about $\varepsilon = 0$ arriving at the asymptotic expansion

$$
\begin{aligned}
w_1 \quad &= w_H + \xi_1 w_{H\xi_1} \varepsilon + \left(\frac{Gl^2 \xi_1^2}{2D_1} + \xi_1 w_{H\xi_1} \right) \varepsilon^2 + w_{H\xi_1} \xi_1 \varepsilon^3 \\
&+ \left(\frac{1}{24} \xi_1^2 \left(\left(\xi_1 - 4 \right) \xi_1 + 6 \right) \Omega^4 w_H + \xi_1 w_{H\xi_1} \right) \varepsilon^4 + O\left(\varepsilon^5\right). \qquad (2.51)
\end{aligned}
$$

It can be easily checked that formula (2.51) coincides with asymptotic solution (2.44) which is an extra validation of the presented derivation.

Let us now test the moment and shear force on the interface at $\xi_1 = 0$, substituting

(2.51) into (2.45) and (2.46) we obtain (2.47) and (2.48).

Now, we seek to find the asymptotic solution for the left component. We rewrite the equation of motion (2.26) in the following form

$$\frac{\partial^4 w_2}{\partial \xi_2^4} - \upsilon^4 \rho^4 \Omega^4 (1-\varepsilon)^4 w_2 = 0, \tag{2.52}$$

subject to

$$\begin{aligned}
&w_2 = 0, \quad at \;\; \xi_2 = 0,\\
&\frac{\partial w_2}{\partial \xi_2} = 0, \quad at \;\; \xi_2 = 0,\\
&\frac{\partial^2 w_2}{\partial \xi_2^2} - \frac{1}{2}\upsilon^4 \Omega^4 w_2 (1-\varepsilon)^2 \varepsilon^2 = \frac{Gl^2}{D_2}(1-\varepsilon)^2, \quad at \;\; \xi_2 = 1,\\
&\frac{\partial^3 w_2}{\partial \xi_2^3} + \upsilon^4 \Omega^4 w_2 (1-\varepsilon)^3 \varepsilon = 0, \quad at \;\; \xi_2 = 1.
\end{aligned} \tag{2.53}$$

We also rewrite the general solution (2.31) in dimensionless variables for $j = 2$ as

$$\begin{aligned}
w_2 \quad &= \alpha_1^{(2)} \sinh(\rho\upsilon\Omega(1-\varepsilon)\xi_2) + \alpha_2^{(2)} \cosh(\rho\upsilon\Omega(1-\varepsilon)\xi_2) + \alpha_3^{(2)} \cos(\rho\upsilon\Omega(1-\varepsilon)\xi_2)\\
&+ \alpha_4^{(2)} \sin(\rho\upsilon\Omega(1-\varepsilon)\xi_2).
\end{aligned} \tag{2.54}$$

Substituting (2.54) into the boundary conditions (2.53) leads to the fourth order system

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \widetilde{m} & \widetilde{m_1} & -\widetilde{m_2} & -\widetilde{m_3} \\ \widetilde{m_4} & \widetilde{m_5} & \widetilde{m_6} & \widetilde{m_7} \end{pmatrix} \begin{pmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \\ \alpha_3^{(2)} \\ \alpha_4^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{Gl^2}{D_2} \\ 0 \end{pmatrix}, \tag{2.55}$$

where

$$\begin{aligned}
\widetilde{m} &= (\rho^2\upsilon^2\Omega^2 - \frac{1}{2}\Omega^4\varepsilon^2)\sinh(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_1} &= (\rho^2\upsilon^2\Omega^2 - \frac{1}{2}\Omega^4\varepsilon^2)\cosh(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_2} &= (\rho^2\upsilon^2\Omega^2 + \frac{1}{2}\Omega^4\varepsilon^2)\cos(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_3} &= (\rho^2\upsilon^2\Omega^2 + \frac{1}{2}\Omega^4\varepsilon^2)\sin(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_4} &= (\rho^3\upsilon^3\cosh(\rho\upsilon\Omega(1-\varepsilon)) + \Omega\varepsilon\sinh(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_5} &= (\rho^3\upsilon^3\sinh(\rho\upsilon\Omega(1-\varepsilon)) + \Omega\varepsilon\cosh(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_6} &= (\rho^3\upsilon^3\sin(\rho\upsilon\Omega(1-\varepsilon)) + \Omega\varepsilon\cos(\rho\upsilon\Omega(1-\varepsilon)),\\
\widetilde{m_7} &= (\rho^3\upsilon^3\cos(\rho\upsilon\Omega(1-\varepsilon)) - \Omega\varepsilon\sin(\rho\upsilon\Omega(1-\varepsilon)).
\end{aligned}$$

Above system has non-trivial solution provided that the related determinant equals zero, using Cramer's rule, we get the asymptotic solution for the left component as

$$\begin{aligned}
w_2 &= \Bigg(\left(\frac{1}{2} - \frac{i}{2}\right) G l^2 \Bigg(\rho^3 \upsilon^3 \Bigg(\cos\left(\left(1 - i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) \\
&+ i \left(\cos\left(\left(1 + i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) - (1 - i)\cos\left(\left(\xi_2 - 1\right)\rho\upsilon\Omega(\varepsilon - 1)\right)\right)\Bigg) \\
&- \rho^3 \upsilon^3 \Bigg(\cosh\left(\left(1 - i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) + i \Bigg(\cosh\left(\left(1 + i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) \\
&- (1 - i)\cosh\left(\left(\xi_2 - 1\right)\rho\upsilon\Omega(\varepsilon - 1)\right)\Bigg)\Bigg) + \Omega\varepsilon \Bigg(\sin\left(\left(1 - i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) \\
&+ i \sin\left(\left(1 + i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) + (1 + i)\sin\left(\left(\xi_2 - 1\right)\rho\upsilon\Omega(\varepsilon - 1)\right)\Bigg) \\
&+ \Omega\varepsilon \Bigg(\sinh\left(\left(1 - i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) + i \sinh\left(\left(1 + i\xi_2\right)\rho\upsilon\Omega(\varepsilon - 1)\right) \\
&+ (1 + i)\sinh\left(\left(\xi_2 - 1\right)\rho\upsilon\Omega(\varepsilon - 1)\right)\Bigg)\Bigg)\Bigg) \Bigg(D_2 \rho^2 \upsilon^2 \Omega^2 \Bigg(2\rho^3 \upsilon^3 \\
&+ 2\cosh(\rho\upsilon\Omega(\varepsilon - 1)) \left(\rho^3 \upsilon^3 \cos(\rho\upsilon\Omega(\varepsilon - 1)) + \Omega\varepsilon \sin(\rho\upsilon\Omega(\varepsilon - 1))\right) \\
&- \Omega\varepsilon \sinh(\rho\upsilon\Omega(\varepsilon - 1))(\rho\upsilon\Omega\varepsilon \sin(\rho\upsilon\Omega(\varepsilon - 1)) + 2\cos(\rho\upsilon\Omega(\varepsilon - 1)))\Bigg)\Bigg)^{-1}.
\end{aligned} \tag{2.56}$$

Now, we introduce new dimensionless variable

$$\tilde{w_2} = \frac{w_2}{l^2}\frac{D_2}{G}.$$

Thus, we can rewrite the exact solution (2.38) and the asymptotic solution (2.56) for the left component as

$$
\begin{aligned}
\tilde{w_2} &= \Big(\Big(\left(\sinh\left(\upsilon\xi_2\rho\Omega(1-\varepsilon)\right)-\sin\left(\upsilon\xi_2\rho\Omega(1-\varepsilon)\right)\right)\Big(\upsilon^3\rho^3(\cos(\Omega\varepsilon)\\
&\quad+\cosh(\Omega\varepsilon))(\sin(\upsilon\rho\Omega(1-\varepsilon))-\sinh(\upsilon\rho\Omega(1-\varepsilon)))+\upsilon^2\rho^2(\sin(\Omega\varepsilon)\\
&\quad-\sinh(\Omega\varepsilon))(\cos(\upsilon\rho\Omega(\varepsilon-1))+\cosh(\upsilon\rho\Omega(\varepsilon-1)))\\
&\quad+\upsilon^5\rho(\cos(\Omega\varepsilon)-\cosh(\Omega\varepsilon))(\sin(\upsilon\rho\Omega(1-\varepsilon))+\sinh(\upsilon\rho\Omega(1-\varepsilon)))\\
&\quad+\upsilon^4(\sin(\Omega\varepsilon)+\sinh(\Omega\varepsilon))(\cos(\upsilon\rho\Omega(\varepsilon-1))-\cosh(\upsilon\rho\Omega(\varepsilon-1)))\Big)\\
&\quad-\left(\cosh\left(\upsilon\xi_2\rho\Omega(\varepsilon-1)\right)-\cos\left(\upsilon\xi_2\rho\Omega(\varepsilon-1)\right)\right)\Big(-\upsilon^3D_2\rho^3(\cos(\Omega\varepsilon)\\
&\quad+\cosh(\Omega\varepsilon))(\cos(\upsilon\rho\Omega(\varepsilon-1))+\cosh(\upsilon\rho\Omega(\varepsilon-1)))+\upsilon^2\rho^2(\sin(\Omega\varepsilon)\\
&\quad-\sinh(\Omega\varepsilon))(\sin(\upsilon\rho\Omega(1-\varepsilon))+\sinh(\upsilon\rho\Omega(1-\varepsilon)))\\
&\quad+\upsilon^4(\sin(\Omega\varepsilon)+\sinh(\Omega\varepsilon))(\sin(\upsilon\rho\Omega(1-\varepsilon))-\sinh(\upsilon\rho\Omega(1-\varepsilon)))\\
&\quad-\upsilon^5\rho(\cos(\Omega\varepsilon)-\cosh(\Omega\varepsilon))(\cos(\upsilon\rho\Omega(\varepsilon-1))-\cosh(\upsilon\rho\Omega(\varepsilon-1)))\Big)\Big)\Big)\\
&\quad\times\Big(2\upsilon\rho\Omega^2\Big(\upsilon^4D_2^2\rho^4(\cos(\Omega\varepsilon)\cosh(\Omega\varepsilon)+1)(\cos(\upsilon\rho\Omega(\varepsilon-1))\\
&\quad\times\cosh(\upsilon\rho\Omega(\varepsilon-1))+1)+\upsilon^5\rho\Big(-\sinh(\Omega\varepsilon)\Big(\sinh(\upsilon\rho\Omega(\varepsilon-1))\Big(\left(\upsilon^2\rho^2+1\right)\\
&\quad\times\cos(\Omega\varepsilon)\cos(\upsilon\rho\Omega(\varepsilon-1))+2\upsilon\rho\sin(\Omega\varepsilon)\sin(\upsilon\rho\Omega(\varepsilon-1))\Big)\\
&\quad+\left(\upsilon^2\rho^2-1\right)\cos(\Omega\varepsilon)\sin(\upsilon\rho\Omega(\varepsilon-1))\cosh(\upsilon\rho\Omega(\varepsilon-1))\Big)\\
&\quad+\left(\frac{1}{2}+\frac{i}{2}\right)\sin(\Omega\varepsilon)\cosh(\Omega\varepsilon)\Big(\left(1-i\upsilon^2\rho^2\right)\sin((1+i)\upsilon\rho\Omega(\varepsilon-1))\\
&\quad+\left(-1-i\upsilon^2\rho^2\right)\sinh((1+i)\upsilon\rho\Omega(\varepsilon-1))\Big)\Big)+\upsilon^8(\cos(\Omega\varepsilon)\cosh(\Omega\varepsilon)-1)\\
&\quad\times(\cos(\upsilon\rho\Omega(\varepsilon-1))\cosh(\upsilon\rho\Omega(\varepsilon-1))-1)\Big)\Big)^{-1}, \qquad (2.57)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{w_2} \quad &= \Bigg(\left(\frac{1}{2} - \frac{i}{2}\right) \bigg(\rho^3 \upsilon^3 \Big(\cos\left(\left(1 - i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) \\
&+ i \left(\cos\left(\left(1 + i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) - (1 - i)\cos\left(\left(\xi_2 - 1\right) \rho\upsilon\Omega(\varepsilon - 1)\right)\right)\Big) \\
&- \rho^3 \upsilon^3 \Big(\cosh\left(\left(1 - i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) + i \Big(\cosh\left(\left(1 + i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) \\
&- (1 - i)\cosh\left(\left(\xi_2 - 1\right) \rho\upsilon\Omega(\varepsilon - 1)\right)\Big)\Big) + \Omega\varepsilon \Big(\sin\left(\left(1 - i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) \\
&+ i \sin\left(\left(1 + i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) + (1 + i)\sin\left(\left(\xi_2 - 1\right) \rho\upsilon\Omega(\varepsilon - 1)\right)\Big) \\
&+ \Omega\varepsilon \Big(\sinh\left(\left(1 - i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) + i \sinh\left(\left(1 + i\xi_2\right) \rho\upsilon\Omega(\varepsilon - 1)\right) \\
&+ (1 + i)\sinh\left(\left(\xi_2 - 1\right) \rho\upsilon\Omega(\varepsilon - 1)\right)\Big)\bigg)\Bigg) \bigg(\rho^2 \upsilon^6 \Omega^2 \Big(2\rho^3 \upsilon^3 \\
&+ 2\cosh(\rho\upsilon\Omega(\varepsilon - 1)) \left(\rho^3 \upsilon^3 \cos(\rho\upsilon\Omega(\varepsilon - 1)) + \Omega\varepsilon \sin(\rho\upsilon\Omega(\varepsilon - 1))\right) \\
&- \Omega\varepsilon \sinh(\rho\upsilon\Omega(\varepsilon - 1))(\rho\upsilon\Omega\varepsilon \sin(\rho\upsilon\Omega(\varepsilon - 1)) + 2\cos(\rho\upsilon\Omega(\varepsilon - 1)))\Big)\bigg)^{-1} \\
&+ O\left(\varepsilon\right). \qquad (2.58)
\end{aligned}
$$

Figures 2.14-2.18 demonstrate the exact solution of the left component $\tilde{w_2}$ (2.57) and the asymptotic solution (2.58) for $\rho = 1$, $\upsilon = 1$ and several values of Ω and ε. It can be seen that with reasonably small values of epsilon, the asymptotic solutions are providing a good approximation. The following Figures 2.19-2.23 showing the maximum error over $0 \leq \xi_2 \leq 1$ between the exact solution (2.57) and the asymptotic solution (2.58) with respect to Ω and ε. All Figures 2.19-2.23 demonstrate that the maximum error is monotonically increasing for increasing frequency.

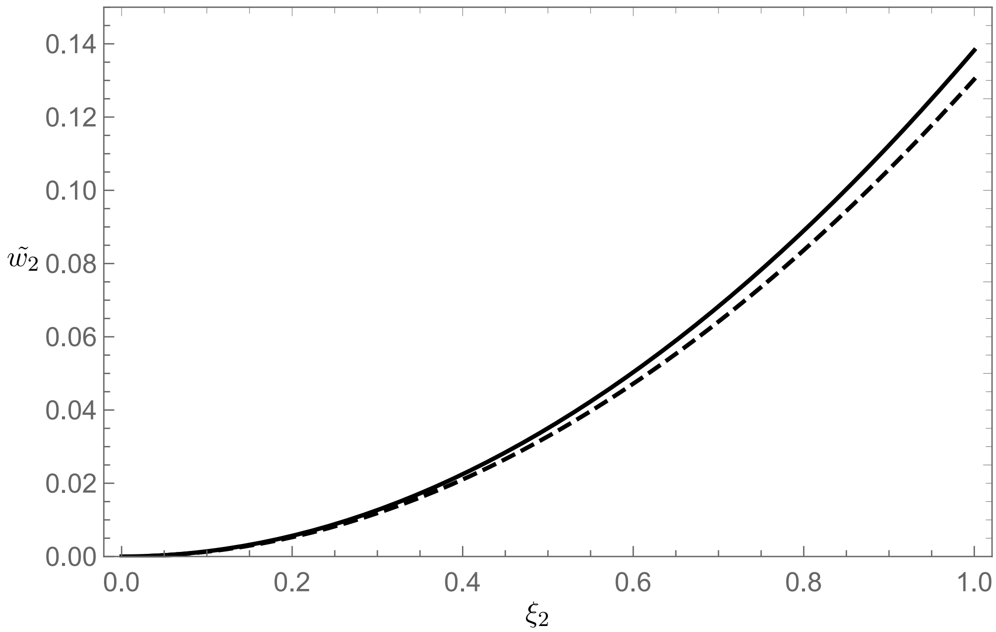

FIGURE 2.14: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1$, $\varepsilon = 0.5$.

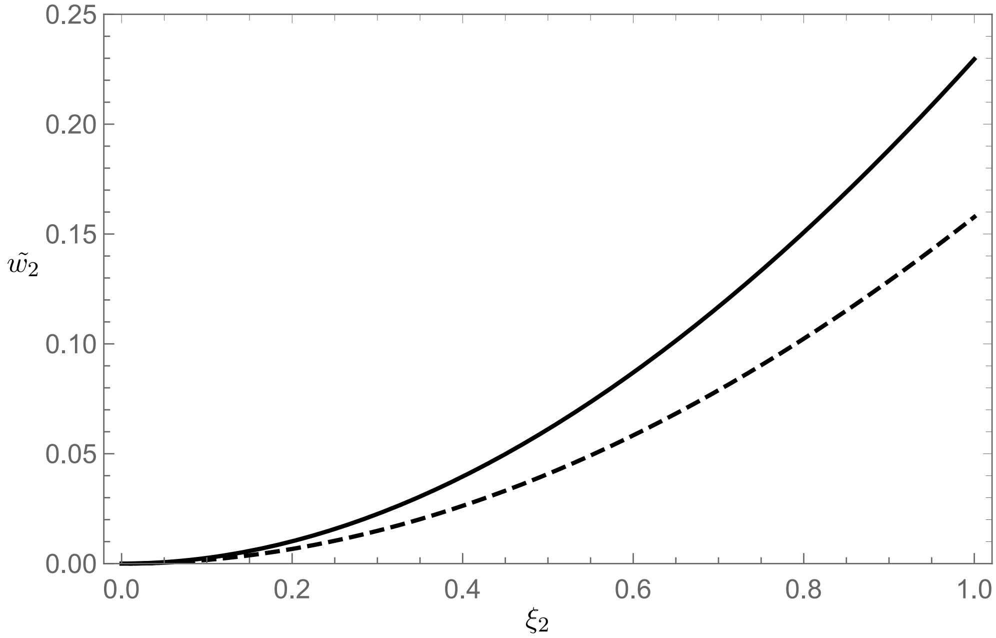

FIGURE 2.15: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1.5$, $\varepsilon = 0.5$.

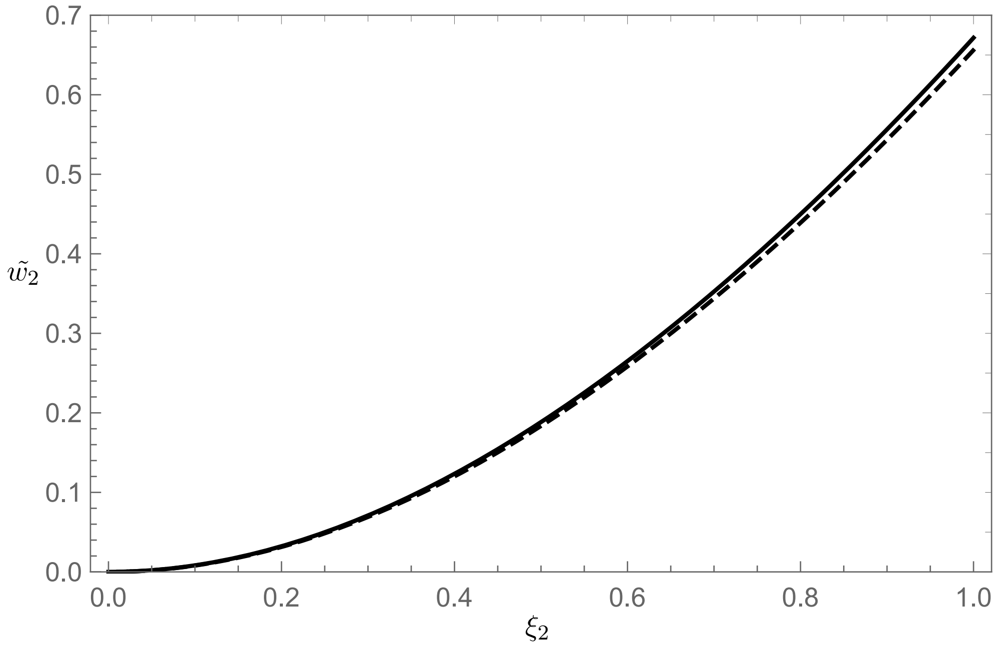

FIGURE 2.16: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1.5$, $\varepsilon = 0.1$.

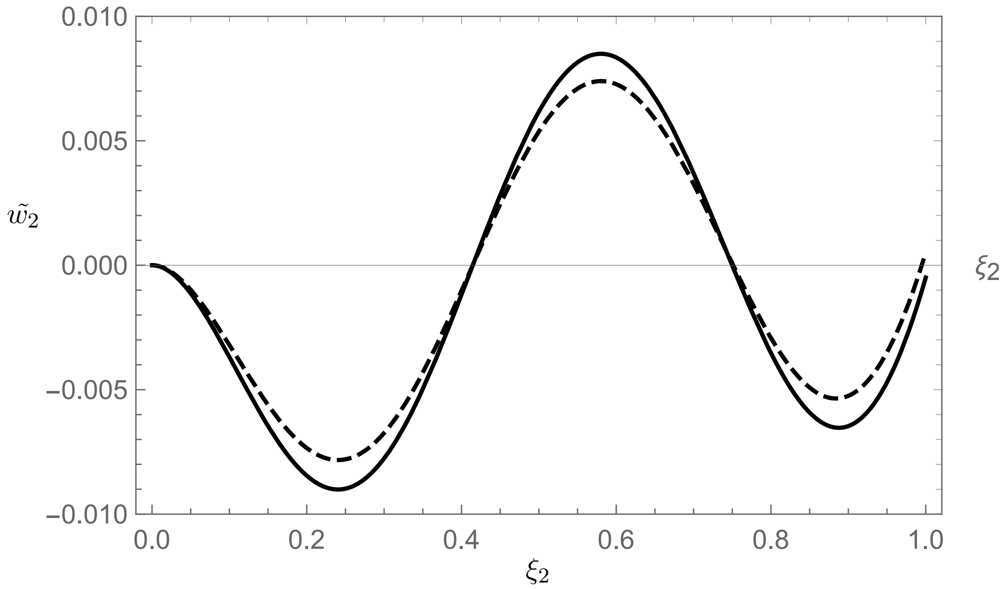

FIGURE 2.17: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 10$, $\varepsilon = 0.05$.

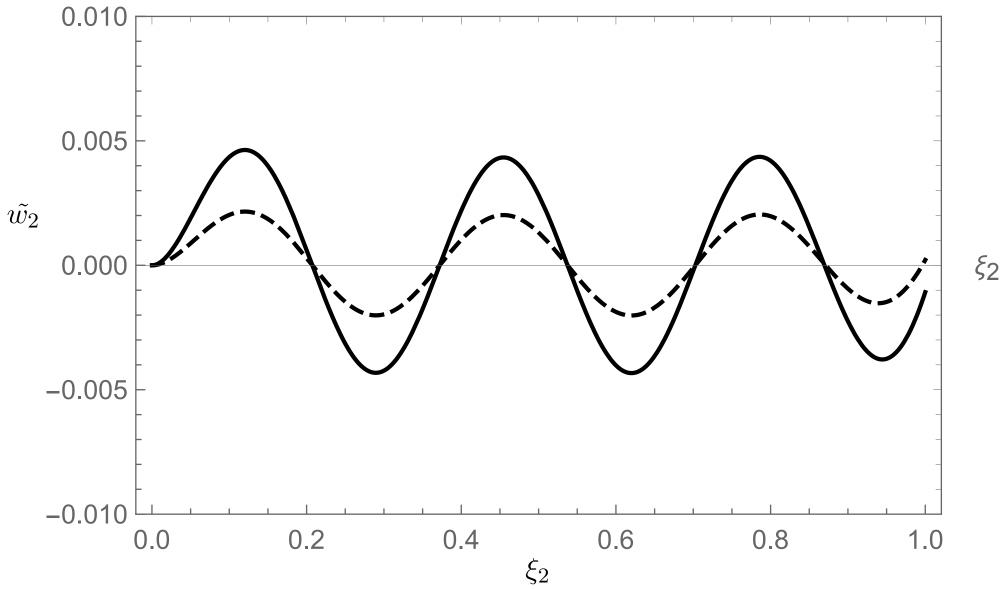

FIGURE 2.18: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 20$, $\varepsilon = 0.05$.

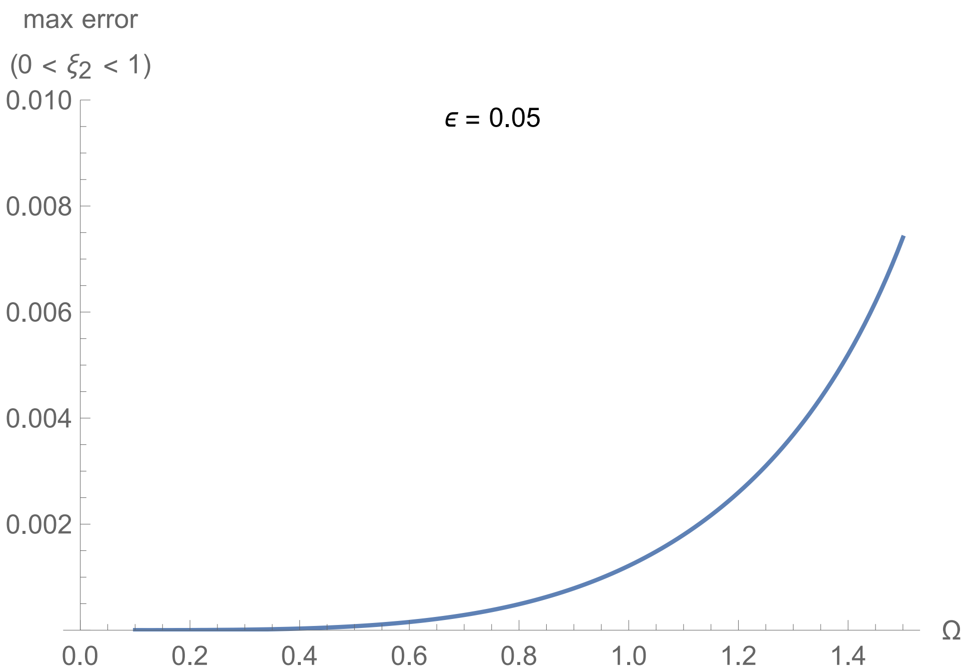

FIGURE 2.19: The maximum error between asymptotic solution (2.58) and exact solution (2.57) for $\varepsilon = 0.05$.

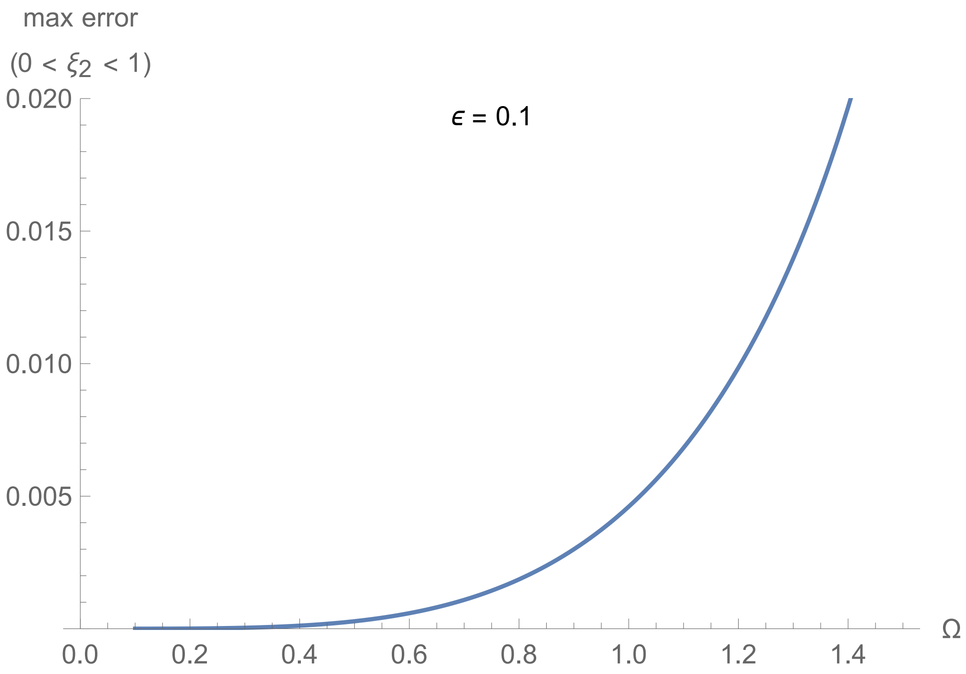

FIGURE 2.20: The maximum error between asymptotic solution (2.58) and exact solution (2.57) for $\varepsilon = 0.1$.

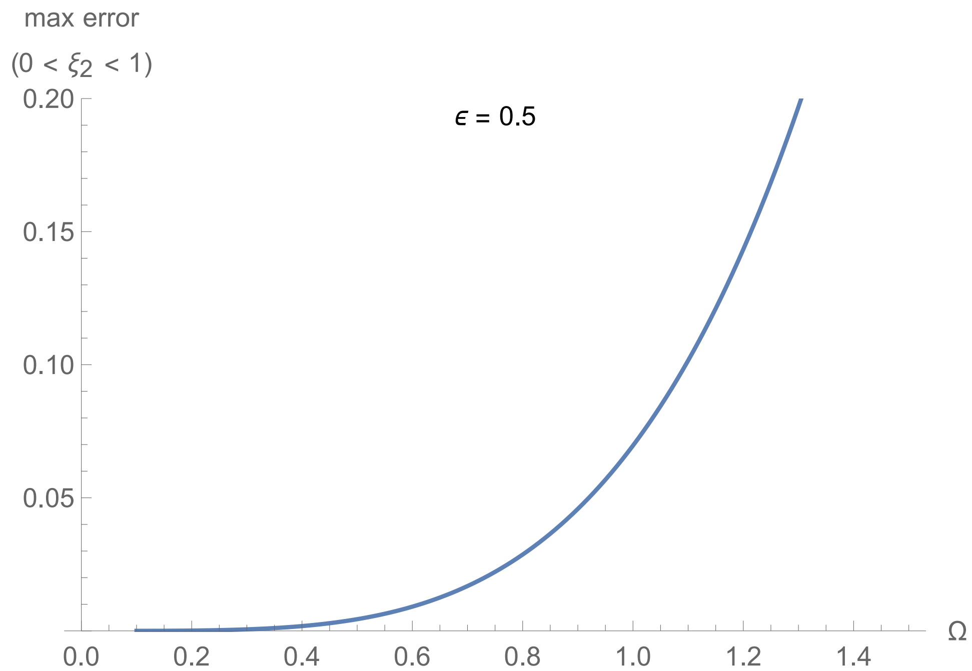

FIGURE 2.21: The maximum error between asymptotic solution (2.58) and exact solution (2.57) for $\varepsilon = 0.5$.

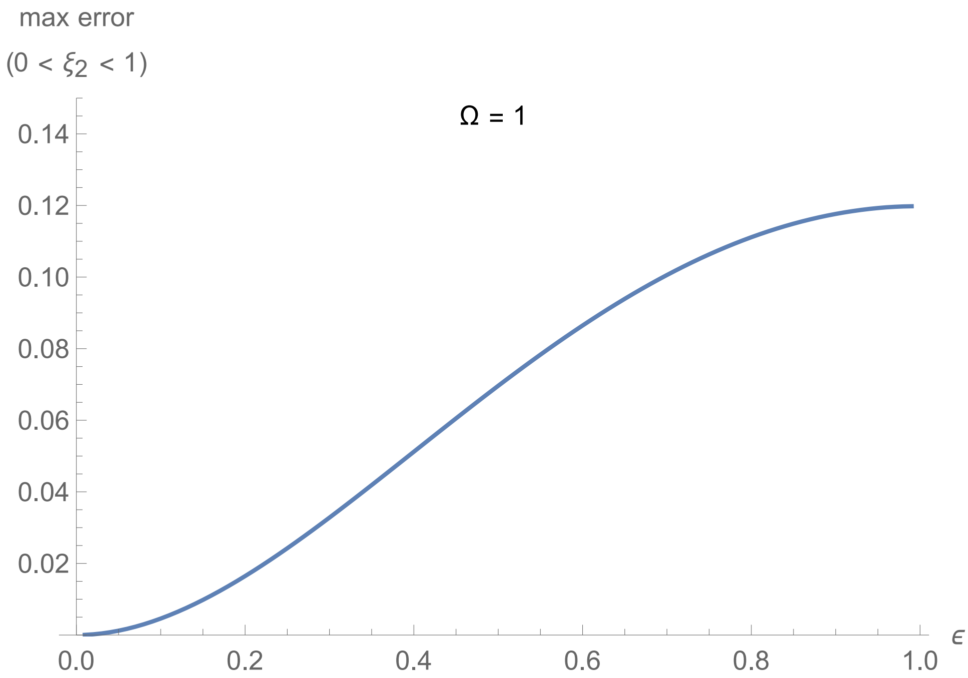

FIGURE 2.22: The maximum error between asymptotic solution (2.58) and exact solution (2.57) for $\Omega = 1$.

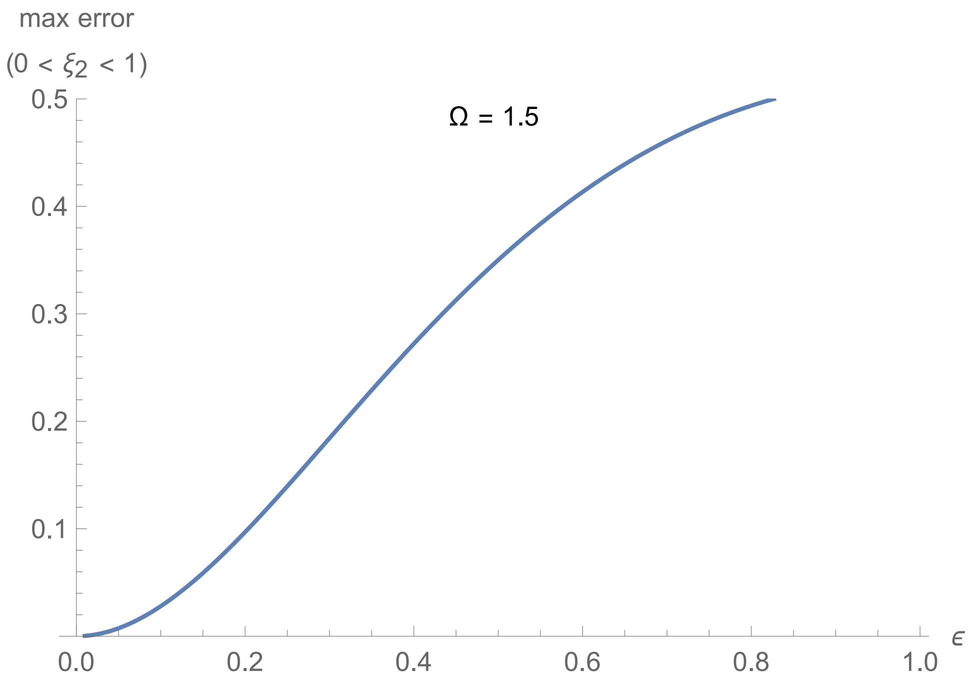

FIGURE 2.23: The maximum error between asymptotic solution (2.58) and exact solution (2.57) for $\Omega = 1.5$.

2.2.3 Second case (excitation by modified shear force)

Substituting (2.31) into the boundary conditions (2.29) and (2.30), we finally arrive at a set of six linear algebraic equations which can be written in a matrix form as

$$\mathrm{Q^b}\cdot\gamma=\mathrm{U^b}, \tag{2.59}$$

where $\gamma=\left(\gamma_1^{(1)},\gamma_2^{(1)},\gamma_3^{(1)},\gamma_4^{(1)},\gamma_1^{(2)},\gamma_2^{(2)}\right)^T$, $\mathrm{U^b}=\left(\dfrac{N}{D_1\beta_1^3},0,0,0,0,0\right)^T$ are vectors and $\mathrm{Q^b}$ is a 6×6 matrix with the non-zero components given as (2.33).

Using Cramer's rule to solve (2.59) and using the dimensionless variables (2.36), we get the exact solution as

$$\begin{aligned}
w_1 \quad &= -\Bigg(\left(\frac{1}{4}+\frac{i}{4}\right)Nl^3\Big(\upsilon^4(2\cos(\upsilon(\epsilon-1)\rho\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)+2)\\
&\times\Big(\sin\left(\epsilon\Omega\left(\xi_1-1\right)\right)+\cosh(\epsilon\Omega)\sin\left(\epsilon\Omega\xi_1\right)-\cos\left(\epsilon\Omega\xi_1\right)\sinh(\epsilon\Omega)\\
&+\cosh\left(\epsilon\Omega\xi_1\right)\left(\sin(\epsilon\Omega)+\sinh(\epsilon\Omega)\right)-\left(\cos(\epsilon\Omega)+\cosh(\epsilon\Omega)\right)\sinh\left(\epsilon\Omega\xi_1\right)\Big)D_2^2\rho^4\\
&-4\upsilon\Big(\cosh(\epsilon\Omega)\Big(\cosh(\upsilon(\epsilon-1)\rho\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\\
&\times\Big(\upsilon^2\sin(\epsilon\Omega)\sinh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\rho^2+\cos\left(\epsilon\Omega\xi_1\right)+\cos(\epsilon\Omega)\cosh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\Big)\\
&-\sinh(\upsilon(\epsilon-1)\rho\Omega)\Big(\cos(\upsilon(\epsilon-1)\rho\Omega)\cos\left(\epsilon\Omega\xi_1\right)\\
&+\cosh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\left(\cos(\epsilon\Omega)\cos(\upsilon(\epsilon-1)\rho\Omega)\right.\\
&+\upsilon\rho\sin(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega))+\upsilon\rho\sin(\upsilon(\epsilon-1)\rho\Omega)\sin\left(\epsilon\Omega\xi_1\right)\\
&+\upsilon\rho(\cos(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)-\upsilon\rho\cos(\upsilon(\epsilon-1)\rho\Omega)\sin(\epsilon\Omega))\sinh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sinh(\epsilon\Omega)\Big(\cosh(\upsilon(\epsilon-1)\rho\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\Big(\upsilon^2\Big(\cosh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\sin(\epsilon\Omega)\\
&+\sin\left(\epsilon\Omega\xi_1\right)\Big)\rho^2+\cos(\epsilon\Omega)\sinh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\Big)\\
&+\sinh(\upsilon(\epsilon-1)\rho\Omega)\Big(\upsilon\rho\Big(-\cos\left(\epsilon\Omega\xi_1\right)\sin(\upsilon(\epsilon-1)\rho\Omega)\\
&+\cosh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\left(\upsilon\rho\cos(\upsilon(\epsilon-1)\rho\Omega)\sin(\epsilon\Omega)-\cos(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\right)\\
&+\upsilon\rho\cos(\upsilon(\epsilon-1)\rho\Omega)\sin\left(\epsilon\Omega\xi_1\right)\Big)-(\cos(\epsilon\Omega)\cos(\upsilon(\epsilon-1)\rho\Omega)\\
&+\upsilon\rho\sin(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega))\sinh\left(\epsilon\Omega\left(\xi_1-1\right)\right)\Big)\Big)\Big)D_1D_2\rho\\
&+(2\cos(\upsilon(\epsilon-1)\rho\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)-2)\Big(-\sin\left(\epsilon\Omega\left(\xi_1-1\right)\right)\\
&+\cosh(\epsilon\Omega)\sin\left(\epsilon\Omega\xi_1\right)+\cosh\left(\epsilon\Omega\xi_1\right)\left(\sin(\epsilon\Omega)-\sinh(\epsilon\Omega)\right)-\cos\left(\epsilon\Omega\xi_1\right)\sinh(\epsilon\Omega)\\
&+(\cosh(\epsilon\Omega)-\cos(\epsilon\Omega))\sinh\left(\epsilon\Omega\xi_1\right)\Big)D_1^2\Big)\Big)\Big(\Omega^3D_1\Big((1+i)\upsilon^4\\
&\times(\cos(\epsilon\Omega)\cosh(\epsilon\Omega)+1)(\cos(\upsilon(\epsilon-1)\rho\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)+1)D_2^2\rho^4\\
&+\upsilon\Big(\cosh(\epsilon\Omega)\sin(\epsilon\Omega)\Big(\left(\upsilon^2\rho^2+i\right)\sin((1+i)\upsilon(\epsilon-1)\rho\Omega)\\
&+\left(\upsilon^2\rho^2-i\right)\sinh((1+i)\upsilon(\epsilon-1)\rho\Omega)\Big)\\
&-(1+i)\sinh(\epsilon\Omega)\Big(\left(\upsilon^2\rho^2-1\right)\cos(\epsilon\Omega)\cosh(\upsilon(\epsilon-1)\rho\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\\
&+\left(\left(\upsilon^2\rho^2+1\right)\cos(\epsilon\Omega)\cos(\upsilon(\epsilon-1)\rho\Omega)+2\upsilon\rho\sin(\epsilon\Omega)\sin(\upsilon(\epsilon-1)\rho\Omega)\right)\\
&\times\sinh(\upsilon(\epsilon-1)\rho\Omega)\Big)\Big)D_1D_2\rho+(1+i)(\cos(\epsilon\Omega)\cosh(\epsilon\Omega)-1)(\cos(\upsilon(\epsilon-1)\rho\Omega)\\
&\times\cosh(\upsilon(\epsilon-1)\rho\Omega)-1)d_1^2\Big)\Big)^{-1},
\end{aligned}
\tag{2.60}
$$

and

$$
\begin{aligned}
w_2 \quad &= \Bigg(N l^3 \Bigg(\left(\cosh\left(\upsilon\xi_2\rho\Omega(\epsilon-1)\right) - \cos\left(\upsilon\xi_2\rho\Omega(\epsilon-1)\right)\right) \Bigg(-\upsilon^3 D_2\rho^3(\sin(\Omega\epsilon) \\
&+ \sinh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) + \cosh(\upsilon\rho\Omega(\epsilon-1))) \\
&- \upsilon^2 D_2\rho^2(\cos(\Omega\epsilon) + \cosh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) + \sinh(\upsilon\rho\Omega(1-\epsilon))) \\
&+ \upsilon D_1\rho(\sinh(\Omega\epsilon) - \sin(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) - \cosh(\upsilon\rho\Omega(\epsilon-1))) \\
&- D_1(\cos(\Omega\epsilon) - \cosh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) - \sinh(\upsilon\rho\Omega(1-\epsilon)))\Bigg) \\
&- \Bigg(\sinh\left(\upsilon\xi_2\rho\Omega(1-\epsilon)\right) - \sin\left(\upsilon\xi_2\rho\Omega(1-\epsilon)\right)\Bigg)\Bigg(\upsilon^3 D_2\rho^3(\sin(\Omega\epsilon) \\
&+ \sinh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) - \sinh(\upsilon\rho\Omega(1-\epsilon))) \\
&- \upsilon^2 D_2\rho^2(\cos(\Omega\epsilon) + \cosh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) + \cosh(\upsilon\rho\Omega(\epsilon-1))) \\
&+ \upsilon D_1\rho(\sin(\Omega\epsilon) - \sinh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) + \sinh(\upsilon\rho\Omega(1-\epsilon))) \\
&- D_1(\cos(\Omega\epsilon) - \cosh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) - \cosh(\upsilon\rho\Omega(\epsilon-1)))\Bigg)\Bigg)\Bigg) \\
&\times \Bigg(2\upsilon\rho\Omega^3 \Bigg(\upsilon^4 D_2^2\rho^4(\cos(\Omega\epsilon)\cosh(\Omega\epsilon) + 1)(\cos(\upsilon\rho\Omega(\epsilon-1))\cosh(\upsilon\rho\Omega(\epsilon-1)) + 1) \\
&+ \upsilon D_1 D_2\rho\Bigg(-\sinh(\Omega\epsilon)\Bigg(\sinh(\upsilon\rho\Omega(\epsilon-1))\Bigg(\left(\upsilon^2\rho^2+1\right)\cos(\Omega\epsilon)\cos(\upsilon\rho\Omega(\epsilon-1)) \\
&+ 2\upsilon\rho\sin(\Omega\epsilon)\sin(\upsilon\rho\Omega(\epsilon-1))\Bigg) + \left(\upsilon^2\rho^2-1\right)\cos(\Omega\epsilon)\sin(\upsilon\rho\Omega(\epsilon-1)) \\
&\times \cosh(\upsilon\rho\Omega(\epsilon-1))\Bigg) + \left(\frac{1}{2} + \frac{i}{2}\right)\sin(\Omega\epsilon)\cosh(\Omega\epsilon)\Bigg(\left(1 - i\upsilon^2\rho^2\right) \\
&\times \sin((1+i)\upsilon\rho\Omega(\epsilon-1)) + \left(-1 - i\upsilon^2\rho^2\right)\sinh((1+i)\upsilon\rho\Omega(\epsilon-1))\Bigg)\Bigg) \\
&+ D_1^2(\cos(\Omega\epsilon)\cosh(\Omega\epsilon) - 1)(\cos(\upsilon\rho\Omega(\epsilon-1))\cosh(\upsilon\rho\Omega(\epsilon-1)) - 1)\Bigg)\Bigg)^{-1}, \quad (2.61)
\end{aligned}
$$

where $\upsilon = \left(\frac{D_1}{D_2}\right)^{\frac{1}{4}}$ and $\rho = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{4}}$.

In order to check our result we are setting $x = l - H$ which implies $\xi_1 = 0$ and $\xi_2 = 1$ when $\varepsilon \to 0$ into (2.60) and (2.61), we obtain

$$w_1 = w_2 = \frac{Nl^3(\cos(\upsilon\rho\Omega)\sinh(\upsilon\rho\Omega) - \sin(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega))}{\upsilon^3 D_2 \rho^3 \Omega^3(\cos(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega) + 1)},$$

which is an additional verification of the solutions.

2.2.3.1 Asymptotic analysis a composite beam

Consider the equation of motion (2.39) subject to

$$\begin{aligned} & w_1 = w_H, && (1-\varepsilon)\frac{\partial w_1}{\partial \xi_1} = \varepsilon w_{H\xi_1}, \quad \text{at} \quad \xi_1 = 0, \\ & \frac{\partial^2 w_1}{\partial \xi_1^2} = 0, && D_1 \frac{\partial^3 w_1}{\partial \xi_1^3} = \varepsilon^3 N l^3, \quad \text{at} \quad \xi_1 = 1, \end{aligned} \tag{2.62}$$

where functions

$$w_H = \frac{Nl^3(\cos(\upsilon\rho\Omega)\sinh(\upsilon\rho\Omega) - \sin(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega))}{\upsilon^3 D_2 \rho^3 \Omega^3(\cos(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega) + 1)}$$

and

$$w_{H\xi_1} = -\frac{Nl^3 \sin(\upsilon\rho\Omega)\sinh(\upsilon\rho\Omega)}{\upsilon^2 D_2 \rho^2 \Omega^2(\cos(\upsilon\rho\Omega)\cosh(\upsilon\rho\Omega) + 1)}$$

are given on the interface.

The deflection w_1 can be expanded into an asymptotic series in terms of ε as (2.41).

Substituting expansion (2.41) into the boundary value problem (2.39)-(2.62), we arrive at the problem formulated at the various asymptotic orders $n = 0, 1, 2, \ldots,$

namely

$$\frac{\partial^4 w_1^{(n)}}{\partial \xi_1^4} - \Omega^4 w_1^{(n-4)} = 0, \tag{2.63}$$

subject to

$$\begin{aligned}
& w_1^{(n)} = w_H^{(n)}, \qquad \xi_1 = 0, \\
& \frac{\partial w_1^{(n)}}{\partial \xi_1} - \frac{\partial w_1^{(n-1)}}{\partial \xi_1} = w_{H\xi_1}^{(n)} \quad \text{at} \quad \xi_1 = 0, \\
& \frac{\partial^2 w_1^{(n)}}{\partial \xi_1^2} = 0 \quad \text{at} \quad \xi_1 = 1, \\
& D_1 \frac{\partial^3 w_1^{(n)}}{\partial \xi_1^3} = N^{(n)} l^{3(n)} \quad \text{at} \quad \xi_1 = 1,
\end{aligned} \tag{2.64}$$

where quantities with the negative superscript are set to be equal to zero. The only non-zero components $w_H^{(n)}$, $w_{H\xi_1}^{(n)}$ and $N^{(n)} l^{3(n)}$ are $w_H^{(0)} = w_H$, $w_{H\xi_1}^{(1)} = w_{H\xi_1}$ and $N^{(3)} l^{3(3)} = Nl^3$, respectively.

Substituting subsequently $n = 0, 1, 2, 3, 4$ into (2.63)-(2.64) we obtain

$$\begin{aligned}
& w_1^{(0)} = w_H, \\
& w_1^{(1)} = w_{H\xi_1} \xi_1, \\
& w_1^{(2)} = w_{H\xi_1} \xi_1, \\
& w_1^{(3)} = w_{H\xi_1} \xi_1 - \frac{1}{2} \frac{Nl^3}{D_1} \xi_1^2 + \frac{1}{6} \frac{Nl^3}{D_1} \xi_1^3, \\
& w_1^{(4)} = w_{H\xi_1} \xi_1 + \frac{1}{4} \Omega^4 w_H \xi_1^2 - \frac{1}{6} \Omega^4 w_H \xi_1^3 + \frac{1}{24} \Omega^4 w_H \xi_1^4.
\end{aligned} \tag{2.65}$$

Finally, using expansion (2.65) together with the relations (2.45) and (2.46) to obtain moment and shear force on the interface at $\xi_1 = 0$ in the form

$$M_1 = -Nl\varepsilon + \frac{D_1\Omega^4}{2l^2} w_H \varepsilon^2 + ..., \tag{2.66}$$

$$N_1 = N - \frac{D_1\Omega^4}{l^3} w_H \varepsilon + ... \tag{2.67}$$

Note that the formula above which present expansion of moment and shear force on the interface will use later to find the asymptotic solution for the left component.

2.2.3.2 Testing of asymptotic formulae

In order to validate the asymptotic results obtained in the previous section, consider the right component over the domain $l - H \leqslant x \leqslant l$. We take equation of motion (2.39) subject to boundary conditions (2.62). The solution of the formulated problem is then sought for in the form (2.31) for $j = 1$, and we finally arrive at a set of four linear algebraic equations which can be written in a matrix form as

$$\bar{\mathrm{Q}}^{\mathrm{b}} \cdot \bar{\gamma} = \bar{\mathrm{U}}^{\mathrm{b}}, \tag{2.68}$$

where $\bar{\gamma} = \left(\bar{\gamma}_1^{(1)}, \bar{\gamma}_2^{(1)}, \bar{\gamma}_3^{(1)}, \bar{\gamma}_4^{(1)}\right)^T$, $\bar{\mathrm{U}}^{\mathrm{b}} = (Nl^3, 0, w_H, w_{H\xi_1})^T$ are vectors and $\bar{\mathrm{Q}}^{\mathrm{b}}$ is a 4×4 matrix with its non-zero components given as (2.50) and the sought for constants $\gamma_i^{(1)}$, $i = 1, 2, 3, 4$ are presented in Appendix B.3.

Next, we rewrite solution (2.31) for w_1 in terms of dimensionless variables and expand it into Taylor series about $\varepsilon = 0$ arriving at the asymptotic expansion

$$\begin{aligned} w_1 \qquad &= w_H + \xi_1 w_{\mathrm{H}\xi_1}\varepsilon + \xi_1 w_{\mathrm{H}\xi_1}\varepsilon^2 + \left(\frac{l^3 N\left(\xi_1 - 3\right)\xi_1^2}{6D_1} + \xi_1 w_{\mathrm{H}\xi_1}\right)\varepsilon^3 \\ &+ \left(\frac{1}{24}\xi_1^2\left(\left(\xi_1 - 4\right)\xi_1 + 6\right)\Omega^4 w_H + \xi_1 w_{\mathrm{H}\xi_1}\right)\varepsilon^4 + O\left(\varepsilon^5\right). \end{aligned} \tag{2.69}$$

It can be easily checked that formula (2.69) coincides with asymptotic solution (2.65) which is an extra validation of the presented derivation.

Let us now test the moment and shear force on the interface at $\xi_1 = 0$, substituting (2.69) into (2.45) and (2.46) we obtain (2.66) and (2.67).

Now, we seek to find the asymptotic solution for the left component. We rewrite the equation of motion (2.26) as (2.52) subject to

$$\begin{aligned} &w_2 = 0, \quad at \;\; \xi_2 = 0, \\ &\frac{\partial w_2}{\partial \xi_2} = 0, \quad at \;\; \xi_2 = 0, \\ &\frac{\partial^2 w_2}{\partial \xi_2^2} - \frac{1}{2}\Omega^4 \upsilon^4 w_2 (1-\varepsilon)^2 \varepsilon^2 = \frac{N l^3}{D_2}(1-\varepsilon)^2 \varepsilon, \quad at \;\; \xi_2 = 1, \\ &\frac{\partial^3 w_2}{\partial \xi_2^3} + \Omega^4 \upsilon^4 w_2 (1-\varepsilon)^3 \varepsilon = \frac{N l^3}{D_2}(1-\varepsilon)^3, \quad at \;\; \xi_2 = 1. \end{aligned} \tag{2.70}$$

We also rewrite the general solution (2.31) in dimensionless variables for $j = 2$ as (2.54).

Substituting (2.54) into the boundary conditions (2.70) leads to the fourth order system

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \widetilde{m} & \widetilde{m_1} & -\widetilde{m_2} & -\widetilde{m_3} \\ \widetilde{m_4} & \widetilde{m_5} & \widetilde{m_6} & \widetilde{m_7} \end{pmatrix} \begin{pmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \\ \alpha_3^{(2)} \\ \alpha_4^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{Nl^3}{D_2}\varepsilon \\ \dfrac{Nl^3}{D_2} \end{pmatrix}, \tag{2.71}$$

where

$$\begin{aligned}
\widetilde{m} &= (\rho^2 \upsilon^2 \Omega^2 - \frac{1}{2}\Omega^4 \upsilon^4 \varepsilon^2) \sinh(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_1} &= (\rho^2 \upsilon^2 \Omega^2 - \frac{1}{2}\Omega^4 \upsilon^4 \varepsilon^2) \cosh(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_2} &= (\rho^2 \upsilon^2 \Omega^2 + \frac{1}{2}\Omega^4 \upsilon^4 \varepsilon^2) \cos(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_3} &= (\rho^2 \upsilon^2 \Omega^2 + \frac{1}{2}\Omega^4 \upsilon^4 \varepsilon^2) \sin(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_4} &= (\rho^3 \cosh(\rho \upsilon \Omega(1-\varepsilon)) + \Omega \upsilon \varepsilon \sinh(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_5} &= (\rho^3 \sinh(\rho \upsilon \Omega(1-\varepsilon)) + \Omega \upsilon \varepsilon \cosh(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_6} &= (\rho^3 \sin(\rho \upsilon \Omega(1-\varepsilon)) + \Omega \upsilon \varepsilon \cos(\rho \upsilon \Omega(1-\varepsilon)), \\
\widetilde{m_7} &= (\rho^3 \cos(\rho \upsilon \Omega(1-\varepsilon)) - \Omega \upsilon \varepsilon \sin(\rho \upsilon \Omega(1-\varepsilon)).
\end{aligned}$$

Above system has non-trivial solution provided that the related determinant is non-zero. Then, using Cramer's rule, we get the asymptotic solution for the left component as

$$
\begin{aligned}
w_2 \quad &= \Bigg(\Big(\frac{1}{4}+\frac{i}{4}\Big)l^3 N\Big((1-i)\upsilon\Omega\Big(2\rho^2\upsilon\Omega+\epsilon^2\Big(\upsilon^3\Omega^3+2\Big)\Big)\sin\Big(\Big(\xi_2-1\Big)\rho\upsilon\Omega \\
&\times(\epsilon-1)\Big)+(1-i)\upsilon\Omega\Big(\epsilon^2\Big(\upsilon^3\Omega^3+2\Big)-2\rho^2\upsilon\Omega\Big)\sinh\Big(\left(\xi_2-1\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-i\upsilon^4\Omega^4\epsilon^2\sin\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+\upsilon^4\Omega^4\epsilon^2\sin\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-i\upsilon^4\Omega^4\epsilon^2\sinh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+\upsilon^4\Omega^4\epsilon^2\sinh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-2i\upsilon\Omega\epsilon^2\sin\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+2\upsilon\Omega\epsilon^2\sin\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-2i\upsilon\Omega\epsilon^2\sinh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+2\upsilon\Omega\epsilon^2\sinh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&+2\rho^3\epsilon\Big(-i\cos\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+\cos\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-(1-i)\cos\Big(\left(\xi_2-1\right)\rho\upsilon\Omega(\epsilon-1)\Big)\Big)+2i\rho^3\epsilon\cosh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-2\rho^3\epsilon\cosh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+(2-2i)\rho^3\epsilon\cosh\Big(\left(\xi_2-1\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&-2i\rho^2\upsilon^2\Omega^2\sin\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+2\rho^2\upsilon^2\Omega^2\sin\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big) \\
&+2i\rho^2\upsilon^2\Omega^2\sinh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)-2\rho^2\upsilon^2\Omega^2\sinh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega \\
&\times(\epsilon-1)\Big)\Big)\Bigg)\Big(D_2\rho^2\upsilon^2\Omega^2\Big(2\rho^3+2\cosh(\rho\upsilon\Omega(\epsilon-1))\Big(\rho^3\cos(\rho\upsilon\Omega(\epsilon-1)) \\
&+\upsilon\Omega\epsilon\sin(\rho\upsilon\Omega(\epsilon-1))\Big)-\upsilon\Omega\epsilon\sinh(\rho\upsilon\Omega(\epsilon-1))(\rho\upsilon\Omega\epsilon\sin(\rho\upsilon\Omega(\epsilon-1)) \\
&+2\cos(\rho\upsilon\Omega(\epsilon-1)))\Big)\Big)^{-1}.
\end{aligned}
\tag{2.72}
$$

Now, we introduce new dimensionless variable

$$
\widetilde{w_2} = \frac{w_2}{l^3}\frac{D_2}{N}.
$$

Thus, we can rewrite the exact solution (2.61) and the asymptotic solution (2.72) for the left component as

$$
\begin{aligned}
\widetilde{w_2} \quad &= \Bigg(\Bigg((\cosh(\upsilon\xi_2\rho\Omega(\epsilon-1)) - \cos(\upsilon\xi_2\rho\Omega(\epsilon-1))) \Bigg(- \upsilon^3\rho^3(\sin(\Omega\epsilon) \\
&+ \sinh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) + \cosh(\upsilon\rho\Omega(\epsilon-1))) - \upsilon^2\rho^2(\cos(\Omega\epsilon) \\
&+ \cosh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) + \sinh(\upsilon\rho\Omega(1-\epsilon))) \\
&+\upsilon^5\rho(\sinh(\Omega\epsilon) - \sin(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) - \cosh(\upsilon\rho\Omega(\epsilon-1))) \\
&-\upsilon^4(\cos(\Omega\epsilon) - \cosh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) - \sinh(\upsilon\rho\Omega(1-\epsilon)))\Bigg) \\
&-\Bigg(\sinh(\upsilon\xi_2\rho\Omega(1-\epsilon)) - \sin(\upsilon\xi_2\rho\Omega(1-\epsilon)) \Bigg)\Bigg(\upsilon^3\rho^3(\sin(\Omega\epsilon) + \sinh(\Omega\epsilon)) \\
&\times(\sin(\upsilon\rho\Omega(1-\epsilon)) - \sinh(\upsilon\rho\Omega(1-\epsilon))) - \upsilon^2 D_2\rho^2(\cos(\Omega\epsilon) \\
&+ \cosh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) + \cosh(\upsilon\rho\Omega(\epsilon-1))) + \upsilon^5\rho(\sin(\Omega\epsilon) \\
&- \sinh(\Omega\epsilon))(\sin(\upsilon\rho\Omega(1-\epsilon)) + \sinh(\upsilon\rho\Omega(1-\epsilon))) \\
&-\upsilon^4(\cos(\Omega\epsilon) - \cosh(\Omega\epsilon))(\cos(\upsilon\rho\Omega(\epsilon-1)) - \cosh(\upsilon\rho\Omega(\epsilon-1)))\Bigg)\Bigg)\Bigg) \\
&\times\Bigg(2\upsilon\rho\Omega^3\Bigg(\upsilon^4\rho^4(\cos(\Omega\epsilon)\cosh(\Omega\epsilon) + 1)(\cos(\upsilon\rho\Omega(\epsilon-1))\cosh(\upsilon\rho\Omega(\epsilon-1)) + 1) \\
&+\upsilon^5\rho\Bigg(- \sinh(\Omega\epsilon)\Bigg(\sinh(\upsilon\rho\Omega(\epsilon-1))\Bigg(\left(\upsilon^2\rho^2 + 1\right)\cos(\Omega\epsilon)\cos(\upsilon\rho\Omega(\epsilon-1)) \\
&+2\upsilon\rho\sin(\Omega\epsilon)\sin(\upsilon\rho\Omega(\epsilon-1))\Bigg) + \left(\upsilon^2\rho^2 - 1\right)\cos(\Omega\epsilon)\sin(\upsilon\rho\Omega(\epsilon-1)) \\
&\times\cosh(\upsilon\rho\Omega(\epsilon-1))\Bigg) + \left(\frac{1}{2} + \frac{i}{2}\right)\sin(\Omega\epsilon)\cosh(\Omega\epsilon)\Bigg(\left(1 - i\upsilon^2\rho^2\right) \\
&\times\sin((1+i)\upsilon\rho\Omega(\epsilon-1)) + \Bigg(- 1 - i\upsilon^2\rho^2\Bigg)\sinh((1+i)\upsilon\rho\Omega(\epsilon-1))\Bigg)\Bigg) \\
&+\upsilon^8(\cos(\Omega\epsilon)\cosh(\Omega\epsilon) - 1)(\cos(\upsilon\rho\Omega(\epsilon-1))\cosh(\upsilon\rho\Omega(\epsilon-1)) - 1)\Bigg)\Bigg)^{-1},
\end{aligned}
\tag{2.73}
$$

and

$$
\begin{aligned}
\widetilde{w_2} \quad &= \Bigg(\Big(\frac{1}{4}+\frac{i}{4}\Big)\Big((1-i)\upsilon\Omega\Big(2\rho^2\upsilon\Omega+\epsilon^2\Big(\upsilon^3\Omega^3+2\Big)\Big)\sin\Big(\Big(\xi_2-1\Big)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&+(1-i)\upsilon\Omega\Big(\epsilon^2\Big(\upsilon^3\Omega^3+2\Big)-2\rho^2\upsilon\Omega\Big)\sinh\Big(\left(\xi_2-1\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-i\upsilon^4\Omega^4\epsilon^2\sin\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+\upsilon^4\Omega^4\epsilon^2\sin\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-i\upsilon^4\Omega^4\epsilon^2\sinh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+\upsilon^4\Omega^4\epsilon^2\sinh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-2i\upsilon\Omega\epsilon^2\sin\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+2\upsilon\Omega\epsilon^2\sin\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-2i\upsilon\Omega\epsilon^2\sinh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+2\upsilon\Omega\epsilon^2\sinh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&+2\rho^3\epsilon\Big(-i\cos\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+\cos\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-(1-i)\cos\Big(\left(\xi_2-1\right)\rho\upsilon\Omega(\epsilon-1)\Big)\Big)+2i\rho^3\epsilon\cosh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-2\rho^3\epsilon\cosh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+(2-2i)\rho^3\epsilon\cosh\Big(\left(\xi_2-1\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&-2i\rho^2\upsilon^2\Omega^2\sin\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)+2\rho^2\upsilon^2\Omega^2\sin\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\\
&+2i\rho^2\upsilon^2\Omega^2\sinh\Big(\left(1-i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)-2\rho^2\upsilon^2\Omega^2\sinh\Big(\left(1+i\xi_2\right)\rho\upsilon\Omega(\epsilon-1)\Big)\Big)\\
&\Big(\rho^2\upsilon^2\Omega^2\Big(2\rho^3+2\cosh(\rho\upsilon\Omega(\epsilon-1))\left(\rho^3\cos(\rho\upsilon\Omega(\epsilon-1))+\upsilon\Omega\epsilon\sin(\rho\upsilon\Omega(\epsilon-1))\right)\\
&-\upsilon\Omega\epsilon\sinh(\rho\upsilon\Omega(\epsilon-1))(\rho\upsilon\Omega\epsilon\sin(\rho\upsilon\Omega(\epsilon-1))+2\cos(\rho\upsilon\Omega(\epsilon-1)))\Big)\Big)^{-1}. \quad (2.74)
\end{aligned}
$$

Figures 2.24-2.27 demonstrate the exact solution of the left component $\widetilde{w_2}$ (2.73) and asymptotic solution (2.74) for $\rho = 1$, $\upsilon = 1$ and several values of Ω and ε. As an example Figure 2.28 shows the maximum error over $0 \leq \xi_2 \leq 1$ between the exact solution (2.73) and the asymptotic solution (2.74) for $\Omega = 1$. Clearly, as in the previous case of a rod the maximum error is monotonically increasing for increasing ε.

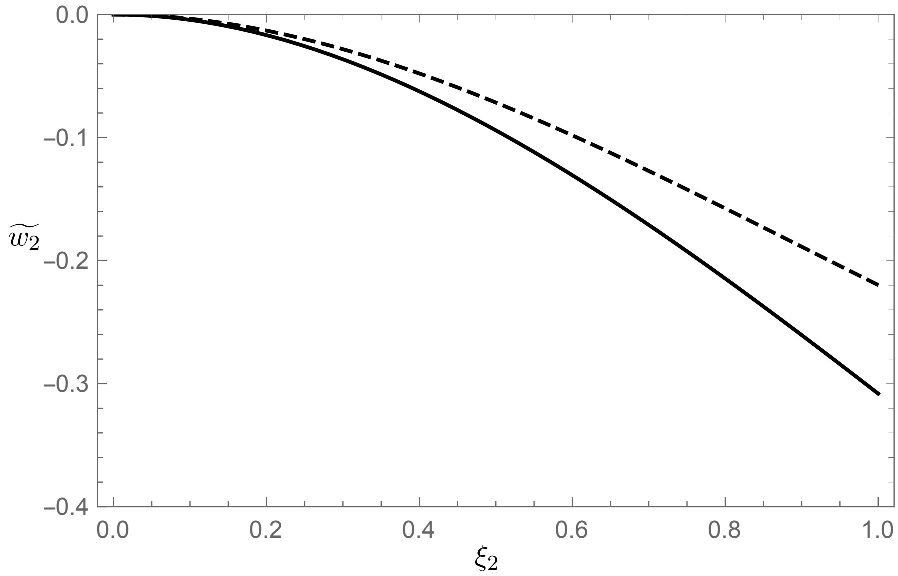

FIGURE 2.24: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1$, $\varepsilon = 0.1$.

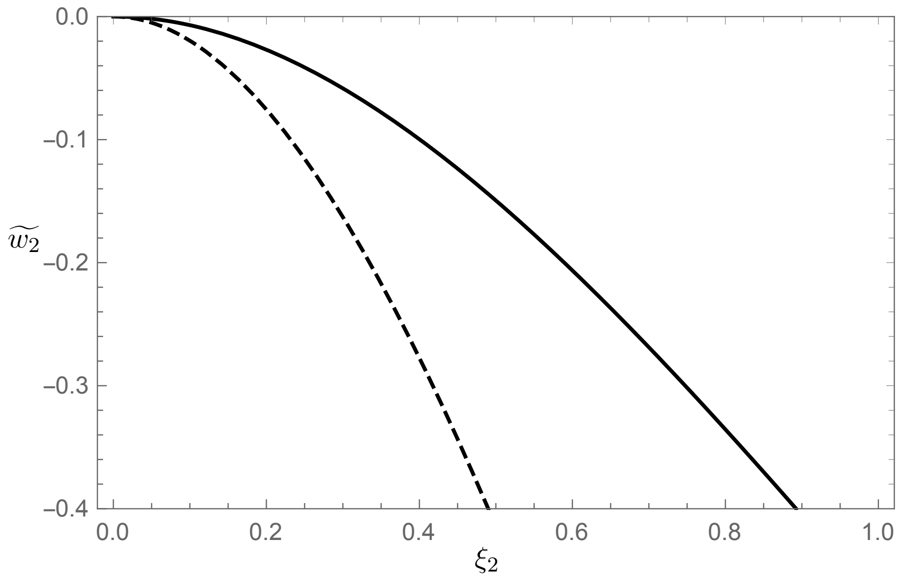

FIGURE 2.25: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1.5$, $\varepsilon = 0.1$.

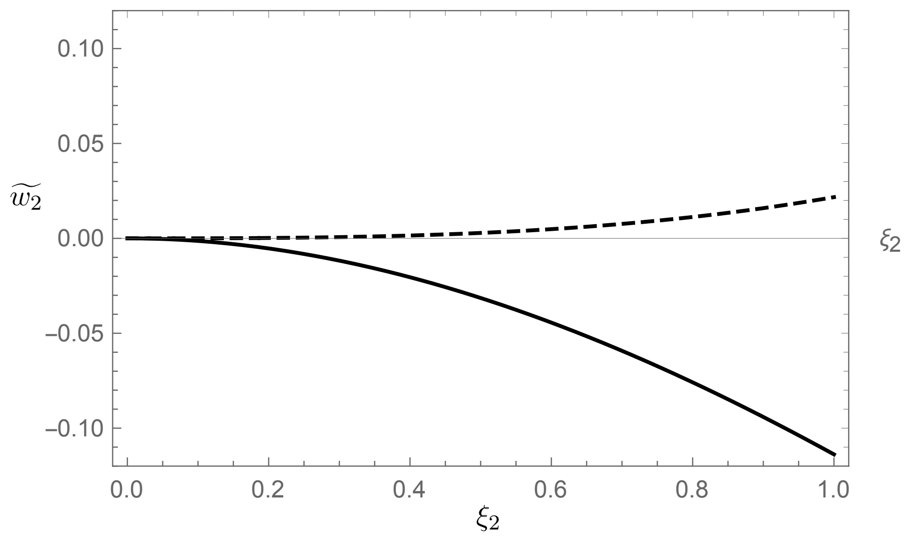

FIGURE 2.26: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1$, $\varepsilon = 0.5$.

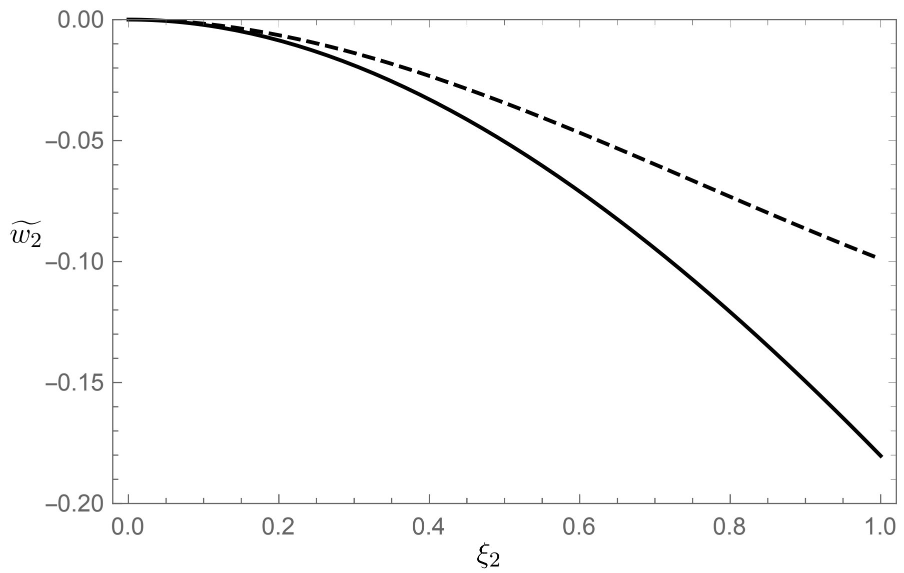

FIGURE 2.27: Comparison of asymptotic solution (2.58) (dashed line) and exact solution (2.57) (solid line) for $\Omega = 1.5$, $\varepsilon = 0.5$.

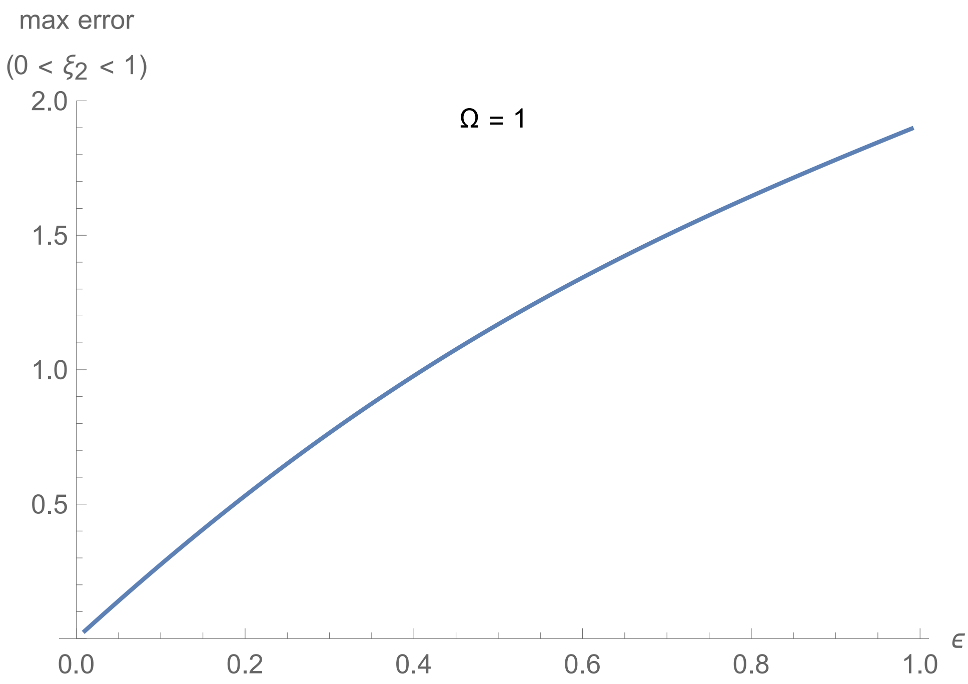

FIGURE 2.28: The maximum error between asymptotic solution (2.58) and exact solution (2.57) for $\Omega = 1$.

In conclusion, 1D problems for a composite rod and composite beam have been considered in this chapter. In section 2.1, we studied harmonic axial vibrations of a composite rod. We assumed the boundary conditions (2.2), corresponding to the fixed left end and the right end subject to external loading, and also the continuity conditions (2.3) are assumed, corresponding to the perfect contact of two components. Then, we obtained the exact solutions (2.8), (2.9) for a two component rod in dimensionless variables. The asymptotic integration method has been used to obtain the effective stress (2.17) on the interface between the components. Comparison of asymptotic solution (2.25) and exact solution (2.24) has been performed, showing a good agreement. In section 2.2, harmonic vibrations of a composite beam have been investigated. Two cases of the boundary conditions have been imposed, one corresponding to absence of the modified transverse shear force at the right end and another one with no bending moment at the same end. Exact solutions have been obtained for both sections. Then, a perturbation scheme has been established. The effective moment and shear force (2.47), (2.48),(2.66), (2.67) on the interface between the components have been derived. Finally, comparisons between the asymptotic solutions and the exact solutions have been presented for both cases.

Chapter 3

The elastic bending wave on the edge of a semi-infinite plate reinforced by a free strip plate

In this chapter, elastic waves localised near the edge of a semi-infinite plate reinforced by a strip plate are considered within the framework of the 2D classical theory for plate bending. In Section 3.1, the governing relations are presented, and then the exact dispersion relation for a composite plate is derived. In Section 3.2, the boundary value problem for the strip plate is subject to asymptotic analysis, assuming that a typical wavelength is much greater than the strip thickness. As a result, effective conditions along the interface corresponding to a plate reinforced by a beam with a narrow rectangular cross-section are established. In Section 3.3, the asymptotic results are validated by considering a model boundary value problem for a strip plate. Finally, in Section 3.4, the approximate dispersion relation is derived. The accuracy of the approximate dispersion relation is tested by comparison with the numerical data obtained from the 'exact' matrix relation for a composite plate. The effect of the problem parameters on the localisation rate is also studied.

3.1 Statement of the problem

Consider a thin isotropic elastic semi-infinite plate of thickness $2h$, reinforced by a strip plate of the same thickness and width H with $h << H$. The origin of the Cartesian coordinate system is chosen to be on the midplane of the composite plate with the $x-$axis directed along the edge of a strip plate and $y-$axis directed into the interior as shown in Figure 3.1.

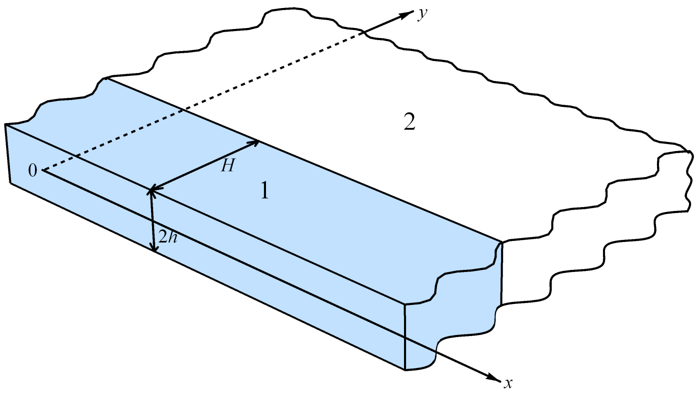

FIGURE 3.1: A semi-infinite plate with the edge coated by a strip plate.

The governing equation of motion in the classical Kirchhoff theory can be written as [9]

$$D_j \left(\frac{\partial^4 w_j}{\partial x^4} + 2 \frac{\partial^4 w_j}{\partial x^2 \partial y^2} + \frac{\partial^4 w_j}{\partial y^4} \right) + 2\rho_j h \frac{\partial^2 w_j}{\partial t^2} = 0, \qquad j = 1, 2, \tag{3.1}$$

where t is time, w_j are midplane deflections, ρ_j are mass densities, and D_j are bending stiffnesses given by

$$D_j = \frac{2E_j h^3}{3(1-\nu_j)},$$

where E_j are the Young's moduli and ν_j are the Poisson's ratios; hereinafter index 1 is used to denote parameters corresponding to the strip plate, whereas index 2 stays for the semi-infinite plate.

The boundary conditions on the free edge $y = 0$ are imposed in such a way that both bending moment and modified shear force are set to zero, i.e.

$$\frac{\partial^2 w_1}{\partial y^2} + \nu_1 \frac{\partial^2 w_1}{\partial x^2} = 0, \quad \frac{\partial^3 w_1}{\partial y^3} + (2-\nu_1)\frac{\partial^3 w_1}{\partial x^2 \partial y} = 0. \tag{3.2}$$

The continuity conditions along the interface $y = H$ for perfectly bonded plates are taken as

$$\begin{aligned}
&w_1 = w_2,\\
&\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial y},\\
&D_1\Big(\frac{\partial^2 w_1}{\partial y^2} + \nu_1\frac{\partial^2 w_1}{\partial x^2}\Big) = D_2\Big(\frac{\partial^2 w_2}{\partial y^2} + \nu_2\frac{\partial^2 w_2}{\partial x^2}\Big),\\
&D_1\Big(\frac{\partial^3 w_1}{\partial y^3} + (2-\nu_1)\frac{\partial^3 w_1}{\partial x^2 \partial y}\Big) = D_2\Big(\frac{\partial^3 w_2}{\partial y^3} + (2-\nu_2)\frac{\partial^3 w_2}{\partial x^2 \partial y}\Big).
\end{aligned} \tag{3.3}$$

The conventional harmonic travelling wave solution of plate bending equation (3.1) is given by

$$w_j(x, y, t) = w_j(y)e^{i(kx-\omega t)}, \quad j = 1, 2, \tag{3.4}$$

where ω is the frequency and k is wave number.

Substituting the latter into (3.1), we have

$$\frac{d^4 w_j(y)}{dy^4} - 2k^2 \frac{d^2 w_j(y)}{dy^2} + (k^4 - \frac{2\rho h \omega^2}{D_j}) w_j(y) = 0. \tag{3.5}$$

Then

$$w_j(y) = C_1^{(j)} e^{k\lambda_{1j} y} + C_2^{(j)} e^{-k\lambda_{1j} y} + C_3^{(j)} e^{k\lambda_{2j} y} + C_4^{(j)} e^{-k\lambda_{2j} y}, \tag{3.6}$$

where $\lambda_{1j} = \sqrt{1+\gamma_j}$, $\lambda_{2j} = \sqrt{1-\gamma_j}$, $\gamma_j = \dfrac{\omega}{k^2}\sqrt{\dfrac{2\rho_j h}{D_j}}$.

As a result, the deflection of each of the plate may be presented as

$$w_j(x,y,t) = e^{i(kx-wt)}\Big(C_1^{(j)} e^{k\lambda_{1j} y} + C_2^{(j)} e^{-k\lambda_{1j} y} + C_3^{(j)} e^{k\lambda_{2j} y} + C_4^{(j)} e^{-k\lambda_{2j} y}\Big), \tag{3.7}$$

where $C_1^{(j)}$, $C_2^{(j)}$, $C_3^{(j)}$ and $C_4^{(j)}$ are arbitrary constants. For the decaying at infinity solution corresponding to the sought for edge bending wave, we set $C_1^{(2)} = C_3^{(2)} = 0$. From the definition of λ_{2j} it follows that $1-\gamma_j \geq 0$, $j = 1,2$ and using $\gamma_1 = \sqrt{\dfrac{\rho}{D}}\gamma_2$, where $D = \dfrac{D_1}{D_2}$, $\rho = \dfrac{\rho_1}{\rho_2}$, resulting in a condition for material parameters of the plates

$$\frac{\rho_1}{\rho_2} \leq \frac{D_1}{D_2}.$$

This condition ensures localized waves.

Next, we insert formulae (3.7) into the boundary conditions (3.2) and continuity relations (3.3) leads to a homogeneous system of order six with the non-zero components of 6×6 matrix $\mathbf{M}$ given by

$$M_{11} = M_{12} = \lambda_{11}^2 - \nu_1, \quad M_{13} = M_{14} = \lambda_{21}^2 - \nu_1$$

$$M_{21} = -M_{22} = \lambda_{11}(\lambda_{11}^2 + \nu_1 - 2),$$

$$M_{23} = -M_{24} = \lambda_{21}(\lambda_{21}^2 + \nu_1 - 2),$$

$$M_{31} = \lambda_{11}e^{\lambda_{11}\delta}, \quad M_{32} = -\lambda_{11}e^{-\lambda_{11}\delta}, \quad M_{33} = \lambda_{21}e^{\lambda_{21}\delta},$$

$$M_{34} = -\lambda_{21}e^{-\lambda_{21}\delta}, \quad M_{35} = \lambda_{12}e^{-\lambda_{12}\delta}, \quad M_{36} = \lambda_{22}e^{-\lambda_{22}\delta},$$

$$M_{41} = e^{\lambda_{11}\delta}, \quad M_{42} = e^{-\lambda_{11}\delta}, \quad M_{43} = e^{\lambda_{21}\delta},$$

$$M_{44} = e^{-\lambda_{21}\delta}, \quad M_{45} = -e^{-\lambda_{12}\delta}, \quad M_{46} = -e^{-\lambda_{22}\delta},$$

$$M_{51} = D(\lambda_{11}^2 - \nu_1)e^{\lambda_{11}\delta}, \quad M_{52} = D(\lambda_{11}^2 - \nu_1)e^{-\lambda_{11}\delta},$$

$$M_{53} = D(\lambda_{21}^2 - \nu_1)e^{\lambda_{21}\delta}, \quad M_{54} = D(\lambda_{21}^2 - \nu_1)e^{-\lambda_{21}\delta},$$

$$M_{55} = -(\lambda_{12}^2 - \nu_2)e^{-\lambda_{12}\delta}, \quad M_{56} = -(\lambda_{22}^2 - \nu_2)e^{-\lambda_{22}\delta},$$

$$M_{61} = D\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2)e^{\lambda_{11}\delta}, \quad M_{62} = -D\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2)e^{-\lambda_{11}\delta},$$

$$M_{63} = D\lambda_{21}(\lambda_{21}^2 + \nu_1 - 2)e^{\lambda_{21}\delta}, \quad M_{64} = -D\lambda_{21}(\lambda_{21}^2 + \nu_1 - 2)e^{-\lambda_{21}\delta},$$

$$M_{65} = \lambda_{12}(\lambda_{12}^2 + \nu_2 - 2)e^{-\lambda_{12}\delta}, \quad M_{66} = \lambda_{22}(\lambda_{22}^2 + \nu_2 - 2)e^{-\lambda_{22}\delta},$$

where

$$D = \frac{D_1}{D_2}, \qquad \rho = \frac{\rho_1}{\rho_2}, \qquad \delta = kH, \tag{3.8}$$

and a relation

$$\gamma_1 = \sqrt{\frac{\rho}{D}}\gamma_2$$

has been used, which has non-zero solutions provided that det $(\mathbf{M}) = 0$. As a result, we deduce the dispersion relation

$$
\begin{aligned}
&D^2\Big(- \sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})(\nu_1 - \lambda_{11}^2)^2\lambda_{21}^6 + 2\Big(\cosh(\delta\lambda_{11})\cosh(\delta\lambda_{21}) - 1\Big)\\
&\times\lambda_{11}(\lambda_{11}^2 - \nu_1)(\lambda_{11}^2 + \nu_1 - 2)\lambda_{21}^5 - \sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})\\
&\times\Big(\lambda_{11}^6 + 4(\nu_1 - 2)\lambda_{11}^4 - (\nu_1 - 2)(3\nu_1 + 2)\lambda_{11}^2 + 2(\nu_1 - 2)\nu_1^2\Big)\lambda_{21}^4\\
&-4\Big(\cosh(\delta\lambda_{11})\cosh(\delta\lambda_{21}) - 1\Big)\lambda_{11}(\lambda_{11}^2 - \nu_1)(\lambda_{11}^2 + \nu_1 - 2)\lambda_{21}^3\\
&+\sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})\Big(2\nu_1\lambda_{11}^6 + (\nu_1 - 2)(3\nu_1 + 2)\lambda_{11}^4\\
&+4(\nu_1 - 2)^2\nu_1\lambda_{11}^2 - (\nu_1 - 2)^2\nu_1^2\Big)\lambda_{21}^2 + 2\Big(\cosh(\delta\lambda_{11})\cosh(\delta\lambda_{21}) - 1\Big)\\
&\times(\nu_1 - 2)\nu_1\lambda_{11}(\nu_1 - \lambda_{11}^2)(\lambda_{11}^2 + \nu_1 - 2)\lambda_{21} - \sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})\\
&\times\nu_1^2\lambda_{11}^2(\lambda_{11}^2 + \nu_1 - 2)^2\Big) + D\Big(\lambda_{11}\lambda_{21}\Big(\Big((\lambda_{22}^2 - 2)\lambda_{11}^2 + 2\nu_2\Big)\lambda_{21}^4\\
&+\Big((\lambda_{22}^2 - 2)\lambda_{11}^2(\lambda_{11}^2 - 4) - 4\nu_2\Big)\lambda_{21}^2 + 2\nu_1^3(\lambda_{22}^2 + 2\nu_2 - 2) + 2\nu_2\lambda_{11}^2(\lambda_{11}^2 - 2)\\
&+\lambda_{12}^2(\nu_1 - \lambda_{11}^2)(\nu_1 - \lambda_{21}^2)\Big(\lambda_{11}^2 + \lambda_{21}^2 + 2\nu_1 - 4\Big) + 2\lambda_{12}\lambda_{22}(\nu_1 - 1)\\
&\times\Big(- \lambda_{11}^4 + 2\lambda_{11}^2 - \lambda_{21}^4 + 2\lambda_{21}^2 + 2(\nu_1 - 2)\nu_1\Big) - \nu_1^2\Big((\lambda_{11}^2 + \lambda_{21}^2 + 4)\lambda_{22}^2\\
&+12\nu_2 - 2(\lambda_{11}^2 + \lambda_{21}^2 + 4)\Big) + \nu_1\Big(- (\lambda_{22}^2 - 2)\Big(\lambda_{11}^4 - 4\lambda_{11}^2 + \lambda_{21}^4 - 4\lambda_{21}^2\Big)\\
&-2\nu_2\Big(\lambda_{11}^4 - 2\lambda_{11}^2 + \lambda_{21}^4 - 2\lambda_{21}^2 - 4\Big)\Big)\Big) + \sinh(\delta\lambda_{21})\Big(\cosh(\delta\lambda_{11})(\lambda_{12} + \lambda_{22})\lambda_{11}\\
&\times(\lambda_{11} - \lambda_{21})(\lambda_{11} + \lambda_{21})\Big(\lambda_{12}\lambda_{22}(\lambda_{11}^2 + \nu_1 - 2)(\nu_1 - \lambda_{21}^2) - (\nu_1 - \lambda_{11}^2)\lambda_{21}^2\\
&\times(\lambda_{21}^2 + \nu_1 - 2)\Big) + \sinh(\delta\lambda_{11})\Big(\nu_2(\nu_1 - \lambda_{11}^2)\lambda_{21}^6 + \Big(2(\lambda_{22}^2 + \nu_2 - 2)\lambda_{11}^4\\
&+\Big((2 - 3\nu_1)\nu_2 - (\lambda_{22}^2 - 2)(\nu_1 + 2)\Big)\lambda_{11}^2 + \nu_1\Big(\nu_1(\lambda_{22}^2 + 3\nu_2 - 2) - 4\nu_2\Big)\Big)\lambda_{21}^4\\
&+\Big(- \nu_2\lambda_{11}^6 + \Big((2 - 3\nu_1)\nu_2 - (\lambda_{22}^2 - 2)(\nu_1 + 2)\Big)\lambda_{11}^4 - 2(\nu_1 - 2)\Big(\nu_1(2\lambda_{22}^2
\end{aligned}
$$

$$
\begin{aligned}
&+3\nu_2-4)-2\nu_2\Big)\lambda_{11}^2+(\nu_1-2)\nu_1\Big(\nu_1(\lambda_{22}^2+2\nu_2-2)-2\nu_2\Big)\Big)\lambda_{21}^2\\
&+\nu_1\lambda_{11}^2(\lambda_{11}^2+\nu_1-2)\Big(\nu_1(\lambda_{22}^2+2\nu_2-2)+\nu_2(\lambda_{11}^2-2)\Big)+\lambda_{12}^2\Big(2\lambda_{11}^4\\
&-(\nu_1+2)\lambda_{11}^2+\nu_1^2\Big)\lambda_{21}^4+\Big(-(\nu_1+2)\lambda_{11}^4-4(\nu_1-2)\nu_1\lambda_{11}^2+(\nu_1-2)\nu_1^2\Big)\lambda_{21}^2\\
&+\nu_1^2\lambda_{11}^2(\lambda_{11}^2+\nu_1-2)\Big)+\lambda_{12}\lambda_{22}\Big((\nu_1-\lambda_{11}^2)\lambda_{21}^6+\Big(2\lambda_{11}^4+(2-3\nu_1)\lambda_{11}^2\\
&+\nu_1(3\nu_1-4)\Big)\lambda_{21}^4+\Big(-\lambda_{11}^6+(2-3\nu_1)\lambda_{11}^4-2(\nu_1-2)(3\nu_1-2)\lambda_{11}^2\\
&+2(\nu_1-2)(\nu_1-1)\nu_1\Big)\lambda_{21}^2+\nu_1\lambda_{11}^2(\lambda_{11}^2+\nu_1-2)(\lambda_{11}^2+2\nu_1-2)\Big)\Big)\Big)\\
&+\cosh(\delta\lambda_{21})\lambda_{21}\Big(\sinh(\delta\lambda_{11})(\lambda_{12}+\lambda_{22})(\lambda_{11}-\lambda_{21})(\lambda_{11}+\lambda_{21})\Big(\lambda_{11}^2(\lambda_{11}^2+\nu_1-2)\\
&\times(\nu_1-\lambda_{21}^2)-\lambda_{12}\lambda_{22}(\nu_1-\lambda_{11}^2)(\lambda_{21}^2+\nu_1-2)\Big)+\cosh(\delta\lambda_{11})\lambda_{11}\Big(-\Big((\lambda_{22}^2-2)\\
&\times\lambda_{11}^2+2\nu_2\Big)\lambda_{21}^4+\Big(4\nu_2-(\lambda_{22}^2-2)\lambda_{11}^2(\lambda_{11}^2-4)\Big)\lambda_{21}^2-2\nu_1^3(\lambda_{22}^2+2\nu_2-2)\\
&-2\nu_2\lambda_{11}^2(\lambda_{11}^2-2)-\lambda_{12}^2(\nu_1-\lambda_{11}^2)(\nu_1-\lambda_{21}^2)(\lambda_{11}^2+\lambda_{21}^2+2\nu_1-4)-2\lambda_{12}\lambda_{22}\\
&\times(\nu_1-1)\Big(-\lambda_{11}^4+2\lambda_{11}^2-\lambda_{21}^4+2\lambda_{21}^2+2(\nu_1-2)\nu_1\Big)+\nu_1^2\Big((\lambda_{11}^2+\lambda_{21}^2+4)\lambda_{22}^2\\
&+12\nu_2-2(\lambda_{11}^2+\lambda_{21}^2+4)\Big)+\nu_1\Big((\lambda_{22}^2-2)(\lambda_{11}^4-4\lambda_{11}^2+\lambda_{21}^4-4\lambda_{21}^2)\\
&+2\nu_2(\lambda_{11}^4-2\lambda_{11}^2+\lambda_{21}^4-2\lambda_{21}^2-4)\Big)\Big)\Big)\Big)+\Big(-\nu_2^2-\Big((\lambda_{12}+\lambda_{22})^2-2\Big)\nu_2\\
&+\lambda_{12}\lambda_{22}(\lambda_{12}\lambda_{22}+2)\Big)\Big(-\lambda_{11}\lambda_{21}^5+\sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})\\
&(\nu_1-\lambda_{11}^2)\lambda_{21}^4+\lambda_{11}\Big(2\cosh(\delta\lambda_{11})\cosh(\delta\lambda_{21})(\lambda_{11}^2-1)+2\Big)\lambda_{21}^3\\
&-\sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})\Big(\lambda_{11}^4+2(\nu_1-2)\lambda_{11}^2-(\nu_1-2)\nu_1\Big)\lambda_{21}^2\\
&+\lambda_{11}\Big(-\lambda_{11}^4+2\lambda_{11}^2+2\nu_1^2-4\nu_1-2\cosh(\delta\lambda_{11})\cosh(\delta\lambda_{21})\\
&\Big(\lambda_{11}^2+(\nu_1-2)\nu_1\Big)\Big)\lambda_{21}+\sinh(\delta\lambda_{11})\sinh(\delta\lambda_{21})\nu_1\lambda_{11}^2(\lambda_{11}^2+\nu_1-2)\Big)=0. \qquad (3.9)
\end{aligned}
$$

Equation (3.9) at $H = 0$ coincides with Konenkov's dispersion relation [79], [34]. Indeed, we obtain from (3.9)

$$\lambda_{12}^2\lambda_{22}^2 + 2(1-\nu_2)\lambda_{12}\lambda_{22} - \nu_2^2 = 0,$$

resulting in

$$\lambda_{12}\lambda_{22} = \sqrt{1-c^4}, \tag{3.10}$$

with

$$c = \left[(1-\nu_2)\left(3\nu_2 - 1 + 2\sqrt{2\nu_2^2 - 2\nu_2 + 1}\right)\right]^{\frac{1}{4}}.$$

The last dispersion relation can be re-written as

$$D_2 k^4 c^4 = 2\rho_2 h \omega^2. \tag{3.11}$$

The goal of the chapter is to derive a perturbation to the above mentioned Konenkov's edge bending wave on a homogeneous plate, assuming that $H \ll l$, where l is a typical wave length for a travelling harmonic wave. Instead of studying a pretty tedious dispersion relation (3.9), in what follows we reduce the influence of the plate strip to effective boundary conditions along the interface $y = H$, similarly to those for a coated elastic half-space, e.g. see [34].

3.2 Effective Boundary Conditions

In order to obtain effective boundary conditions we first aim at expressing the bending moment and the modified shear force at the interface $y = H$ through given deflection and angle of rotation. The strip plate is considered separately as shown in Figure 3.2, having traction free upper face and prescribed displacement and rotation angle on the lower surface of the plate. Thus, for the strip plate we are solving the equation of motion (3.1) subject to the traction free boundary conditions (3.2) at $y = 0$. At the interface $y = H$ we have

$$w_1\big|_{y=H} = w_H, \qquad \frac{\partial w_1}{\partial y}\bigg|_{y=H} = \frac{1}{l} G_H, \tag{3.12}$$

where functions $w_H = w_H(x, t)$ and $G_H = G_H(x, t)$ are assumed to be known.

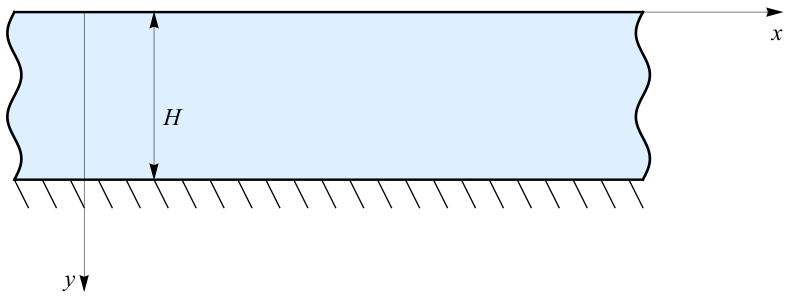

FIGURE 3.2: A strip plate with free upper side and loaded lower side considered separately

Below, we adapt the asymptotic methodology similar to that for thin elastic structures, e.g. see [1], [34] and references therein. First, introduce the following dimensionless variables

$$\xi = \frac{x}{l}, \qquad \eta_1 = \frac{y}{H}, \qquad \tau = \sqrt{\frac{D_1}{2\rho_1 h}}\,\frac{t}{l^2}, \tag{3.13}$$

where τ has been introduce in such a way that it would asymptotically balance a fourth order plate bending equation.

In terms of new variables the governing equation (3.1) becomes

$$\frac{\partial^4 w_1}{\partial \eta_1^4} + 2\varepsilon^2 \frac{\partial^4 w_1}{\partial \xi^2 \partial \eta_1^2} + \varepsilon^4 \left(\frac{\partial^4 w_1}{\partial \xi^4} + \frac{\partial^2 w_1}{\partial \tau^2} \right) = 0, \tag{3.14}$$

subject to the boundary conditions

$$\begin{aligned} &\frac{\partial^2 w_1}{\partial \eta_1^2} + \varepsilon^2 \nu_1 \frac{\partial^2 w_1}{\partial \xi^2} = 0\,, \quad \frac{\partial^3 w_1}{\partial \eta_1^3} + \varepsilon^2 (2-\nu_1) \frac{\partial^3 w_1}{\partial \xi^2 \partial \eta_1} = 0 \quad \text{at} \quad \eta_1 = 0, \\ &w_1 = w_H, \qquad \frac{\partial w_1}{\partial \eta_1} = \varepsilon G_H \quad \text{at} \quad \eta_1 = 1, \end{aligned} \tag{3.15}$$

where a small parameter ε has been introduced as

$$\varepsilon = \frac{H}{l}.$$

A deflection w_1 can be expanded into an asymptotic series in terms of ε as

$$w_1 = w_1^{(0)} + w_1^{(1)}\varepsilon + w_1^{(2)}\varepsilon^2 + w_1^{(3)}\varepsilon^3 + w_1^{(4)}\varepsilon^4 + \dots \tag{3.16}$$

Substituting expansion (3.16) into the boundary value problem (3.14)-(3.15), we arrive at the problem formulated for various asymptotic orders $n = 0, 1, 2, \dots,$ namely

$$\frac{\partial^4 w_1^{(n)}}{\partial \eta_1^4} + 2 \frac{\partial^4 w_1^{(n-2)}}{\partial \xi^2 \partial \eta_1^2} + \frac{\partial^4 w_1^{(n-4)}}{\partial \xi^4} + \frac{\partial^2 w_1^{(n-4)}}{\partial \tau^2} = 0, \tag{3.17}$$

subject to

$$\begin{aligned}
&\frac{\partial^2 w_1^{(n)}}{\partial \eta_1^2} + \nu_1 \frac{\partial^2 w_1^{(n-2)}}{\partial \xi^2} = 0\,, \quad \frac{\partial^3 w_1^{(n)}}{\partial \eta_1^3} + (2-\nu_1)\frac{\partial^3 w_1^{(n-2)}}{\partial \xi^2 \partial \eta_1} = 0 \quad \text{at} \quad \eta_1 = 0, \\
&w_1^{(n)} = w_H^{(n)}, \qquad \frac{\partial w_1^{(n)}}{\partial \eta_1} = G_H^{(n)} \quad \text{at} \quad \eta_1 = 1,
\end{aligned} \tag{3.18}$$

where quantities with the negative superscript are set to be equal to zero. The only non-zero components $w_H^{(n)}$ and $G_H^{(n)}$ are $w_H^{(0)} = w_H$ and $G_H^{(1)} = G_H$, respectively.

Substituting subsequently $n = 0, 1, 2, 3, 4$ into (3.17)-(3.18) we obtain

$$\begin{aligned}
w_1^{(0)} &= w_H, \\
w_1^{(1)} &= G_H(\eta_1 - 1), \\
w_1^{(2)} &= -\frac{\nu_1(\eta_1-1)^2}{2}\frac{\partial^2 w_H}{\partial \xi^2}, \\
w_1^{(3)} &= \frac{\partial^2 G_H}{\partial \xi^2}\left(\frac{(\nu_1-2)}{6}\eta_1^3 + \frac{\nu_1}{2}\eta_1^2 + \frac{(2-3\nu_1)}{2}\eta_1 - \frac{4-5\nu_1}{6}\right), \\
w_1^{(4)} &= \frac{1}{24}\Big((2\nu_1-1)\frac{\partial^4 w_H}{\partial \xi^4} - \frac{\partial^2 w_H}{\partial \tau^2}\Big)\eta_1^4 + \frac{\nu_1(\nu_1-2)}{6}\frac{\partial^4 w_H}{\partial \xi^4}\eta_1^3 \\
&\quad + \frac{\nu_1^2}{4}\frac{\partial^4 w_H}{\partial \xi^4}\eta_1^2 + \frac{1}{6}\Big(\frac{\partial^2 w_H}{\partial \tau^2} - (6\nu_1^2 - 4\nu_1 - 1)\frac{\partial^4 w_H}{\partial \xi^4}\Big)\eta_1 \\
&\quad + \frac{1}{24}(14\nu_1^2 - 10\nu_1 - 3)\frac{\partial^4 w_H}{\partial \xi^4} - \frac{1}{8}\frac{\partial^2 w_H}{\partial \tau^2}.
\end{aligned} \tag{3.19}$$

Now, using expansion (3.16) together with continuity conditions for moments and shear forces on the interface, we obtain effective boundary conditions for the semi-infinite plate at $\eta_2 = \varepsilon$ in the form

$$\begin{aligned}
&D_2\Big((2-\nu_2)\frac{\partial^3 w_2}{\partial \xi^2 \partial \eta_2} + \frac{\partial^3 w_2}{\partial \eta_2^3}\Big) = -\varepsilon D_1\Big((1-\nu_1^2)\frac{\partial^4 w_2}{\partial \xi^4} + \frac{\partial^2 w_2}{\partial \tau^2}\Big), \\
&D_2\Big(\nu_2\frac{\partial^2 w_2}{\partial \xi^2} + \frac{\partial^2 w_2}{\partial \eta_2^2}\Big) = -\varepsilon D_1 2(1-\nu_1)\frac{\partial^3 w_2}{\partial \xi^2 \partial \eta_2}.
\end{aligned} \tag{3.20}$$

Returning back to the original variables, effective boundary conditions at $y = H$ can be re-written as

$$\begin{aligned} &D_2\Big((2-\nu_2)\frac{\partial^3 w_2}{\partial x^2 \partial y} + \frac{\partial^3 w_2}{\partial y^3}\Big) = -D_1 H \left((1-\nu_1^2)\frac{\partial^4 w_2}{\partial x^4} + \frac{2\rho_1 h}{D_1}\frac{\partial^2 w_2}{\partial t^2}\right), \\ &D_2\Big(\nu_2 \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\Big) = -D_1 H \; 2(1-\nu_1)\frac{\partial^3 w_2}{\partial x^2 \partial y}. \end{aligned} \tag{3.21}$$

The right hand sides of the above equations can be re-written to demonstrate analogy with the problem for a semi-infinite plate reinforced by a beam with a narrow rectangular cross-section $2h \times H$, taking the form

$$\begin{aligned} &D_2\Big((2-\nu_2)\frac{\partial^3 w_2}{\partial x^2 \partial y} + \frac{\partial^3 w_2}{\partial y^3}\Big) = -E_1 I_y \frac{\partial^4 w_2}{\partial x^4} - \rho_1 A \frac{\partial^2 w_2}{\partial t^2}, \\ &D_2\Big(\nu_2 \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\Big) = -G_1 J_t \frac{\partial^3 w_2}{\partial x^2 \partial y}, \end{aligned} \tag{3.22}$$

where A is the area of the beam's cross section, $G_1 = \dfrac{E_1}{2(1+\nu_1)}$ is the shear modulus of the beam, I_y is the area moment of inertia, and J_t is the torsional constant, given by

$$A = 2hH, \qquad I_y = \frac{2}{3}Hh^3, \qquad J_t = \frac{8}{3}Hh^3.$$

To conclude, the formulae (3.22) are, to within the inertial terms, identical to the boundary conditions in [38],[39],[91],[108], written for a beam with an arbitrary cross-section. It is worth noting that the rotation inertia does not appear at the second equation (3.22) at the leading order approximation.

3.3 Testing of effective boundary conditions

For validating the asymptotic results obtained in the previous section, consider a model boundary value problem for a strip plate over the domain $0 \leqslant y \leqslant H$. We take an equation of motion (3.1) for $j = 1$ subject to homogeneous boundary conditions (3.2) at $y = 0$ and impose the boundary conditions

$$w_1\big|_{y=H} = w_H, \qquad \left.\frac{\partial w_1}{\partial y}\right|_{y=H} = kG_H, \tag{3.23}$$

at $y = H$ with functions $w_H(x,t)$ and $G_H(x,t)$ specified as travelling waves $w_H = Ae^{i(kx-\omega t)}$ and $G_H = Be^{i(kx-\omega t)}$, respectively.

The solution of the formulated problem is then sought for in the form (3.7) for $j = 1$, and we finally arrive at a set of four linear algebraic equations which can be written in a matrix form as

$$\mathrm{Q} \cdot \mathrm{C} = \mathrm{U}, \tag{3.24}$$

where $\mathrm{C} = \left(C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}\right)^T$, $\mathrm{U} = (0, 0, A, kB)^T$ are vectors and Q is a 4×4 matrix with the components given by

$$
\begin{aligned}
&Q_{11} = Q_{12} = (\lambda_{11}^2 - \nu_1), \;\; Q_{13} = Q_{14} = (\lambda_{21}^2 - \nu_1),\\
&Q_{21} = -Q_{22} = \lambda_{11}^3 - (2 - \nu_1)\lambda_{11},\\
&Q_{23} = -Q_{24} = \lambda_{21}^3 - (2 - \nu_1)\lambda_{21},\\
&Q_{31} = e^{\lambda_{11}\delta}, \;\; Q_{32} = e^{-\lambda_{11}\delta}, \;\; Q_{33} = e^{\lambda_{21}\delta}, \;\; Q_{34} = e^{-\lambda_{21}\delta},\\
&Q_{41} = \lambda_{11} e^{\lambda_{11}\delta}, \;\; Q_{42} = -\lambda_{11} e^{-\lambda_{11}\delta},\\
&Q_{43} = \lambda_{21} e^{\lambda_{21}\delta}, \;\; Q_{44} = -\lambda_{21} e^{-\lambda_{21}\delta}.
\end{aligned}
$$

The sought for constants $C_i^{(1)}$, $i = 1, 2, 3, 4$ are presented in Appendix C.1.

Next, re-write a solution (3.7) for w_1 in terms of dimensionless variables and expand it into Taylor series about $\delta = 0$, where δ is defined in (3.8), arriving at the asymptotic expansion

$$
\begin{aligned}
w_1 = e^{i(\xi - \gamma_1\tau)}\Big(&A + \delta B(\eta_1 - 1) + \delta^2 A \frac{\nu_1}{2}(\eta_1 - 1)^2\\
&- \delta^3 \frac{B}{6}(\eta_1 - 1)^2\left((\nu_1 - 2)\eta_1 + 5\nu_1 - 4\right)\\
&+ \delta^4 \frac{A}{24}(\eta_1 - 1)^2\big((\gamma_1^2 + 2\nu_1 - 1)\eta_1^2 + (2\gamma_1^2 + 4\nu_1^2 - 4\nu_1 - 2)\eta_1\\
&\qquad + 3\gamma_1^2 + 14\nu_1^2 - 10\nu_1 - 3\big)\Big) + \ldots.
\end{aligned}
\tag{3.25}
$$

It can be easily demonstrated, that at $\varepsilon = \delta$, i.e. at $l = 1/k$ the last formula coincides with the expansion (3.16) with the coefficients (3.19), in which functions w_H and G_H are defined as $w_H = Ae^{i(\xi - \gamma_1\tau)}$ and $G_H = Be^{i(\xi - \gamma_1\tau)}$.

Let us now test the derived effective boundary conditions for the chosen w_H and G_H.

To this end, express first the continuity conditions (3.3) as

$$
\begin{aligned}
&D_2\Big((2-\nu_2)\frac{\partial^3 w_2}{\partial \xi^2 \partial \eta_2}+\frac{\partial^3 w_2}{\partial \eta_2^3}\Big)=\delta^{-1}D_1\Big((2-\nu_1)\frac{\partial^3 w_1}{\partial \xi^2 \partial \eta_1}+\delta^{-2}\frac{\partial^3 w_1}{\partial \eta_1^3}\Big),\\
&D_2\Big(\nu_2\frac{\partial^2 w_2}{\partial \xi^2}+\frac{\partial^2 w_2}{\partial \eta_2^2}\Big)=D_1\Big(\nu_1\frac{\partial^2 w_1}{\partial \xi^2}+\delta^{-2}\frac{\partial^2 w_1}{\partial \eta_1^2}\Big).
\end{aligned}
\tag{3.26}
$$

Substituting expansion (3.25) into the right hand sides of the above equations, and keeping only $O(\delta)$ terms, we obtain effective boundary conditions (3.20), where $w_2 = w_H = Ae^{i(\xi-\gamma_1\tau)}$ and $\dfrac{\partial w_2}{\partial \eta_2} = G_H = Be^{i(\xi-\gamma_1\tau)}$ are inserted into the right hand side of equations (3.20), namely

$$
\begin{aligned}
&D_2\Big((2-\nu_2)\frac{\partial^3 w_2}{\partial \xi^2 \partial \eta_2}+\frac{\partial^3 w_2}{\partial \eta_2^3}\Big)=\delta D_1 A(\gamma_1^2+\nu_1^2-1)+O(\delta^2),\\
&D_2\Big(\nu_2\frac{\partial^2 w_2}{\partial \xi^2}+\frac{\partial^2 w_2}{\partial \eta_2^2}\Big)=-\delta 2D_1 B(\nu_1-1)+O(\delta^2).
\end{aligned}
$$

Clearly, the right hand side of the last equations is small as $\delta \longrightarrow 0$, showing the effect of a thin coating plate. The purpose of approach is restricted to the long wave domain given by the strong inequality $\delta \ll 1$.

3.4 Approximate dispersion relation

We aim at finding an asymptotic dispersion relation over the domain $H \leqslant y \leqslant \infty$ by using the derived effective boundary conditions (3.21) together with equation of motion (3.1). The solution is sought for in the form of a travelling harmonic wave (3.7). As a result, we deduce an approximate dispersion relation, which could be re-written in dimensionless variables as

$$\begin{aligned} &2(1-\nu_2)\sqrt{1-\gamma_2^2}+1-\gamma_2^2-\nu_2^2 \\ &+\delta\sqrt{2\left(1+\sqrt{1-\gamma_2^2}\right)}\left(D\left(1-\nu_1^2+2(1-\nu_1)\sqrt{1-\gamma_2^2}\right)-\rho\gamma_2^2\right)=0. \end{aligned} \tag{3.27}$$

Setting $\delta = 0$ in the above equation, we obtain the well-known dispersion relation of the bending wave on a free edge of a semi-infinite homogeneous plate [79]. Introducing new notation

$$\phi = \frac{\sqrt{1-\gamma_2^2}}{\nu_2^2}, \tag{3.28}$$

above equation (3.27) can be presented in the form

$$\begin{aligned} &\nu_2^2\left(\phi^2\nu_2^2+2\phi(1-\nu_2)-1\right) \\ &+\delta\sqrt{2\left(1+\phi\nu_2^2\right)}\left(D(1-\nu_1)(1+\nu_1+2\phi\nu_2^2)-\rho\left(1-\phi^2\nu_2^4\right)\right)=0. \end{aligned} \tag{3.29}$$

In this equation $\delta = kH \sim \dfrac{H}{l} \ll 1$ where $l \sim \dfrac{1}{k}$ is a typical wavelength which is much greater due the original assumption. Expanding ϕ into a series about $\delta = 0$

$$\phi = \phi_0 + \phi_1\delta + \ldots \tag{3.30}$$

and substituting it into dispersion relation (3.29), we obtain

$$\phi_0 = \frac{-1+\nu_2+\sqrt{2\nu_2^2-2\nu_2+1}}{\nu_2^2},$$
$$\phi_1 = -\sqrt{1+\phi_0\nu_2^2}\,\frac{\left(D(1-\nu_1)(1+\nu_1+2\phi_0\nu_2^2)-\rho(1-\phi_0^2\nu_2^4)\right)}{\sqrt{2}\nu_2^2(\phi_0\nu_2^2-\nu_2+1)},$$

where ϕ_0 corresponds to Konenkov's wave on a free edge of a homogeneous semi-infinite plate, while ϕ_1 is associated with the correction due to the effect of a strip plate.

Figures 3.3 and 3.4 demonstrate the solutions of the exact dispersion relation (3.9) and its asymptotic expansion (3.30) for several values of the relative stiffness D and relative density ρ of the strip plate. In Figure 3.3 the function ϕ is plotted at $\rho = 1.0$ and $D = 1.0,\ 1.1,\ 1.3$. In Figure 3.4 $D = 1.0$ and $\rho = 1.0,\ 0.95,\ 0.8$. Figure 3.3 shows that decay of ϕ is greater at larger D and, hence, the localisation of the edge wave becomes more pronounced. Similarly, in Figure 3.4, the decay rate decreases at larger ρ, and as a result, the edge wave has a greater spread over the interior.

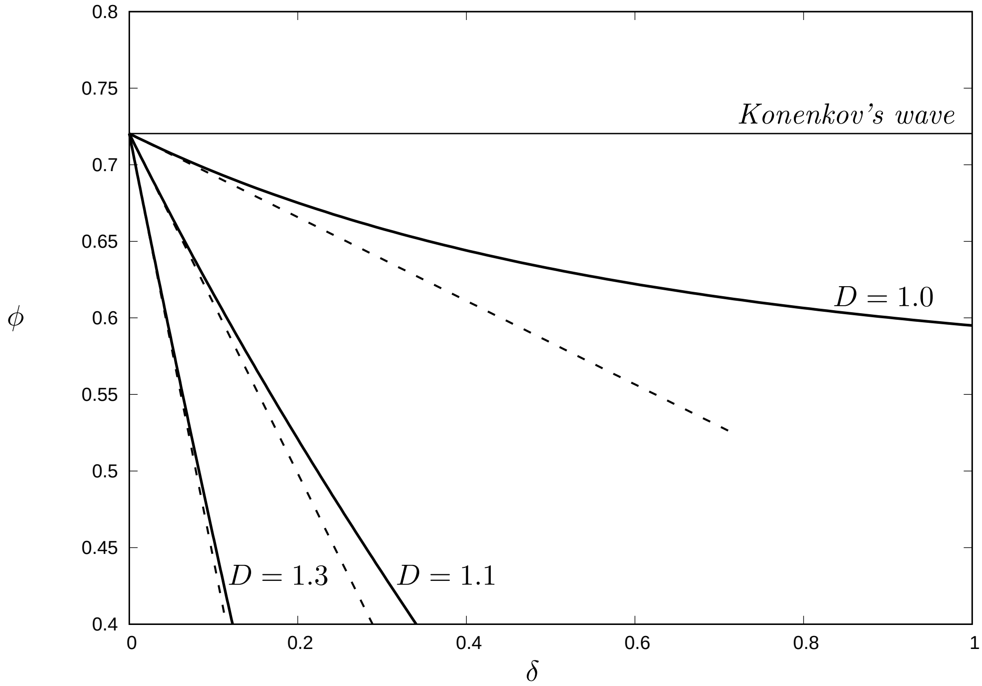

FIGURE 3.3: Comparison of asymptotic expansion (3.30) (dashed line) and exact dispersion relation (3.9) (solid line) for $\rho = 1.0$ and $D = 1.0$, 1.1, 1.3.

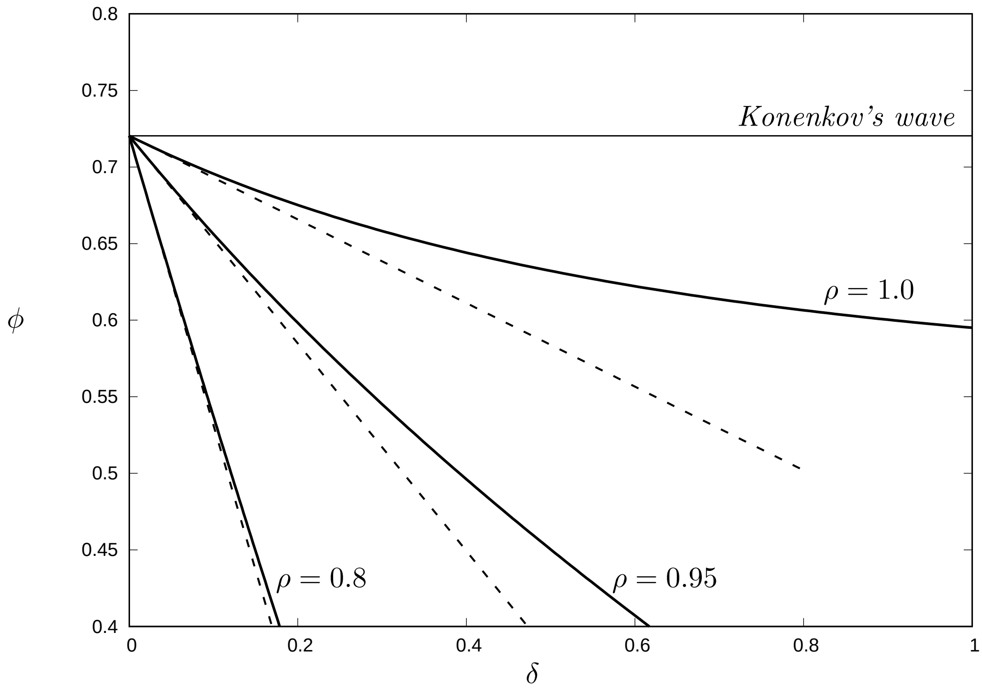

FIGURE 3.4: Comparison of asymptotic expansion (3.30) (dashed line) and exact dispersion relation (3.9) (solid line) for $D = 1.0$ and $\rho = 1.0$, 0.95, 0.8.

To summarize this chapter, a semi-infinite plate with the edge reinforced by a strip plate has been studied. For a strip plate, an asymptotic procedure has been developed to derive effective boundary conditions at the interface. The associated approximate dispersion relation for edge bending waves have been analysed. The obtained results have been compared with the exact solution of the original problem for a composite plate. The results obtained in this chapter motivated us for considering a simpler setup, namely, a plate with edge reinforced by a beam, which is exposed in the next chapter 4.

Chapter 4

The edge bending wave on a plate reinforced by a beam

The chapter is organized as follows. In Section 4.1, the edge bending wave on a thin isotropic semi-infinite plate reinforced by a beam is considered within the framework of the classical plate and beam theories. The boundary conditions at the plate edge incorporate both dynamic bending and twisting of the beam. Then, in Section 4.2, a dispersion relation is derived along with its long-wave approximation. The effect of the problem parameters on the cutoffs of the wave in question is studied asymptotically. Finally, in Section 4.3, an illustrative example of comparison of the dispersion curves for a composite plate-plate structure and a plate reinforced by a beam is presented.

4.1 Statement of the problem

Consider a thin isotropic elastic plate stiffened by an elastic beam along the edge. The Cartesian coordinate system is chosen in such a way that x and y are in the midplane of the plate with x going along the interface, see Figure 4.1. The equation

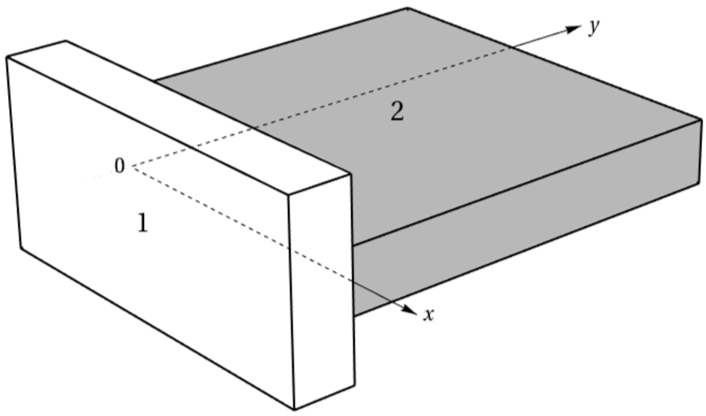

FIGURE 4.1: Plate reinforced by a beam

of motion for the midplane deflection w_2 in the classical theory for plate bending is

$$D_2\left(\frac{\partial^4 w_2}{\partial x^4}+2\frac{\partial^4 w_2}{\partial x^2\partial y^2}+\frac{\partial^4 w_2}{\partial y^4}\right)+2\rho_2 h\frac{\partial^2 w_2}{\partial t^2}=0, \tag{4.1}$$

where D_2 is bending stiffness of the plate, h is the half thickness of the plate, and t is time. Also, in what follows ρ_j are mass densities, E_j are Young's moduli, G_j are shear moduli, ν_j are Poisson's ratios, $j = 1, 2$. Indexes 1 and 2 correspond to the beam and plate, respectively.

The boundary conditions for the plate edge $y = 0$ maybe obtained by considering the beam flexure and twisting, see for example [108], resulting in

$$\begin{aligned} E_1 I_y \frac{\partial^4 w_2}{\partial x^4} + \rho_1 A \frac{\partial^2 w_2}{\partial t^2} &= -D_2 \Big(\frac{\partial^3 w_2}{\partial y^3} + (2 - \nu_2) \frac{\partial^3 w_2}{\partial x^2 \partial y} \Big), \\ G_1 J_t \frac{\partial^3 w_2}{\partial x^2 \partial y} - \rho_1 J \frac{\partial^3 w_2}{\partial t^2 \partial y} &= -D_2 \Big(\frac{\partial^2 w_2}{\partial y^2} + \nu_2 \frac{\partial^2 w_2}{\partial x^2} \Big), \end{aligned} \tag{4.2}$$

where I_y and J are the area and polar moments of inertia of the beam's cross section, J_t is the torsional constant, and A is the area of the beam's cross section.

4.2 Dispersion relation

The solution of the equation (4.1) is sought for in the form of a conventional harmonic travelling wave as

$$w_2(x, y, t) = w_2(y) e^{i(kx - \omega t)}, \tag{4.3}$$

where ω is frequency, and k is wave number. Substituting (4.3) into (4.1), we arrive at the edge wave in the form

$$w_2(y) = C_1 e^{-k\lambda_1 y} + C_2 e^{-k\lambda_2 y},$$

where C_1 and C_2 are arbitrary constants, and

$$\lambda_1 = \sqrt{1 + \gamma_2}, \quad \lambda_2 = \sqrt{1 - \gamma_2}, \quad \gamma_2 = \frac{\omega}{k^2} \sqrt{\frac{2\rho_2 h}{D_2}}.$$

Now, on substituting (4.3) into the boundary conditions (4.2) we arrive at the 2×2 set of linear equations, leading to the general exact dispersion relation

$$\begin{aligned}
&(\lambda_1\lambda_2 + 1)^2 - \nu_2(\lambda_1 + \lambda_2)^2 - (1-\nu_2)^2 \\
&- (\lambda_1 + \lambda_2)(\alpha_1\gamma_2^2\rho - \beta_2\lambda_1\lambda_2 - \beta_1)\delta_h \\
&- \beta_2(\alpha_1\gamma_2^2\rho - \beta_1)\delta_h^2 - \alpha_2\gamma_2^2\rho\lambda_1\lambda_2(\lambda_1+\lambda_2)\delta_h^3 \\
&+ \alpha_2\gamma_2^2\rho(\alpha_1\gamma_2^2\rho - \beta_1)\delta_h^4 = 0,
\end{aligned} \tag{4.4}$$

where

$$\beta_1 = \frac{E_1 I_y}{hD_2}, \quad \beta_2 = \frac{G_1 J_t}{hD_2}, \quad \alpha_1 = \frac{A}{2h^2}, \quad \alpha_2 = \frac{J}{2h^4},$$

and $\delta_h = kh$, $\rho = \dfrac{\rho_1}{\rho_2}$.

Setting $\delta_h = 0$ in (4.4) and returning back to original variables we get the well known relation for a free plate edge, see e.g. [79]

$$D_2 k^4 c^4 = 2\rho_2 h\omega^2,$$

where

$$c = \left[(1-\nu_2)\left(3\nu_2 - 1 + 2\sqrt{2\nu_2^2 - 2\nu_2 + 1}\right)\right]^{1/4}.$$

Let us next introduce a new unknown function by

$$\phi = \frac{\sqrt{1-\gamma_2^2}}{\nu_2^2}, \tag{4.5}$$

corresponding to the appropriately normalised attenuation rate which is not sensitive to the value of a Poison's ratio. This is seemingly the most relevant characteristic of slowly decaying edge bending waves. Hence, equation (4.4) can be re-written as

$$\begin{aligned}
&(1+\nu_2^2\phi)^2 - 2\nu_2(1+\nu_2^2\phi) - (1-\nu_2)^2 \\
&- \sqrt{2(1+\nu_2^2\phi)}\left(\alpha_1\rho\left(1-\nu_2^4\phi^2\right) - \beta_2\nu_2^2\phi - \beta_1\right)\delta_h \\
&- \beta_2\left(\alpha_1\rho(1-\nu_2^4\phi^2) - \beta_1\right)\delta_h^2 \\
&- \alpha_2\nu_2^2\rho(1-\nu_2^4\phi^2)\phi\sqrt{2(1+\nu_2^2\phi)}\delta_h^3 \\
&+ \alpha_2\rho\left(\alpha_1\rho(1-\nu_2^4\phi^2) - \beta_1\right)\left(1-\nu_2^4\phi^2\right)\delta_h^4 = 0.
\end{aligned} \tag{4.6}$$

At $\phi = 0$ ($\gamma_2 = 1$) we have for cut-off values

$$\nu_2^2 + (\alpha_1\rho - \beta_1)\left(\sqrt{2}\delta_h + \beta_2\delta_h^2 - \alpha_2\rho\delta_h^4\right) = 0. \tag{4.7}$$

Over the range of validity of thin plate theory ($\delta_h \ll 1$) we get at leading order

$$\delta_h^* \approx \frac{\nu_2^2}{\sqrt{2}(\beta_1 - \alpha_1\rho)}. \tag{4.8}$$

Thus, in the considered case of no contrast in material parameters ($\alpha_1 \sim \beta_1 \sim \rho \sim 1$), the cut-offs under investigation (at zero wave number) should satisfy $0 < \delta_h^* \ll 1$, provided that $\nu_2 \ll 1$ and $\beta_1 > \alpha_1\rho$. Higher-order cut-offs, considered in paper [17] are outside of the validity of the present consideration.

Next, expanding ϕ into a series about $\delta_h = 0$

$$\phi = \phi_0 + \phi_1 \delta_h + \dots \tag{4.9}$$

and substituting into the dispersion relation (4.6), we obtain

$$\phi_0 = \frac{\nu_2 - 1 + \sqrt{2\nu_2^2 - 2\nu_2 + 1}}{\nu_2^2}, \tag{4.10}$$

and

$$\phi_1 = \frac{\left((1 - \nu_2^4 \phi_0^2)\rho\alpha_1 - \beta_1 - \beta_2 \nu_2^2 \phi_0\right)\sqrt{2(1 + \nu_2^2 \phi_0)}}{2\nu_2^4 \phi_0 - 2(\nu_2 - 1)\nu_2^2}. \tag{4.11}$$

It is worth noting that (4.8) and (4.9)-(4.11) do not contain the parameter α_2 involving rotational inertia of the beam. This is inline with the asymptotic analysis of a similar problem for the edge reinforcement in the form of a plate strip in [9]. In addition, (4.8) also does not depend on parameter β_2, expressing the effect of torsional rigidity.

4.3 Example (Comparison of the dispersion curves for a composite plate-plate structure and a plate reinforced by a beam)

In this section we present the results of numerical comparison of the dispersion curves for a plate reinforced by a beam and a composite 'plate-plate' structure, in order to validate the 'plate-beam' model in the previous section. To this end, consider bending of a semi-infinite Kirchhoff plate reinforced by a strip plate along the edge as shown in Figure 4.2, assuming that for the strip plate $H \gg h$. For the latter, the equation of motion follows from (4.1) by substituting 1 instead of 2 in all the suffices.

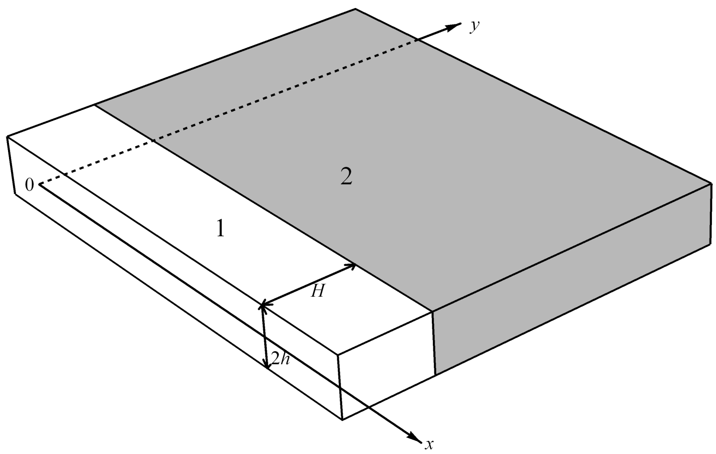

FIGURE 4.2: Plate reinforced by a strip plate

Traction free boundary conditions on the edge $y = 0$ are given by

$$\frac{\partial^2 w_1}{\partial y^2} + \nu_1 \frac{\partial^2 w_1}{\partial x^2} = 0, \quad \frac{\partial^3 w_1}{\partial y^3} + (2 - \nu_1) \frac{\partial^3 w_1}{\partial x^2 \partial y} = 0. \tag{4.12}$$

The continuity conditions at $y = H$ are

$$
\begin{aligned}
& w_1 = w_2, \\
& \frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial y}, \\
& D_1\Big(\frac{\partial^2 w_1}{\partial y^2} + \nu_1 \frac{\partial^2 w_1}{\partial x^2}\Big) = D_2\Big(\frac{\partial^2 w_2}{\partial y^2} + \nu_2 \frac{\partial^2 w_2}{\partial x^2}\Big), \\
& D_1\Big(\frac{\partial^3 w_1}{\partial y^3} + (2-\nu_1)\frac{\partial^3 w_1}{\partial x^2 \partial y}\Big) = D_2\Big(\frac{\partial^3 w_2}{\partial y^3} + (2-\nu_2)\frac{\partial^3 w_2}{\partial x^2 \partial y}\Big).
\end{aligned}
$$

The related dispersion equation is

$$\det \mathbf{M} = 0, \tag{4.13}$$

with the non-zero components of the 6×6 matrix $\mathbf{M}$ given in Appendix C.2, where the notation

$$D = \frac{D_1}{D_2}$$

is introduced.

For a plate reinforced by a beam with a narrow rectangular cross section the quantities I_y, J, J_t, and A in (4.2) are defined as

$$I_y = \frac{2}{3}h^3 H, \quad J = \frac{1}{6}hH^3, \quad J_t = \frac{8}{3}h^3 H, \quad A = 2hH.$$

Taking into account the relations

$$D_j = \frac{2E_j h^3}{3(1-\nu_j^2)}, \qquad G_j = \frac{E_j}{2(1+\nu_j)}, \quad j = 1, 2,$$

we have

$$\alpha_1 = \eta, \quad \alpha_2 = \frac{1}{12}\eta^3, \quad \beta_1 = D(1-\nu_1^2)\eta, \quad \beta_2 = 2D(1-\nu_1)\eta, \tag{4.14}$$

where $\eta = H/h$. Substituting the above formulae into (4.6) we obtain dispersion equation

$$\begin{aligned}
&(1+\nu_2^2\phi)^2 - 2\nu_2(1+\nu_2^2\phi) - (1-\nu_2)^2 - \sqrt{2(1+\nu_2^2\phi)}\times \\
&\qquad \left(\rho(1-\nu_2^4\phi^2) - D(1-\nu_1)(1+\nu_1+2\nu_2^2\phi)\right)\delta_H \\
&- 2D(1-\nu_1)\left(\rho(1-\nu_2^4\phi^2) - D(1-\nu_1^2)\right)\delta_H^2 \\
&- \frac{1}{12}(1-\nu_2^4\phi^2)\nu_2^2\rho\phi\sqrt{2(1+\nu_2^2\phi)}\delta_H^3 \\
&+ \frac{1}{12}(1-\nu_2^4\phi^2)\rho(\rho(1-\nu_2^4\phi^2) - D(1-\nu_1^2))\delta_H^4 = 0,
\end{aligned} \tag{4.15}$$

where $\delta_H = kH \ll 1$. Now, the cut-off at leading order is given by the formula

$$\delta_H^* \approx \frac{\nu_2^2}{\sqrt{2}(D(1-\nu_1^2)-\rho)}, \tag{4.16}$$

which readily follows from (4.8) and is valid provided that $\nu_2^2 \ll 1$ and $D(1-\nu_1^2) > \rho$.

Also, the asymptotic expansion for ϕ, analogous to (4.9), becomes

$$\phi = \widetilde{\phi}_0 + \widetilde{\phi}_1\delta_H + \dots, \tag{4.17}$$

where $\widetilde{\phi}_0 = \phi_0$ in (4.10) and

$$\widetilde{\phi}_1 = \sqrt{2(1+\nu_2^2\phi_0)} \times \frac{\Big(\rho(1-\nu_2^4\phi_0^2) - D(1-\nu_1)(1+\nu_1+\nu_2^2\phi_0)\Big)}{2\nu_2^2(1-\nu_2+\nu_2^2\phi_0)}.$$

In Figures 4.3 and 4.4 the function ϕ is plotted against dimensionless wave number δ_H. In these figures the dispersion curves for a plate reinforced by a beam (4.15) and by a strip plate (4.13) are plotted together with those corresponding to the two term asymptotic approximations (4.17). Numerical examples are presented for $\nu_1 = 0.31$ and $\nu_2 = 0.35$.

As might be expected, both beam approximation (4.15) and its two-term asymptotics (4.17) are robust only over the long wave range ($\delta_H \ll 1$), see the curves for $D = 2.3$ in Figure 4.3 and $\rho = 0.2$ in Figure 4.4, for which the asymptotic formulae (4.16) gives $\delta_H^* = 0.08$ and $\delta_H^* = 0.12$, respectively. Outside the long wave range, the deviation between the results for plate and beam edge reinforcement become more substantial. In particular, as follows from formula (4.7) with (4.14) the beam reinforcement does not assume a cut-off under the condition $D(1-\nu_1^2) - \rho = 0$ since the analysis is in the long-wave low-frequency region, which is satisfied for the curves corresponding to $D = 1.11$ in Figure 4.3 and $\rho = 0.9$ in Figure 4.4. At the same time, for both of these scenarios the strip plate reinforcement predicts cut-offs at $\delta_H^* \sim 1$.

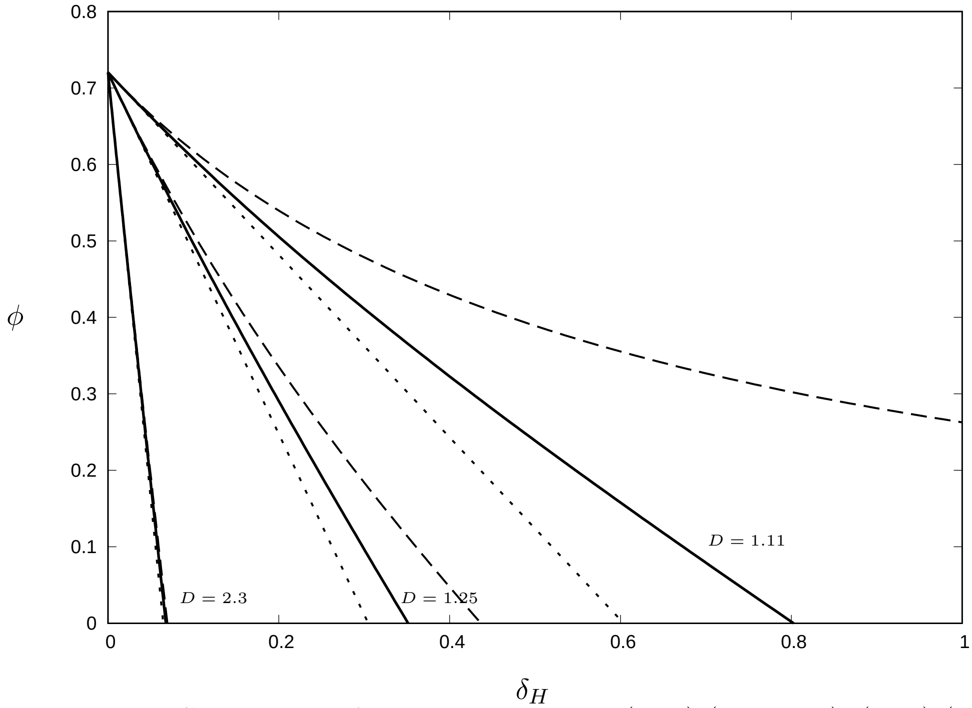

FIGURE 4.3: Comparison of dispersion relations (4.13) (solid line), (4.15) (dashed line) and asymptotic expansion (4.17) (dotted line) for $\rho = 1.0$ and $D = 2.3$, 1.25, 1.11.

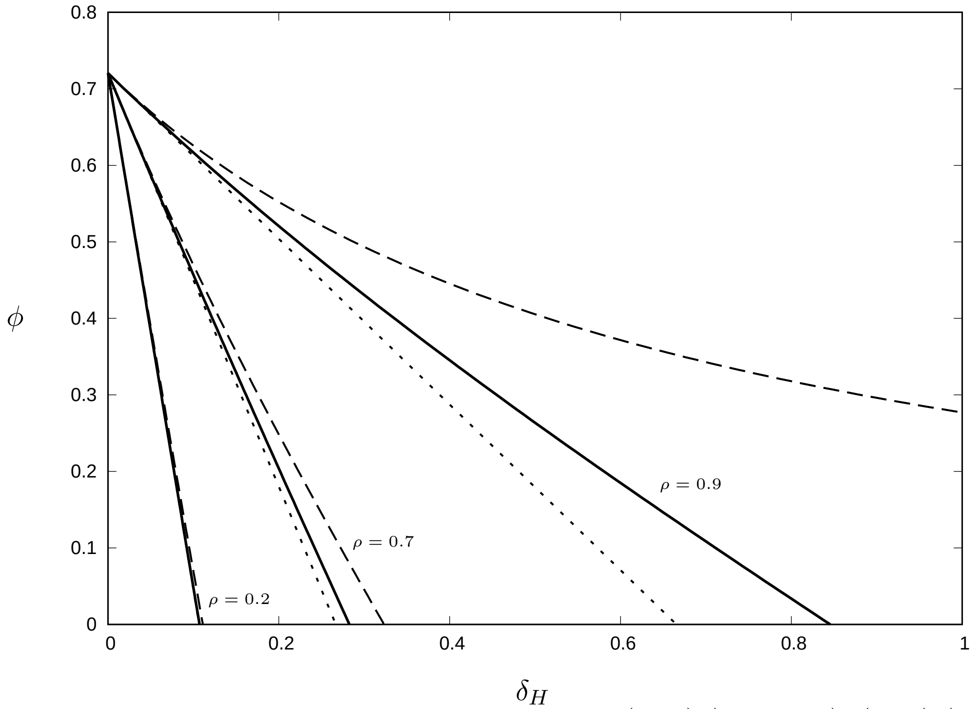

FIGURE 4.4: Comparison of dispersion relations (4.13) (solid line), (4.15) (dashed line) and asymptotic expansion (4.17) (dotted line) for $D = 1.0$ and $\rho = 0.2$, 0.7, 0.9.

In this chapter we have studied edge bending waves in a plate with an edge reinforced by a beam. We have also compared these results with those achieved in chapter 3 for a more general setup of reinforcement by a strip plate. Predictably, it is shown that the beam model is a good approximation in case of narrow strip reinforcement.

Chapter 5

The elastic bending wave on the edge of a semi-infinite circular plate reinforced by an annular plate

This chapter is developing the previous results taking into account the effect of curvature. More specifically, it is concerned with the propagation of bending edge waves on a thin isotropic elastic circular plate perfectly bonded with a narrow annular plate of the same thickness. We focus on the asymptotic treatment of a narrow plate with a free outer edge and its inner edge subject to prescribed deflection and angle of rotation. In Section 5.1, a review of the equations of motion and statement of the problem are presented. Then, the exact dispersion relation is obtained. In Section 5.2, we introduce appropriate scaling for the space variables and derive the effective boundary conditions. Finally, in Section 5.3, the approximate dispersion relation is deduced and comparison of approximate dispersion relation and exact dispersion relation is demonstrated.

5.1 Statement of the problem

Consider a thin isotropic elastic circular plate of thickness $2h$ and radius R, reinforced by an annular plate of the same thickness and width H with $h << H$ as shown in Figure 5.1.

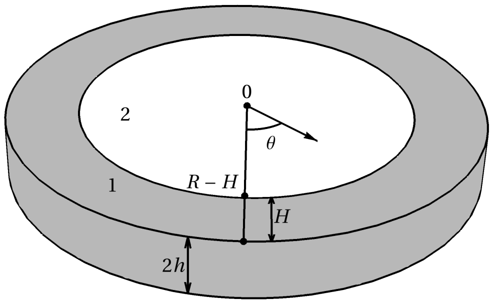

FIGURE 5.1: A semi-infinite circular plate with the edge coated by an annular plate.

The equation of motion can be written as

$$D_j\left(\frac{\partial^4 w_j}{\partial r^4}+\frac{2}{r^2}\frac{\partial^4 w_j}{\partial r^2\partial\theta^2}+\frac{1}{r^4}\frac{\partial^4 w_j}{\partial\theta^4}+\frac{2}{r^3}\frac{\partial^3 w_j}{\partial r\partial\theta^2}+\frac{2}{r}\frac{\partial^3 w_j}{\partial r^3}+\frac{1}{r^2}\frac{\partial^2 w_j}{\partial r^2}\right)$$
$$+2\rho_j h\frac{\partial^2 w_j}{\partial t^2}=0,\quad j=1,2, \tag{5.1}$$

where t is time, w_j are deflections, ρ_j are mass densities, and D_j are bending stiffnesses given by

$$D_j=\frac{2E_jh^3}{3(1-\nu_j)},$$

where E_j are the Young's moduli and ν_j are the Poisson's ratios; hereinafter index 1 is used to denote parameters corresponding to the annular plate, whereas index 2 stays for the inner plate.

The boundary conditions on the free edge $r = R$ are imposed in such a way that both bending moment and modified shear force are set to zero,

$$\frac{\partial^2 w_1}{\partial r^2} + \frac{\nu_1}{r}\frac{\partial w_1}{\partial r} + \frac{\nu_1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2} = 0, \quad \frac{\partial^3 w_1}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r^2}\frac{\partial^3 w_1}{\partial r \partial \theta^2} = 0. \tag{5.2}$$

The continuity conditions along the interface $r = R - H$ for perfectly bonded plates are taken as

$$\begin{aligned}
&w_1 = w_2,\\
&\frac{\partial w_1}{\partial r} = \frac{\partial w_2}{\partial r},\\
&D_1\Big(\frac{\partial^2 w_1}{\partial r^2} + \frac{\nu_1}{r}\frac{\partial w_1}{\partial r} + \frac{\nu_1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\Big) = D_2\Big(\frac{\partial^2 w_2}{\partial r^2} + \frac{\nu_2}{r}\frac{\partial w_2}{\partial r} + \frac{\nu_2}{r^2}\frac{\partial^2 w_2}{\partial \theta^2}\Big),\\
&D_1\Big(\frac{\partial^3 w_1}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r^2}\frac{\partial^3 w_1}{\partial r \partial \theta^2}\Big) = D_2\Big(\frac{\partial^3 w_2}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^3 w_2}{\partial r \partial \theta^2}\Big).
\end{aligned} \tag{5.3}$$

The solutions of plate bending equation (5.1) is given by

$$w_j(r, \theta, t) = w_j(r)\cos(k\theta)e^{iwt}, \quad j = 1, 2, \tag{5.4}$$

where ω is the frequency and k is circumferential wave number.

Substituting the latter into (5.1), we have

$$\Big[\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k^2}{r^2}\Big)^2 - \lambda_j^4\Big] w_j(r) = 0. \tag{5.5}$$

Then we get [145]

$$w_j(r) = C_1^{(j)} J_k(\lambda_j r) + C_2^{(j)} Y_k(\lambda_j r) + C_3^{(j)} I_k(\lambda_j r) + C_4^{(j)} K_k(\lambda_j r), \tag{5.6}$$

where $\lambda_j^2 = \omega\sqrt{\dfrac{2\rho_j h}{D_j}}$, and $J_k(\lambda_j r),\ Y_k(\lambda_j r)$ are Bessel functions and $I_k(\lambda_j r),\ K_k(\lambda_j r)$ are modified Bessel functions.

As a result, the deflection of each of the plate may be presented as

$$w_j(r,\theta,t) = \Big(C_1^{(j)} J_k(\lambda_j r) + C_2^{(j)} Y_k(\lambda_j r) + C_3^{(j)} I_k(\lambda_j r) + C_4^{(j)} K_k(\lambda_j r)\Big) \cos(k\theta) e^{iwt}, \tag{5.7}$$

where $C_1^{(j)}$, $C_2^{(j)}$, $C_3^{(j)}$ and $C_4^{(j)}$ are arbitrary constants. For a solid plate with no hole at $r = 0$, one requires that $C_2^{(2)} = C_4^{(2)} = 0$, Since $Y_k(\lambda_j r)$ and $K_k(\lambda_j r)$ becomes unbounded as $r \longrightarrow 0$.

Next, we insert formulae (5.7) into the boundary conditions (5.2) and continuity relations (5.3) leads to a homogeneous system of order six with the non-zero components of 6×6 matrix $\mathbf{S}$ given by

$$S_{11} = \Omega^2(\varepsilon-1)^2(J_{k-2}(\Omega) - 2J_k(\Omega) + J_{k+2}(\Omega)) - 4\nu_1(\Omega(\varepsilon-1)J_{k-1}(\Omega)$$

$$+ k(k-\varepsilon+1)J_k(\Omega)),$$

$$S_{12} = \Omega^2(\varepsilon-1)^2(Y_{k-2}(\Omega) - 2Y_k(\Omega) + Y_{k+2}(\Omega)) - 4\nu_1(\Omega(\varepsilon-1)Y_{k-1}(\Omega)$$

$$+ k(k-\varepsilon+1)Y_k(\Omega)),$$

$$S_{13} = \Omega^2(\varepsilon-1)^2(I_{k-2}(\Omega) + 2I_k(\Omega) + I_{k+2}(\Omega)) - 4\nu_1(\Omega(\varepsilon-1)I_{k-1}(\Omega)$$

$$+ k(k-\varepsilon+1)I_k(\Omega)),$$

$$S_{14} = 4\nu_1(\Omega(\varepsilon-1)K_{k-1}(\Omega) + k(-k+\varepsilon-1)K_k(\Omega)) + \Omega^2(\varepsilon-1)^2(K_{k-2}(\Omega)$$

$$+ 2K_k(\Omega) + K_{k+2}(\Omega)),$$

$$S_{21} = 4k^2(J_{k+1}(\Omega) - J_{k-1}(\Omega)) + \Omega(\varepsilon-1)(\Omega(\varepsilon-1)(J_{k-3}(\Omega) - 3J_{k-1}(\Omega)$$

$$+ 3J_{k+1}(\Omega) - J_{k+3}(\Omega)) - 2(J_{k-2}(\Omega) - 2J_k(\Omega) + J_{k+2}(\Omega))),$$

$$S_{22} = 4k^2(Y_{k+1}(\Omega) - Y_{k-1}(\Omega)) + \Omega(\varepsilon-1)(\Omega(\varepsilon-1)(Y_{k-3}(\Omega) - 3Y_{k-1}(\Omega)$$

$$+ 3Y_{k+1}(\Omega) - Y_{k+3}(\Omega)) - 2(Y_{k-2}(\Omega) - 2Y_k(\Omega) + Y_{k+2}(\Omega))),$$

$$S_{23} = \Omega(\varepsilon-1)(\Omega(\varepsilon-1)(I_{k-3}(\Omega) + 3(I_{k-1}(\Omega) + I_{k+1}(\Omega)) + I_{k+3}(\Omega))$$

$$- 2(I_{k-2}(\Omega) + 2I_k(\Omega) + I_{k+2}(\Omega))) - 4k^2(I_{k-1}(\Omega) + I_{k+1}(\Omega)),$$

$$S_{24} = 4k^2(K_{k-1}(\Omega) + K_{k+1}(\Omega)) - \Omega(\varepsilon-1)(2(K_{k-2}(\Omega) + 2K_k(\Omega) + K_{k+2}(\Omega))$$

$$+ \Omega(\varepsilon-1)(K_{k-3}(\Omega) + 3(K_{k-1}(\Omega) + K_{k+1}(\Omega)) + K_{k+3}(\Omega))),$$

$$S_{31} = J_k(\Omega(1-\varepsilon)), \quad S_{32} = Y_k(\Omega(1-\varepsilon)),$$

$$S_{33} = I_k(\Omega(1-\varepsilon)), \quad S_{34} = K_k(\Omega(1-\varepsilon)),$$

$$S_{35} = -J_k(\rho D\Omega(1-\varepsilon)), \quad S_{36} = -I_k(\rho D\Omega(1-\varepsilon)),$$

$$S_{41} = -\frac{1}{2}\Omega(\varepsilon-1)\varepsilon(J_{k-1}(\Omega-\varepsilon\Omega) - J_{k+1}(\Omega-\varepsilon\Omega)),$$

$$S_{42} = -\frac{1}{2}\Omega(\varepsilon-1)\varepsilon(Y_{k-1}(\Omega-\varepsilon\Omega) - Y_{k+1}(\Omega-\varepsilon\Omega)),$$

$$S_{43} = -\frac{1}{2}\Omega(\varepsilon-1)\varepsilon(I_{k-1}(\Omega-\varepsilon\Omega) + I_{k+1}(\Omega-\varepsilon\Omega)),$$

$$S_{44} = \frac{1}{2}\Omega(\varepsilon-1)\varepsilon(K_{k-1}(\Omega-\varepsilon\Omega) + K_{k+1}(\Omega-\varepsilon\Omega)),$$

$$S_{45} = \varepsilon(D\rho\Omega(\varepsilon-1)J_{k-1}(-D(\varepsilon-1)\rho\Omega) + kJ_k(-D(\varepsilon-1)\rho\Omega)),$$

$$S_{46} = \varepsilon(D\rho\Omega(\varepsilon-1)I_{k-1}(-D(\varepsilon-1)\rho\Omega) + kI_k(-D(\varepsilon-1)\rho\Omega)),$$

$$\begin{aligned}S_{51} = D^4\varepsilon^2\Big(&(\nu_1-1)(\Omega-\Omega\varepsilon)J_{k-1}(\Omega-\varepsilon\Omega)\\ &-\left(k(k+1)\nu_1 - k(k+1) + \Omega^2(\varepsilon-1)^2\right)J_k(\Omega-\varepsilon\Omega)\Big),\end{aligned}$$

$$\begin{aligned}S_{52} = \frac{1}{4}D^4\varepsilon^2\Big(&4\nu_1((\Omega-\Omega\varepsilon)Y_{k-1}(\Omega-\varepsilon\Omega) - k(k+1)Y_k(\Omega-\varepsilon\Omega))\\ &+\Omega^2(\varepsilon-1)^2(Y_{k-2}(\Omega-\varepsilon\Omega) - 2Y_k(\Omega-\varepsilon\Omega) + Y_{k+2}(\Omega-\varepsilon\Omega))\Big),\end{aligned}$$

$$\begin{aligned}S_{53} = \frac{1}{4}D^4\varepsilon^2\Big(&4\nu_1((\Omega-\Omega\varepsilon)I_{k-1}(\Omega-\varepsilon\Omega) - k(k+1)I_k(\Omega-\varepsilon\Omega))\\ &+\Omega^2(\varepsilon-1)^2(I_{k-2}(\Omega-\varepsilon\Omega) + 2I_k(\Omega-\varepsilon\Omega) + I_{k+2}(\Omega-\varepsilon\Omega))\Big),\end{aligned}$$

$$\begin{aligned}S_{54} = \frac{1}{4}D^4\varepsilon^2\Big(&4\nu_1(\Omega(\varepsilon-1)K_{k-1}(\Omega-\varepsilon\Omega) - k(k+1)K_k(\Omega-\varepsilon\Omega))\\ &+\Omega^2(\varepsilon-1)^2(K_{k-2}(\Omega-\varepsilon\Omega) + 2K_k(\Omega-\varepsilon\Omega) + K_{k+2}(\Omega-\varepsilon\Omega))\Big),\end{aligned}$$

$$\begin{aligned}S_{55} = \varepsilon^2\Big(&\left(D^2\rho^2\Omega^2(\varepsilon-1)^2 + k(k+1)\nu_2 - k(k+1)\right)J_k(-D(\varepsilon-1)\rho\Omega)\\ &+D(\nu_2-1)\rho\Omega(\varepsilon-1)J_{k-1}(-D(\varepsilon-1)\rho\Omega)\Big),\end{aligned}$$

$$\begin{aligned}S_{56} = \varepsilon^2\Big(&D(\nu_2-1)\rho\Omega(\varepsilon-1)I_{k-1}(-D(\varepsilon-1)\rho\Omega)\\ &-\left(D^2\rho^2\Omega^2(\varepsilon-1)^2 + k^2 - (k+1)k\nu_2 + k\right)I_k(-D(\varepsilon-1)\rho\Omega)\Big),\end{aligned}$$

$$
\begin{aligned}
S_{61} &= D^4\varepsilon^3\Big(k\left(-2k+\Omega^2(\varepsilon-1)^2-1\right)J_k(\Omega-\varepsilon\Omega)\\
&+\Omega(\varepsilon-1)(\Omega(\varepsilon-1)-1)(\Omega(\varepsilon-1)+1)J_{k-1}(\Omega-\varepsilon\Omega)\Big),\\
S_{62} &= -\frac{1}{8\Omega(\varepsilon-1)}D^4\varepsilon^3\Big(\Big(16k\left(-2k^2+k+1\right)+8(2(k-1)k+1)\Omega^2(\varepsilon-1)^2\\
&-7\Omega^4(\varepsilon-1)^4\Big)Y_{k-1}(\Omega-\varepsilon\Omega)+\Omega^4(\varepsilon-1)^4Y_{k-3}(\Omega-\varepsilon\Omega)\\
&-2\Omega(\varepsilon-1)\left(4k(2k+1)-(5k-2)\Omega^2(\varepsilon-1)^2\right)Y_{k-2}(\Omega-\varepsilon\Omega)\Big),\\
S_{63} &= D^4\varepsilon^3\Big(\Omega(1-\varepsilon)\left(\Omega^2(\varepsilon-1)^2+1\right)I_{k-1}(\Omega-\varepsilon\Omega)\\
&-k\left(2k+\Omega^2(\varepsilon-1)^2+1\right)I_k(\Omega-\varepsilon\Omega)\Big),\\
S_{64} &= \frac{1}{8\Omega(\varepsilon-1)}D^4\varepsilon^3\Big(\Omega^4(\varepsilon-1)^4K_{k-3}(\Omega-\varepsilon\Omega)-2\Omega(\varepsilon-1)\Big((5k-2)\Omega^2(\varepsilon-1)^2\\
&+4k(2k+1)\Big)K_{k-2}(\Omega-\varepsilon\Omega)+\Big(8(2(k-1)k+1)\Omega^2(\varepsilon-1)^2+16(k-1)k(2k+1)\\
&+7\Omega^4(\varepsilon-1)^4\Big)K_{k-1}(\Omega-\varepsilon\Omega)\Big),\\
S_{65} &= -\frac{1}{2}\varepsilon^3\Big(-D\rho\Omega(\varepsilon-1)\left(-2D^2\rho^2\Omega^2(\varepsilon-1)^2+k^2+2\right)J_{k-1}(-D(\varepsilon-1)\rho\Omega)\\
&-k(2(-D\rho\Omega(\varepsilon-1)+k+1)(D\rho\Omega(\varepsilon-1)+k+1)J_k(-D(\varepsilon-1)\rho\Omega)\\
&+Dk\rho\Omega(\varepsilon-1)J_{k+1}(-D(\varepsilon-1)\rho\Omega))\Big),\\
S_{66} &= \frac{1}{2}\varepsilon^3\Big(D\rho\Omega(\varepsilon-1)\left(D^2\rho^2\Omega^2(\varepsilon-1)^2+1\right)(I_{k-1}(-D(\varepsilon-1)\rho\Omega)\\
&+I_{k+1}(-D(\varepsilon-1)\rho\Omega))+4k^2I_k(-D(\varepsilon-1)\rho\Omega)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega &= \lambda_1 l,\\
\lambda_2 &= \lambda_1\rho D,\\
D &= \left(\frac{D_1}{D_2}\right)^{1/4},\\
\rho &= \left(\frac{\rho_2}{\rho_1}\right)^{1/4},
\end{aligned}
\tag{5.8}
$$

which has non-zero solutions provided that

$$\det(\mathbf{S}) = 0. \tag{5.9}$$

5.2 Effective Boundary Conditions

The goal of the section is to obtain effective boundary conditions by expressing the bending moment and the modified shear force at the interface $r = R - H$ through given deflection and angle of rotation. First, let us take in a separate consideration the annular plate, having traction free outer edge and prescribed displacement and rotation angle on the inner edge. Thus, for the annular plate we are solving the equation of motion (5.1) subject to the traction free boundary conditions (5.2) at $r = R$. At the interface $r = R - H$ we have

$$w_1|_{r=l-H} = w_H, \qquad \left.\frac{\partial w_1}{\partial r}\right|_{r=R-H} = \frac{1}{R-H} G_H, \tag{5.10}$$

where functions $w_H = w_H(\theta, t)$ and $G_H = G_H(\theta, t)$ are assumed to be known. Accordingly, we introduce the following variables

$$R_1 = \left(\frac{r}{R} - 1\right)\frac{1}{\varepsilon} + 1, \quad R_2 = \frac{r}{R-H}, \ \tau = \sqrt{\frac{D_1}{2\rho_1 h}}\,\frac{t}{R^2}, \ \varepsilon = \frac{H}{R} << 1. \tag{5.11}$$

In terms of new variables the governing equation (5.1) becomes

$$\begin{aligned}
&\frac{\partial^4 w_1}{\partial R_1^4}+\varepsilon\Big(4(R_1-1)\frac{\partial^4 w_1}{\partial R_1^4}+2\frac{\partial^3 w_1}{\partial R_1^3}\Big)+\varepsilon^2\Big(6(R_1-1)^2\frac{\partial^4 w_1}{\partial R_1^4}+2\frac{\partial^4 w_1}{\partial R_1^2\partial\theta^2}\\
&+6(R_1-1)\frac{\partial^3 w_1}{\partial R_1^3}+\frac{\partial^2 w_1}{\partial R_1^2}\Big)+\varepsilon^3\Big(4(R_1-1)^3\frac{\partial^4 w_1}{\partial R_1^4}+4(R_1-1)\frac{\partial^4 w_1}{\partial R_1^2\partial\theta^2}\\
&+2\frac{\partial^3 w_1}{\partial R_1\partial\theta^2}+6(R_1-1)^2\frac{\partial^3 w_1}{\partial R_1^3}+2(R_1-1)\frac{\partial^2 w_1}{\partial R_1^2}\Big)+\varepsilon^4\Big((R_1-1)^4\frac{\partial^4 w_1}{\partial R_1^4}\\
&+2(R_1-1)^2\frac{\partial^4 w_1}{\partial R_1^2\partial\theta^2}+\frac{\partial^4 w_1}{\partial\theta^4}+2(R_1-1)\frac{\partial^3 w_1}{\partial R_1\partial\theta^2}+2(R_1-1)^3\frac{\partial^3 w_1}{\partial R_1^3}\\
&+(R_1-1)^2\frac{\partial^2 w_1}{\partial R_1^2}+\frac{\partial^2 w_1}{\partial\tau^2}\Big)+\varepsilon^5\Big(4(R_1-1)\frac{\partial^2 w_1}{\partial\tau^2}\Big)+\varepsilon^6\Big(6(R_1-1)^2\frac{\partial^2 w_1}{\partial\tau^2}\Big)\\
&+\varepsilon^7\Big(4(R_1-1)^3\frac{\partial^2 w_1}{\partial\tau^2}\Big)+\varepsilon^8\Big((R_1-1)^4\frac{\partial^2 w_1}{\partial\tau^2}\Big)=0, \qquad (5.12)
\end{aligned}$$

subject to the boundary conditions

$$\begin{aligned}
&\frac{\partial^2 w_1}{\partial R_1^2}+\varepsilon\nu_1\frac{\partial w_1}{\partial R_1}+\varepsilon^2\nu_1\frac{\partial^2 w_1}{\partial\theta^2}=0\,,\quad \frac{\partial^3 w_1}{\partial R_1^3}+\varepsilon\frac{\partial^2 w_1}{\partial R_1^2}+\varepsilon^2\frac{\partial^3 w_1}{\partial R_1\partial\theta^2}=0\quad\text{at}\quad R_1=1,\\
&w_1=w_H,\qquad \frac{\partial w_1}{\partial R_1}-\varepsilon\frac{\partial w_1}{\partial R_1}=\varepsilon G_H\quad\text{at}\quad R_1=0.
\end{aligned}\qquad (5.13)$$

A deflection w_1 can be expanded into an asymptotic series in terms of ε as

$$w_1=w_1^{(0)}+w_1^{(1)}\varepsilon+w_1^{(2)}\varepsilon^2+w_1^{(3)}\varepsilon^3+w_1^{(4)}\varepsilon^4+\dots \qquad (5.14)$$

Substituting expansion (5.14) into the boundary value problem (5.12)-(5.13), we arrive at the problem formulated for various asymptotic orders $n = 0, 1, 2, \ldots$, namely

$$\begin{aligned}
&\frac{\partial^4 w_1^{(n)}}{\partial R_1^4} + 2\frac{\partial^3 w_1^{(n-1)}}{\partial R_1^3} + 2\frac{\partial^4 w_1^{(n-2)}}{\partial R_1^2 \partial \theta^2} \\
&+\frac{\partial^2 w_1^{(n-2)}}{\partial R_1^2} + 2\frac{\partial^3 w_1^{(n-3)}}{\partial R_1 \partial \theta^2} + \frac{\partial^4 w_1^{(n-4)}}{\partial \theta^4} + \frac{\partial^2 w_1^{(n-4)}}{\partial \tau^2} \\
&+(R_1 - 1)\Big(4\frac{\partial^4 w_1^{(n-1)}}{\partial R_1^4} + 6\frac{\partial^3 w_1^{(n-2)}}{\partial R_1^3} + 4\frac{\partial^4 w_1^{(n-3)}}{\partial R_1^2 \partial \theta^2} \\
&+2\frac{\partial^2 w_1^{(n-3)}}{\partial R_1^2} + 2\frac{\partial^3 w_1^{(n-4)}}{\partial R_1 \partial \theta^2} + 4\frac{\partial^2 w_1^{(n-5)}}{\partial \tau^2}\Big) \\
&+(R_1 - 1)^2\Big(6\frac{\partial^4 w_1^{(n-2)}}{\partial R_1^4} + 6\frac{\partial^3 w_1^{(n-3)}}{\partial R_1^3} + 2\frac{\partial^4 w_1^{(n-4)}}{\partial R_1^2 \partial \theta^2} + \frac{\partial^2 w_1^{(n-4)}}{\partial R_1^2} + 6\frac{\partial^2 w_1^{(n-6)}}{\partial \tau^2}\Big) \\
&+(R_1 - 1)^3\Big(4\frac{\partial^4 w_1^{(n-3)}}{\partial R_1^4} + 2\frac{\partial^3 w_1^{(n-4)}}{\partial R_1^3} + 4\frac{\partial^2 w_1^{(n-7)}}{\partial \tau^2}\Big) \\
&+(R_1 - 1)^4\Big(\frac{\partial^4 w_1^{(n-4)}}{\partial R_1^4} + \frac{\partial^2 w_1^{(n-8)}}{\partial \tau^2}\Big) = 0,
\end{aligned} \tag{5.15}$$

subject to

$$\begin{aligned}
&\frac{\partial^2 w_1^{(n)}}{\partial R_1^2} + \nu_1 \frac{\partial w_1^{(n-1)}}{\partial R_1} + \nu_1 \frac{\partial^2 w_1^{(n-2)}}{\partial \theta^2} = 0\,, \quad \frac{\partial^3 w_1^{(n)}}{\partial R_1^3} + \frac{\partial^2 w_1^{(n-1)}}{\partial R_1^2} + \frac{\partial^3 w_1^{(n-2)}}{\partial R_1 \partial \theta^2} = 0 \quad \text{at} \quad R_1 = 1, \\
&w_1^{(n)} = w_H^{(n)}, \qquad \frac{\partial w_1^{(n)}}{\partial R_1} - \frac{\partial w_1^{(n-1)}}{\partial R_1} = G_H^{(n)} \quad \text{at} \quad R_1 = 0,
\end{aligned} \tag{5.16}$$

where quantities with the negative superscript are set to be equal to zero. The only non-zero components $w_H^{(n)}$ and $G_H^{(n)}$ are $w_H^{(0)} = w_H$ and $G_H^{(1)} = G_H$, respectively.

Substituting subsequently $n = 0, 1, 2, 3, 4$ into (5.15)-(5.16) we obtain

$$
\begin{aligned}
w_1^{(0)} &= w_H, \\
w_1^{(1)} &= G_H R_1, \\
w_1^{(2)} &= -\frac{1}{2}\nu_1\Big(\frac{\partial^2 w_H}{\partial\theta^2} + G_H\Big)R_1^2 + G_H R_1, \\
w_1^{(3)} &= \frac{1}{6}\Big(\nu_1\frac{\partial^2 w_H}{\partial\theta^2} + \nu_1 G_H - \frac{\partial^2 G_H}{\partial\theta^2}\Big)R_1^3 \\
&+ \frac{1}{2}\Big(\nu_1(\nu_1-1)\frac{\partial^2 w_H}{\partial\theta^2} + \nu_1(\nu_1-2)G_H + (1-\nu_1)\frac{\partial^2 G_H}{\partial\theta^2}\Big)R_1^2 + G_H R_1, \\
w_1^{(4)} &= \frac{1}{24}\Big((2\nu_1-1)\frac{\partial^4 w_H}{\partial\theta^4} - \nu_1\frac{\partial^2 w_H}{\partial\theta^2} - \nu_1 G_H + 2\nu_1\frac{\partial^2 G_H}{\partial\theta^2} - \frac{\partial^2 w_H}{\partial\tau^2}\Big)R_1^4 \\
&+ \frac{1}{6}\Big((1-\nu_1)\frac{\partial^4 w_H}{\partial\theta^4} + \nu_1(1-\nu_1)\frac{\partial^2 w_H}{\partial\theta^2} + \nu_1(2-\nu_1)G_H - \frac{\partial^2 G_H}{\partial\theta^2} + \frac{\partial^2 w_H}{\partial\tau^2}\Big)R_1^3 \\
&+ \frac{1}{2}\Big(\frac{1}{2}(\nu_1^2-1)\frac{\partial^4 w_H}{\partial\theta^4} - \nu_1(\nu_1^2 - \frac{3}{2}\nu_1 + \frac{1}{2})\frac{\partial^2 w_H}{\partial\theta^2} - \nu_1(\nu_1^2 - \frac{5}{2}\nu_1 + \frac{5}{2})G_H \\
&- (\frac{5}{2}\nu_1 - \frac{3}{2}\nu_1^2 - 1)\frac{\partial^2 G_H}{\partial\theta^2} - \frac{1}{2}\frac{\partial^2 w_H}{\partial\tau^2}\Big)R_1^2 + G_H R_1.
\end{aligned}
\tag{5.17}
$$

Now, using expansion (5.14) together with the continuity conditions for moments and shear forces on the interface, we obtain effective boundary conditions for the infinite plate at $R_2 = 1$ in the form

$$
\begin{aligned}
D_2\Big(\frac{\partial^2 w_2}{\partial R_2^2} + \nu_2\frac{\partial w_2}{\partial R_2} + \nu_2\frac{\partial^2 w_2}{\partial \theta^2}\Big) &= D_1\Big(\nu_1(\nu_1-3)\frac{\partial^2 w_2}{\partial \theta^2} + \nu_1(\nu_1-4)\frac{\partial w_2}{\partial R_2} \\
&\quad + (1-\nu_1)\frac{\partial^3 w_2}{\partial \theta^2 \partial R_2}\Big)\varepsilon, \\
D_2\Big(\frac{\partial^3 w_2}{\partial R_2^3} + \frac{\partial^2 w_2}{\partial R_2^2} + \frac{\partial^3 w_2}{\partial \theta^2 \partial R_2}\Big) &= D_1\Big((1-\nu_1)\frac{\partial^4 w_2}{\partial \theta^4} - \nu_1\Big(\frac{\partial^2 w_2}{\partial \theta^2} + \frac{\partial w_2}{\partial R_2}\Big) \\
&\quad + (3-\nu_1)\frac{\partial^3 w_2}{\partial \theta^2 \partial R_2} + \frac{\partial^2 w_2}{\partial \tau^2}\Big)\varepsilon.
\end{aligned}
\tag{5.18}
$$

Thus, at the interface we have above the effective boundary conditions in terms of displacements. We note that there is an additional term at the right hands at these formula, demonstrating the influence of the coating. Clearly, in case of a soft annular plate, when D_1 is small, effect of the coating is also diminished.

5.3 Approximate dispersion relation

We aim at finding an asymptotic dispersion relation over the domain $0 \leqslant r \leqslant R - H$ by using the derived effective boundary conditions (5.18) together with equation of motion (5.1). The solution is sought for in the form of (5.7). As a result, we deduce an approximate dispersion relation, which could be re-written in dimensionless variables as

$$
\begin{aligned}
&\left(2D^4k^4\nu_1\varepsilon - k\left(2D^4k^3\varepsilon - k^2 + k + 1\right) + D^4\Omega^4\varepsilon - (k+1)k^2\nu_2\right) J_k(D(\varepsilon-1)\rho\Omega) \\
&\times I_k(D(\varepsilon-1)\rho\Omega) + \Big(- D^8k^6\nu_1^2\varepsilon^2 + D^8\left(-k^6\right)\varepsilon^2 - D^4k^4\varepsilon + D^4\Omega^4\Big(\varepsilon\left(D^4k^2\varepsilon - 1\right) \\
&+ \rho^4(\varepsilon-1)^4\Big) + D^2\rho^2\Omega^2(\varepsilon-1)^2\left(2D^4k^3\varepsilon - k(k+2) + 1\right) + \nu_2\Big(-k^2\Big(D^4k^2\varepsilon + 1\Big) \\
&+ D^4\Omega^4\varepsilon + D^2k(k+2)\rho^2\Omega^2(\varepsilon-1)^2\Big) + D^4k^2\nu_1\varepsilon\Big(2D^4k^4\varepsilon - D^4\Omega^4\varepsilon - 2D^2k\rho^2\Omega^2 \\
&\times(\varepsilon-1)^2 + k^2\nu_2 + k^2\Big) + k^2\Big)J_k(D(\varepsilon-1)\rho\Omega)I_{k-1}(D(\varepsilon-1)\rho\Omega) + J_{k-1}(D(\varepsilon-1)\rho\Omega) \\
&\times\Big(2D^3\rho^3\Omega^3(\varepsilon-1)^3\left(D^4k^2\nu_1\varepsilon + D^4\left(-k^2\right)\varepsilon - \nu_2 + 1\right) I_{k-1}(D(\varepsilon-1)\rho\Omega) \\
&+ \Big(D^8k^6\nu_1^2\varepsilon^2 - D^4\Omega^4\left(\varepsilon\left(D^4k^2\varepsilon - 1\right) + \rho^4(\varepsilon-1)^4\right) + k^2\left(D^8k^4\varepsilon^2 + D^4k^2\varepsilon - 1\right) \\
&+ D^2\rho^2\Omega^2(\varepsilon-1)^2\left(2D^4k^3\varepsilon - k(k+2) + 1\right) - D^4k^2\nu_1\varepsilon\Big(2D^4k^4\varepsilon - D^4\Omega^4\varepsilon \\
&+ 2D^2k\rho^2\Omega^2(\varepsilon-1)^2 + k^2\nu_2 + k^2\Big) + \nu_2\Big(D^4k^4\varepsilon - D^4\Omega^4\varepsilon \\
&+ D^2(k+2)k\rho^2\Omega^2(\varepsilon-1)^2 + k^2\Big)\Big)I_k(D(\varepsilon-1)\rho\Omega)\Big) = 0.
\end{aligned}
\tag{5.19}
$$

Figures 5.2-5.5 demonstrate the exact dispersion relation (5.9) and the approximate dispersion relation (5.19) for $\varepsilon = 0.05$, $\nu_1 = 0.31$, $\nu_2 = 0.35$ and several values of the relative stiffness D and relative density ρ of the annular plate. Since the solution is periodic along the angle, we only plot sets of discrete points, depicting the frequency Ω over the wave number k. Clearly the presented sequence show monotonic increase behaviour.

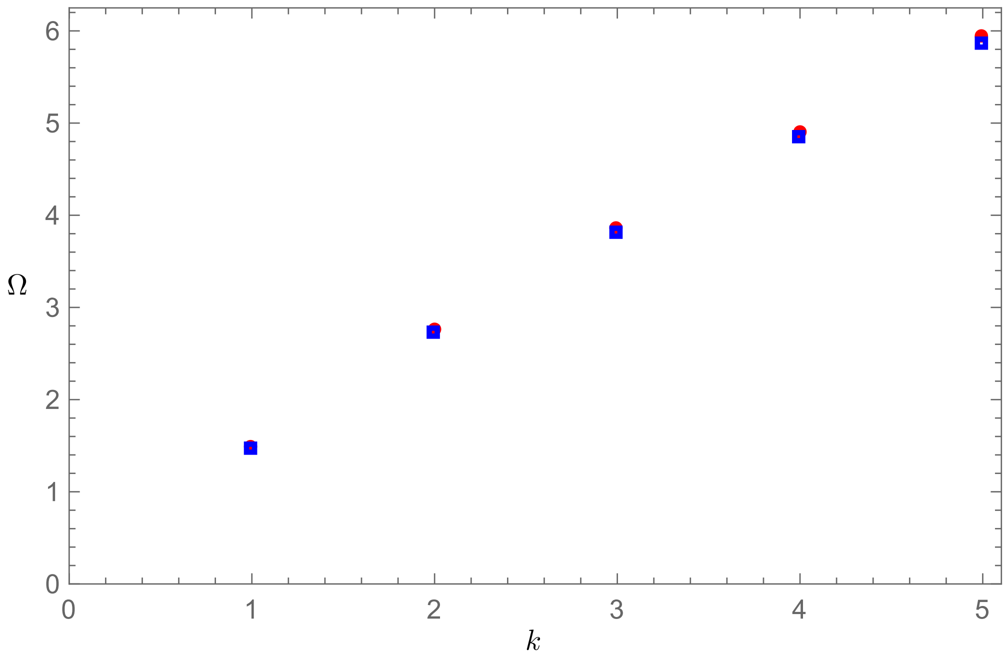

FIGURE 5.2: Comparison of approximate dispersion relation (5.19) (blue square) and exact dispersion relation (5.9) (red circle) for $\rho = 1$ and $D = 1$.

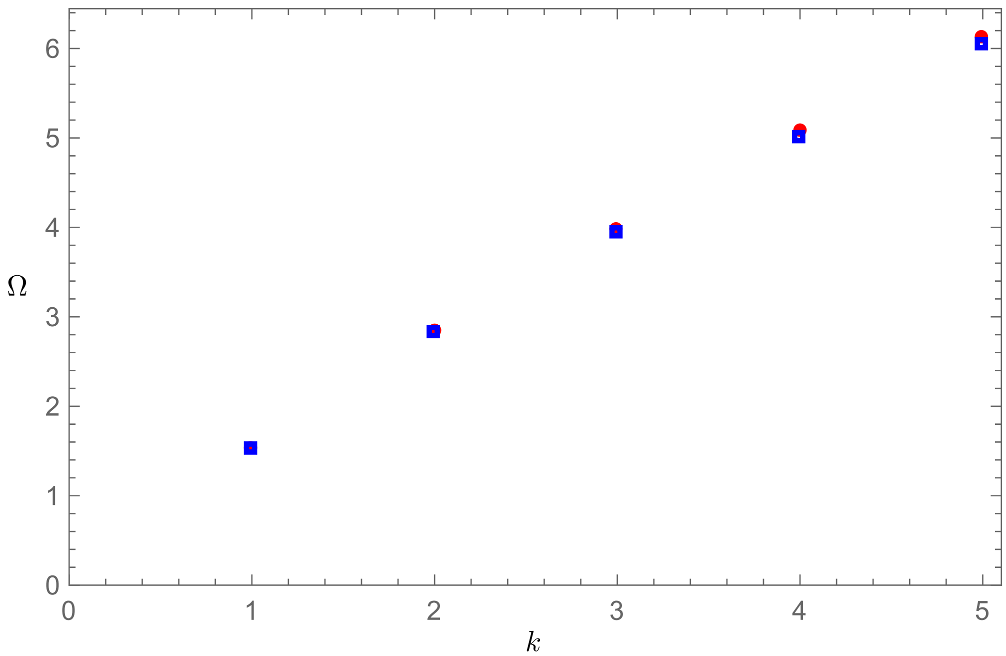

FIGURE 5.3: Comparison of approximate dispersion relation (5.19) (blue square) and exact dispersion relation (5.9) (red circle) for $\rho = 0.95$ and $D = 1$.

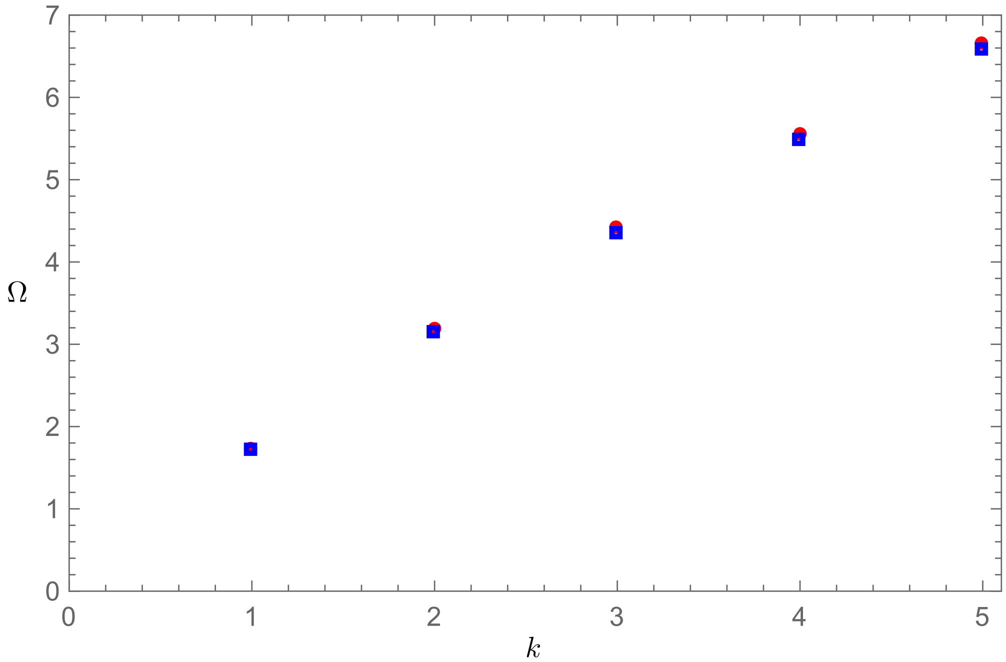

FIGURE 5.4: Comparison of approximate dispersion relation (5.19) (blue square) and exact dispersion relation (5.9) (red circle) for $\rho = 0.8$ and $D = 1$.

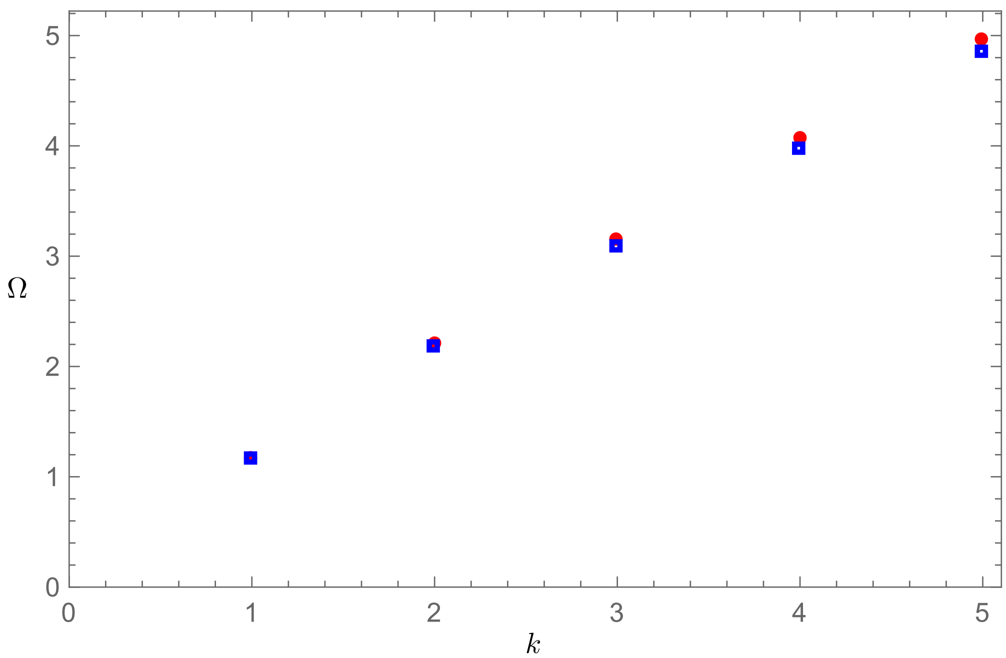

FIGURE 5.5: Comparison of approximate dispersion relation (5.19) (blue square) and exact dispersion relation (5.9) (red circle) for $\rho = 1$ and $D = 1.3$.

In this chapter the previously achieved results have been developed further, for the case of a circular plate with reinforcement of annular shape. Effective boundary conditions were derived, leading to an approximate dispersion relation. The exact and asymptotic results have been compared, illustrating the validity of the derived explicit approximations.

Chapter 6

Conclusions

In this thesis, low-frequency vibrations of coated elastic structures have been investigated. First, harmonic vibrations of a composite rod and a composite beam have been considered. Exact and approximate solutions found using asymptotic methods have been analysed. Effective boundary conditions for thin end attachments have been derived. They appeared to be useful for tackling more general 2D problems for thin plates in the main body of the thesis.

An asymptotic procedure for a strip plate leading to effective boundary conditions (3.21) is established. Along with the traditional long-wavelength assumption, it adapts the time scale specific to bending waves, see (3.13). It is also worth noting that the derivation of leading-order effective boundary condition relies on a fourth-order expansion of the deflection, see (3.16), because of a peculiarity of the studied boundary value problem (3.1),(4.12), and (3.12) for a plate strip. The proposed effective conditions are tested in Section 4 by asymptotic analysis of the exact solution for a plate strip subject to a kinematic loading in the form of travelling harmonic waves along its lower face.

The aforementioned effective boundary conditions may be interpreted in terms of a beam with a narrow rectangular cross-section perfectly bonded to the edge of a semi-infinite plate, see equations (3.22). Therefore, these conditions may also be derived using a less formal physical approach similarly to the derivation in [131] for a coated half-space, starting from the classical theory for plate extension.

The approximate dispersion relation (3.27) is derived starting from the effective boundary conditions (3.21). It perturbs the well-known dispersion relation (3.11) for the edge bending wave on a homogeneous plate. A good agreement between exact and approximate solutions is demonstrated numerically using the dispersion relation (3.9) for a composite plate. In addition, the influence of the relative stiffness and density of a strip plate on the localisation of the edge wave is investigated, indicating a possibility for the edge wave control.

We studied the edge wave problem for a semi-infinite plate reinforced by a beam taking into account both bending and twisting vibrations of the beam. The explicit asymptotic formulae for the cut-offs of the edge waves are presented. The validity of the chosen approximate formulation starting from the classical plate and beam theories is also addressed. A detailed dispersion relation is obtained and the long-wave approximation is derived. The numerical results are validated by comparison with the more general dispersion relation for a reinforcement in the form of a strip plate, which is also treated on the basis of the 2D Kirchhoff theory. The developed framework may be extended to more general setups including anisotropic structures as well as more elaborated structure models, e.g. see [104, 148].

We also studied the elastic bending wave on the edge of a circular plate reinforced by an annular plate. We focused on the asymptotic treatment of a narrow plate with a free outer edge and its inner edge subject to prescribed deflection and rotation. We derived the effective boundary conditions along with approximate dispersion relations.

The developed setup allows various extensions and generalisations. In particular, a similar problem may be formulated for elastic waves localised near a reinforced edge of a thin shell, e.g. see [76]. Also, strong contrast in the material properties of the components of a composite structure may be analysed. It is of particular interest to consider the high-contrast setup, having ”soft” coating subject to clamped edge boundary conditions, developing further results achieved recently for the Rayleigh wave, see [75]. Finally, the derived effective conditions appear to be of interest for a broad range of problems for thin plates.

Appendix A.1

The constants in the system of equations (2.5) are

$$A^{(1)} = \frac{\Delta_1}{\Delta}, \quad B^{(1)} = \frac{\Delta_2}{\Delta}, \quad A^{(2)} = \frac{\Delta_3}{\Delta}, \quad B^{(2)} = \frac{\Delta_4}{\Delta}, \tag{1}$$

where

$$\Delta = \frac{l\omega \left(E_1 c_2 \sin\left(\frac{H\omega}{c_1}\right) \sin\left(\frac{\omega(H-l)}{c_2}\right) + E_2 c_1 \cos\left(\frac{H\omega}{c_1}\right) \cos\left(\frac{\omega(H-l)}{c_2}\right)\right)}{c_1^2 c_2}, \tag{2}$$

$$\Delta_1 = Fl \left(\frac{\sin\left(\frac{\omega(l-H)}{c_2}\right) \cos\left(\frac{\omega(l-H)}{c_1}\right)}{c_1} - \frac{E_2 \sin\left(\frac{\omega(l-H)}{c_1}\right) \cos\left(\frac{\omega(l-H)}{c_2}\right)}{E_1 c_2} \right), \tag{3}$$

$$\Delta_2 = Fl \left(\frac{\sin\left(\frac{\omega(l-H)}{c_1}\right) \sin\left(\frac{\omega(l-H)}{c_2}\right)}{c_1} + \frac{E_2 \cos\left(\frac{\omega(l-H)}{c_1}\right) \cos\left(\frac{\omega(l-H)}{c_2}\right)}{E_1 c_2} \right), \tag{4}$$

$$\Delta_3 = 0, \tag{5}$$

$$\Delta_4 = \frac{Fl}{c_1}. \tag{6}$$

Appendix A.2

The constants in the system of equations (2.18) are

$$A^{(1)} = \frac{\Delta_1^*}{\Delta^*}, \quad B^{(1)} = \frac{\Delta_2^*}{\Delta^*}, \tag{7}$$

where

$$\Delta^* = -\cos\left(\Omega_1 \epsilon\right), \tag{8}$$

$$\Delta_1^* = \frac{F \sin\left(\Omega_1(1-\epsilon)\right)}{e_1 \Omega_1} - u_H \cos\left(\Omega_1\right), \tag{9}$$

$$\Delta_2^* = -\frac{F \cos\left(\Omega_1(\epsilon-1)\right)}{e_1 \Omega_1} - u_H \sin\left(\Omega_1\right). \tag{10}$$

Appendix B.1

The constants in (2.32) are

$$\alpha_1^{(1)} = \frac{\bar{\Delta}_1}{\bar{\Delta}}, \quad \alpha_2^{(1)} = \frac{\bar{\Delta}_2}{\bar{\Delta}}, \quad \alpha_3^{(1)} = \frac{\bar{\Delta}_3}{\bar{\Delta}}, \quad \alpha_4^{(1)} = \frac{\bar{\Delta}_4}{\bar{\Delta}}, \quad \alpha_1^{(2)} = \frac{\bar{\Delta}_5}{\bar{\Delta}}, \quad \alpha_2^{(2)} = \frac{\bar{\Delta}_6}{\bar{\Delta}}$$

where

$$\begin{aligned}
\bar{\Delta} = -4\beta_1^3\beta_2 D_1 \Big(&\frac{1}{2}\beta_2\beta_1 D_1 D_2 \Big(\beta_1^2 \left(\sin\left((1+i)\beta_1 H\right) + \sinh\left((1+i)\beta_1 H\right)\right) \\
&\quad \times \left(\sin\left((1+i)\beta_2(H-l)\right) - \sinh\left((1+i)\beta_2(H-l)\right)\right) \\
&- 4\beta_2\beta_1 \sin\left(\beta_1 H\right) \sinh\left(\beta_1 H\right) \sin\left(\beta_2(H-l)\right) \sinh\left(\beta_2(H-l)\right) \\
&+ \beta_2^2 \left(\sin\left((1+i)\beta_1 H\right) - \sinh\left((1+i)\beta_1 H\right)\right) \\
&\quad \times \left(\sin\left((1+i)\beta_2(H-l)\right) + \sinh\left((1+i)\beta_2(H-l)\right)\right)\Big) \\
&\quad + \beta_1^4 D_1^2 \left(\cos\left(\beta_1 H\right) \cosh\left(\beta_1 H\right) - 1\right) \left(\cos\left(\beta_2(H-l)\right) \cosh\left(\beta_2(H-l)\right) - 1\right) \\
&+ \beta_2^4 D_2^2 \left(\cos\left(\beta_1 H\right) \cosh\left(\beta_1 H\right) + 1\right) \left(\cos\left(\beta_2(H-l)\right) \cosh\left(\beta_2(H-l)\right) + 1\right)\Big),
\end{aligned}$$

$$\bar{\Delta_1} = (1+i)\beta_1\beta_2 G\Big(i\beta_1^4 D_1^2\Big(\sin(\beta_1((1+i)H-il))$$

$$+\sinh(\beta_1(-l+(1+i)H))+(1+i)\sinh(\beta_1 l)\Big)$$

$$\times(\cos(\beta_2(l-H))\cosh(\beta_2(l-H))-1)+2\beta_2\beta_1 D_1 D_2\Big(\beta_1^2\sin(\beta_1 H)\sinh(\beta_1(l-H))$$

$$\times(\sinh((1+i)\beta_2(l-H))-\sin((1+i)\beta_2(l-H)))$$

$$+\beta_2\beta_1\sin(\beta_2(l-H))\sinh(\beta_2(l-H))(\sin(\beta_1((1+i)H-il))-\sinh(\beta_1(-l+(1+i)H)))$$

$$+i\beta_2^2\cos(\beta_1 H)\cosh(\beta_1(l-H))(\sin((1+i)\beta_2(l-H))+\sinh((1+i)\beta_2(l-H)))\Big)$$

$$+i\beta_2^4 D_2^2(\sin(\beta_1((1+i)H-il))+\sinh(\beta_1(-l+(1+i)H))-(1+i)\sinh(\beta_1 l))$$

$$\times(\cos(\beta_2(l-H))\cosh(\beta_2(l-H))+1)\Big),$$

$$\bar{\Delta_2} = (-1-i)\beta_1\beta_2 G\Big(\beta_1^4 D_1^2\Big(\cos(\beta_1((1+i)H-il))$$

$$-i\cosh(\beta_1(-l+(1+i)H))-(1-i)\cosh(\beta_1 l)\Big)$$

$$\times(\cos(\beta_2(l-H))\cosh(\beta_2(l-H))-1)$$

$$+\beta_2^4 D_2^2\Big(\cos(\beta_1((1+i)H-il))-i\cosh(\beta_1(-l+(1+i)H))$$

$$+(1-i)\cosh(\beta_1 l)\Big)(\cos(\beta_2(l-H))\cosh(\beta_2(l-H))+1)$$

$$+2\beta_2\beta_1 D_1 D_2\Big(i\beta_2^2\cos(\beta_1 H)\sinh(\beta_1(l-H))$$

$$\times(\sin((1+i)\beta_2(l-H))+\sinh((1+i)\beta_2(l-H)))$$

$$+\beta_1^2\sin(\beta_1 H)\cosh(\beta_1(l-H))\Big(\sinh((1+i)\beta_2(l-H))$$

$$-\sin((1+i)\beta_2(l-H))\Big)+\beta_2\beta_1\sin(\beta_2(l-H))\sinh(\beta_2(l-H))$$

$$\times(\cosh(\beta_1(-l+(1+i)H))-i\cos(\beta_1((1+i)H-il)))\Big)\Big),$$

$$
\begin{aligned}
\bar{\Delta}_3 = {} & \beta_1\beta_2(-G)\Big((-1+i)\beta_1^4 D_1^2\Big(\cos\left(\beta_1(-l+(1+i)H)\right)+i\cosh\left(\beta_1((1+i)H-il)\right)\\
& -(1+i)\cos\left(\beta_1 l\right)\Big)\left(\cos\left(\beta_2(l-H)\right)\cosh\left(\beta_2(l-H)\right)-1\right)\\
& -(1-i)\beta_2^4 D_2^2\left(\cos\left(\beta_1(-l+(1+i)H)\right)+i\cosh\left(\beta_1((1+i)H-il)\right)+(1+i)\cos\left(\beta_1 l\right)\right)\\
& \times\left(\cos\left(\beta_2(l-H)\right)\cosh\left(\beta_2(l-H)\right)+1\right)+(2+2i)\beta_2\beta_1 D_1 D_2\Big(\beta_1^2\sinh\left(\beta_1 H\right)\\
& \times\cos\left(\beta_1(l-H)\right)\left(\sin\left((1+i)\beta_2(l-H)\right)-\sinh\left((1+i)\beta_2(l-H)\right)\right)\\
& +i\beta_2^2\cosh\left(\beta_1 H\right)\sin\left(\beta_1(l-H)\right)\left(\sin\left((1+i)\beta_2(l-H)\right)+\sinh\left((1+i)\beta_2(l-H)\right)\right)\\
& +\beta_2\beta_1\sin\left(\beta_2(l-H)\right)\sinh\left(\beta_2(l-H)\right)\Big(\cos\left(\beta_1(-l+(1+i)H)\right)\\
& -i\cosh\left(\beta_1((1+i)H-il)\right)\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\bar{\Delta}_4 = {} & (-1-i)\beta_1\beta_2 G\Big(-i\beta_1^4 D_1^2\Big(\sin\left(\beta_1(-l+(1+i)H)\right)+\sinh\left(\beta_1((1+i)H-il)\right)\\
& +(1+i)\sin\left(\beta_1 l\right)\Big)\left(\cos\left(\beta_2(l-H)\right)\cosh\left(\beta_2(l-H)\right)-1\right)\\
& +2\beta_2\beta_1 D_1 D_2\Big(\beta_1^2\sinh\left(\beta_1 H\right)\sin\left(\beta_1(l-H)\right)\left(\sin\left((1+i)\beta_2(l-H)\right)-\sinh\left((1+i)\beta_2(l-H)\right)\right)\\
& +\beta_2\beta_1\sin\left(\beta_2(l-H)\right)\sinh\left(\beta_2(l-H)\right)\left(\sinh\left(\beta_1((1+i)H-il)\right)-\sin\left(\beta_1(-l+(1+i)H)\right)\right)\\
& -i\beta_2^2\cosh\left(\beta_1 H\right)\cos\left(\beta_1(l-H)\right)\left(\sin\left((1+i)\beta_2(l-H)\right)+\sinh\left((1+i)\beta_2(l-H)\right)\right)\Big)\\
& -i\beta_2^4 D_2^2\left(\sin\left(\beta_1(-l+(1+i)H)\right)+\sinh\left(\beta_1((1+i)H-il)\right)-(1+i)\sin\left(\beta_1 l\right)\right)\\
& \times\left(\cos\left(\beta_2(l-H)\right)\cosh\left(\beta_2(l-H)\right)+1\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\bar{\Delta_5} = &-2\beta_1^3 D_1 G\Big(\beta_1^2 D_1 \Big(\beta_1\Big(\sin(\beta_1 H) + \sinh(\beta_1 H)\Big) \\
&\times (\cos(\beta_2(H-l)) - \cosh(\beta_2(H-l))) \\
&- \beta_2 (\cos(\beta_1 H) - \cosh(\beta_1 H)) (\sin(\beta_2(H-l)) + \sinh(\beta_2(H-l)))\Big) \\
&+ \beta_2^2 D_2\Big(\beta_1 (\sin(\beta_1 H) - \sinh(\beta_1 H)) (\cos(\beta_2(H-l)) + \cosh(\beta_2(H-l))) \\
&+ \beta_2 (\cos(\beta_1 H) + \cosh(\beta_1 H)) (\sinh(\beta_2(H-l)) - \sin(\beta_2(H-l)))\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\bar{\Delta_6} = &\beta_1^3 D_1 G\Big(2\beta_1^2 D_1\Big(\beta_1 (\sin(\beta_1 H) + \sinh(\beta_1 H)) \Big(\sinh(\beta_2(H-l)) \\
&- \sin(\beta_2(H-l))\Big) - \beta_2 (\cos(\beta_1 H) - \cosh(\beta_1 H)) \\
&\times (\cos(\beta_2(H-l)) - \cosh(\beta_2(H-l)))\Big) \\
&- 2\beta_2^2 D_2\Big(\beta_1 (\sin(\beta_1 H) - \sinh(\beta_1 H)) \\
&\times (\sin(\beta_2(H-l)) + \sinh(\beta_2(H-l))) \\
&+ \beta_2 (\cos(\beta_1 H) + \cosh(\beta_1 H)) (\cos(\beta_2(H-l)) + \cosh(\beta_2(H-l)))\Big)\Big).
\end{aligned}
$$

Appendix B.2

The constants in (2.49) are

$$\bar{\alpha}_1^{(1)} = \frac{\bar{\Delta}_1^*}{\bar{\Delta}^*}, \quad \bar{\alpha}_2^{(1)} = \frac{\bar{\Delta}_2^*}{\bar{\Delta}^*}, \quad \bar{\alpha}_3^{(1)} = \frac{\bar{\Delta}_3^*}{\bar{\Delta}^*}, \quad \bar{\alpha}_4^{(1)} = \frac{\bar{\Delta}_4^*}{\bar{\Delta}^*}, \quad \bar{\alpha}_1^{(2)} = \frac{\bar{\Delta}_5^*}{\bar{\Delta}^*}, \quad \bar{\alpha}_2^{(2)} = \frac{\bar{\Delta}_6^*}{\bar{\Delta}^*}$$

where

$$\bar{\Delta}^* = -2D_1\Omega^3(\epsilon - 1)(\cos(\Omega\epsilon)\cosh(\Omega\epsilon) + 1),$$

$$\begin{aligned}
\bar{\Delta}_1^* = &\left(-\frac{1}{2} + \frac{i}{2}\right)\Omega\Big(- D_1\Omega\Big(\Omega(\epsilon - 1)w_H(\sin(\Omega(\epsilon + i)) + (1+i)\sinh(\Omega - \Omega\epsilon) \\
&+ \sinh(\Omega + i\Omega\epsilon)) + (1+i)w_{H\xi_1}(\cosh(\Omega - \Omega\epsilon) - \sinh(\Omega)\sin(\Omega\epsilon) \\
&+ \cosh(\Omega)\cos(\Omega\epsilon))\Big) - Gl^2(\epsilon - 1)((1+i)\sinh(\Omega) + \sinh(\Omega - (1+i)\Omega\epsilon) \\
&+ i\sinh(\Omega - (1-i)\Omega\epsilon))\Big),
\end{aligned}$$

$$\bar{\Delta_2^*} = D_1\Omega^2\Big(w_{H\xi_1}(-\sinh(\Omega-\Omega\epsilon)-\sinh(\Omega)\cos(\Omega\epsilon)+\cosh(\Omega)\sin(\Omega\epsilon))$$

$$-\Omega(\epsilon-1)w_H(\cosh(\Omega-\Omega\epsilon)+\sinh(\Omega)\sin(\Omega\epsilon)+\cosh(\Omega)\cos(\Omega\epsilon))\Big)$$

$$-\Big(\frac{1}{2}-\frac{i}{2}\Big)Gl^2\Omega(\epsilon-1)((1+i)\cosh(\Omega)+\cosh(\Omega-(1+i)\Omega\epsilon)$$

$$+i\cosh(\Omega-(1-i)\Omega\epsilon)),$$

$$\bar{\Delta_3^*} = D_1\Omega^2\Big(\Omega(-(\epsilon-1))w_H(\cos(\Omega-\Omega\epsilon)-\sin(\Omega)\sinh(\Omega\epsilon)$$

$$+\cos(\Omega)\cosh(\Omega\epsilon))-w_{H\xi_1}(\sin(\Omega-\Omega\epsilon)-\cos(\Omega)\sinh(\Omega\epsilon)+\sin(\Omega)\cosh(\Omega\epsilon))\Big)$$

$$+\left(\frac{1}{2}-\frac{i}{2}\right)Gl^2\Omega(\epsilon-1)((1+i)\cos(\Omega)+\cos(\Omega-(1+i)\Omega\epsilon)$$

$$+i\cos(\Omega-(1-i)\Omega\epsilon)),$$

$$\bar{\Delta_4^*} = D_1\Omega^2\Big(w_{H\xi_1}(\cos(\Omega-\Omega\epsilon)+\sin(\Omega)\sinh(\Omega\epsilon)+\cos(\Omega)\cosh(\Omega\epsilon))\epsilon)$$

$$-\Omega(\epsilon-1)w_H(\sin(\Omega-\Omega+\cos(\Omega)\sinh(\Omega\epsilon)+\sin(\Omega)\cosh(\Omega\epsilon))\Big)$$

$$+\left(\frac{1}{2}-\frac{i}{2}\right)Gl^2\Omega(\epsilon-1)((1+i)\sin(\Omega)+\sin(\Omega-(1+i)\Omega+i\sin(\Omega-(1-i)\Omega\epsilon)).$$

Appendix B.3

The constants in (2.68) are

$$\bar{\gamma}_1^{(1)} = \frac{\bar{\Delta}_1^{**}}{\bar{\Delta}^{**}}, \quad \bar{\gamma}_2^{(1)} = \frac{\bar{\Delta}_2^{**}}{\bar{\Delta}^{**}}, \quad \bar{\gamma}_3^{(1)} = \frac{\bar{\Delta}_3^{**}}{\bar{\Delta}^{**}}, \quad \bar{\gamma}_4^{(1)} = \frac{\bar{\Delta}_4^{**}}{\bar{\Delta}^{**}}, \quad \bar{\gamma}_1^{(2)} = \frac{\bar{\Delta}_5^{**}}{\bar{\Delta}^{**}}, \quad \bar{\gamma}_2^{(2)} = \frac{\bar{\Delta}_6^{**}}{\bar{\Delta}^{**}}$$

where

$$\bar{\Delta}^{**} = -2D_1\Omega^3(\epsilon-1)(\cos(\Omega\epsilon)\cosh(\Omega\epsilon)+1),$$

$$\begin{aligned}
\bar{\Delta}_1^{**} = {} & Nl^3\left(-\frac{1}{2}+\frac{i}{2}\right)\Omega\Big(-D_1\Omega\Big(\Omega(\epsilon-1)w_H(\sin(\Omega(\epsilon+i))+(1+i)\sinh(\Omega-\Omega\epsilon) \\
& +\sinh(\Omega+i\Omega\epsilon))+(1+i)w_{H\xi_1}(\cosh(\Omega-\Omega\epsilon)-\sinh(\Omega)\sin(\Omega\epsilon) \\
& +\cosh(\Omega)\cos(\Omega\epsilon))\Big)-(\epsilon-1)((1+i)\sinh(\Omega)+\sinh(\Omega-(1+i)\Omega\epsilon) \\
& +i\sinh(\Omega-(1-i)\Omega\epsilon))\Big),
\end{aligned}$$

$$\bar{\Delta_2^{**}} = D_1\Omega^2 N l^3 \Big(w_{H\xi_1}(-\sinh(\Omega-\Omega\epsilon) - \sinh(\Omega)\cos(\Omega\epsilon) + \cosh(\Omega)\sin(\Omega\epsilon))$$

$$- \Omega(\epsilon-1) w_H(\cosh(\Omega-\Omega\epsilon) + \sinh(\Omega)\sin(\Omega\epsilon) + \cosh(\Omega)\cos(\Omega\epsilon))\Big)$$

$$- \Big(\frac{1}{2} - \frac{i}{2}\Big)\Omega(\epsilon-1)((1+i)\cosh(\Omega) + \cosh(\Omega-(1+i)\Omega\epsilon)$$

$$+ i\cosh(\Omega-(1-i)\Omega\epsilon)),$$

$$\bar{\Delta_3^{**}} = D_1\Omega^2 N l^3 \Big(\Omega(-(\epsilon-1)) w_H(\cos(\Omega-\Omega\epsilon) - \sin(\Omega)\sinh(\Omega\epsilon)$$

$$+ \cos(\Omega)\cosh(\Omega\epsilon)) - w_{H\xi_1}(\sin(\Omega-\Omega\epsilon) - \cos(\Omega)\sinh(\Omega\epsilon) + \sin(\Omega)\cosh(\Omega\epsilon))\Big)$$

$$+ \left(\frac{1}{2} - \frac{i}{2}\right)\Omega(\epsilon-1)((1+i)\cos(\Omega) + \cos(\Omega-(1+i)\Omega\epsilon)$$

$$+ i\cos(\Omega-(1-i)\Omega\epsilon)),$$

$$\bar{\Delta_4^{**}} = D_1\Omega^2 N l^3 \Big(w_{H\xi_1}(\cos(\Omega-\Omega\epsilon) + \sin(\Omega)\sinh(\Omega\epsilon) + \cos(\Omega)\cosh(\Omega\epsilon))\epsilon)$$

$$- \Omega(\epsilon-1) w_H(\sin(\Omega-\Omega + \cos(\Omega)\sinh(\Omega\epsilon) + \sin(\Omega)\cosh(\Omega\epsilon))\Big)$$

$$+ \left(\frac{1}{2} - \frac{i}{2}\right)\Omega(\epsilon-1)((1+i)\sin(\Omega) + \sin(\Omega-(1+i)\Omega + i\sin(\Omega-(1-i)\Omega\epsilon)).$$

Appendix C.1

The constants in (3.24) are

$$C_1^{(1)} = \frac{N_1}{N}, \quad C_2^{(1)} = \frac{N_2}{N}, \quad C_3^{(1)} = \frac{N_3}{N}, \quad C_4^{(1)} = \frac{N_4}{N}, \tag{11}$$

where

$$\begin{aligned}
N = {} & 2\cosh(\delta(\lambda_{11} - \lambda_{21}))(\lambda_{11} + \lambda_{21})^2 \\
& \quad \times ((-\lambda_{11}^2 + \nu_1)\lambda_{21}^2 - 2\lambda_{11}(\nu_1 - 1)\lambda_{21} + \nu_1(\lambda_{11}^2 + \nu_1 - 2)) \\
& - 2\cosh(\delta(\lambda_{11} + \lambda_{21}))(\lambda_{11} - \lambda_{21})^2 \\
& \quad \times ((-\lambda_{11}^2 + \nu_1)\lambda_{21}^2 + 2\lambda_{11}(\nu_1 - 1)\lambda_{21} + \nu_1(\lambda_{11}^2 + \nu_1 - 2)) \\
& - 4\lambda_{21}\lambda_{11}(-\lambda_{11}^4 - \lambda_{21}^4 + 2\nu_1^2 + 2\lambda_{11}^2 + 2\lambda_{21}^2 - 4\nu_1),
\end{aligned}$$

$$
\begin{aligned}
N_1 = {} & (A\lambda_{21}+B)(\lambda_{11}+\lambda_{21})e^{-\delta\lambda_{21}} \\
& \quad \times ((-\lambda_{11}^2+\nu_1)\lambda_{21}^2 - 2\lambda_{11}(\nu_1-1)\lambda_{21} + \nu_1(\lambda_{11}^2+\nu_1-2)) \\
& + (A\lambda_{21}-B)(\lambda_{11}-\lambda_{21})e^{\delta\lambda_{21}} \\
& \quad \times ((-\lambda_{11}^2+\nu_1)\lambda_{21}^2 + 2\lambda_{11}(\nu_1-1)\lambda_{21} + \nu_1(\lambda_{11}^2+\nu_1-2)) \\
& - 2(A\lambda_{11}+B)\lambda_{21}(\lambda_{21}^2+\nu_1-2)(-\lambda_{21}^2+\nu_1)e^{-\delta\lambda_{11}},
\end{aligned}
$$

$$
\begin{aligned}
N_2 = {} & (A\lambda_{21}+B)(\lambda_{11}-\lambda_{21})e^{-\delta\lambda_{21}} \\
& \quad \times ((-\lambda_{11}^2+\nu_1)\lambda_{21}^2 + 2\lambda_{11}(\nu_1-1)\lambda_{21} + \nu_1(\lambda_{11}^2+\nu_1-2)) \\
& + (A\lambda_{21}-B)(\lambda_{11}+\lambda_{21})e^{\delta\lambda_{21}} \\
& \quad \times ((-\lambda_{11}^2+\nu_1)\lambda_{21}^2 - 2\lambda_{11}(\nu_1-1)\lambda_{21} + \nu_1(\lambda_{11}^2+\nu_1-2)) \\
& - 2(A\lambda_{11}-B)\lambda_{21}(\lambda_{21}^2+\nu_1-2)(-\lambda_{21}^2+\nu_1)e^{\delta\lambda_{11}},
\end{aligned}
$$

$$
\begin{aligned}
N_3 = {} & (A\lambda_{11} + B)(\lambda_{11} + \lambda_{21})e^{-\delta\lambda_{11}} \\
& \times ((-\lambda_{21}^2 + \nu_1)\lambda_{11}^2 - 2\lambda_{11}(\nu_1 - 1)\lambda_{21} + \nu_1(\lambda_{21}^2 + \nu_1 - 2)) \\
& - (A\lambda_{11} - B)(\lambda_{11} - \lambda_{21})e^{\delta\lambda_{11}} \\
& \times ((-\lambda_{21}^2 + \nu_1)\lambda_{11}^2 + 2\lambda_{11}(\nu_1 - 1)\lambda_{21} + \nu_1(\lambda_{21}^2 + \nu_1 - 2)) \\
& - 2(A\lambda_{21} + B)\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2)(-\lambda_{11}^2 + \nu_1)e^{-\delta\lambda_{21}},
\end{aligned}
$$

$$
\begin{aligned}
N_4 = {} & -(A\lambda_{11} + B)(\lambda_{11} - \lambda_{21})e^{-\delta\lambda_{11}} \\
& \times ((-\lambda_{21}^2 + \nu_1)\lambda_{11}^2 + 2\lambda_{11}(\nu_1 - 1)\lambda_{21} + \nu_1(\lambda_{21}^2 + \nu_1 - 2)) \\
& + (A\lambda_{11} - B)(\lambda_{11} + \lambda_{21})e^{\delta\lambda_{11}} \\
& \times ((-\lambda_{21}^2 + \nu_1)\lambda_{11}^2 - 2\lambda_{11}(\nu_1 - 1)\lambda_{21} + \nu_1(\lambda_{21}^2 + \nu_1 - 2)) \\
& - 2(A\lambda_{21} - B)\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2)(-\lambda_{11}^2 + \nu_1)e^{\delta\lambda_{21}}.
\end{aligned}
$$

Appendix C.2

The entries of the matrix M in (4.13) are given by

$$M_{11} = M_{12} = \lambda_{11}^2 - \nu_1, \quad M_{13} = M_{14} = \lambda_{21}^2 - \nu_1$$

$$M_{21} = -M_{22} = \lambda_{11}(\lambda_{11}^2 + \nu_1 - 2),$$

$$M_{23} = -M_{24} = \lambda_{21}(\lambda_{21}^2 + \nu_1 - 2),$$

$$M_{31} = \lambda_{11} e^{\lambda_{11}\delta_H}, \quad M_{32} = -\lambda_{11} e^{-\lambda_{11}\delta_H},$$

$$M_{33} = \lambda_{21} e^{\lambda_{21}\delta_H}, \quad M_{34} = -\lambda_{21} e^{-\lambda_{21}\delta_H},$$

$$M_{35} = \lambda_{12} e^{-\lambda_{12}\delta_H}, \quad M_{36} = \lambda_{22} e^{-\lambda_{22}\delta_H},$$

$$M_{41} = e^{\lambda_{11}\delta_H}, \quad M_{42} = e^{-\lambda_{11}\delta_H}, \quad M_{43} = e^{\lambda_{21}\delta_H},$$

$$M_{44} = e^{-\lambda_{21}\delta_H}, \quad M_{45} = -e^{-\lambda_{12}\delta_H}, \quad M_{46} = -e^{-\lambda_{22}\delta_H},$$

$$M_{51} = D(\lambda_{11}^2 - \nu_1) e^{\lambda_{11}\delta_H}, \quad M_{52} = D(\lambda_{11}^2 - \nu_1) e^{-\lambda_{11}\delta_H},$$

$$M_{53} = D(\lambda_{21}^2 - \nu_1) e^{\lambda_{21}\delta_H}, \quad M_{54} = D(\lambda_{21}^2 - \nu_1) e^{-\lambda_{21}\delta_H},$$

$$M_{55} = -(\lambda_{12}^2 - \nu_2) e^{-\lambda_{12}\delta_H}, \quad M_{56} = -(\lambda_{22}^2 - \nu_2) e^{-\lambda_{22}\delta_H},$$

$$M_{61} = D\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2) e^{\lambda_{11}\delta_H},$$

$$M_{62} = -D\lambda_{11}(\lambda_{11}^2 + \nu_1 - 2)e^{-\lambda_{11}\delta_H},$$

$$M_{63} = D\lambda_{21}(\lambda_{21}^2 + \nu_1 - 2)e^{\lambda_{21}\delta_H},$$

$$M_{64} = -D\lambda_{21}(\lambda_{21}^2 + \nu_1 - 2)e^{-\lambda_{21}\delta_H},$$

$$M_{65} = \lambda_{12}(\lambda_{12}^2 + \nu_2 - 2)e^{-\lambda_{12}\delta_H},$$

$$M_{66} = \lambda_{22}(\lambda_{22}^2 + \nu_2 - 2)e^{-\lambda_{22}\delta_H},$$

where

$$\lambda_{1j} = \sqrt{1+\gamma_j}, \quad \lambda_{2j} = \sqrt{1-\gamma_j},$$

and

$$\gamma_j = \frac{\omega}{k^2}\sqrt{\frac{2\rho_j h}{D_j}}, \quad j = 1, 2.$$

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