Asymptotic models for elastic solids taking into account nonlocal boundary layers

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Declaration

I certify that this thesis submitted for the degree of Doctor of Philosophy is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Abstract

The thesis is aimed at asymptotic analysis of the near-surface boundary layers in non-locally elastic solids. The dynamic response of a homogeneous half-space with a traction-free surface is analysed for a nonlocal exponential kernel. A typical wavelength is assumed to be much greater than the lengthscale associated with internal properties of the elastic medium. The dominant effect of the boundary layer is revealed. The leading order long-wave approximations are shown to coincide with the 'local' problem for a half-space having a vertical inhomogeneity localised near the surface. An explicit correction to the classical boundary conditions on the surface of a 'locally' elastic half-space is obtained by asymptotic analysis of the near-surface behaviour. The order of the derived correction exceeds that of the well-known correction to the governing differential equations of Eringen's model, e.g., see [44]. The obtained refined boundary conditions enable evaluating the interior stress-strain solution outside a narrow boundary layer localised near the surface. As examples, the effect of nonlocal elastic phenomena on the Rayleigh wave speed and also a plane strain problem of a moving load on the surface of a half-space are studied. In addition, a thin layer with a traction-free upper face, subjected to prescribed displacements along its lower face, is investigated. Further, the 3D dynamic equations in nonlocal elasticity for a thin plate are considered, assuming the plate thickness to be much greater than a typical microscale size. The long-wave low-frequency approximations are obtained for both plate bending and extension. Boundary layers characteristic of nonlocal behaviour are revealed near the plate faces. It is established that taking into account the effect of the boundary layers results in first-order corrections to the bending and extensional stiffness in the classical 2D plate theory.

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1 Introduction

For investigation of physical phenomena such as dispersion of waves, or for solving problems such as Rayleigh surface waves or Rayleigh-Lamb waves in plates, vibration, bending, buckling or other types of loads in solid structures, the classical elasticity theory is a ready-to-use macroscale model. However, when it is necessary to consider micro- or nanoscopic materials, it is important to account for small particle interaction (e.g., molecular or atomic), such as Van der Waals forces, which are significant at the micro- and nanoscale, although can be neglected in respect of the macroscale. One of the most popular theories, which can account for such internal structural forces, was developed by Eringen [44]. It evolved from, and is based on, the earlier fundamental work by the same author [40]. It is called the nonlocal elasticity theory, which can be described by introducing nonlocal elements into the governing equations of the classic elasticity theory.

The essence of Eringen's nonlocality principle is that the stress at any reference point $\boldsymbol{x}$ is a functional of the strain field at every point $\boldsymbol{x}'$ in the body. In the case of homogeneous, isotropic materials, Eringen's nonlocal linear theory leads to a set of integropartial differential equations for the displacement field. These are difficult to solve, but there are a selection of kernels that reduce these equations to singular PDEs. Moreover, these kernels proved to give excellent experimental results with good approximations obtained for many physical problems with a wide range of characteristic lengths, from the micro- to the macroscale [44]. Also, nonlocal theory tends to classical ('local') elasticity theory in the long-wave limit (for example, see [119]) and atomic lattice dynamics theory in the short-wave limit ([43] cited in [44]).

The nonlocal theory has proved to be in a good agreement with lattice dynamics theories. It can be used, for instance, to solve problems for solids having impurities and dislocations (so-called imperfect elastic solids for which it is difficult to characterise their internal, atomic state) and for solids with boundaries at finite regions (i.e., surface physics) [43].

It has been established by the modern scientific community that the concept of nonlocality is of fundamental importance in nanomechanics, which is rapidly becoming a cutting-edge area of mathematical, physical, and engineering science. Several examples of nanomechanical applications are provided in this thesis. At this stage, we remark that nanotechnology is actually being used in a wide variety of fields, e.g., medical and biophysical research, naval engineering, and electronics, including computer technologies. The research described in this thesis is concerned with mostly physical applied mathematics, as it is necessary first to establish all the necessary models and verify them, both theoretically and experimentally, then prove the efficiency of the newly proposed models. Only after this they can be practically implemented within the mentioned areas.

The present thesis aims to tackle a nonlocal elasticity problem of Eringen's 'integral' type using asymptotic methods, rather than numerically. Details of the 'integral' nonlocal models and numerical methods for solving nonlocal problems are outlined in Section 1.1. The use of asymptotic methods allows one to understand the main significant elements and patterns of a physical process, as well as providing a more fundamental understanding of the underlying physics by elucidating essential features. The motivation is rather clear, to the best of the author's knowledge, no convenient to use closed-form or asymptotic solutions have been previously obtained for nonlocal elasticity problems in either the 2D or 3D cases. There have been a considerable number of publications, see Section 1.1, however the fundamental effect of boundaries on the implementation of nonlocal elasticity models has yet to be properly addressed.

Let us reiterate that the main concept of nonlocal elasticity theories is that the intervals of integration involved in the nonlocal constitutive relations are dependent on the distance from a reference point to the boundary, e.g., see [44]. This results in the appearance of the so-called boundary layers which correspond to localised nonhomogeneous stress and strain fields. This concept is an essential part of the thesis, as it was necessary to fill this gap to understand the influence of localised boundary layers on overall dynamic behaviour. There are authors who emphasised the important role of boundary layers, e.g., see Huang [61], Bazant *et al.* [15], Borino & Polizzotto [21], and Abdollahi & Boroomand [2]. However, we are not aware of any cases within which the problems were investigated using the asymptotic methods.

1.1 Literature review

In this section, a literature review of nonlocal theories is presented, together with a discussion of methods utilised to solve some associated problems. The first part of the review contains a discussion on the features of nonlocal continuum models and their evolution, starting from the pioneering works and spanning to the latest developments. In the second part, there is a review on various applications of modern nonlocal models. Finally, in the third part, a brief historical account of the development of asymptotic methods is provided, as these methods were utilised as the main tool for derivation of all results in the thesis.

1.1.1 Nonlocal continuum elastic models

One of the first attempts to account for the nonlocal phenomenon in the form of Van der Waals cohesive forces, affecting properties of an elastic material under certain inhomogeneous stress conditions, such as defect interaction and diffusion, was made by Kroner [80]. More historical references on the development of continuum elasticity, starting from Cosserat brothers' theory [25] and Bohr's concept of the atom [20], alongside the main conceptual changes in understanding of elasticity, can be found in Krumhansl [82], who derived the continuum theory using lattice theory. This author states that for heterogeneous materials, as well as for those having some level of milli-structure, non-classical (such as Eringen's nonlocal theory) continuum theories are very useful in application to real materials [82]. In addition, there is a good summary of what has been developed in elasticity theory presented by Kunin *et al.*, see [83], [84], and references therein. A detailed discussion on microstructure theories and their complexity in comparison with classical local elasticity theory are also provided in the mentioned works by Kunin *et al.* They also contain an in-depth analysis of the theoretical foundations of media with microstructure and some applications of such theories.

The classic theory of elasticity is based on the assumption that the internal forces of the body are of contact type, thus, having a zero range. The cohesive forces in materials have finite or infinite range, but not necessarily zero range. This represents certain limitations. Nonlocal elasticity theory accounts for the finite range of the cohesive forces [79]. For instance, in atomic lattice theories, long-range, cohesive forces, unlike contact forces, are well known. The classical elasticity model, the long-wave limit of the atomic theory, does not capture these cohesive forces. In order to improve the efficiency and representativeness of the elasticity theory, a variety of theories were based on the granular nature of materials [41].

The contact interactions between small particles of the material could be called 'local', with distant interactions termed 'nonlocal'. The nonlocal theories can give better approximation of the real physics of solid media, providing better description of materials at micro- and nanoscales. At present, two popular conceptual models of material media are the material continuum (where matter is spread continuously over a region) and the molecular model (where matter is made of separate moving molecules). In the macroworld, the continuum model is considered an excellent approximation in comparison to the molecular model. The latter model assumes that there are no zero-range interactions between particles (other than collisions between molecules), therefore interaction can be treated as nonlocal. If, in a certain material, characteristic distances are much greater than a typical granular (intermolecular) distance, but still much smaller than a typical range of intermolecular forces, any local theory is not valid. The molecular model is therefore not appropriate, a nonlocal continuum theory should be used instead. This and other aspects of nonlocal theory of elastic continua, related to boundary value problems (BVP) and initial and boundary value problems (IBVP), as well as conditions for existence of its fundamental solutions, were investigated, for instance, by Rogula [119].

Eringen's nonlocal elasticity theory is different to the other theories mentioned previously and is essentially based on thermodynamic considerations. Nonlocality is a powerful concept for describing and analysing the classical effects from macroscale to very small scales, such as molecular and atomic [42]. The description of many theories of polar media, granular aspects, and internal structure of materials can be found in a conference proceedings edited by Kroner [81]. Generally speaking, polar theories are nonlocal theories where nonlocality is described via moment tensors for each point of the body [42]. Briefly, Eringen *et al.* derived nonlocal elasticity theories by investigating the presence of nonlocal residuals of field quantities (such as body force, mass, internal energy, etc.) and found the residuals using thermodynamic theory. Later, by the early 1980s, the nonlocal theory was simplified and it became possible to introduce nonlocal elastic moduli in the stress-strain constitutive relations of the classic ('local') elasticity theory, where elastic moduli are functions of the Euclidean distance between strain and stress points. Such a type of Eringen & Edelen [41], Edelen *et al.* [38], and Eringen & Laws [37] nonlocal elasticity models are termed 'strongly nonlocal' (or 'integral'), with the stress at any point expressed as a weighted function of the entire strain field. There is another type of model of nonlocal elasticity (e.g., see [80] and [119]), which is called 'weakly nonlocal' (or 'gradient'), with the stress considered a function of the strain and its gradient at the same point of the body. A more detailed review on nonlocal elasticity theory evolution, alongside applications and examples of problems, which became feasible with the introduction of nonlocal theories, can be found in [109].

The current state of art in the area of nonlocal elasticity, and its applications in small-size devices and materials possessing microstructure, has been presented by a number of authors during the scientific development of the area, see, for example, [34]. In addition, the paper by dell'Isola *et al.* [31] reminds us about Piola's contribution, which was not widely known for a long time, see also references to Piola's papers in [31].

Some critics of Eringen's nonlocal theory can be found, for instance, in [127]. Eringen's theory assumes the same attenuation function for all material moduli, but it is shown not possible to simultaneously fit both the longitudinal and shear acoustic dispersion curves for certain materials, for example, Si, Au, and Pt. In mitigation, a general form of the nonlocal theory was developed in the above mentioned paper, which works with different attenuation functions for the distinct material moduli and is able to reflect both hardening and softening behaviours of the material. This work also reports elastic moduli and nonlocal parameters of various materials.

Let us stress that it is possible to account for the internal structure of materials via, for instance, an internal characteristic length. This means that, unlike the classic elasticity theory that does not employ internal characteristic length, nonlocal elasticity can be utilised when it is essential to take into account the influence of microstructure, which can be significant in micro- and sub-microscale [45]. Small scale effects play an important role in micro- and nanoscale materials, but this effect is ignored in the classical 'local' continuum theory, therefore the latter should be refined [129].

Let us reiterate that the length scales in nanotechnology are very small, so classical continuum models often cannot be employed. Atomic and molecular models are computationally expensive and difficult to formulate. Therefore, nonlocal elasticity can be used to extend the continuum models to the nanoscale. The necessary accuracy of the analysis of the dynamic behaviour of various nanostructures cannot be achieved using the classical local elasticity theory. Moreover, conducting experiments with materials of nanoscale size can be both extremely challenging and expensive. It is therefore crucial to develop appropriate nonlocal elasticity mathematical models for nanomaterials, e.g., see [91] and [92].

A review of micromechanical approaches, known in the literature as 'structured deformations', can be found in [98], [30], and [99]. As an example, in [98] structured deformations are considered as a multi-scale geometrical setting that would allow us to put the fields in continuum mechanics into two groups: ones due to smooth changes at smaller length scales and ones due to disarrangements (i.e., slips and separations) at smaller length scales. As an example, see [35] within which a variational formulation is used to derive a micromechanical, explicit nonlocal constitutive equation, relating the stress and strain ensemble averages for a class of linear elastic composite materials. Similarly, in [29], a fully nonlocal effective response of prestressed composites is found to be perfectly analogous to the unstressed case dealt with in the previously mentioned paper by Drugan & Willis [35].

As an example of nonlocal model applications, the problem of Rayleigh surface waves in a nonlocal setting, which are dealt with in the current thesis, can be found, for example, in [97] and [91]. Nowinski *et al.* studied the propagation of longitudinal waves in isotropic homogeneous elastic plates using the linear theory of nonlocal continuum mechanics of Eringen. In this study, it is concluded that the effect of lattice defects (vacancies and interstitials) creates a significant strain of the material medium as a result of the difference between lattice atoms radii and the defects. The classical continuum theory's underlying assumption is that stresses are of the local type (i.e., stress at a point is dependent only on the strain at the same point), not involving any internal length scales. The absence of the internal size parameter creates discrepancies in the prediction of mechanical response. One example is non-dispersive wave behaviour (where wave velocity does not depend on frequency). Within the framework of the classical elasticity, Rayleigh surface waves, which propagate along the surface of a isotropic elastic half-space, are not dispersive. However, both the atomic theory of lattice and experiments contradict this [91].

In the paper by Abdollahi & Boroomand [2], the application of various numerical techniques for solving nonlocal problems is discussed, including the finite element method (FEM) (for example, see [109]), boundary element method (BEM) (see [126], where a numerical solution method for 3D nonlocal elastic problems is proposed). However, it is difficult to obtain numerical results because of high computational costs (numerical integration, etc.), therefore a mesh reduction approach can be considered to decrease the computational time. For instance, Abdollahi & Boroomand [1] presented a low-residual solutions for 1D and 2D problems using Chebyshev polynomials. Trefftz's approach, using the method of fundamental solutions (MFS) or BEM (see [47]), can give low-residual numerical solutions, but is computationally expensive as it evaluates Green's functions. Later, the method of solving 1D (and partly 2D) problems using Chebyshev polynomials was improved by using exponential basis functions and a boundary layer approach [2]. A concept of structural symmetry, with its theoretical issues and computational methods, was considered as a nonlocal version of the FEM in [108].

1.1.2 Nonlocal theories for thin elastic structures

Micro- and nanoscale technology has seen rapid development lately. Because of this, small scale effects (inter-molecular interactions, atomic forces) must be accounted for in order to achieve good accuracy. Ignoring these interactions may sometimes lead to incorrect results [5]. In this paper, methods of molecular dynamics dealing with size effects and atomic-scale length are discussed. Note that in [131] a semicontinuum model for nanostructured materials with a plate-like geometry is considered (e.g., ultra-thin films). In contrast to the classical continuum theory, the semicontinuum model can account for the discrete nature in the transverse direction. In [131], it is found that the Young's modulus and Poisson's ratio are dependent on the number of atom layers in the transverse direction, tending to the respective bulk values as the number of atom layers increases. Similarly, in [138] a plate model is developed for nanomaterials, such as ultrathin films, and the effectiveness of the model is established by analysis of the dispersion relations of a 3-layered nanomaterial. The main result shows that if the continuum plate theory is used to obtain the Young's modulus of a nanomaterial, its value may be underestimated significantly. Additionally, in [139] and [87], and in respect of single-walled carbon nanotubes (CNTs), studies using simulation within the framework of the molecular dynamics establish results in line with the predictions in [87]. All the mentioned methods of molecular dynamics require the solving of a large number of equations. It would therefore be difficult to handle systems having large length and time scales, which is thus left to continuum mechanics theories. One of the most popular theories is the nonlocal theory of Eringen, a theory exploited in this thesis, see [41]. Let us reiterate that Eringen's theory is capable of predicting the behaviour of large micro- and nanoscale structures without the need to solve large numbers of equations; this is due to the fact that small-scale interactions are accounted for via material parameters, see [46].

A review on recent research on the application of nonlocal continuum theory for the modelling of carbon nanotubes and graphene sheets can be found in [12] by Arash & Wang. These authors also introduce nonlocal beam, plate, and shell models. The paper is a good brief introduction of the evolution of nonlocal continuum theories, especially their application to modelling of nanomaterials.

Let us now discuss some examples of how and where nonlocal theories can be applied. It is a very challenging task to find exact solutions of problems stated using nonlocal elasticity theory. For instance, Pisano & Fuschi [107] present an analytical solution for a simple 1D nonlocal elasticity problem, namely a nonlocal elastic bar in tension. The problem of an Euler-Bernoulli cantilever nanobeam with a point load, with application to microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS), is tackled in [22]. In the latter article, it is shown that a paradox exists, with some beam bending solutions for nonlocal elastic beams found to be identical to the classical ('local') solutions. This means that the small scale effect is redundant in the nonlocal solution and can be overcome with a gradient ('weakly' nonlocal) elastic model or with an integral ('strongly' nonlocal) elastic model. A mixed model, which represents a combination of the gradient model and Eringen's integral model, is also proposed. In [16] it is mentioned that the models being discussed in the current paragraph are defined as 'local/nonlocal' stress-strain models, where the stress is defined as a weighted sum of local and nonlocal stress components. In [121], bending of a nanobeam is considered within the framework of a newly proposed 'stress-driven' nonlocal model and a stiffer elastic response, attributed to normalisation of the kernel, is observed. This theory provides an effective methodology to account for small-scale effects in nanobeams, by well-posed problems. In addition, recent research in [120] reveals the requirement to include constitutive boundary conditions in the nonlocal stress-driven model. The solution procedure for the stress-driven nonlocal law is adopted and the effectiveness of this stress-driven model when used in structural design of nanodevices is shown. In order to analyse the static response of a nanobeam for different types of loadings and boundary conditions, nonlocal integral beam models for different attenuation functions are considered in [78]. Finite element analysis of Timoshenko nanobeams, using the nonlocal integral model, is carried out in [96]. Within this study, some case studies demonstrating the influence of boundary conditions, nonlocal microscale parameter, and loading factor are presented. The paradox related to cantilevers is tackled and said to be resolved using the integral nonlocal model.

MEMS/NENS (micro- and nanoelectromechanical systems) nanostructures found applications in, among others, communications, machinery, information technology and biotechnology, e.g., see [63]. In general, nanostructures possess superior mechanical, electrical, electronic, and thermal properties in comparison with the conventional structural materials. The list of potential applications include aerospace, nanocomposites, biomedical and bioelectrical microelectronics, e.g., see [95].

As an example of the above mentioned types of applications, Bernoulli-Euler beam model is extended by including nonlocal elastic material response. In the case of cantilever actuators, a model predicting that MEMS-scale devices do not show nonlocal effects, while NEMS-scale (i.e., nanoscale) devices do, see [101]). In [126] it is mentioned that classical elasticity cannot be applied, for instance, to the modelling of carbon nanotubes dynamics, where size effects become significant, again the nonlocal theory should be employed. In addition, in [130], it is discussed that one can find a model for the column buckling of multiwalled carbon nanotubes based on a nonlocal continuum theory, in which it is shown that small scale effects make a significant effect on the mechanical behaviour of multiwalled carbon nanotubes and cannot be ignored.

It is shown in [32], in the context of mechanically based nonlocal elasticity, that the direct substitution of an attenuation function into the Eringen's integral model can lead to inconsistencies at the boundaries of a finite bar in respect of the boundary conditions. The authors of this work dealt with this inconsistency and proposed a model incorporating the action of long-range forces for a nonlocal 1D bar, both for unbounded and bounded domains. Similar inconsistencies are also discussed in [33] and [140]. In the latter paper, an analysis in bounded domains showed that the natural frequencies of a cantilever bar increase as the internal length scale of the material increases. Also, a numerical analysis showed that a mechanically based model of nonlocal elasticity can be applied to the problems of elastic or thermal wave propagation in nanosystems.

Nanoplates find applications, for example, in energy storage, chemical and biological sensors, renewable energy (solar cells), field emission nanodevices and transporting of nanocars, see references and some discussion in [75] and [13]. In addition to being used in sensors, actuators, and MEMS/NEMS, nanoplates are used to model graphene sheets embedded in an elastic matrix or stiff thin films resting on an elastomeric substrate, see [129] and [128] and references therein. Research into the mechanisms of nonlocal effect on the transverse vibration for 2D nanoplates (for instance, a single layer graphene sheets) can also be found in [136].

Due to their potentially remarkable mechanical properties, nanoplates made from nanomaterials have been widely used as the building blocks for ultrasensitive and ultrafine resolution applications within NEMS area, see [9].

Various applications of micro- and nanoplates have been studied within the framework of classical and first-order plate theories. In respect of the first-order shear deformation theory, the transverse shear strain and stress are assumed independent of the thickness co-ordinate (transverse direction), which is a rough approximation of the actual variation which could vanish on the faces of a plate. In order to validate the mentioned discrepancy, the shear correction factor is introduced, see [5]. Third-order shear deformation theories, where the displacement field has a cubic term in the transverse co-ordinate, have been developed, see [114]. These theories are able to predict the deflections and stresses more accurately than the first-order theory.

In [115], beam theories (of Euler-Bernoulli, Timoshenko, Reddy, and Levinson) are reformulated using the nonlocal differential constitutive relations established by Eringen, where the theoretical developments and numerical results are applied in nonlocal theories of beams, plates, and shells. In addition, generalised governing equilibrium equations of nonlinear nonlocal Kirchhoff and Mindlin plate theories are derived. Some discussion on nonlocal shell theory, applied to study the scale effect on carbon nanotubes wave propagation, can be found in [135]. The main conclusion is that the nonlocal shell theory is crucial to account for the small-scale effect on phonon dispersion relations at larger wavenumbers in the longitudinal direction. It also plays a major role in revealing the small-scale effects at larger wavenumbers in the circumferential direction of carbon nanotubes.

Bending of a nanoplate, subjected to different in-plane loads and taking into account the small-scale effects, is considered employing a nonlocal continuum theory by Kananipour [63]. Within this work, governing equations and displacements for nonlocal Mindlin and Kirchhoff plate models are derived and their application discussed while employing numerical two-dimensional differential quadrature method (DQM) for bending analysis. Vibration analysis of a nanoplate within the framework of the 3D theory of elasticity, using nonlocal continuum mechanics, can be found in [9]. In this paper, a closed-form solution for the natural frequencies of a rectangular simply supported nanoplate is obtained and the effect of the nonlocal parameter on frequency behaviour is examined. Closed form solutions for the axisymmetric bending of micro- and nanoscale circular plates (based on the nonlocal plate theory) are obtained in [36]. Specifically, in this work the nonlocal solutions confirm that taking into account the small scale effect leads to larger deflections, bending moments, shear force, and a lower bending stiffness for the plate. A nonlocal plate model for bending, buckling, and free vibration of micro- and nanoscale plates, based on the nonlocal differential constitutive relations of Eringen, is also developed in [132]. The authors establish that the inclusion of small-scale and shear deformation effects makes the plate more flexible and, thus, predict an increased deflection and decreased buckling load and natural frequency.

In [62], small scale effects on the transient analysis of nanoscale plates are investigated. The nanoscale plate theory is reformulated with use of Eringen's nonlocal differential constitutive relations. It is shown that Eringen's nonlocal elasticity is able to capture various small scale effects. In particular, solutions of the transient dynamic response of a nanoscale plate are presented to illustrate the effect of nonlocal theory on dynamic response of the nanoscale plates. A part of this work is based on nonlocal continuum-based modelling of a nanoplate subjected to a moving nanoparticle, see [75] and [76]. These papers deal with nanoplates applications from a nonlocal elastodynamic point of view. The fully simply supported nanoplate is modelled based on the nonlocal Kirchhoff, Mindlin, and higher-order plate theories. The second part, [76], is a more detailed parametric study concerning the effects of influential parameters (e.g., small-scale parameters) on the dynamic response of the above mentioned nonlocal plate models.

In Eringen's nonlocal elasticity model, the governing equations for a thin plate can be derived by integrating the equations of motion for the nonlocal linear elasticity through the thickness of the plate. With some assumptions for displacement components, several plate theories have been obtained. Analogues of the well-known Kirchhoff and Mindlin plate theories are derived and analysed in [88]. The Kirchhoff plate theory neglects the effect of transverse shear deformation. The Mindlin plate theory is a first-order shear-deformable plate theory that incorporates this effect which becomes significant in thick plates and shear-deformable plates. Bending and vibration problems for a rectangular plate with simply supported edges are investigated in [88], based on the two mentioned plate theories. These theories allow us to examine the effect of small scale on the bending and vibration solutions.

In [137], a derivation of Kirchhoff and Mindlin plate models based on generalised gradient elasticity (with both stress gradient and strain gradient parameters) is presented. In this work, a variational formulation of the gradient Kirchhoff plate model is established to deal with a plate at nanoscale with complex geometries and boundary conditions. Static bending and free vibration of rectangular Kirchhoff and Mindlin simply supported plates are also obtained analytically, with the solutions demonstrating the influence of the generalized gradient parameters on both the static and dynamic behaviour of plates.

An analytical solution for free vibration of nanoplates, using first-order shear deformation plate theory (FSDT) as well as the classical plate theory (CLPT) (both reformulated using the nonlocal differential constitutive relation of Eringen [44]) can be found in [111]. Again nonlocal theories are used to demonstrate the effect of the nonlocal parameter (e.g., internal material properties) on the natural frequencies of nanoplates.

Analysis of functionally graded (FG) materials at micro/nanoscale using a modified Eringen's nonlocal theory is provided in [124]. The new, modified nonlocal theory is also compared with Eringen's theory by analysing free vibration of FG rectangular micro/nanoplates with simply supported boundary conditions using a first-order plate theory and 3D elasticity and assumed to be functionally graded properties only along the plate thickness. In addition, some exact solutions for buckling of heated functionally graded (FG) annular nanoplates resting on an elastic foundation were derived in [13]. It may be concluded that the small scale effects significantly affect the thermal stability characteristics of FG annular nanoplates.

A new method called 'full modified nonlocal (FMNL) theory', based on variational principle, for bending and buckling analysis of simply supported rectangular nanoplates was proposed in [93]. It is shown that an advantage of the FMNL theory is that one defect of nonlocal theory, in vanishing of small scale effect, can be resolved for some problems.

Vibrational characteristics of multi-layered graphene sheets with different boundary conditions are studied in [10]. Within this work it is concluded that the small-scale effects in the nonlocal continuum model affect small size multi-layered graphene sheets making them more flexible. Specifically, the classical continuum model usually overestimates the resonant frequencies of small size graphene sheets and in order to reduce the error, nonlocal theories should be employed. The importance of the small length scales is affected by the boundary conditions of multi-layered graphene sheets. For instance, the effect of the small length scales on a fully clamped graphene sheet is much more significant than on its freely supported counterpart. It is also found that the small size effect becomes more noticeable for smaller values of moduli of the elastic medium, see [10].

The small scale effect on the thermal buckling behaviour of arbitrary straight-sided quadrilateral orthotropic nanoplates embedded in elastic medium is presented in [90]. The derived formulation in this work is based on the classical plate theory, with the small scale effect accounted for by using the nonlocal elasticity constitutive relations. The effects of nonlocal parameter, as well as the elastic medium parameters, geometrical shapes, and boundary conditions, on the critical buckling temperature rise of orthotropic nanoplates are analysed in detail.

A nonlocal elastic plate model that accounts for the scale effects is developed for wave propagations in graphene sheets in [11] using the finite element method (FEM).

The applicability of this is verified by molecular dynamics simulations and the nonlocal finite element plate model is shown to be crucial in predicting graphene phonon dispersion relations, especially at very small wavelengths (less than 1 nm) when the small-scale effects become significant. As an example, an application of graphene sheets as nanosensors for noble gas atoms is considered.

A closed-form solution for 3D static deformation and free vibrational response of a simply supported and multilayered quasicrystal nanoplate with incorporation of the nonlocal effect in Eringen's form can be found in [134]. It is remarkable that the long wave assumption fails near the boundaries of a non-locally elastic solid, where it is essential to account for the effect of boundary layers. The importance of boundary layers in non-locally elastic bodies is emphasised by a number of authors, for example, see [15] and [2].

Summarising, the existing 2D nonlocal models for thin elastic plates are usually derived using the differential constitutive relations of Eringen [44], e.g., see previously mentioned papers by Lu *et al.* [88], Duan & Wang [36], Aghababaei & Reddy [5], Pradhan & Phadikar[111], Malekzadeh *et al.* [90], Xu *et al.* [137], Thai *et al.* [132], Jung & Han [62], and Mousavi *et al.* [93]. In these models a reduction from 3D to 2D is performed using ad-hoc assumptions and neglecting the variation of nonlocal properties across the thickness of a plate. This results in the occurrence of nonlocal corrections only at second-order in the microscale parameter related to the external longitudinal wavelength, which is assumed much greater than the plate thickness. In [123], the authors take into account the variation of several nonlocal integral kernels along the thickness and it seems, to the best of our knowledge, to be the only exception. In addition, in [123] nonlocal bending moments and shear forces are calculated using the conventional engineering hypotheses underlying the classical theory for plate bending and extension, e.g., see [52].

1.1.3 Asymptotic methods for thin plates and coatings

Asymptotic methods started developing in the field of statics and low-frequency dynamics, for instance, see Friedrichs & Dressler [48], Reiss & Locke [116], Green [56], Aksentian & Vorovich [8], Reissner [117], Goldenveizer [49], [51], Goldenveizer *et al.* [52], and references therein. We also mention here papers of Kaplunov *et al.* [67] and Pichugin & Rogerson [105] dealing with pre-stressed structures. Asymptotic methods were also applied in more general dynamic problems where not only low-frequency approximations, which as a rule generalise statics (e.g., see [52]), but high-frequency theories were also considered, including both long- and short-wave cases. We reiterate that in contrast to low-frequency approximations, high-frequency is characterised by sinusoidal variation across the thickness. Here and below, further details on the full classification of the shell vibrations are apprently for the first time presented in [66], see also [18], [86], and [17]. For the short-wave approximations, a typical wave length is of order of the thickness, while for long-waves it is much greater than the thickness. High-frequency long-wave modes, arising near thickness resonant frequencies, were found in various set-ups, including pre-stressed and anisotropic structures (see [71], [68], [103], [104], and [7]), structures interacting with media ([64]), structures with clamped faces ([65] and [69]), layered bodies ([89], [122], and [27]), localisation of high-frequency long-wave modes (e.g., see [68], [72], and [59]). High-frequency short-wave approximations are also important for analysis of transient wave fronts ([77], [70], and [118]).

Asymptotic analysis of boundary and initial conditions is of particular importance. In fact, most researchers only derive the differential equations of motion, ignoring the effect of boundary and initial conditions. For analysis of boundary conditions, see the Saint-Venant's principle expressing the decaying of edge data as a starting point, see [53], [57], [58], [14], and numerous references therein. Asymptotic analysis of initial conditions requires considering of high-frequency long-wave modes alongside the classical, low-frequency long-wave ones, see [73].

Asymptotic analysis of refined theories such as of Timoshenko-Reissner, can be found e.g., in [52], [110], [39], and [133].

1.1.4 Outline of thesis

The structure of the thesis is as follows. Chapter 1 is the introductory part of the thesis outlining essential concepts of nonlocal elasticity. Section 1.1 provides a literature review on the topic. In Subsection 1.1.1, basic facts, importance, and evolution of nonlocal elasticity theories are discussed, starting from the very first significant works in this area of mechanics and leading to the latest modern developments. Examples of applications are also provided and numerical methods in nonlocal models mentioned. Plate-like structures, plate models in nonlocal context alongside their applications to modelling of micro- and nanomaterials such as nanobeams, carbon nanotubes, graphene sheets and especially nanoplates, among others, are reviewed in Subsection 1.1.2, treated both analytically and numerically. Areas of scientific and industrial applications and the growing popularity of nanostructures are also discussed. Next, Subsection 1.1.3 contains a brief history of the development of asymptotic methods, which will be the main mathematical tool utilised to derive all the results in the present thesis.

In Chapter 2, essential relations in classical elasticity are provided in Section 2.1, including the derivations of Rayleigh transcendental equation in Subsection 2.1.1 and Rayleigh-Lamb dispersion relation in Subsection 2.1.2. Important relations in nonlocal linear elasticity theory are shown and explained in Subsection 2.2.

Chapter 3 is structured as follows. In Section 3.1, an elastic half-space governed by the nonlocal equations given in [44] is considered. For the sake of definiteness, it is assumed that the nonlocal behaviour is modelled by an exponential kernel involving a small internal lengthscale. Next, a long-wave asymptotic scheme, originated from Goldenveizer [52] and later developed by, for instance, Dai *et al.* [28] and Aghalovyan [6] is adapted, where the characteristic wavelength is assumed to considerably exceed a typical microscale parameter. The original nonlocal problem is reduced to a formulation identical to the classical problem for an elastic half-space with a vertical inhomogeneity localised near the surface. The effect of the inhomogeneity is then reduced to effective boundary conditions imposed at a near-surface interface. To this end, it is only possible to asymptotically evaluate the interval which yields the location of the interface. A more attractive option is exploited in Section 3.2, where the stresses along the 'virtual' interface are transformed into effective boundary conditions that need to be imposed on the surface of a homogeneous half-space, enabling one to evaluate the interior stress and strain outside the narrow boundary layer. In Section 3.3, a more general setup is analysed, when specific assumptions in nonlocal elasticity are not employed; instead, the inhomogeneity of the near-surface layer is considered in the general form of variable longitudinal and transverse wave speeds, see [23]. As examples of the application of the effective boundary conditions, the nonlocal correction to the Rayleigh surface wave is obtained in Section 3.4 and the effect of nonlocal elastic behaviour in a problem of a moving load on the surface of a half-space is analysed in Section 3.5.

In Chapter 4, plate bending theory is considered within the nonlocal elasticity framework. The variation of nonlocal properties across the plate thickness using the integral constitutive relations in [44] is analysed in Section 4.1. Direct asymptotic integration through the thickness is employed and adapted (see [52], [66], [67], and [73]) in Section 4.2 to derive appropriate long-wave low-frequency approximations of the nonlocal 3D dynamic elasticity equations for plate bending. For the sake of simplicity, a single small parameter is specified: it is equal to the ratio of the thickness to a characteristic longitudinal lengthscale and, at the same time, equal to the ratio of the thickness to a microscale internal size. A commonly used exponential nonlocal kernel (see [44]) is utilised. It is revealed that nonlocal stresses contain specific components corresponding to the effect of boundary layers adjacent to the plate faces (alongside the counterparts of classical, 'local' stresses demonstrating a polynomial variation across the thickness), which is numerically presented in Section 4.3.

In Chapter 5, the theory of plate extension is analysed in the nonlocal context. The variation of nonlocal properties across the thickness of a plate is investigated in Section 5.1. Asymptotic integration through the thickness is then exploited in Section 5.2, with long-wave low-frequency approximations of the governing equations for nonlocal elasticity in respect of plate extension obtained. The effect of boundary layers adjacent to the plate faces is analysed numerically and corresponding graphs are plotted in Section 5.3.

The nonlocal equations of motion obtained in Chapters 4 and 5 are expressed in the form of associated 'local' equations with modified bending and extensional stiffness, respectively. Hence, these equations involve nonlocal first-order corrections to their 'local' versions, see [24]. Chapter 6 presents a discussion on the essential results obtained in the thesis and on potential developments, extensions, and applications of the derived asymptotic models.

2 Preliminaries

In the current chapter, some essential relations such as governing equations in classical and nonlocal linear elasticity are shown and explained. Derivations for Rayleigh and Rayleigh-Lamb equations are provided and discussed briefly.

2.1 Classical linear elasticity

The main relations we will need in this thesis, alongside their detailed derivations can be found in [4] and [55]. Within this chapter, we provide only essential details. First, the stress equations of motion for a homogeneous isotropic linearly elastic solid are given by

$$\sigma_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2} \ , \tag{2.1}$$

where σ_{mn}, $m, n = 1, 2, 3$, are the components of the stress tensor $\boldsymbol{\sigma}$, u_n the components of the displacement vector $\boldsymbol{u}$, ρ the volume density, and t time. We also note that Einstein's summation convention is employed here. Hooke's law, relating stress and strain, can be conveniently written in the well-known form

$$\sigma_{mn} = \lambda e_{ll} \delta_{mn} + 2\mu e_{mn} \ , \tag{2.2}$$

where δ_{mn} is the Kronecker's delta and λ and μ the Lamé constants; the components of the linear elastic strain tensor $\boldsymbol{e}$ can be expressed through displacements as

$$e_{mn} = \frac{1}{2} \left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right) \ . \tag{2.3}$$

It is clear that $e_{mn} = e_{nm}$ and $\sigma_{mn} = \sigma_{nm}$. The system of equations governing the motion of a homogeneous, isotropic, linearly elastic body consists of the equations (2.1)-(2.3).

2.1.1 Rayleigh transcendental equation

A wave traveling along the free surface of an elastic, linear, isotropic half-space, in such a way that the disturbance can be detected mostly in the vicinity of the boundary, was for the first time considered by Lord Rayleigh [112]. The derivation of the classical Rayleigh surface wave equation can be found, for example, in [4], and is briefly discussed in this subsection.

Let us consider a half-space ($x_3 \geq 0$) in the state of plane strain, where $u_m = u_m(x_1, x_3),\ m = 1, 3$, and $u_2 = 0$, see Figure 2.1.

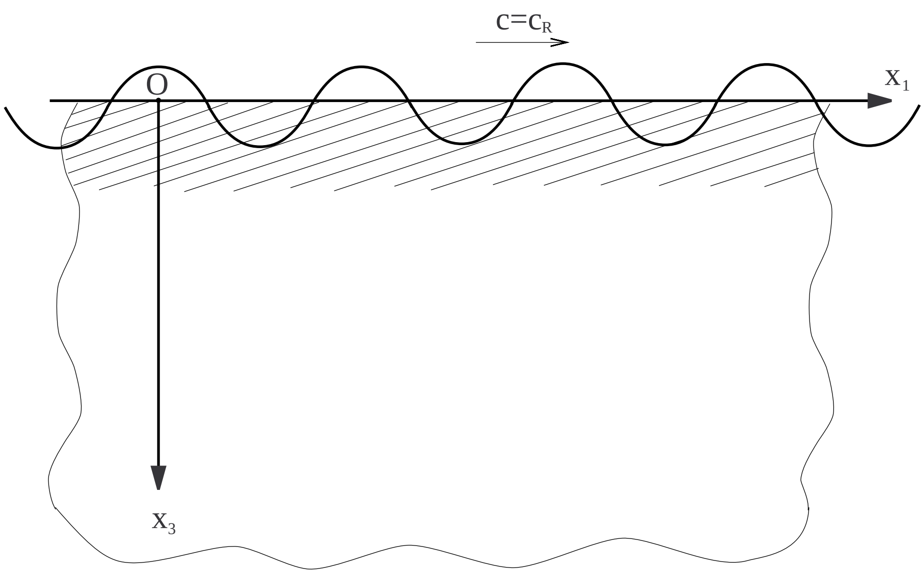

Figure 2.1: Rayleigh wave on the surface of a half-space.

We further assume a traction-free surface, i.e., the boundary conditions at the surface, in the case of plane strain, become

$$\sigma_{31} = 0\ ,\ \sigma_{33} = 0 \quad \text{at } x_3 = 0\ . \tag{2.4}$$

The classical equation of motion, in plane strain, in terms if displacements can be written in the vector form as

$$\frac{E}{2(1+\nu)}\Delta \boldsymbol{u} + \frac{E}{2(1+\nu)(1-2\nu)}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = 0\ , \tag{2.5}$$

where $\boldsymbol{u} = (u_1, 0, u_3)$ and its components are independent of x_2 and Δ is the Laplacian. Using Helmholtz's theorem, the displacement vector $\boldsymbol{u}$ may be decomposed as follows, e.g., see [4] and [55],

$$\boldsymbol{u} = \mathrm{grad}\,\varphi + \mathrm{curl}\,\boldsymbol{\psi}\ , \tag{2.6}$$

where φ and $\boldsymbol{\psi}$ are wave potentials. In the case of plane strain, $\boldsymbol{\psi} = (0, \psi_2, 0)$.

Hence, the equations of motion, expressed in terms of wave potentials φ and ψ_2, can be written as

$$\Delta_1 \varphi - \frac{1}{c_1^2}\frac{\partial^2 \varphi}{\partial t^2} = 0\ , \tag{2.7}$$

$$\Delta_1 \psi_2 - \frac{1}{c_2^2}\frac{\partial^2 \psi_2}{\partial t^2} = 0\ , \tag{2.8}$$

where $\Delta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$ and $\psi_m = 0, m = 1, 3$, e.g., see [66].

One of the main attributes of Rayleigh surface waves is that the displacement decays exponentially with distance from the free surface. Therefore, let us look for travelling wave solutions of the following form

$$\begin{aligned} \varphi &= A e^{-r x_3 + ik(x_1 - ct)}\ , \\ \psi_2 &= B e^{-q x_3 + ik(x_1 - ct)}\ , \end{aligned} \tag{2.9}$$

where c is the phase velocity and the attenuation coefficients r and q, by use of (2.1) - (2.3), may be shown to be given by

$$r = k\sqrt{1 - \frac{c^2}{c_1^2}}$$

and

$$q = k\sqrt{1 - \frac{c^2}{c_2^2}}\ .$$

We can now express the displacements in terms of potentials, thus obtaining

$$
\begin{aligned}
u_1 &= \varphi_{,1} + \psi_{2,3} = \left(ikAe^{-rx_3} - qBe^{-qx_3}\right) e^{ik(x_1-ct)} \, , \\
u_3 &= \varphi_{,3} - \psi_{2,1} = \left(-rAe^{-rx_3} - ikBe^{-qx_3}\right) e^{ik(x_1-ct)} \, .
\end{aligned}
\tag{2.10}
$$

Next, on substituting (2.10) into the surface boundary conditions (2.4), we obtain, after taking into account the plane strain forms of (2.2) and (2.3), the following system of equations

$$
\begin{aligned}
&\left[2 - \frac{c^2}{c_2^2}\right] A + \left[2i\sqrt{1 - \frac{c^2}{c_2^2}}\right] B = 0 \, , \\
&\left[-2i\sqrt{1 - \frac{c^2}{c_1^2}}\right] A + \left[2 - \frac{c^2}{c_2^2}\right] B = 0 \, .
\end{aligned}
\tag{2.11}
$$

The condition for existence of a non-trivial solution of the system of equations (2.11) yields the classical Rayleigh equation

$$
R(\gamma) = (2-\gamma^2)^2 - 4\sqrt{(1-\gamma^2)(1-\kappa^2\gamma^2)} = 0 \, ,
$$

where $\gamma = \frac{c}{c_2}$ is the normalised dimensionless phase velocity and

$$
\kappa = \frac{c_2}{c_1} = \sqrt{\frac{1-2\nu}{2-2\nu}} = \sqrt{\frac{\mu}{\lambda + 2\mu}} \, , \tag{2.12}
$$

with ν Poisson's ratio.

2.1.2 Rayleigh-Lamb dispersion relation

Lord Rayleigh [113] and Lamb [85] obtained a frequency equation for waves in elastic plates with traction-free surfaces. A brief derivation of the Rayleigh-Lamb dispersion relation is provided below. For further details the reader is referred to, for example, Graff [55].

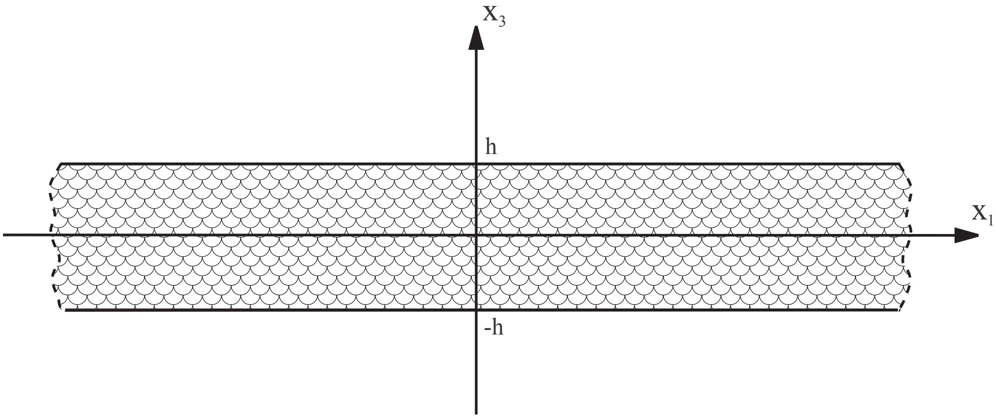

Figure 2.2: Geometry for the Rayleigh-Lamb problem.

The classical equation of motion in terms of displacements, in plane strain, for an infinite layer of thickness $2h$, see Figure 2.2, was given by (2.5) earlier. The stresses can be expressed in terms of displacements in the following form

$$\begin{aligned}
\sigma_{11} &= \frac{E}{2(1+\nu)\kappa^2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\nu}{1-\nu}\frac{\partial u_3}{\partial x_3}\right) \ , \\
\sigma_{22} &= \frac{E}{2(1-\nu^2)\kappa^2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \ , \\
\sigma_{33} &= \frac{E}{2(1+\nu)\kappa^2}\left(\frac{\nu}{1-\nu}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \ , \\
\sigma_{31} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \ , \\
&(\sigma_{21} = \sigma_{23} \equiv 0) \ .
\end{aligned} \tag{2.13}$$

Classical boundary conditions in the case of plane strain are

$$\sigma_{31} = 0 \ , \ \sigma_{33} = 0 \quad \text{at } x_3 = \pm h \ . \tag{2.14}$$

We specify the displacements in terms of wave potentials φ and ψ_2 as in (2.10)

$$u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi_2}{\partial x_3} \,, \qquad u_3 = \frac{\partial \varphi}{\partial x_3} - \frac{\partial \psi_2}{\partial x_1} \,. \tag{2.15}$$

On substituting of (2.15) into (2.5), we obtain equations (2.7) and (2.8).

Let use introduce the following dimensionless quantities

$$\xi_1 = \frac{x_1}{h} \,,\ \zeta = \frac{x_3}{h} \,,\ \tau = \frac{tc_2}{h} \,,$$

and

$$K = kh \,,\ \Omega = \frac{\omega h}{c_2} \,,$$

and seek the solutions of equations (2.7) and (2.8) in the form

$$\varphi = f(\zeta) \exp\left[i(K\xi_1 - \Omega\tau)\right] \,, \qquad \psi_2 = g(\zeta) \exp\left[i(K\xi_1 - \Omega\tau)\right] \,. \tag{2.16}$$

Substituting (2.16) into (2.7) and (2.8) , we obtain

$$\frac{\partial^2 f}{\partial \zeta^2} - \alpha^2 f = 0 \,, \qquad \frac{\partial^2 g}{\partial \zeta^2} - \beta^2 f = 0 \,, \tag{2.17}$$

where $\alpha^2 = K^2 - \kappa^2\Omega^2$, $\beta^2 = K^2 - \Omega^2$, with κ defined by (2.12).

The vibration modes described by the equations (2.17) can be divided into two groups: symmetric with respect to the midplane of the layer ($\zeta = 0$) and antisymmetric. Let us first obtain the solution for the antisymmetric, so-called bending modes, for which the displacement u_1 and stresses σ_{11}, σ_{22}, and σ_{33} are odd with respect to ζ, whereas u_3 and σ_{31} are even. The solutions of the equations (2.17) can be written as

$$f = A \sinh(\alpha\zeta) \,,\ g = B \cosh(\beta\zeta) \,, \tag{2.18}$$

where A and B are arbitrary constants.

Next, on substituting (2.16) into (2.15), having found f and g in the form (2.18), the displacements may be written as

$$\begin{aligned} u_1 &= \frac{1}{h}\left(A\sinh(\alpha\zeta)iK + B\sinh(\beta\zeta)\beta\right)\exp\left[i(K\xi_1 - \Omega\tau)\right] , \\ u_3 &= \frac{1}{h}\left(A\cosh(\alpha\zeta)\alpha - B\cosh(\beta\zeta)iK\right)\exp\left[i(K\xi_1 - \Omega\tau)\right] . \end{aligned} \tag{2.19}$$

On substituting (2.19) into $(2.13)_{3,4}$ and satisfying the boundary conditions (2.14), we have a system of two linear equations in A and B, namely

$$A\left[\alpha iK\cosh\alpha\right] + B\left[\gamma^2\cosh\beta\right] = 0 ,$$

$$A[\gamma^2\sinh\alpha] - B\left[\beta iK\sinh\beta\right] = 0 ,$$

and equating the determinant of this system of equations to zero yields the classical Rayleigh-Lamb dispersion equation

$$\gamma^4\frac{\sinh\alpha}{\alpha}\cosh\beta - \beta^2K^2\cosh\alpha\frac{\sinh\beta}{\beta} = 0 , \tag{2.20}$$

where $\gamma = K^2 - \frac{\Omega^2}{2}$, e.g., [4] and [55].

Let us now consider the low-frequency long-wave case, with $K \ll 1, \Omega \ll 1$. From the classical Rayleigh-Lamb dispersion relation (2.20) it is known that asymptotically, after we expand the trigonometric functions sinh and cosh in Taylor series up to the first order, we have, see [66],

$$K^4 = \frac{3}{4(1-\kappa^2)}\Omega^2(1 + O(\Omega)) , \tag{2.21}$$

therefore $K \sim \sqrt{\Omega}$. Alternatively, the same relation can be derived directly from the classical Kirchhoff equation for plate bending by substituting a travelling wave solution into it.

For symmetric modes, which express extension and compression of a layer, the displacement u_1 and stresses σ_{11}, σ_{22}, and σ_{33} are even with respect to the thickness variable ζ, with u_3 and σ_{31} odd. This means that the solutions to the equations (2.17) in this case are given by

$$f = A\cosh(\alpha\zeta) \ , \ g = B\sinh(\beta\zeta) , \tag{2.22}$$

where A and B are arbitrary constants. The Rayleigh-Lamb dispersion equation for the symmetric case can therefore be derived from the equation for the antisymmetric case (2.20) by substituting 'sinh' instead of 'cosh' and vice versa, yielding

$$\gamma^4 \cosh\alpha \frac{\sinh\beta}{\beta} - \alpha^2 K^2 \frac{\sinh\alpha}{\alpha} \cosh\beta = 0 \ , \tag{2.23}$$

where, as before, $\gamma = K^2 - \frac{\Omega^2}{2}$, see [66].

We again consider the low-frequency long-wave case ($K \ll 1$, $\Omega \ll 1$). From the classical Rayleigh-Lamb dispersion relation for the symmetric case (2.23) it is known that asymptotically, after we expand the trigonometric functions 'sinh' and 'cosh' in Taylor series up to the leading order, we obtain

$$K^2 = \frac{1}{4(1-\kappa^2)} \Omega^2 (1 + O(\Omega^2)) \ , \tag{2.24}$$

therefore $K \sim \Omega$. Similarly, this relation can be derived from the classical equation for plate extension by substituting a travelling wave solution into it, e.g., see [66].

2.2 Nonlocal linear elasticity

We shall now consider nonlocal elasticity and start with the nonlocal elasticity equations, see [44]. For a homogeneous isotropic elastic solid, we have (2.25)-(2.26) below

$$s_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2} \ , \tag{2.25}$$

with $u_n, n = 1, 2, 3$, the components of the displacement vector, ρ volume density, t time and

$$s_{mn}(\boldsymbol{x}) = \int\limits_V K\left(|\boldsymbol{x}' - \boldsymbol{x}|, a\right) \sigma_{mn}(\boldsymbol{x}')\, dv(\boldsymbol{x}') \ , \tag{2.26}$$

where s_{mn} and σ_{mn} are the nonlocal and classical stress tensors, respectively, considered at time t, $\boldsymbol{x} = (x_1, x_2, x_3)$ is a reference point, V the domain occupied by the body, $K(\boldsymbol{x}, a)$ the so-called nonlocal modulus, and a is an internal characteristic length, for example, lattice parameter or granular distance. Throughout the paper we assume that the internal size a is asymptotically small in comparison with a typical wavelength. This long-wave assumption provides validity for the adapted nonlocal integral model. We remark that this follows, in particular, from lattice dynamics [43].

The nonlocal kernel K in (2.26) is normalised over 3D space, so that

$$\int\limits_{V_\infty} K\left(|\boldsymbol{x}'|, a\right)\, dv(\boldsymbol{x}') = 1\ . \tag{2.27}$$

The two equations (2.25) and (2.26) are accompanied by classical equations (2.2) and (2.3).

We select the 3D exponential nonlocal modulus as follows, see [44],

$$K(|\boldsymbol{x}|, a) = \frac{1}{\pi^{3/2}a^3} \exp\left[-\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{a^2}\right]\ , \tag{2.28}$$

and in the case of a half-space ($-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, and $0 \leq x_3 < \infty$), (2.26) becomes

$$s_{mn}(\boldsymbol{x}) = \frac{1}{\pi^{3/2}a^3} \int\limits_0^\infty dx_3' \int\limits_{-\infty}^\infty dx_1' \int\limits_{-\infty}^\infty dx_2'\, \exp\left[-\frac{(\boldsymbol{x}'-\boldsymbol{x})^2}{a^2}\right]\sigma_{mn}(\boldsymbol{x}')\ , \tag{2.29}$$

while in the case of a plate of thickness $2h$ ($-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, and $-h \leq x_3 \leq h$), (2.26) takes the form

$$s_{mn}(\boldsymbol{x}) = \frac{1}{\pi^{3/2}a^3} \int\limits_{-h}^h dx_3' \int\limits_{-\infty}^\infty dx_1' \int\limits_{-\infty}^\infty dx_2'\, \exp\left[-\frac{(\boldsymbol{x}'-\boldsymbol{x})^2}{a^2}\right]\sigma_{mn}(\boldsymbol{x}')\ . \tag{2.30}$$

Let us expand the stresses σ_{mn} in Taylor series about the specific reference point $\boldsymbol{x'} = (x_1, x_2, x_3')$, assuming, as in [23], that the typical wavelength characterising the classical stress field is much greater than the internal size a. Thus, we establish from (2.30) that

$$s_{mn}(\boldsymbol{x}) = \frac{1}{a\sqrt{\pi}} \int\limits_{-h}^h \exp\left[-\frac{(x_3'-x_3)^2}{a^2}\right]\sigma_{mn}(x_1, x_2, x_3')dx_3'\ . \tag{2.31}$$

Note that the limits of integration are from $-h$ to h. It holds true for a layer or plate of thickness $2h$, while for a half-space, i.e., $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, and $0 \leq x_3 < \infty$, the expression (2.29) becomes

$$s_{mn}(\boldsymbol{x}) = \frac{1}{a\sqrt{\pi}} \int\limits_{0}^{\infty} \exp\left[-\frac{(x_3' - x_3)^2}{a^2}\right] \sigma_{mn}(x_1, x_2, x_3') dx_3' \; . \tag{2.32}$$

3 Effective boundary conditions on the free surface of a half-space accounting for nonlocal effects

In this chapter, a homogeneous layer with free upper face and prescribed displacements at the lower face is considered within the framework of nonlocal elasticity. We start from the equations (2.25) and (2.26) with the nonlocal modulus satisfying the equations (2.27) and (2.28). The leading order long-wave approximation of the aforementioned problem is found. It is implemented for the formulation of the effective boundary conditions which allow one to evaluate the interior solution outside a narrow boundary layer localised near the surface of an elastic half-space treated within the classical local elasticity. In addition, an illustration of the effect of nonlocal elastic phenomena on the Rayleigh wave speed is provided.

3.1 Near-surface non-locally elastic layer

First, let us take in a separate consideration a thin, elastic near-surface layer of thickness h starting from the relations in Section 2.2. Assume that the upper face of the layer, $x_3 = 0$, is traction-free while at the lower face, $x_3 = h$, the displacements are prescribed, where x_3 is the transverse co-ordinate. The equations of motion in 3D nonlocal elasticity, using Einstein's summation convention, are given by

$$s_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2} \, , \tag{3.1}$$

which can be rewritten for convenience as

$$\begin{aligned} \frac{\partial s_{3i}}{\partial x_3} &= -\frac{\partial s_{ii}}{\partial x_i} - \frac{\partial s_{ij}}{\partial x_j} + \rho \frac{\partial^2 u_i}{\partial t^2} \, , \\ \frac{\partial s_{33}}{\partial x_3} &= -\frac{\partial s_{3i}}{\partial x_i} - \frac{\partial s_{3j}}{\partial x_j} + \rho \frac{\partial^2 u_3}{\partial t^2} \, , \end{aligned} \tag{3.2}$$

where $i \neq j = 1, 2$ and Einstein's summation convention is not employed.

Using the integral expression (2.32), and Hooke's law in the form (2.2), we can express the nonlocal stresses as follows

$$s_{mn}(\boldsymbol{x}) = \frac{1}{a\sqrt{\pi}} \int\limits_{0}^{\infty} \exp\left[-\frac{(x_3' - x_3)^2}{a^2}\right] (\lambda e_{ll}(x_1, x_2, x_3')\delta_{mn} + 2\mu e_{mn}(x_1, x_2, x_3'))\, dx_3' \, , \tag{3.3}$$

where a is an internal characteristic length, δ_{mn} the Kronecker delta, λ and μ the Lamé constants, and the linear elastic strains are given by (2.3).

The boundary conditions on the upper and lower faces are respectively given by

$$s_{3n} = 0 \quad \text{at } x_3 = 0 \, , \tag{3.4}$$

and

$$u_n = U_n \quad \text{at } x_3 = h \, , \tag{3.5}$$

where $U_n = U_n(x_1, x_2, t)$ denotes the prescribed displacements, with $n = 1, 2, 3$. The formulated problem for a layer, see Figure 3.1, is also of considerable interest, see [74] by Kaplunov & Chebakov.

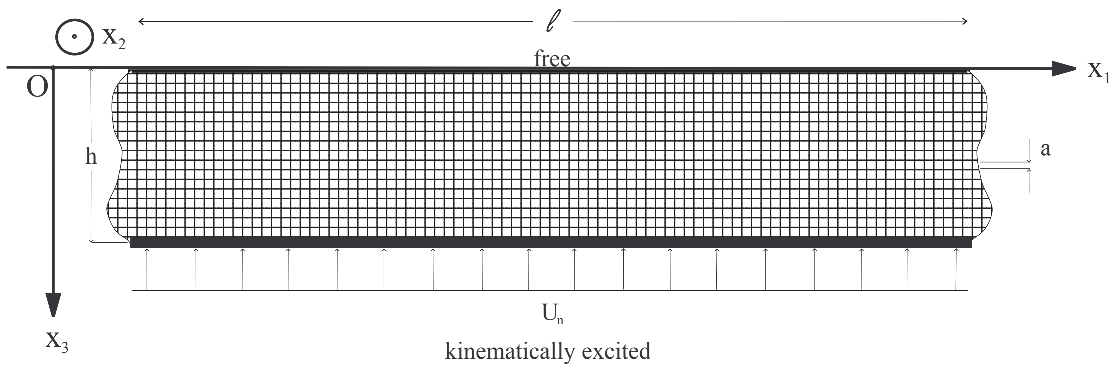

Figure 3.1: A non-locally elastic layer of thickness h; $a \ll h \ll \ell$.

Let us introduce two small geometrical parameters. Assume that the thickness of the near-surface layer h is much smaller than a typical wavelength ℓ, therefore denote $\eta = \frac{h}{\ell} \ll 1$, but h is in turn also much greater then the internal microscale a, thus denote $\theta = \frac{a}{h} \ll 1$, see Figure 3.2.

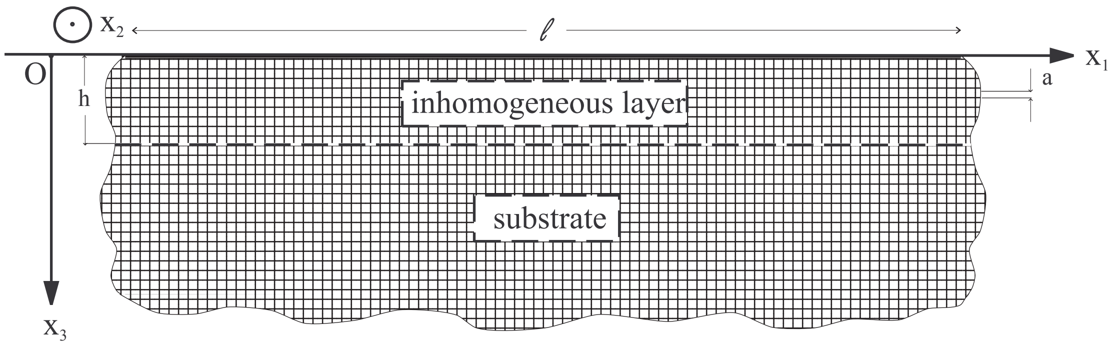

Figure 3.2: A homogeneous substrate coated by a vertically inhomogeneous layer of thickness h; $a \ll h \ll \ell$.

Also assume, for the sake of simplicity, that ratio θ coincides with $\sqrt{\eta}$, hence we have the following relation

$$\theta = \frac{a}{h} = \sqrt{\frac{h}{\ell}}\,. \tag{3.6}$$

In fact, this choice can be generalised as $\theta = \eta^k$, where $k > 0$, resulting, however, in cumbersome formulae which would lead virtually to the same final conclusions not bringing much novelty.

The classical stresses σ_{mn} were earlier expressed in terms of strains e_{mn}, see (2.2). Let us also express, for convenience, the nonlocal stresses s_{mn} as a sum of their classical counterparts $p_{mn}(\zeta_p)$ and nonlocal counterparts $q_{mn}(\zeta_q)$, where the latter is associated with the influence of a nonlocal boundary layer localised near a free surface. This convenient notation allows us to separately analyse classical and nonlocal behaviour.

We now adapt the well-known asymptotic integration method, e.g., see [50], [52], and [6], in order to find the nonlocal stresses s_{3n} at $x_3 = h$. First, we scale the original variables as

$$x_i = \ell \xi_i \ , \quad x_3 = h\zeta_p = a\zeta_q \ , \quad \text{and} \quad t = \frac{\ell}{c_2}\tau \ , \tag{3.7}$$

Note that the novelty of this approach is in its multi-scale nature, including both a transverse dimensionless variable ζ_p normalised by the macroscale h and its analogue ζ_q normalised by the microscale a. This is a natural consequence of the existence of an internal microscale inside a thin layer.

We now define the dimensionless quantities, denoting, as agreed earlier, the dimensionless nonlocal stresses s_{mn} as sums of their classical and nonlocal constituents, p_{mn} and q_{mn}, respectively. Thus, we have the following scaling for displacements

$$u_n = \ell v_n \ , \ U_n = \ell w_n \ , \tag{3.8}$$

strains

$$e_{ii} = \varepsilon_{ii} \ , \ e_{ij} = \varepsilon_{ij} \ , \ e_{33} = \varepsilon_{33} \ , \tag{3.9}$$

and nonlocal stresses

$$\begin{aligned} s_{ii} &= \mu(p_{ii} + q_{ii}) \ , \\ s_{ij} &= \mu(p_{ij} + q_{ij}) \ , \\ s_{3i} &= \theta^2\mu(p_{3i} + \theta q_{3i}) \ , \\ s_{33} &= \theta^2\mu(p_{33} + \theta^2 q_{33}) \ , \end{aligned} \tag{3.10}$$

and it is also necessary to introduce the following quantity

$$\gamma_{i3} = \frac{\partial u_i}{\partial x_3} \ , \tag{3.11}$$

where v, w, ε, p and q, and γ_{i3} are assumed to be of the same asymptotic order. Formula (3.11) is typical for the asymptotic integration method, e.g., see [52], and serves for analysing quantities with uniform variation along the thickness, see also formula (3.16).

Let us again recall that in (3.10) above, $p_{mn}(\zeta_p)$ components of the nonlocal stresses s_{mn} effectively represent those in classical linear elasticity, while $q_{mn}(\zeta_q)$ components represent the effect of boundary (near-surface) layers characteristic of nonlocal elasticity only.

Now using (3.3) let us write down the equations for nonlocal stresses s_{mn} expressed in terms of strains e_{mn}

$$
\begin{aligned}
s_{ij} &= \frac{2\mu}{a\sqrt{\pi}} \int_0^\infty \exp\left[\frac{-(x_3' - x_3)^2}{a^2}\right] e_{ij} dx_3' \ , \\
s_{ii} &= \frac{\mu}{a\sqrt{\pi}} \int_0^\infty \exp\left[\frac{-(x_3' - x_3)^2}{a^2}\right] \left(\kappa^{-2} e_{ii} + (\kappa^{-2} - 2)(e_{jj} + e_{33})\right) dx_3' \ , \\
s_{3i} &= \frac{2\mu}{a\sqrt{\pi}} \int_0^\infty \exp\left[\frac{-(x_3' - x_3)^2}{a^2}\right] e_{3i} dx_3' \ , \\
s_{33} &= \frac{\mu}{a\sqrt{\pi}} \int_0^\infty \exp\left[\frac{-(x_3' - x_3)^2}{a^2}\right] \left(\kappa^{-2} e_{33} + (\kappa^{-2} - 2)(e_{ii} + e_{jj})\right) dx_3' \ ,
\end{aligned}
\tag{3.12}
$$

where κ is defined by (2.12).

The reader can notice that we cannot find e_{33} which we need to be able to calculate s_{ii} in $(3.12)_2$. Therefore let us first express e_{33} through e_{ii}, e_{jj}, and s_{33} from $(3.12)_4$ and then substitute it into $(3.12)_2$, yielding

$$
\begin{aligned}
s_{ii} = \frac{\mu}{a\sqrt{\pi}} \int_0^\infty \exp\left[\frac{-(x_3' - x_3)^2}{a^2}\right] \left(4(1-\kappa^2) e_{ii} + 2(1 - 2\kappa^2) e_{jj}\right) dx_3' \\
+ (1 - 2\kappa^2) s_{33} \ .
\end{aligned}
\tag{3.13}
$$

This is another key trick in asymptotic theory for thin solids originated from A. L. Goldenveizer, e.g., see [52] and references therein. Now we are in a position to write down non-dimensional equations for the nonlocal stresses s_{mn}; these were expressed earlier in the form of dimensionless components p and q, see (3.10). We express these stresses in terms of dimensionless strains ε_{mn} from (3.12) and (3.13), obtaining

$$p_{ij} + q_{ij} = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij} d\zeta_q' \ ,$$

$$\begin{aligned} p_{ii} + q_{ii} &= \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(4(1-\kappa^2)\varepsilon_{ii} + 2(1-2\kappa^2)\varepsilon_{jj}\right) d\zeta_q' \\ &\qquad + \theta^2(1-2\kappa^2)(p_{33} + \theta^2 q_{33}) \ , \\ \theta^2(p_{33} + \theta^2 q_{33}) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{33}\right. \\ &\qquad \left. + (\kappa^{-2} - 2)(\varepsilon_{ii} + \varepsilon_{jj})\right) d\zeta_q' \ . \end{aligned} \tag{3.14}$$

It is now possible to express the strains e_{mn} in terms of displacements u_n, $n = 1, 2, 3$, as in (2.3), resulting in the dimensionless forms

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_i}\right) \ , \\ \varepsilon_{ii} &= \frac{\partial v_i}{\partial \xi_i} \ , \\ \theta^2 \varepsilon_{33} &= \frac{\partial v_3}{\partial \zeta_p} \ , \end{aligned} \tag{3.15}$$

and, accordingly, equation (3.11) becomes

$$\theta^2 \gamma_{i3} = \frac{\partial v_i}{\partial \zeta_p} \ . \tag{3.16}$$

The nonlocal equations of motion (3.2), in dimensionless form in terms of p and q stress components, become

$$\begin{aligned} \frac{\partial p_{3i}}{\partial \zeta_p} + \frac{\partial q_{3i}}{\partial \zeta_q} &= -\frac{\partial (p_{ii} + q_{ii})}{\partial \xi_i} - \frac{\partial (p_{ij} + q_{ij})}{\partial \xi_j} + \frac{\partial^2 v_i}{\partial \tau^2} \ , \\ \frac{\partial p_{33}}{\partial \zeta_p} + \theta \frac{\partial q_{33}}{\partial \zeta_q} &= -\theta^2 \left[\frac{\partial (p_{3i} + \theta q_{3i})}{\partial \xi_i} + \frac{\partial (p_{3j} + \theta q_{3j})}{\partial \xi_j}\right] + \frac{\partial^2 v_3}{\partial \tau^2} \ . \end{aligned} \tag{3.17}$$

The corresponding dimensionless form of the boundary conditions (3.4) becomes

$$p_{3i} + \theta q_{3i} = 0 \ , \ p_{33} + \theta^2 q_{33} = 0 \quad \text{at } \zeta_p = 0 \ (\zeta_q = 0) \ , \tag{3.18}$$

with boundary conditions (3.5) taking the form

$$v_n = w_n \quad \text{at } \zeta_p = 1 \; (\zeta_q = \theta^{-1}) \; . \tag{3.19}$$

Now we expand all the dimensionless quantities, namely the displacements v_n, strains ε_{mn}, and nonlocal stress constituents p_{mn} and q_{mn}, in asymptotic series in terms of the small parameter θ, given by (3.6), as follows

$$\begin{pmatrix} v_n \\ p_{mn} \\ q_{mn} \\ \varepsilon_{mn} \end{pmatrix} = \begin{pmatrix} v_n^{(0)} \\ p_{mn}^{(0)} \\ q_{mn}^{(0)} \\ \varepsilon_{mn}^{(0)} \end{pmatrix} + \theta \begin{pmatrix} v_n^{(1)} \\ p_{mn}^{(1)} \\ q_{mn}^{(1)} \\ \varepsilon_{mn}^{(1)} \end{pmatrix} + \ldots \tag{3.20}$$

Substitution of the above expansions into equations $(3.14)_{1,2}$, (3.15), (3.16), and (3.17) results, at leading order, in the following set of equations

$$
\begin{aligned}
&\frac{\partial v_i^{(0)}}{\partial \zeta_p} = 0 \, , \\
&\frac{\partial v_3^{(0)}}{\partial \zeta_p} = 0 \, , \\
&\varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^{(0)}}{\partial \xi_j} + \frac{\partial v_j^{(0)}}{\partial \xi_i}\right) \, , \\
&\varepsilon_{ii}^{(0)} = \frac{\partial v_i^{(0)}}{\partial \xi_i} \, , \\
&p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij}^{(0)} d\zeta_q' \, , \\
&p_{ii}^{(0)} + q_{ii}^{(0)} = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(4(1-\kappa^2)\varepsilon_{ii}^{(0)} + 2(1-2\kappa^2)\varepsilon_{jj}^{(0)}\right) d\zeta_q' \, , \\
&\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(0)} + q_{ii}^{(0)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(0)} + q_{ij}^{(0)})}{\partial \xi_j} + \frac{\partial^2 v_i^{(0)}}{\partial \tau^2} \, , \\
&\frac{\partial p_{33}^{(0)}}{\partial \zeta_p} = \frac{\partial^2 v_3^{(0)}}{\partial \tau^2} \, ,
\end{aligned}
\tag{3.21}
$$

along with the boundary conditions

$$
p_{3i}^{(0)} = 0 \, , \; p_{33}^{(0)} = 0 \quad \text{at } \zeta_p = 0 \; (\zeta_q = 0) \, , \tag{3.22}
$$

and

$$
v_n^{(0)} = w_n \quad \text{at } \zeta_p = 1 \; (\zeta_q = \theta^{-1}) \, . \tag{3.23}
$$

On integrating equations $(3.21)_{1,2}$ with respect to ζ_p, satisfying the boundary conditions (3.23), and then using the obtained results in $(3.21)_{3,4}$, we are able to establish that

$$
\begin{aligned}
v_n^{(0)} &= w_n \ , \\
\varepsilon_{ij}^{(0)} &= \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i}\right) \ , \\
\varepsilon_{ii}^{(0)} &= \frac{\partial w_i}{\partial \xi_i} \ ,
\end{aligned}
\tag{3.24}
$$

where we recall that the prescribed displacements $w_n = w_n(\xi_i, \xi_j, \tau)$.

We are now in a position to show that equation $(3.21)_5$ may be represented through

$$
p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{1}{\sqrt{\pi}}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i}\right)\int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \ . \tag{3.25}
$$

Next, on making the substitution $t = \zeta_q' - \zeta_q$ in equation (3.25), at leading order we have

$$
p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{1}{\sqrt{\pi}}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i}\right)\int_{-\zeta_q}^\infty \exp\left[-t^2\right] dt \ . \tag{3.26}
$$

We can rewrite the integral in equation (3.26) in the following form

$$
\int_{-\zeta_q}^\infty \exp\left[-t^2\right] dt = \int_{-\infty}^\infty \exp\left[-t^2\right] dt - \int_{-\infty}^{-\zeta_q} \exp\left[-t^2\right] dt \tag{3.27}
$$

and note that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ (e.g., see [54]). We are now in a position to separate the classical stress components $p_{ij}^{(0)}$ in (3.26), which contains no nonlocal constituents, from the nonlocal stress components $q_{ij}^{(0)}$, which involves the integral reflecting the effect of the boundary layer, thus

$$
\begin{aligned}
p_{ij}^{(0)} &= \frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} \ , \\
q_{ij}^{(0)} &= -\frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i}\right) \operatorname{erfc}(\zeta_q) \ ,
\end{aligned}
\tag{3.28}
$$

where complementary error function, e.g., see [3], is defined by

$$
\operatorname{erfc}(\zeta_q) = \frac{2}{\sqrt{\pi}}\int_{\zeta_q}^\infty e^{-t^2} dt \ . \tag{3.29}
$$

We are now in a position to note that equation $(3.21)_6$ becomes

$$p_{ii}^{(0)} + q_{ii}^{(0)} = \frac{1}{\sqrt{\pi}} \left(4(1-\kappa^2)\frac{\partial w_i}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j}{\partial \xi_j} \right) \int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \, . \quad (3.30)$$

Similarly, following the same steps used to derive (3.28) above, yielding the same integral as in (3.25)), and using the same idea to separate $p_{ii}^{(0)}$ and $q_{ii}^{(0)}$, we can readily establish that

$$\begin{aligned} p_{ii}^{(0)} &= 4(1-\kappa^2)\frac{\partial w_i}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j}{\partial \xi_j} \, , \\ q_{ii}^{(0)} &= -\left(2(1-\kappa^2)\frac{\partial w_i}{\partial \xi_i} + (1-2\kappa^2)\frac{\partial w_j}{\partial \xi_j} \right) \operatorname{erfc}(\zeta_q) \, . \end{aligned} \quad (3.31)$$

We now separate p and q stress components of the governing equation $(3.21)_7$. Integrating the p part of $(3.21)_7$, i.e., the following equation

$$\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_i^{(0)}}{\partial \tau^2} \, , \quad (3.32)$$

with respect to ζ_p, we obtain

$$p_{3i}^{(0)} = -\zeta_p \left[4(1-\kappa^2)\frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} - \frac{\partial^2 w_i}{\partial \tau^2} \right] + C \, , \quad (3.33)$$

where $C = C(\xi_1, \xi_2, \tau)$ is an arbitrary function independent of the transverse coordinate, and satisfying the corresponding boundary condition $(3.22)_1$, i.e.,

$$p_{3i}^{(0)} = 0 \quad \text{at } \zeta_p = 0 \, , \quad (3.34)$$

we can find the arbitrary function C to conclude that

$$p_{3i}^{(0)} = -\zeta_p \left[4(1-\kappa^2)\frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} - \frac{\partial^2 w_i}{\partial \tau^2} \right] \, . \quad (3.35)$$

On integrating $(3.21)_8$, with respect to ζ_p, we obtain

$$p_{33}^{(0)} = \zeta_p \frac{\partial^2 w_3}{\partial \tau^2} + D \, , \quad (3.36)$$

where $D = D(\xi_1, \xi_2, \tau)$ is an arbitrary function independent of the transverse coordinate. Satisfying the corresponding boundary condition $(3.22)_2$, i.e.,

$$p_{33}^{(0)} = 0 \quad \text{at } \zeta_p = 0 \, , \tag{3.37}$$

we can now find the arbitrary function D, arriving at

$$p_{33}^{(0)} = \zeta_p \frac{\partial^2 w_3}{\partial \tau^2} \, . \tag{3.38}$$

In what follows, we need to find $q_{3i}^{(0)}$. Therefore, we now need to integrate the q part of $(3.21)_7$, i.e., the following equation

$$\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial q_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial q_{ij}^{(0)}}{\partial \xi_j} \, , \tag{3.39}$$

with respect to ζ_q. On substituting $q_{ij}^{(0)}$ from $(3.28)_2$ and $q_{ii}^{(0)}$ from $(3.31)_2$ into equation (3.39) above, we obtain

$$\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = \frac{1}{2} \left[4(1-\kappa^2) \frac{\partial^2 w_i}{\partial \xi_i^2} + (3 - 4\kappa^2) \frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} \right] \operatorname{erfc}(\zeta_q) \, , \tag{3.40}$$

and on integration, similarly to derivation of (3.28), we establish that

$$q_{3i}^{(0)} = \frac{1}{2} \left[4(1-\kappa^2) \frac{\partial^2 w_i}{\partial \xi_i^2} + (3 - 4\kappa^2) \frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} \right] \int_{\zeta_q}^{\infty} \operatorname{erfc}(\zeta_q') d\zeta_q' \, . \tag{3.41}$$

Then, on integrating by parts we have

$$q_{3i}^{(0)} = \frac{1}{2} \left[4(1-\kappa^2) \frac{\partial^2 w_i}{\partial \xi_i^2} + (3 - 4\kappa^2) \frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} \right] Q(\zeta_q) \, , \tag{3.42}$$

where

$$Q(\zeta_q) = \zeta_q \operatorname{erfc}(\zeta_q) - \frac{1}{\sqrt{\pi}} \exp\left[-\zeta_q^2\right] . \tag{3.43}$$

Similarly, at the next order we have the following equation set

$$
\begin{aligned}
&\frac{\partial v_i^{(1)}}{\partial \zeta_p} = 0 \ , \\
&\frac{\partial v_3^{(1)}}{\partial \zeta_p} = 0 \ , \\
&\varepsilon_{ij}^{(1)} = \frac{1}{2}\left(\frac{\partial v_i^{(1)}}{\partial \xi_j} + \frac{\partial v_j^{(1)}}{\partial \xi_i}\right) \ , \\
&\varepsilon_{ii}^{(1)} = \frac{\partial v_i^{(1)}}{\partial \xi_i} \ , \\
&p_{ij}^{(1)} + q_{ij}^{(1)} = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right]\varepsilon_{ij}^{(1)} d\zeta_q' \ , \\
&p_{ii}^{(1)} + q_{ii}^{(1)} = \frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left[-(\zeta_q' - \zeta_q)^2\right]\left(4(1-\kappa^2)\varepsilon_{ii}^{(1)} + 2(1-2\kappa^2)\varepsilon_{jj}^{(1)}\right) d\zeta_q' \ , \\
&\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(1)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(1)} + q_{ii}^{(1)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(1)} + q_{ij}^{(1)})}{\partial \xi_j} + \frac{\partial^2 v_i^{(1)}}{\partial \tau^2} \ , \\
&\frac{\partial p_{33}^{(1)}}{\partial \zeta_p} + \frac{\partial q_{33}^{(0)}}{\partial \zeta_p} = 0 \ ,
\end{aligned}
\tag{3.44}
$$

along with the boundary conditions

$$
p_{3i}^{(1)} + q_{3i}^{(0)} = 0 \ , \ p_{33}^{(1)} = 0 \quad \text{at } \zeta_p = 0 \ (\zeta_q = 0) \ , \tag{3.45}
$$

and

$$
v_n^{(1)} = 0 \quad \text{at } \zeta_p = 1 \ (\zeta_q = \theta^{-1}) \ . \tag{3.46}
$$

In similar spirit to the leading order calculations above, on integrating of the equations $(3.44)_{1,2}$ with respect to ζ_p while satisfying the boundary conditions (3.46) and then using the obtained results in $(3.44)_{3,4}$, yields

$$
\begin{aligned}
\upsilon_n^{(1)} &= 0 \;, \\
\varepsilon_{ij}^{(1)} &= 0 \;, \\
\varepsilon_{ii}^{(1)} &= 0 \;.
\end{aligned}
\tag{3.47}
$$

Equation $(3.44)_5$ then becomes

$$
p_{ij}^{(1)} + q_{ij}^{(1)} = 0 \;, \tag{3.48}
$$

indicating that

$$
\begin{aligned}
p_{ij}^{(1)} &= 0 \;, \\
q_{ij}^{(1)} &= 0 \;.
\end{aligned}
\tag{3.49}
$$

Next, equation $(3.44)_6$ becomes

$$
p_{ii}^{(1)} + q_{ii}^{(1)} = 0 \;, \tag{3.50}
$$

which also yields

$$
\begin{aligned}
p_{ii}^{(1)} &= 0 \;, \\
q_{ii}^{(1)} &= 0 \;.
\end{aligned}
\tag{3.51}
$$

Now we separate the p and q stress components of the governing equation $(3.44)_7$. After substituting (3.47), (3.49), and (3.51) into it, the p part of $(3.44)_7$ becomes

$$
\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} = 0 \;, \tag{3.52}
$$

and on integrating with respect to ζ_p we obtain

$$
p_{3i}^{(1)} = E \;, \tag{3.53}
$$

where $E = E(\xi_1, \xi_2, \tau)$ is an arbitrary function independent of the transverse coordinate.

On satisfying of the corresponding boundary condition $(3.45)_1$, i.e.,

$$
p_{3i}^{(1)} + q_{3i}^{(0)} = 0 \quad \text{at } \zeta_p = 0 \;, \tag{3.54}
$$

we can find the arbitrary function E and conclude that

$$p_{3i}^{(1)} = \frac{1}{2\sqrt{\pi}}\left[4(1-\kappa^2)\frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2}\right] . \tag{3.55}$$

As previously, we can now consider the p part of the equation $(3.44)_8$, i.e.,

$$\frac{\partial p_{33}^{(1)}}{\partial \zeta_p} = 0 , \tag{3.56}$$

and integrate it with respect to ζ_p to obtain

$$p_{33}^{(1)} = F , \tag{3.57}$$

where $F = F(\xi_1, \xi_2, \tau)$ is an arbitrary function independent of on the transverse coordinate. Satisfying the boundary condition $(3.45)_2$, i.e.,

$$p_{33}^{(1)} = 0 \quad \text{at } \zeta_p = 0 , \tag{3.58}$$

we can find the arbitrary function F to conclude that

$$p_{33}^{(1)} = 0 . \tag{3.59}$$

We may now write down the sought for stresses, s_{3i} and s_{33}, in dimensionless form (in terms of p and q) up to $O(\theta)$ terms. In particular, the stresses s_{3i} take the following form

$$\begin{aligned} p_{3i} + \theta q_{3i} &= p_{3i}^{(0)} + \theta p_{3i}^{(1)} + \theta q_{3i}^{(0)} \\ &= \zeta_p \frac{\partial^2 w_i}{\partial \tau^2} + \left[4(1-\kappa^2)\frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2}\right] \\ &\times \left[-\zeta_p + \frac{\theta}{2}\left(\frac{1}{\sqrt{\pi}} + Q\right)\right] . \end{aligned} \tag{3.60}$$

The second stress of interest, s_{33}, up to $O(\theta)$ terms is written as

$$p_{33} = p_{33}^{(0)} = \zeta_p \frac{\partial^2 w_3}{\partial \tau^2} . \tag{3.61}$$

Hence, in terms of the original variables, the expressions (3.60) and (3.61) may be rewritten as

$$
\begin{aligned}
s_{3i} &= \rho x_3 \frac{\partial^2 U_i}{\partial t^2} \\
&\quad + \rho c_2^2 \left[4(1-\kappa^2) \frac{\partial^2 U_i}{\partial x_i^2} + (3-4\kappa^2) \frac{\partial^2 U_j}{\partial x_i \partial x_j} + \frac{\partial^2 U_i}{\partial x_j^2} \right] \left(-x_3 + \frac{a}{2} \left(\frac{1}{\sqrt{\pi}} + Q \right) \right) , \\
s_{33} &= \rho x_3 \frac{\partial^2 U_3}{\partial t^2} \, .
\end{aligned}
\tag{3.62}
$$

where $U_n = \ell w_n$. Note that at $a = 0$ equations (3.62) coincide with those without taking into account nonlocal effects, e.g., see [28].

For what follows, we need to find expressions for the stresses s_{3i} and s_{33} at $x_3 = h$ ($\zeta_p = 1$ or $\zeta_q = \theta^{-1}$). As at $x_3 \gg a$ ($\zeta_q \gg 1$) the boundary layer term Q in (3.60) is exponentially small, the stresses at the lower face $x_3 = h$ can be expressed in dimensionless form, to within an exponentially small error, as

$$
\begin{aligned}
p_{3i} + \theta q_{3i} &= \left[4(1-\kappa^2) \frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2) \frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} \right] \left(-1 + \frac{\theta}{2\sqrt{\pi}} \right) \\
&\qquad + \frac{\partial^2 w_i}{\partial \tau^2} , \qquad\qquad (3.63) \\
p_{33} &= \frac{\partial^2 w_3}{\partial \tau^2} ,
\end{aligned}
$$

which in terms of the original variables becomes

$$
\begin{aligned}
s_{3i} &= \rho h \frac{\partial^2 U_i}{\partial t^2} + \rho c_2^2 \left[4(1-\kappa^2) \frac{\partial^2 U_i}{\partial x_i^2} + (3-4\kappa^2) \frac{\partial^2 U_j}{\partial x_i \partial x_j} + \frac{\partial^2 U_i}{\partial x_j^2} \right] \\
&\qquad \times \left[-h + \frac{a}{2\sqrt{\pi}} \right] , \qquad\qquad (3.64) \\
s_{33} &= \rho h \frac{\partial^2 U_3}{\partial t^2} \, .
\end{aligned}
$$

In order to illustrate the effect of the boundary layer localised near the free surface $x_3 = 0$, see Figure 3.1, we now consider the dimensionless classical and 'nonlocal' components of the derived stresses, i.e., p_{mn} and q_{mn}, as functions of the dimensionless transverse co-ordinate ζ_p. First, let us rewrite the p and q components corresponding to the stress s_{ij}, see (3.28), in the following form

$$\begin{aligned} p_{ij}^{(0)} &= \left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} \right) \ , \\ q_{ij}^{(0)} &= \left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} \right) Q_{ij} \ , \end{aligned} \tag{3.65}$$

where

$$Q_{ij} = -\frac{1}{2} \operatorname{erfc}\left(\theta^{-1}\zeta_p\right) \ , \tag{3.66}$$

recalling that $\zeta_q = \theta^{-1}\zeta_p$.

We next complete similar steps for the p and q components, corresponding to the stress s_{ii}, see (3.31), resulting in

$$\begin{aligned} p_{ii}^{(0)} &= \left(4(1-\kappa^2)\frac{\partial w_i}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j}{\partial \xi_j} \right) P_{ii} \ , \\ q_{ii}^{(0)} &= \left(4(1-\kappa^2)\frac{\partial w_i}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j}{\partial \xi_j} \right) Q_{ii} \ , \end{aligned} \tag{3.67}$$

where

$$Q_{ii} = -\frac{1}{2} \operatorname{erfc}\left(\theta^{-1}\zeta_p\right) \ . \tag{3.68}$$

Let us now define $Q' = Q_{ij} = Q_{ii}$. The graph for Q' against ζ_p is plotted in Figure 3.3.

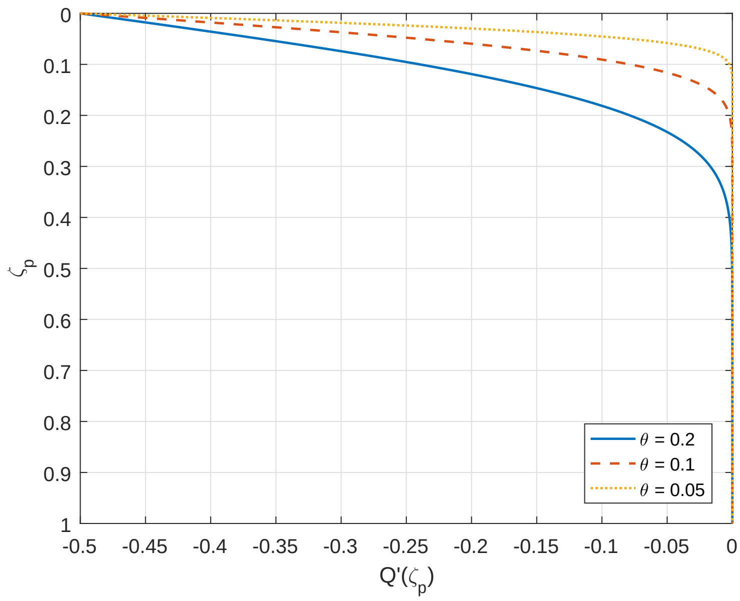

Figure 3.3: The nonlocal component of the stresses s_{ij} and s_{ii}, $i \neq j = 1, 2$.

We note that $Q' = -\frac{1}{2}$ on the upper face of the layer $x_3 = 0$ ($\zeta_p = 0$). In addition, we now explicitly show the main terms leading to exponential type of attenuation of Q'. To this end, let us substitute the complementary error function in the expression for Q' above by the following asymptotic expansion at $x \gg 1$ (e.g., see [3])

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} \right) . \tag{3.69}$$

Keeping the leading order term only, we obtain a simpler expression

$$Q'_{asymp} = -\frac{\exp\left[-(\theta^{-1}\zeta_p)^2\right]}{2\sqrt{\pi}(\theta^{-1}\zeta_p)} , \tag{3.70}$$

where this asymptotic expansion only works for large argument, i.e., when $\zeta_p \gg \theta$, see Figure 3.4.

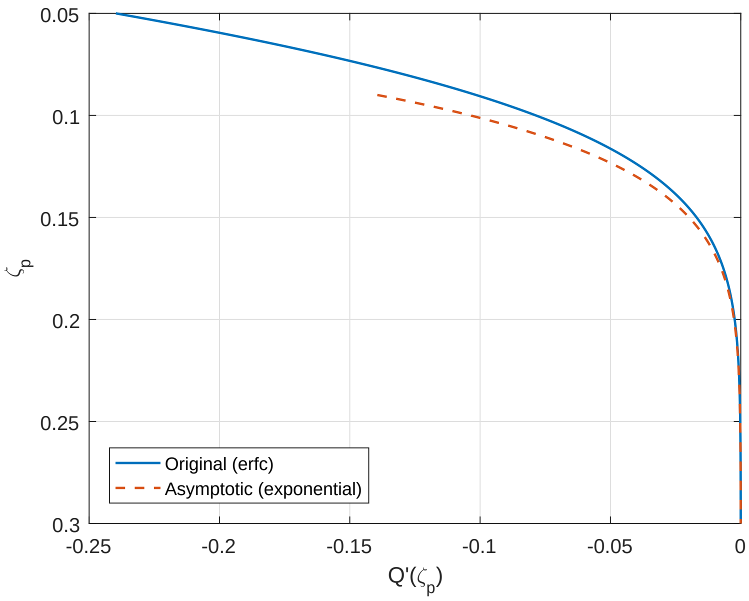

Figure 3.4: Attenuation of Q'; $\theta = 0.1$.

Next, we proceed with p and q components corresponding to the stress s_{3i}, see (3.35) and (3.42), for which we deduce that

$$\begin{aligned} p_{3i}^{(0)} &= \left(4(1-\kappa^2)\frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2} - \frac{\partial^2 w_i}{\partial \tau^2}\right) P_{3i}\ , \\ q_{3i}^{(0)} &= \left(4(1-\kappa^2)\frac{\partial^2 w_i}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 w_j}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 w_i}{\partial \xi_j^2}\right) Q_{3i}\ , \end{aligned} \tag{3.71}$$

where

$$P_{3i} = -\zeta_p \tag{3.72}$$

and

$$Q_{3i} = \frac{1}{2} Q\ , \tag{3.73}$$

where Q is defined by (3.43). Recalling that $\zeta_q = \theta^{-1}\zeta_p$, the expression for Q_{3i} becomes

$$Q_{3i} = \frac{1}{2}\left[\theta^{-1}\zeta_p \operatorname{erfc}(\theta^{-1}\zeta_p) - \frac{1}{\sqrt{\pi}} \exp\left[-(\theta^{-1}\zeta_p)^2\right]\right]\ . \tag{3.74}$$

The graph for Q_{3i} versus ζ_p is plotted in Figure 3.5.

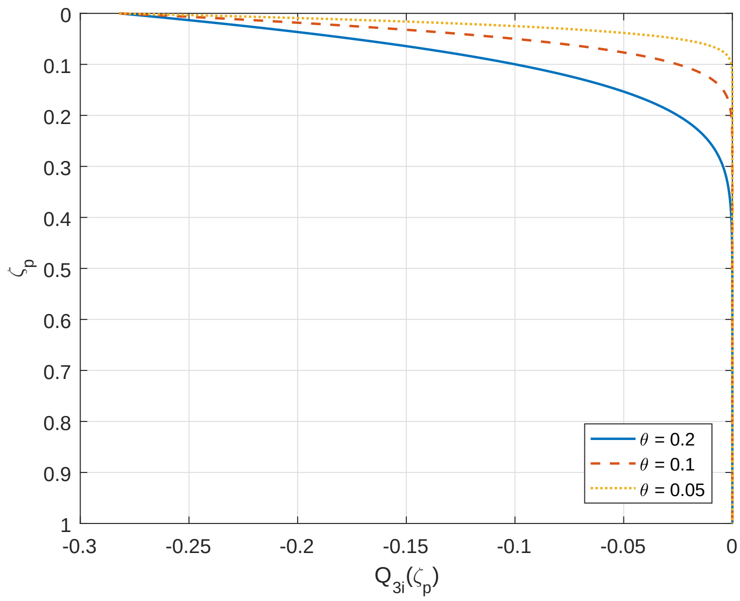

Figure 3.5: The nonlocal part of the stresseses s_{3i}, $i = 1, 2$.

It is worth reiterating that $P_{3i} = 0$ and $Q_{3i} = -\frac{1}{2\sqrt{\pi}}$ on the upper face $x_3 = 0$ ($\zeta_p = 0$). Let us now consider the asymptotics of attenuation for Q_{3i}. On using the asymptotic expansion (3.69) for the complementary error function (for large arguments) in Q_{3i}, up to first order terms, we are able to deduce that

$$Q_{3i}^{asymp} = -\frac{\exp\left[-(\theta^{-1}\zeta_p)^2\right]}{4(\theta^{-1}\zeta_p)^2}\,, \tag{3.75}$$

see Figure 3.6.

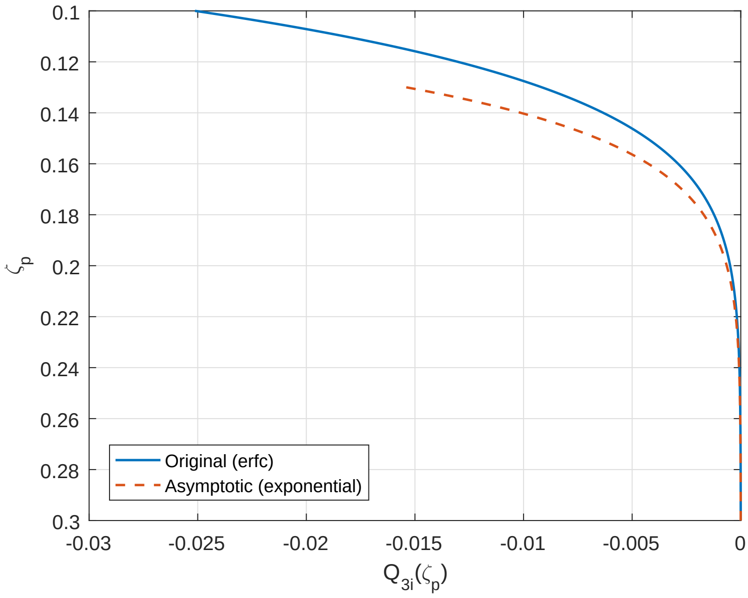

Figure 3.6: Attenuation of Q_{3i}; $\theta = 0.1$.

3.2 Derivation of effective boundary conditions

It is clear that the nonlocal boundary layer stress components q_{mn} are exponentially small at the lower face of the layer $x_3 = h$. At the same time, due to their interaction with the classical stress components p_{mn} via boundary conditions at the free surface, an extra $O(\theta)$ term arises in the expressions (3.62) at the upper face $x_3 = 0$. As a result, in order to find the stresses over the interior of a non-locally elastic half-space, we may think of reformulating the problem in terms of classical 'local' elasticity. In fact, we may formulate an inverse problem for a homogeneous elastic half-space with certain effective boundary conditions along its free surface. Such boundary conditions have to ensure the classical stresses σ_{3i} and σ_{33} at the depth $x_3 = h$ coincide with nonlocal stresses (3.64) derived earlier. Therefore, we are going to obtain effective boundary conditions along the surface accounting for the presence of the nonlocal boundary layer, e.g., see [23].

Let us consider a near-surface layer of thickness h in a homogeneous half-space within the framework of classical elasticity. We assume continuity of displacements and stresses at the 'virtual' interface $x_3 = h$, where, as before, x_3 is the transverse co-ordinate. The classical equations of motion are

$$\sigma_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2} , \tag{3.76}$$

which can be conveniently rewritten as (3.2) with σ_{mn} instead of s_{mn} as follows

$$\begin{aligned} \frac{\partial \sigma_{3i}}{\partial x_3} &= -\frac{\partial \sigma_{ii}}{\partial x_i} - \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial^2 u_i}{\partial t^2} , \\ \frac{\partial \sigma_{33}}{\partial x_3} &= -\frac{\partial \sigma_{3i}}{\partial x_i} - \frac{\partial \sigma_{3j}}{\partial x_j} + \rho \frac{\partial^2 u_3}{\partial t^2} , \end{aligned} \tag{3.77}$$

where $i \neq j = 1, 2$, $n, m = 1, 2, 3$, and, in contrast to 3.76, Einstein's summation convention is not employed.

Let the boundary conditions at the surface of the half-space be given by

$$\sigma_{3n} = \chi_{3n} \quad \text{at } x_3 = 0 , \tag{3.78}$$

where χ_{3n} are the sought for surface stresses, which we need to find in order to obtain the effective boundary conditions mentioned earlier. As before, we assume continuity of displacements

$$u_n = v_n \quad \text{at } x_3 = h , \tag{3.79}$$

where $v_n = v_n(x_1, x_2, t)$, $n = 1, 2, 3$, denotes the prescribed displacements in the 'virtual' substrate defined by $x_3 \geq h$. We further assume that the thickness of the near-surface layer $h \ll \ell$, so $\theta = \sqrt{\frac{h}{\ell}}$ is a small geometric parameter; as above, ℓ denotes a typical macroscale.

Here we again make use of the asymptotic integration method, see [50], [52], and [6], in order to find the classical ('local') stresses σ_{3n} at $x_3 = h$ expressed in terms of the unknown surface stresses χ_{3n}. It is then necessary to equate the obtained σ_{3n} to the nonlocal stresses at $x_3 = h$ (3.64) in order to find the sought for effective boundary conditions, which is in fact equivalent to finding the value of χ_{3n}.

Let us start by scaling the original variables in the form

$$x_i = \ell \xi_i \ , \quad x_3 = h\zeta \ , \quad \text{and} \quad t = \frac{\ell}{c_2}\tau \ . \tag{3.80}$$

Next, we define dimensionless quantities as follows

$$\begin{aligned} &u_n = \ell u_n^* \ , \ v_n = \ell v_n^* \ , \\ &e_{ii} = e_{ii}^* \ , \ e_{ij} = e_{ij}^* \ , \\ &\sigma_{ii} = \mu \sigma_{ii}^* \ , \\ &\sigma_{ij} = \mu \sigma_{ij}^* \ , \\ &\sigma_{3n} = \theta^2 \mu \sigma_{3n}^* \ , \\ &\chi_{3n} = \theta^3 \mu \chi_{3n}^* \ , \end{aligned} \tag{3.81}$$

and also assume

$$\frac{\partial u_i}{\partial x_3} = \gamma_{i3}^* \ , \tag{3.82}$$

where μ is a Lamé constant and all the dimensionless quantities with an asterisk are assumed to be of the same asymptotic order. As usual, classical stresses σ_{mn} are expressed in terms of strains as in (2.2). We use it to write down the classical stresses σ_{mn} as

$$\begin{aligned} &\sigma_{ij} = 2\mu e_{ij} \ , \\ &\sigma_{ii} = (\lambda + 2\mu) e_{ii} + \lambda (e_{jj} + e_{33}) \ , \\ &\sigma_{3i} = 2\mu e_{3i} \ , \\ &\sigma_{33} = (\lambda + 2\mu) e_{33} + \lambda (e_{ii} + e_{jj}) \ . \end{aligned} \tag{3.83}$$

In order to be able to express all the necessary stresses in a form easy-to-use for subsequent asymptotic integration, we need to express e_{33} in $(3.83)_4$ through e_{ii}, e_{jj}, and σ_{33} and then substitute it into $(3.83)_2$ to obtain

$$\sigma_{ii} = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} e_{ii} + \frac{2\lambda\mu}{\lambda + 2\mu} e_{jj} + \frac{\lambda}{\lambda + 2\mu} \sigma_{33} \ . \tag{3.84}$$

Now let us write down non-dimensional equations for the stresses. In what follows, we only need σ_{ij} in (3.83) and σ_{ii} in (3.84). After making the substitutions $\lambda = \rho(c_1^2 - 2c_2^2)$ and $\mu = 2\rho c_2^2$, we obtain the following set of non-dimensional equations

$$\begin{aligned}
&\sigma_{ij}^* = 2e_{ij}^* \ , \\
&\sigma_{ii}^* = 4(1-\kappa^2)e_{ii}^* + 2(1-\kappa^2)e_{jj}^* + \theta^2(1-2\kappa^2)\sigma_{33}^* \ ,
\end{aligned} \tag{3.85}$$

where κ is defined by (2.12). We may now express the strains e_{mn} in terms of the displacements u_n using the formula (2.3), which in dimensionless form becomes

$$\begin{aligned}
&e_{ij}^* = \frac{1}{2}\left(\frac{\partial u_i^*}{\partial \xi_j} + \frac{\partial u_j^*}{\partial \xi_i}\right) \ , \\
&e_{ii}^* = \frac{\partial u_i^*}{\partial \xi_i} \ , \\
&\eta e_{33}^* = \frac{\partial u_3^*}{\partial \zeta} \ ,
\end{aligned} \tag{3.86}$$

with the analogous form of equation (3.82) given by

$$\frac{\partial u_i^*}{\partial \zeta} = \theta^2 \gamma_{i3}^* \ . \tag{3.87}$$

The equations of motion (3.77) take the form

$$\begin{aligned}
&\frac{\partial \sigma_{3i}^*}{\partial \zeta} = -\frac{\partial \sigma_{ii}^*}{\partial \xi_i} - \frac{\partial \sigma_{ij}^*}{\partial \xi_j} + \frac{\partial^2 u_i^*}{\partial \tau^2} \ , \\
&\frac{\partial \sigma_{33}^*}{\partial \zeta} = -\theta^2\left(\frac{\partial \sigma_{3i}^*}{\partial \xi_i} + \frac{\partial \sigma_{3j}^*}{\partial \xi_j}\right) + \frac{\partial^2 u_3^*}{\partial \tau^2} \ .
\end{aligned} \tag{3.88}$$

The boundary conditions (3.78) can be written as

$$\sigma_{3n}^* = \theta \chi_{3n}^* \quad \text{at } \zeta = 0 \ , \tag{3.89}$$

and the continuity of displacements at the 'virtual' interface is given by

$$u_n^* = v_n^* \quad \text{at } \zeta = 1 \ . \tag{3.90}$$

Now we can expand all the dimensionless quantities, namely the displacements u_n^*, stresses σ_{mn}^* and strains e_{mn}^*, in asymptotic series in terms of the previously introduced small parameter $\theta = \sqrt{\frac{h}{\ell}}$, these being given by

$$\begin{pmatrix} u_n^* \\ \\ \sigma_{mn}^* \\ \\ e_{mn}^* \end{pmatrix} = \begin{pmatrix} u_n^{(0)} \\ \\ \sigma_{mn}^{(0)} \\ \\ e_{mn}^{(0)} \end{pmatrix} + \theta \begin{pmatrix} u_n^{(1)} \\ \\ \sigma_{mn}^{(1)} \\ \\ e_{mn}^{(1)} \end{pmatrix} + \ldots \tag{3.91}$$

Substitution of these expansions into equations (3.85), (3.86), (3.87), and (3.88) results, at leading order, in the following set of equations

$$\begin{aligned}
&\frac{\partial u_i^{(0)}}{\partial \zeta} = 0 \ , \\
&\frac{\partial u_3^{(0)}}{\partial \zeta} = 0 \ , \\
&e_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial u_i^{(0)}}{\partial \xi_j} + \frac{\partial u_j^{(0)}}{\partial \xi_i}\right) \ , \\
&e_{ii}^{(0)} = \frac{\partial u_i^{(0)}}{\partial \xi_i} \ , \\
&\sigma_{ij}^{(0)} = 2e_{ij}^{(0)} \ , \\
&\sigma_{ii}^{(0)} = 4(1-\kappa^2)e_{ii}^{(0)} + 2(1-\kappa^2)e_{jj}^{(0)} \ , \\
&\frac{\partial \sigma_{3i}^{(0)}}{\partial \zeta} = -\frac{\partial \sigma_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial \sigma_{ij}^{(0)}}{\partial \xi_j} + \frac{\partial^2 u_i^{(0)}}{\partial \tau^2} \ , \\
&\frac{\partial \sigma_{33}^{(0)}}{\partial \zeta} = \frac{\partial^2 u_3^{(0)}}{\partial \tau^2} \ ,
\end{aligned} \tag{3.92}$$

along with the boundary conditions

$$\sigma_{3n}^{(0)} = 0 \quad \text{at } \zeta = 0 \tag{3.93}$$

and the 'interfacial' conditions

$$u_n^{(0)} = v_n^* \quad \text{at } \zeta = 1 \ . \tag{3.94}$$

On integrating of equations $(3.92)_{1,2}$ with respect to ζ, and satisfying the 'interfacial' conditions (3.94), then using the obtained results in $(3.92)_{3-6}$, we have

$$\begin{aligned}
&u_n^{(0)} = v_n^* \ , \\
&e_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^*}{\partial \xi_j} + \frac{\partial v_j^*}{\partial \xi_i}\right) \ , \\
&e_{ii}^{(0)} = \frac{\partial v_i^*}{\partial \xi_i} \ , \\
&\sigma_{ij}^{(0)} = \frac{\partial v_i^*}{\partial \xi_j} + \frac{\partial v_j^*}{\partial \xi_i} \ , \\
&\sigma_{ii}^{(0)} = 4(1-\kappa^2)\frac{\partial v_i^*}{\partial \xi_i} + 2(1-\kappa^2)\frac{\partial v_j^*}{\partial \xi_j} \ ,
\end{aligned} \tag{3.95}$$

where we recall that v_n^* are dimensionless prescribed displacements and $v_n^* = v_n^*(\xi_i, \xi_j, \tau)$. We remark that these displacements are not dependent of ζ (i.e., independent of the transverse co-ordinate x_3).

On substituting of $(3.95)_{1,4,5}$ into the governing equation $(3.92)_7$, we obtain

$$\frac{\partial \sigma_{3i}^{(0)}}{\partial \zeta} = -\frac{\partial^2 v_i^*}{\partial \xi_j^2} - 4(1-\kappa^2)\frac{\partial^2 v_i^*}{\partial \xi_i^2} - (3-4\kappa^2)\frac{\partial^2 v_j^*}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 v_i^*}{\partial \tau^2} \ , \tag{3.96}$$

and on integrating of the equation above with respect to ζ we arrive at

$$\sigma_{3i}^{(0)} = -\zeta\left[\frac{\partial^2 v_i^*}{\partial \xi_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i^*}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j^*}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 v_i^*}{\partial \tau^2}\right] + C \ , \tag{3.97}$$

where $C = C(\xi_1, \xi_2, \tau)$ is an arbitrary function. Now, satisfying the corresponding boundary condition (3.93), we can find the arbitrary function C and establish that

$$\sigma_{3i}^{(0)} = -\zeta\left[\frac{\partial^2 v_i^*}{\partial \xi_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i^*}{\partial \xi_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j^*}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 v_i^*}{\partial \tau^2}\right] \ . \tag{3.98}$$

Next, on substituting $(3.95)_1$ into the governing equation $(3.92)_8$, we obtain

$$\frac{\partial \sigma_{33}^{(0)}}{\partial \zeta} = \frac{\partial^2 v_3^*}{\partial \tau^2} \ , \tag{3.99}$$

which on integrating with respect to ζ yields

$$\sigma_{33}^{(0)} = \zeta_p \frac{\partial^2 v_3^*}{\partial \tau^2} + D \ , \tag{3.100}$$

where $D = D(\xi_1, \xi_2, \tau)$ is an arbitrary function. We may now satisfy the boundary condition (3.93) to obtain an expression for the arbitrary function D and, thus, arrive at

$$\sigma_{33}^{(0)} = \zeta_p \frac{\partial^2 v_3^*}{\partial \tau^2} \ . \tag{3.101}$$

Similarly to (3.92), at the next order we readily have

$$\begin{aligned}
&\frac{\partial u_i^{(1)}}{\partial \zeta} = 0 \ , \\
&\frac{\partial u_3^{(1)}}{\partial \zeta} = 0 \ , \\
&e_{ij}^{(1)} = \frac{1}{2}\left(\frac{\partial u_i^{(1)}}{\partial \xi_j} + \frac{\partial u_j^{(1)}}{\partial \xi_i}\right) \ , \\
&e_{ii}^{(1)} = \frac{\partial u_i^{(1)}}{\partial \xi_i} \ , \\
&\sigma_{ij}^{(1)} = 2e_{ij}^{(1)} \ , \\
&\sigma_{ii}^{(1)} = 4(1-\kappa^2)e_{ii}^{(1)} + 2(1-\kappa^2)e_{jj}^{(1)} \ , \\
&\frac{\partial \sigma_{3i}^{(1)}}{\partial \zeta} = -\frac{\partial \sigma_{ii}^{(1)}}{\partial \xi_i} - \frac{\partial \sigma_{ij}^{(1)}}{\partial \xi_j} + \frac{\partial^2 u_i^{(1)}}{\partial \tau^2} \ , \\
&\frac{\partial \sigma_{33}^{(1)}}{\partial \zeta} = \frac{\partial^2 u_3^{(1)}}{\partial \tau^2} \ ,
\end{aligned} \tag{3.102}$$

while the boundary conditions take the form

$$\sigma_{3n}^{(1)} = \chi_{3n}^* \quad \text{at } \zeta = 0 \tag{3.103}$$

and the 'interfacial' conditions become

$$u_n^{(1)} = 0 \quad \text{at } \zeta = 1 \, . \tag{3.104}$$

Similarly to the leading order calculations previously carried out, on integrating equations $(3.102)_{1,2}$ with respect to ζ and satisfying the 'interfacial' conditions (3.104) and using the obtained results in $(3.102)_{3-6}$, we readily have

$$\begin{aligned} &u_n^{(1)} = 0 \, , \\ &e_{ij}^{(1)} = 0 \, , \; e_{ii}^{(1)} = 0 \, , \\ &\sigma_{ij}^{(1)} = 0 \, , \; \sigma_{ii}^{(1)} = 0 \, . \end{aligned} \tag{3.105}$$

On substituting of the equations (3.105) into the governing equation $(3.102)_7$, we obtain

$$\frac{\partial \sigma_{3i}^{(1)}}{\partial \zeta} = 0 \, , \tag{3.106}$$

and integrating it with respect to ζ yields

$$\sigma_{3i}^{(1)} = E \, , \tag{3.107}$$

where $E = E(\xi_1, \xi_2, \tau)$ is an arbitrary function. Now, satisfying the corresponding boundary condition (3.103), we can find the arbitrary function E and establish that

$$\sigma_{3i}^{(1)} = \chi_{3i}^{*} \, . \tag{3.108}$$

We now substitute $(3.105)_1$ into the governing equation $(3.102)_8$ to obtain

$$\frac{\partial \sigma_{33}^{(1)}}{\partial \zeta} = 0 \, , \tag{3.109}$$

and after integration with respect to ζ, we have

$$\sigma_{33}^{(1)} = F \, , \tag{3.110}$$

where $F = F(\xi_1, \xi_2, \tau)$ is an arbitrary function. On satisfying of the boundary conditions (3.103), the arbitrary function F may be found, enabling us to deduce that

$$\sigma_{33}^{(1)} = \chi_{33}^{*} \, . \tag{3.111}$$

We are now in a position to write down the dimensionless stresses σ^*_{3i} and σ^*_{33} up to $O(\theta)$ order. The stresses σ^*_{3i} take the form

$$\begin{aligned}\sigma^*_{3i} &= \sigma^{(0)}_{3i} + \theta\sigma^{(1)}_{3i}\\ &= -\zeta\left[\frac{\partial^2 v^*_i}{\partial\xi^2_j} + 4(1-\kappa^2)\frac{\partial^2 v^*_i}{\partial\xi^2_i} + (3-4\kappa^2)\frac{\partial^2 v^*_j}{\partial\xi_i\partial\xi_j} - \frac{\partial^2 v^*_i}{\partial\tau^2}\right] + \theta\chi^*_{3i}\ ,\end{aligned} \tag{3.112}$$

with the stress σ^*_{33} given by

$$\begin{aligned}\sigma^*_{33} &= \sigma^{(0)}_{33} + \theta\sigma^{(1)}_{33}\\ &= \zeta\frac{\partial^2 v^*_3}{\partial\tau^2} + \theta\chi^*_{33}\ .\end{aligned} \tag{3.113}$$

In terms of the original variables, the expressions (3.112) and (3.113) may be presented as

$$\begin{aligned}\sigma_{3i} &= \rho x_3\left(\frac{\partial^2 v_i}{\partial t^2} - c_2^2\left[\frac{\partial^2 v_i}{\partial x^2_j} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x^2_i} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i\partial x_j}\right]\right) + \chi_{3i}\ ,\\ \sigma_{33} &= \rho x_3\frac{\partial^2 v_3}{\partial t^2} + \chi_{33}\ .\end{aligned} \tag{3.114}$$

We now need to obtain expression for stresses σ_{3i} and σ_{33} at the interface. Therefore these stresses can be written as

$$\begin{aligned}\sigma^*_{3i} &= -\left[\frac{\partial^2 v^*_i}{\partial\xi^2_j} + 4(1-\kappa^2)\frac{\partial^2 v^*_i}{\partial\xi^2_i} + (3-4\kappa^2)\frac{\partial^2 v^*_j}{\partial\xi_i\partial\xi_j} - \frac{\partial^2 v^*_i}{\partial\tau^2}\right] + \theta\chi^*_{3i}\ ,\\ \sigma^*_{33} &= \frac{\partial^2 v^*_3}{\partial\tau^2} + \theta\chi^*_{33}\\ &\qquad \text{at } \zeta = 1\ ,\end{aligned} \tag{3.115}$$

which in terms of the original variables become

$$\begin{aligned}\sigma_{3i} &= \rho h\left(\frac{\partial^2 v_i}{\partial t^2} - c_2^2\left[\frac{\partial^2 v_i}{\partial x^2_j} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x^2_i} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i\partial x_j}\right]\right) + \chi_{3i}\ ,\\ \sigma_{33} &= \rho h\frac{\partial^2 v_3}{\partial t^2} + \chi_{33}\\ &\qquad \text{at } x_3 = h\ .\end{aligned} \tag{3.116}$$

Having found the values of the classical ('local') stresses σ_{3n} at the 'virtual' interface $x_3 = h$, the next step is to equate them to the corresponding nonlocal stresses s_{3n} in (3.64). This effectively means that we could complement the considered inverse problem by extra interfacial conditions, namely

$$\sigma_{3n} = s_{3n} \quad \text{at } x_3 = h \; . \tag{3.117}$$

Satisfying these conditions, we obtain the sought for values of χ_{3n} in the form

$$\begin{aligned} \chi_{3i} &= \rho c_2^2 \frac{a}{2\sqrt{\pi}} \left(\frac{\partial^2 U_i}{\partial x_j^2} + 4(1-\kappa^2) \frac{\partial^2 U_i}{\partial x_i^2} + (3-4\kappa^2) \frac{\partial^2 U_j}{\partial x_i \partial x_j} \right) \; , \\ \chi_{33} &= 0 \; , \end{aligned} \tag{3.118}$$

where $U_n = \ell v_n^*$.

We can now conclude that the effective boundary conditions that need to be imposed on the surface of a 'locally' elastic, homogeneous half-space in order to account for nonlocal near-surface effects are given by

$$\begin{aligned} \sigma_{3i} &= \rho c_2^2 \frac{a}{2\sqrt{\pi}} \left(\frac{\partial^2 u_i}{\partial x_j^2} + 4(1-\kappa^2) \frac{\partial^2 u_i}{\partial x_i^2} + (3-4\kappa^2) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \; , \\ \sigma_{33} &= 0 \\ & \qquad\qquad\qquad\qquad \text{at } x_3 = 0 \; . \end{aligned} \tag{3.119}$$

The differential operator in brackets in the right-hand side of $(3.119)_1$ coincides with that in plane elasticity, e.g., see [66]. This operator also appears in the effective boundary conditions for a coated 'local' half-space, see [28]. The difference between the effective boundary conditions in [28] and the derived effective conditions (3.119) is in that the former are imposed along the interface while the latter hold at the surface. It is obvious that this is the only chance within the considered set up since we are dealing with a 'virtual' interface at all times.

3.3 An alternative approach to analysing a near-surface vertically inhomogeneous layer

In this part, a more general setup is analysed. The specific assumptions in nonlocal elasticity as presented in (2.26) are not employed here. Instead, the inhomogeneity of the near-surface layer is taken in the general form of variable longitudinal and transverse wave speeds c_1' and c_2', respectively, for further references see [23].

Let us begin by considering a thin, vertically inhomogeneous layer of thickness $h \ll \ell$, where ℓ is a typical wavelength, see Figure 3.7. Then $\varepsilon = \frac{h}{\ell}$ is a small geometric parameter.

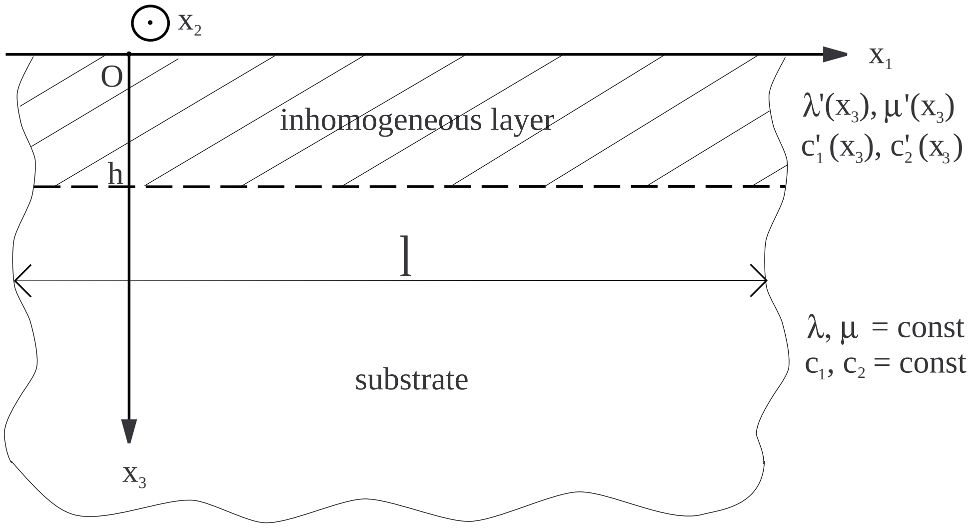

Figure 3.7: A substrate coated with a near-surface vertical inhomogeneity.

Equations of motion are given by

$$\begin{aligned} \frac{\partial s_{ii}}{\partial x_i} + \frac{\partial s_{ij}}{\partial x_j} + \frac{\partial s_{3i}}{\partial x_3} &= \rho \frac{\partial^2 u_i}{\partial t^2} \ , \\ \frac{\partial s_{3i}}{\partial x_i} + \frac{\partial s_{3j}}{\partial x_j} + \frac{\partial s_{33}}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2} \ , \end{aligned} \tag{3.120}$$

and constitutive relations by

$$
\begin{aligned}
s_{ij} &= \rho c_2'^2(x_3)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \ , \\
s_{ii} &= \rho c_1'^2(x_3)\frac{\partial u_i}{\partial x_i} + \rho(c_1'^2(x_3) - 2c_2'^2(x_3))\left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_3}{\partial x_3}\right) \ , \\
s_{3i} &= \rho c_2'^2(x_3)\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) \ , \\
s_{33} &= \rho c_1'^2(x_3)\frac{\partial u_3}{\partial x_3} + \rho(c_1'^2(x_3) - 2c_2'^2(x_3))\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j}\right) \ ,
\end{aligned}
\tag{3.121}
$$

where $i \neq j = 1, 2$ and Einstein's summation convention is not employed. The variable wave speeds in (3.121) are given by

$$
c_1'(x_3) = \sqrt{\frac{\lambda'(x_3) + 2\mu'(x_3)}{\rho}} \quad \text{and} \quad c_2'(x_3) = \sqrt{\frac{\mu'(x_3)}{\rho}} \ . \tag{3.122}
$$

Traction-free boundary conditions at the surface are imposed in the form

$$
s_{3n} = 0 \quad \text{at } x_3 = 0 \ , \tag{3.123}
$$

with assumed continuity of displacement along the interface expressed through

$$
u_n = v_n \quad \text{at } x_3 = h \ , \tag{3.124}
$$

where $v_n = v_n(x_1, x_2, t)$, $n = 1, 2, 3$, are the prescribed displacements in the substrate. We adapt a similar asymptotic approach as in the previous sections in order to express the stresses s_{3n} at the interface $x_3 = h$ in terms of the prescribed substrate displacements v_n.

First, we scale the original variables as follows

$$
x_i = \xi_i \ell \ , \quad x_3 = \eta h \ , \quad \text{and} \quad t = \frac{\ell}{c_2}\tau \ , \tag{3.125}
$$

where $c_2 = c_2'(h)$, and also introduce the dimensionless quantities

$$
\begin{aligned}
& u_n^* = \frac{1}{\ell}u_n \ , \ v_n^* = \frac{1}{\ell}v_n \ , \\
& s_{ij}^* = \frac{1}{\mu}s_{ij} \ , \ s_{ii}^* = \frac{1}{\mu}s_{ii} \ , \ s_{3n}^* = \frac{\ell}{\mu h}s_{3n} = \frac{\varepsilon^{-1}}{\mu}s_{3n} \ ,
\end{aligned}
\tag{3.126}
$$

where all the quantities with an asterisk are assumed to be of the same asymptotic order.

The equations of motion (3.120) and constitutive relations (3.121) can now be rewritten as

$$\begin{aligned}
&\frac{\partial s_{ii}^*}{\partial \xi_i} + \frac{\partial s_{ij}^*}{\partial \xi_j} + \frac{\partial s_{3i}^*}{\partial \eta} = \frac{\partial^2 u_i^*}{\partial \tau^2} \ , \\
&\frac{\partial s_{33}^*}{\partial \eta} + \varepsilon\left(\frac{\partial s_{3i}^*}{\partial \xi_i} + \frac{\partial s_{3j}^*}{\partial \xi_j}\right) = \frac{\partial^2 u_3^*}{\partial \tau^2} \ ,
\end{aligned} \tag{3.127}$$

and

$$\begin{aligned}
&s_{ij}^* = \kappa_2'^2\left(\frac{\partial u_i^*}{\partial \xi_j} + \frac{\partial u_j^*}{\partial \xi_i}\right) \ , \\
&\varepsilon s_{ii}^* = (\kappa_1'^2 - 2\kappa_2'^2)\frac{\partial u_3^*}{\partial \eta} + \varepsilon\left(\kappa_1'^2\frac{\partial u_i^*}{\partial \xi_i} + (\kappa_1'^2 - 2\kappa_2'^2)\frac{\partial u_j^*}{\partial \xi_j}\right) \ , \\
&\varepsilon^2 s_{3i}^* = \kappa_2'^2\left(\frac{\partial u_i^*}{\partial \eta} + \varepsilon\frac{\partial u_3^*}{\partial \xi_i}\right) \ , \\
&\varepsilon^2 s_{33}^* = \kappa_1'^2\frac{\partial u_3^*}{\partial \eta} + \varepsilon(\kappa_1'^2 - 2\kappa_2'^2)\left(\frac{\partial u_i^*}{\partial \xi_i} + \frac{\partial u_j^*}{\partial \xi_j}\right) \ ,
\end{aligned} \tag{3.128}$$

with $\kappa_m' = \frac{c_m'(x_3)}{c_2}$, $m = 1, 2$.

Similarly to the previous section, it is convenient to express $\frac{\partial u_3^*}{\partial \eta}$ in $(3.128)_2$ from $(3.128)_4$, yielding

$$s_{ii}^* = 4\kappa_2'^2\left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right)\frac{\partial u_i^*}{\partial \xi_i} + 2\kappa_2'^2\left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right)\frac{\partial u_j^*}{\partial \xi_j} + \varepsilon\left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right)s_{33}^* \ . \tag{3.129}$$

In the dimensionless form, the boundary conditions (3.123) are

$$s_{3n}^* = 0 \quad \text{at } \eta = 0 \tag{3.130}$$

and the interfacial conditions (3.124) become

$$u_n^* = v_n^* \quad \text{at } \eta = 1 \ . \tag{3.131}$$

We now expand the displacements u_n and stresses s_{mn} in asymptotic series in terms of the previously specified small geometric parameter ε, and therefore have

$$\begin{pmatrix} u_n^* \\ \\ s_{ii}^* \\ \\ s_{ij}^* \\ \\ s_{3i}^* \\ \\ s_{33}^* \end{pmatrix} = \begin{pmatrix} u_n^{(0)} \\ \\ s_{ii}^{(0)} \\ \\ s_{ij}^{(0)} \\ \\ s_{3i}^{(0)} \\ \\ s_{33}^{(0)} \end{pmatrix} + \varepsilon \begin{pmatrix} u_n^{(1)} \\ \\ s_{ii}^{(1)} \\ \\ s_{ij}^{(1)} \\ \\ s_{3i}^{(1)} \\ \\ s_{33}^{(1)} \end{pmatrix} + \dots \tag{3.132}$$

Substitution of the expressions above into equations (3.127) - (3.131) results, at leading order, in the following set of equations of motion

$$\begin{aligned} &\frac{\partial s_{ii}^{(0)}}{\partial \xi_i} + \frac{\partial s_{ij}^{(0)}}{\partial \xi_j} + \frac{\partial s_{3i}^{(0)}}{\partial \eta} = \frac{\partial^2 u_i^{(0)}}{\partial \tau^2} \ , \\ &\frac{\partial s_{33}^{(0)}}{\partial \eta} = \frac{\partial^2 u_3^{(0)}}{\partial \tau^2} \ , \end{aligned} \tag{3.133}$$

and constitutive relations

$$\begin{aligned} &s_{ij}^{(0)} = \kappa_2'^2 \left(\frac{\partial u_i^{(0)}}{\partial \xi_j} + \frac{\partial u_j^{(0)}}{\partial \xi_i} \right) \ , \\ &s_{ii}^{(0)} = 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2} \right) \frac{\partial u_i^{(0)}}{\partial \xi_i} + 2\kappa_2'^2 \left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2} \right) \frac{\partial u_j^{(0)}}{\partial \xi_j} \ , \\ &\frac{\partial u_n^{(0)}}{\partial \eta} = 0 \ , \end{aligned} \tag{3.134}$$

together with the boundary conditions

$$s_{3n}^{(0)} = 0 \quad \text{at } \eta = 0 \tag{3.135}$$

and the interfacial conditions

$$u_n^{(0)} = v_n^* \quad \text{at } \eta = 1 \ . \tag{3.136}$$

On integrating $(3.133)_2$ and $(3.134)_3$ with respect to η, and taking into account the corresponding interfacial conditions (3.136), we obtain

$$u_n^{(0)} = v_n^* \tag{3.137}$$

and

$$s_{33}^{(0)} = \eta \frac{\partial^2 v_3^*}{\partial \tau^2} \,. \tag{3.138}$$

Using (3.137), we obtain from $(3.134)_2$ the following expression

$$s_{ii}^{(0)} = 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) \frac{\partial v_i^*}{\partial \xi_i} + 2\kappa_2'^2 \left(1 - \frac{2\kappa_2'^2}{\kappa_1'^2}\right) \frac{\partial v_j^*}{\partial \xi_j} \,. \tag{3.139}$$

Next, we integrate equation $(3.133)_1$, using $(3.134)_2$ and (3.137), and then satisfy the boundary conditions (3.135), to establish that

$$\begin{aligned} s_{3i}^{(0)} = \eta \frac{\partial^2 v_i^*}{\partial \tau^2} - \frac{\partial^2 v_i^*}{\partial \xi_j^2} \int_0^\eta \kappa_2'^2 \, d\eta' - 4 \frac{\partial^2 v_i^*}{\partial \xi_i^2} \int_0^\eta \kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) d\eta' \\ - \frac{\partial^2 v_j^*}{\partial \xi_i \partial \xi_j} \int_0^\eta \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) d\eta' \,. \end{aligned} \tag{3.140}$$

In terms of the original variables, let us write down the expressions for the stresses s_{3i} and s_{33}, which through use of (3.140) and (3.138) become

$$\begin{aligned} s_{3i} = \rho \Bigg[x_3 \frac{\partial^2 u_i}{\partial t^2} - c_2^2 \frac{\partial^2 u_i}{\partial x_j^2} \int_0^{x_3} \kappa_2'^2 \, dx_3' - c_2^2 \frac{\partial^2 u_i}{\partial x_i^2} \int_0^{x_3} 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \\ - c_2^2 \frac{\partial^2 u_j}{\partial x_i \partial x_j} \int_0^{x_3} \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \Bigg] \,, \end{aligned} \tag{3.141}$$

$$s_{33} = \rho x_3 \frac{\partial^2 u_3}{\partial t^2} \,,$$

where now $u_n = \ell u_n^{(0)}$. Note that for subsequent calculations we also need to use the following expressions for s_{ii} and s_{ij}

$$\begin{aligned} s_{ii} &= 2\rho c_2'^2 \left[2\left(1 - \frac{c_2'^2}{c_1'^2}\right) \frac{\partial u_i}{\partial x_i} + \left(1 - \frac{2c_2'^2}{c_1'^2}\right) \frac{\partial u_j}{\partial x_j} \right] , \\ s_{ij} &= \rho c_2'^2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) . \end{aligned} \tag{3.142}$$

The stresses s_{3i} and s_{33} at the interface $x_3 = h$ may now be expressed through the substrate displacements v_n, yielding

$$\begin{aligned} s_{3i} = \rho \Bigg[h\frac{\partial^2 v_i}{\partial t^2} - c_2^2 \frac{\partial^2 v_i}{\partial x_j^2} \int_0^h \kappa_2'^2 \, dx_3' - \rho c_2^2 \frac{\partial^2 v_i}{\partial x_i^2} \int_0^h 4\kappa_2'^2 \left(1 - \frac{\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \\ - c_2^2 \frac{\partial^2 v_j}{\partial x_i \partial x_j} \int_0^h \kappa_2'^2 \left(3 - \frac{4\kappa_2'^2}{\kappa_1'^2}\right) dx_3' \Bigg] , \end{aligned} \tag{3.143}$$

$$s_{33} = \rho h \frac{\partial^2 v_3}{\partial t^2} ,$$

where, as previously, $\kappa_m' = \frac{c_m'}{c_2}$, $m = 1, 2$.

Let us recall the important expression (2.29) in Section 2.2, namely

$$s_{\alpha\beta}(\boldsymbol{x}) = \frac{1}{\pi^{3/2} a^3} \int_0^\infty dx_3' \int_{-\infty}^\infty dx_1' \int_{-\infty}^\infty dx_2' \, \exp\left[-\frac{(\boldsymbol{x}' - \boldsymbol{x})^2}{a^2} \right] \sigma_{\alpha\beta}(\boldsymbol{x}') . \tag{3.144}$$

Now we expand the stresses $\sigma_{\alpha\beta}$ in Taylor series about the reference point $\boldsymbol{x}' = \boldsymbol{x}$, assuming again that the typical wavelength characterising the classical stress field is much greater than the internal size a. Then (3.144) can be transformed into

$$\begin{aligned} s_{\alpha\beta}(\boldsymbol{x}) = \frac{1}{a\sqrt{\pi}} \Bigg\{ \sigma_{\alpha\beta}(\boldsymbol{x}) \int_0^\infty \exp\left[-\frac{(x_3' - x_3)^2}{a^2} \right] dx_3' \\ + \frac{\partial \sigma_{\alpha\beta}(\boldsymbol{x})}{\partial x_3} \int_0^\infty (x_3' - x_3) \exp\left[-\frac{(x_3' - x_3)^2}{a^2} \right] dx_3' \Bigg\} \\ + \dots , \end{aligned} \tag{3.145}$$

which on integration becomes

$$s_{\alpha\beta}(\boldsymbol{x}) = \frac{\sigma_{\alpha\beta}(\boldsymbol{x})}{2} \operatorname{erfc}\left(-\frac{x_3}{a}\right) + \frac{a}{2\sqrt{\pi}} \frac{\partial \sigma_{\alpha\beta}(\boldsymbol{x})}{\partial x_3} \exp\left[-\frac{x_3^2}{a^2}\right] + \dots \,, \tag{3.146}$$

where $\operatorname{erfc}(x)$ is given by (3.29).

In the current case of a half-space we only keep the term linear in a, which does not occur in the case of 3D space, see [23]. Thus, recalling Hooke's law in the form of (2.2) and using (2.26), we keep the leading order term in (3.146) to obtain

$$s_{\alpha\beta} = \lambda' e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu' e_{\alpha\beta} \,, \tag{3.147}$$

where

$$\begin{aligned} \lambda' &= \frac{1}{2} \operatorname{erfc}\left(-\frac{x_3}{a}\right) \lambda \,, \\ \mu' &= \frac{1}{2} \operatorname{erfc}\left(-\frac{x_3}{a}\right) \mu \,. \end{aligned} \tag{3.148}$$

Note that the problem in nonlocal elasticity expressed by equations (2.25), (3.147), and (3.148) is formally equivalent to a 'locally' elastic problem for a vertically inhomogeneous half-space.

At the interface $x_3 = h$, to within an exponentially small error, $\operatorname{erfc}\left(-\frac{h}{a}\right) = 2$, which leads to

$$\lambda'(h) = \lambda \quad \text{and} \quad \mu'(h) = \mu \tag{3.149}$$

with the nonlocal stresses $s_{\alpha\beta}$ tending to their local counterparts $\sigma_{\alpha\beta}$.

Since the variable elastic moduli are given by (3.148), the variable wave speeds become

$$c_m'^2(x_3) = \frac{1}{2} c_m^2 \operatorname{erfc}\left(-\frac{x_3}{a}\right), m = 1, 2 \,, \tag{3.150}$$

where, recalling (3.149),

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_2 = \sqrt{\frac{\mu}{\rho}} \,.$$

We are now in a position to calculate the integrals in (3.143). Having assumed $a \ll h$, we have, to within $O(ae^{-(h/a)^2})$ error,

$$\int_0^h \kappa_2'^2 \, dx_3' = h\left(1 - \frac{1}{2\sqrt{\pi}} \frac{a}{h}\right) \,,$$

$$\int_0^h 4\kappa_2'^2\left(1-\frac{\kappa_2'^2}{\kappa_1'^2}\right)\,dx_3' = 4h(1-\kappa^2)\left(1-\frac{1}{2\sqrt{\pi}}\frac{a}{h}\right)\,,$$

and

$$\int_0^h \kappa_2'^2\left(3-\frac{4\kappa_2'^2}{\kappa_1'^2}\right)\,dx_3' = h(3-4\kappa^2)\left(1-\frac{1}{2\sqrt{\pi}}\frac{a}{h}\right)\,,$$

where κ is defined by (2.12). Therefore, the stresses s_{3n} at the interface $x_3 = h$ can be expressed as

$$\begin{aligned} s_{3i} = \;& \rho h\left[\frac{\partial^2 v_i}{\partial t^2} - c_2^2\left\{\frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j}\right\}\right. \\ & \left. + c_2^2\frac{a}{2h\sqrt{\pi}}\left\{\frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa^2)\frac{\partial^2 v_i}{\partial x_i^2} + (3-4\kappa^2)\frac{\partial^2 v_j}{\partial x_i \partial x_j}\right\}\right]\,, \\ s_{33} = \;& \rho h\frac{\partial^2 v_3}{\partial t^2}\,. \end{aligned} \tag{3.151}$$

From $(3.151)_1$ we can conclude that nonlocal elastic properties can be taken into account via introduction of an asymptotic correction of the relative asymptotic order $O(\frac{a}{h})$. Note that this correction must exceed the truncation error $O(\varepsilon)$ in asymptotic derivation of (3.143). This implies that for (3.151) to be legitimate, the following double inequality must hold

$$a \ll h \ll \sqrt{a\ell}\,, \tag{3.152}$$

Another important point is that we need to show that the accuracy of the leading order approximation (3.147) is consistent with the $O(\frac{a}{h})$ accuracy of $(3.151)_1$. Therefore, we recall that the stresses s_{3n} are expressed at leading order in terms of the stresses s_{ii} and displacements u_n in (3.133). To this end, the 'local' stresses σ_{ii} and σ_{ij} corresponding to their nonlocal counterparts s_{ii} and s_{ij} given by (3.142), making use of the dimensionless equation $(3.133)_1$, are given by

$$\begin{aligned} \sigma_{ii} &= 2\rho c_2^2\left[2(1-\kappa^2)\frac{\partial u_i}{\partial x_i} + (1-2\kappa^2)\frac{\partial u_j}{\partial x_j}\right]\,, \\ \sigma_{ij} &= \rho c_2^2\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\,. \end{aligned} \tag{3.153}$$

Note that these stresses are uniform across the thickness. Hence, the $O(\frac{a}{h})$ contribution of the second term in (3.146) disappears after differentiation with respect to x_3. In addition, the inertial terms in (3.133) do not make a $O(\frac{a}{h})$ contribution to the nonlocal stresses, for further details see [23].

Outside a near-surface layer all nonlocal stresses, to within an asymptotic error not exceeding $O(\frac{a}{h})$, coincide with their local counterparts, see equations (3.146) - (3.149). Therefore at $x_3 \geq h$ we may proceed with a classical problem with the following boundary conditions

$$\sigma_{3n} = s_{3n} \quad \text{at } x_3 = h \; , \tag{3.154}$$

where the values of s_{3n} are given in (3.151).

Finally, considering the fact that the nonlocal stresses s_{3n} in (3.151) coincide with those derived earlier in (3.64), on formulating and solving exactly the same inverse problem for a thin homogeneous layer in the classical elasticity framework as in Section 3.2, we obtain the effective boundary conditions identical to (3.119). These should be imposed on the surface $x_3 = 0$ so that the stresses σ_{3n} at the interface $x_3 = h$ satisfy the interfacial conditions (3.154). Outside the narrow boundary layer, i.e., at $x_3 \gg a$, the dynamics of the half-space is governed by equations with constant elastic moduli λ and μ, subject to the boundary conditions (3.119). Let us emphasise that these conditions involve a $O(\frac{a}{\ell})$ correction (with ℓ a typical macroscale size as before), which is greater than the $O(\frac{a^2}{\ell^2})$ correction in the differential equations of nonlocal elasticity, e.g., see [44], based on the relations

$$(1 - a^2\Delta)s_{mn} = \sigma_{mn} \; , \tag{3.155}$$

where, as above, σ_{mn} and s_{mn} denote classical and nonlocal stress components.

Thus, the transition from original integral constitutive relations in nonlocal elasticity (2.26) to the simplified differential formulae (3.155) is not justified over near-surface domains. Moreover, taking into consideration the nonlocal phenomena within the equation of motion seems to be a second-order effect in comparison with the nonlocal corrections to boundary conditions. Further observations on the subject are presented in the next section.

3.4 Rayleigh surface wave on a non-locally elastic half-space

As an example application of the effective boundary conditions derived in Section 3.2, we consider the influence of nonlocal elastic behaviour on surface wave propagation in the case of plane strain, where $x_2 \equiv 0$ and $\frac{\partial}{\partial x_2} \equiv 0$, $u_m = u_m(x_1, x_3),\ m = 1, 3$, and $u_2 = 0$. The effective boundary conditions (3.119) in the case of plane strain become

$$\begin{aligned} \sigma_{31} &= \frac{2a}{\sqrt{\pi}} \rho c_2^2 (1 - \kappa^2) \frac{\partial^2 u_1}{\partial x_1^2} \ , \\ \sigma_{33} &= 0 \\ &\text{at } x_3 = 0 \ . \end{aligned} \tag{3.156}$$

The equations of motion, expressed in terms of wave potentials φ and ψ_2 are given in (2.7) and (2.8). We then look for travelling wave solutions in the form (2.9). Displacements can be described in terms of potentials as in (2.10).

Next, on substituting of (2.10) into the surface boundary conditions (3.156), we obtain, after taking into account the plane strain forms of (2.2) and (2.3), the following system of equations

$$\begin{aligned} &\left[2 - \frac{c^2}{c_2^2}\right] A + \left[2i\sqrt{1 - \frac{c^2}{c_2^2}}\right] B = 0 \ , \\ &\left[-i\left(\frac{2ak}{\sqrt{\pi}}(\kappa^2 - 1) + 2\sqrt{1 - \frac{c^2}{c_1^2}}\right)\right] A + \\ &\qquad + \left[\left(2 - \frac{c^2}{c_2^2}\right) + \frac{2ak}{\sqrt{\pi}}(\kappa^2 - 1)\sqrt{1 - \frac{c^2}{c_2^2}}\right] B = 0 \ . \end{aligned} \tag{3.157}$$

The condition for existence of a non-trivial solution of the system of equations (3.157) yields

$$R(\gamma) - 4\sqrt{\pi}\theta(\kappa^2 - 1)\gamma^2\sqrt{1 - \gamma^2} = 0 \ , \tag{3.158}$$

where $\theta = \frac{a}{\ell} = \frac{ak}{2\pi} \ll 1$ is a small parameter, $\gamma = \frac{c}{c_2}$ is the normalised dimensionless phase velocity and $R(\gamma)$ is the classical Rayleigh denominator, i.e.,

$$R(\gamma) = (2-\gamma^2)^2 - 4\sqrt{(1-\gamma^2)(1-\kappa^2\gamma^2)}\ .$$

We may now expand γ as an asymptotic series in terms of the small parameter θ, i.e.,

$$\gamma = \gamma_0 + \theta\gamma_1 + \dots\ . \tag{3.159}$$

In this case, the Taylor series expansion of $R(\gamma)$, about $\gamma = \gamma_0$, are given by

$$R(\gamma) = R(\gamma_0) + R'(\gamma_0)(\gamma - \gamma_0) + \dots\ , \tag{3.160}$$

where γ_0 is the normalised classical Rayleigh wave speed, which means that $R(\gamma_0) = 0$. Then, on substituting (3.159) and (3.160) into (3.158), we obtain the first order term in the asymptotic expansion (3.159)

$$\gamma_1 = \frac{4\sqrt{\pi}(\kappa^2-1)\gamma_0^2\sqrt{1-\gamma_0^2}}{R'(\gamma_0)}\ , \tag{3.161}$$

and, hence, we can conclude the following

$$\gamma = \gamma_0 + \theta\frac{4\sqrt{\pi}(\kappa^2-1)\gamma_0^2\sqrt{1-\gamma_0^2}}{R'(\gamma_0)} + \dots\ , \tag{3.162}$$

where $\theta = \frac{ak}{2\pi}$, as before.

We remark that the constructed $O(\theta)$ correction to the classical Rayleigh wave speed, originating from the effective boundary conditions (3.156) imposed on the surface $x_3 = 0$ of the half-space, substantially exceeds $O(\theta^2)$ correction associated with the 'nonlocal terms' in the differential equations of motion, see explicit formulae (5.10) and (5.11) in [44] containing $O(\theta^2)$ corrections to the Rayleigh wave speed. It would also be of obvious interest that the proposed methodology could be compared with the results arising from the concept of surface stresses originated from [60].

Numerical results are presented in Figure 3.8.

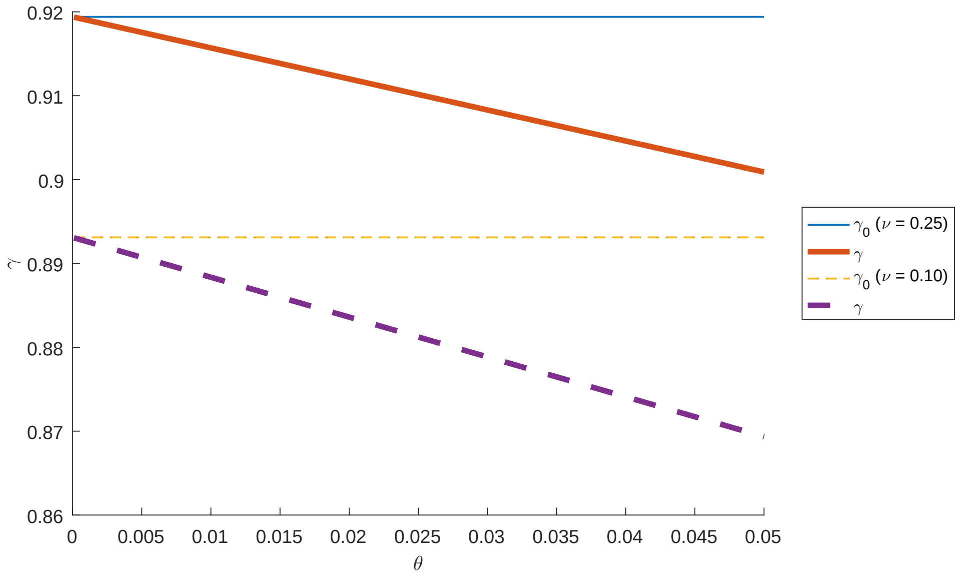

Figure 3.8: Effect of nonlocal phenomena on Rayleigh wave speed.

The classical Rayleigh root γ_0 and the newly derived, 'nonlocal' root γ in (3.162) are plotted as function of the small parameter a for different values of Poisson ratio. The coefficient γ_1 in (3.161) takes the values $\gamma_1 = -0.37$ and -0.48, while its 'local' counterpart (i.e., normalised classical Rayleigh wave speed) is $\gamma_0 = 0.92$ and 0.89 for the Poisson ratios $\nu = 0.25$ and 0.10, respectively. At a fixed value of the microscale parameter a, the presented numerical data may be used for evaluating of the effect of wave number or angular frequency on the sought for nonlocal wave speed. The presence of nonlocal phenomena decreases the Rayleigh wave speed due to low values of the Lamé parameters which denote the stiffness of the system near the surface,

$$\lambda' = \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x_3}{a}\right)\lambda\ ,$$

$$\mu' = \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x_3}{a}\right)\mu\ ,$$

as it was found in [23], where λ' and μ' are nonlocal Lamé parameters which depend on the transverse co-ordinate x_3 and microstructure paramter a, and λ and μ are classical, constant Lamé parameters outside of the near-surface layer. The attenuation of the effect of the nonlocal boundary layer (i.e., $\lambda' \to \lambda$ and $\mu' \to \mu$), when moving away from the surface of a half-space along the vertical direction, can be expressed as

$$Q' = \frac{\lambda'}{\lambda} = \frac{\mu'}{\mu} = \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x_3}{a}\right) \ , \tag{3.163}$$

see Figure 3.9.

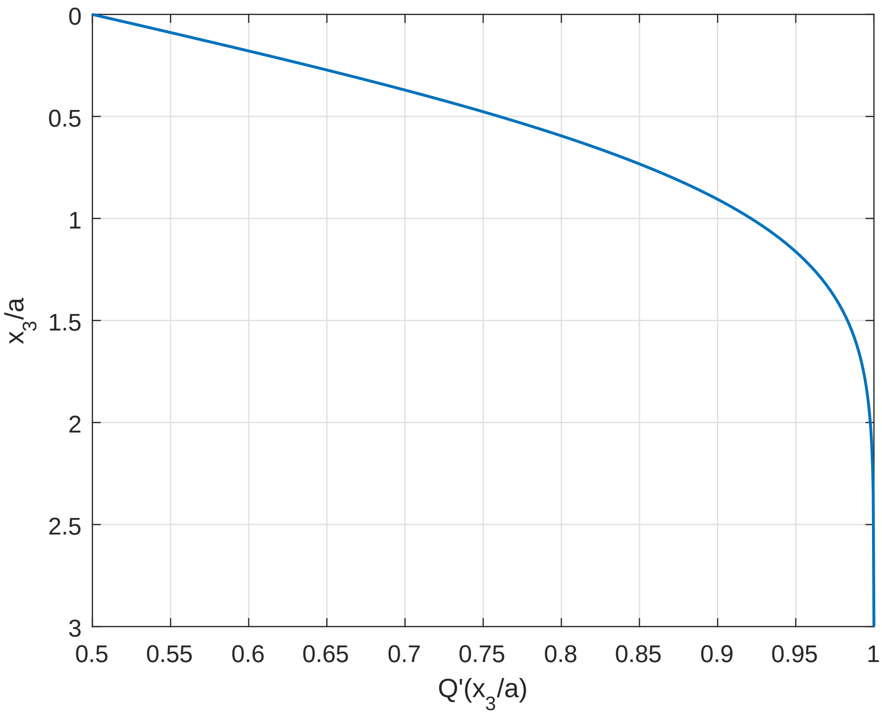

Figure 3.9: Attenuation of the nonlocal boundary layer.

3.5 A moving load on a non-locally elastic half-space

As another example application of the effective boundary conditions (derived in Section 3.2), we analyse the effect of nonlocal elastic behaviour in a problem of a moving load on the surface. Similarly to the problem in the previous section, we discuss this in the case of plane strain ($\frac{\partial}{\partial x_2} \equiv 0$ $u_m = u_m(x_1, x_3),\ m = 1, 3$ and $u_2 = 0$), with the effective boundary conditions (3.119) becoming

$$\begin{aligned} \sigma_{31} &= \frac{2a}{\sqrt{\pi}} \rho c_2^2 (1 - \kappa^2) \frac{\partial^2 u_1}{\partial x_1^2} \ , \\ \sigma_{33} &= P \exp\left[ik(x_1 - ct)\right] \end{aligned} \tag{3.164}$$

$$\text{at } x_3 = 0 \ .$$

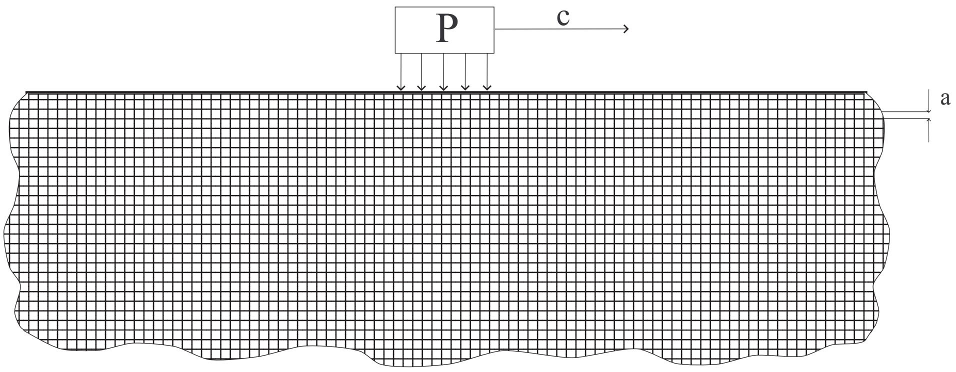

Figure 3.10: A moving load on a non-locally elastic half-space.

Generally, the solution of this problem involves carrying out similar steps as in the previous section and as well as Chapter 2. The equations of motion in terms of wave potentials φ and ψ_2 can be written as (2.7) and (2.8). As previously, we look for travelling wave solutions in the form (2.9). Then we express the displacements in terms of potentials as in (2.10). Next, on substituting of (2.10) into the boundary conditions (3.164), we obtain, after taking into account the plane strain forms of (2.2) and (2.3), the following set of equations

$$
\begin{aligned}
&\left[-i\left(2\sqrt{1-\frac{c^2}{c_1^2}}-\frac{2ak}{\sqrt{\pi}}(1-\kappa^2)\right)\right]A \\
&\qquad\qquad +\left[\left(2-\frac{c^2}{c_2^2}\right)-\frac{2ak}{\sqrt{\pi}}(1-\kappa^2)\right]B=0\ , \\
&\left[\rho c_2^2k^2\left(2-\frac{c^2}{c_2^2}\right)\right]A \\
&\qquad\qquad +\left[2\rho c_2^2 ik^2\sqrt{1-\frac{c^2}{c_2^2}}\right]B=P\ ,
\end{aligned}
\tag{3.165}
$$

Now we need to express A through B in $(3.165)_1$ and substitute it into $(3.165)_2$ to find B as

$$
B=\frac{-2iP\left[2\sqrt{\pi}\theta(1-\kappa^2)-\sqrt{1-\dfrac{c^2}{c_1^2}}\right]}{\rho c_2^2k^2\left[R(\gamma)+4\sqrt{\pi}\theta(1-\kappa^2)\gamma^2\sqrt{1-\gamma^2}\right]}\ , \tag{3.166}
$$

where $\theta=\frac{a}{\ell}=\frac{ak}{2\pi}\ll 1$ is a small parameter, $\gamma=\frac{c}{c_2}$ is the normalised dimensionless phase velocity and $R(\gamma)$ is the classical Rayleigh denominator, i.e.,

$$
R(\gamma)=(2-\gamma^2)^2-4\sqrt{(1-\gamma^2)(1-\kappa^2\gamma^2)}\ . \tag{3.167}
$$

In order to find critical wave speed, denoted by γ_R, we should equate the denominator of (3.166) to zero, i.e.,

$$
R(\gamma)+4\sqrt{\pi}\theta(1-\kappa^2)\gamma^2\sqrt{1-\gamma^2}=0\ , \tag{3.168}
$$

where in the case of classical elasticity (i.e., when the small parameter $\theta=0$) we obtain

$$
R(\gamma)=0\ , \tag{3.169}
$$

resulting in the critical speed $c=c_R$.

As above, we may expand γ in an asymptotic series in terms of the small parameter θ, i.e.,

$$
\gamma=\gamma_0+\theta\gamma_1+\dots\ . \tag{3.170}
$$

Then the Taylor series expansion of $R(\gamma)$ about $\gamma = \gamma_0$ is given by

$$R(\gamma) = R(\gamma_0) + R'(\gamma_0)(\gamma - \gamma_0) + \dots\ , \tag{3.171}$$

where γ_0 is the normalised classical Rayleigh phase velocity ($R(\gamma_0) = 0$, $\gamma_0 = \frac{c_R}{c_2}$). On substituting (3.170) and (3.171) into (3.168), we now obtain the expression for the first order term in the asymptotic expansion (3.170)

$$\gamma_1 = \frac{4\sqrt{\pi}(\kappa^2 - 1)\gamma_0^2\sqrt{1 - \gamma_0^2}}{R'(\gamma_0)}\ , \tag{3.172}$$

and using the result above in (3.170), we can find the critical speed $c = c_2\gamma$, where

$$\gamma = \gamma_0 + \theta\frac{4\sqrt{\pi}(\kappa^2 - 1)\gamma_0^2\sqrt{1 - \gamma_0^2}}{R'(\gamma_0)} + \dots\ , \tag{3.173}$$

with $\theta = \frac{ak}{2\pi}$, as before.

Now we are also in a position to determine A and B from (3.165)in the form

$$\begin{aligned} A &= \frac{P\left[(2 - \gamma^2) - 4\sqrt{\pi}\theta(1 - \kappa^2)\sqrt{1 - \gamma^2}\right]}{\rho c_2^2 k^2\left[R(\gamma) + \theta D(\gamma)\right]}\ , \\ B &= \frac{2iP\left[\sqrt{1 - \kappa^2\gamma^2} - 2\sqrt{\pi}\theta(1 - \kappa^2)\right]}{\rho c_2^2 k^2\left[R(\gamma) + \theta D(\gamma)\right]}\ , \end{aligned} \tag{3.174}$$

where

$$D(\gamma) = 4\sqrt{\pi}(1 - \kappa^2)\gamma^2\sqrt{1 - \gamma^2}\ . \tag{3.175}$$

Next, we express the transverse displacement u_3 on the surface of the half-space ($x_3 = 0$). To this end, first substitute (3.174) into $(2.10)_2$ to obtain

$$\begin{aligned} u_3|_{x_3=0} = &\frac{P\left\{4\sqrt{\pi}\theta(1 - \kappa^2)\left[\sqrt{(1 - \kappa^2\gamma^2)(1 - \gamma^2)} - 1\right] + \gamma^2\sqrt{1 - \kappa^2\gamma^2}\right\}}{\rho c_2^2 k\left[R(\gamma) + \theta D(\gamma)\right]} \\ &\times \exp\left[x_1 - ct\right]\ . \end{aligned} \tag{3.176}$$

We then rewrite this as

$$u_3|_{x_3=0} = \frac{Pa}{2\pi\rho c_2^2}M(\gamma, \theta)\exp\left[x_1 - ct\right]\ , \tag{3.177}$$

where the magnitude is

$$M(\gamma, \theta) = \frac{N(\gamma, \theta)}{\theta\,[R(\gamma) + \theta D(\gamma)]}\ , \tag{3.178}$$

with

$$N(\gamma, \theta) = 4\sqrt{\pi}\theta(1-\kappa^2)\left[\sqrt{(1-\kappa^2\gamma^2)(1-\gamma^2)} - 1\right] + \gamma^2\sqrt{1-\kappa^2\gamma^2}\ , \tag{3.179}$$

and, as previously, $R(\gamma)$ and $D(\gamma)$ given by (3.171) and (3.175), respectively.

Numerical results are presented in Figures 3.11 and 3.12. The magnitude, $M(\gamma, \theta)$, of the vertical displacement $u_3|_{x_3=0}$ on the surface of the half-space is plotted as function of the normalised speed $\gamma = \frac{c}{c_2}$ for different values of Poisson ratio ν and small scale nonlocal parameter θ.

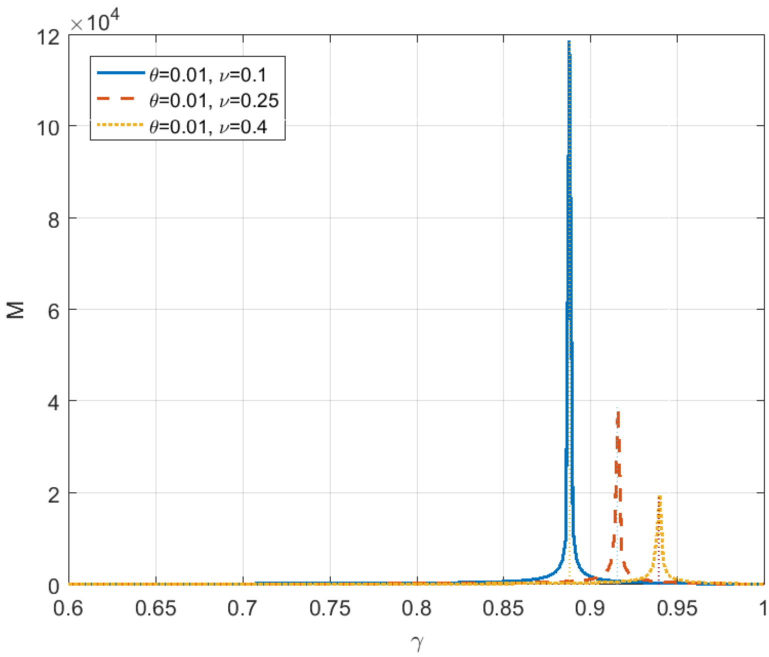

Figure 3.11: A. Effect of nonlocal phenomena in a moving load problem; $\theta = 0.01$.

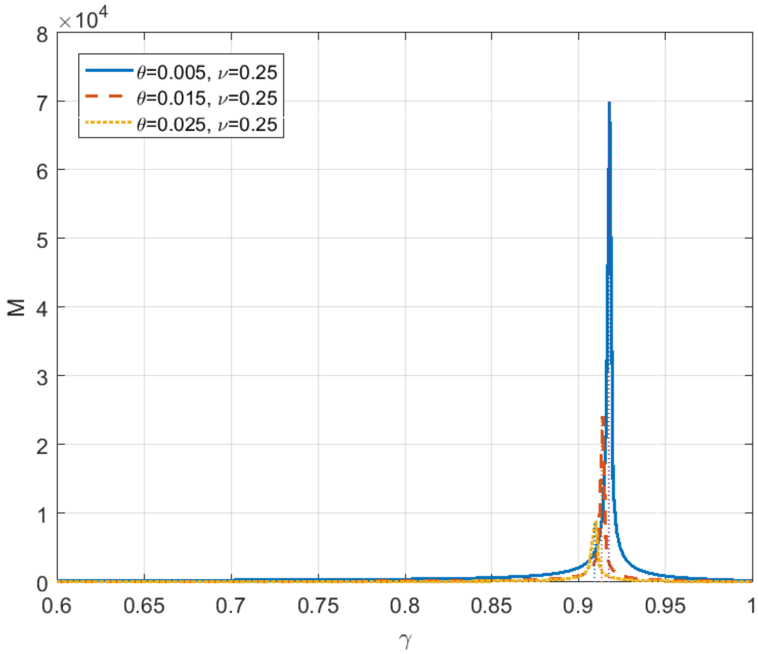

Figure 3.12: B. Effect of nonlocal phenomena in a moving load problem; $\nu = 0.25$

The values of the normalised critical wave speeds (denoted γ_R) can be found by equating the denominator of (3.176) to zero. For $\theta = 0.01$ and different values of Poisson's ratio ν, see Figures 3.11 and 3.12, we obtain the following values: $\gamma_R = 0.8882$ for $\nu = 0.10$ ($\kappa = 0.67$); $\gamma_R = 0.9156$ for $\nu = 0.25$ ($\kappa = 0.58$); and $\gamma_R = 0.9394$ for $\nu = 0.40$ ($\kappa = 0.41$). For $\nu = 0.25$ ($\kappa = 0.58$) and different values of the small scale nonlocal parameter θ, they become: $\gamma_R = 0.9175$ for $\theta = 0.005$; $\gamma_R = 0.9135$ for $\theta = 0.015$; and $\gamma_R = 0.9092$ for $\theta = 0.025$. Note that, for instance, the classical normalised critical wave speed, coinciding with the conventional Rayleigh wave speed, which is a root of the equation (3.169), is equal to 0.9194 for $\nu = 0.25$ ($\kappa = 0.58$). It is obvious that this value may also be obtained from (3.176) by substituting $\theta = 0$. As might be expected, the effect of nonlocal phenomena results in slight deviation of the value of the critical speed from the Rayleigh value.

4 A nonlocal theory for plate bending

In this chapter, the classical Kirchhoff equation for plate bending is refined by introducing a correction to account for nonlocal effects arising from the presence of boundary layers near the plate faces.

4.1 Problem statement and asymptotic scaling

Let us consider an elastic plate of thickness $2h$ with traction-free faces, as shown in Figure 4.1.

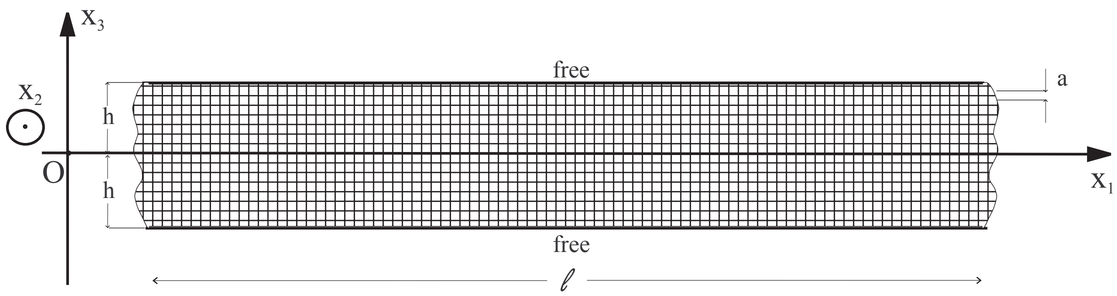

Figure 4.1: A non-locally elastic plate.

We denote x_3 a transverse co-ordinate with the origin coinciding with the mid-plane of the plate, so that the faces of the plate are located at $x_3 = \pm h$. Recalling (2.25) and (2.26), the equations of motion in 3D nonlocal elasticity are given by

$$s_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2} \ , \tag{4.1}$$

where $m, n = 1, 2, 3$, and the constitutive relations for an isotropic material are given by (2.2) and (2.3). The boundary conditions imposed on the traction-free faces are

$$s_{3n} = 0 \quad \text{at } x_3 = \pm h \ . \tag{4.2}$$

Let us assume that the half thickness of the plate, h, is much smaller than a typical wavelength ℓ, and at the same time is much greater then the internal microscale a, i.e., $a \ll h \ll \ell$, see Figure 4.1. In addition, for the sake of definiteness, let us specify a single small geometric parameter given by the following relation

$$\eta = \frac{a}{h} = \frac{h}{\ell} \ll 1 \ . \tag{4.3}$$

As above, see (3.6), this condition can be relaxed. Here we again, as in Chapter 3, express the nonlocal stresses s_{mn} as a sum of its classical counterpart, denoted $p_{mn}(\zeta_p)$, and nonlocal counterpart, $q_{mn}(\zeta_q)$, where the latter originates from the existence of nonlocal boundary layers localised near the plate faces. Such a notation allows to conveniently analyse classical and nonlocal behaviour of the plate separately, see [24].

Next, similarly to Chapter 3, the well-known asymptotic integration method is utilised, e.g., see [50], [52], and [6], in order to derive a refined plate bending equation incorporating a nonlocal correction. A schematic illustration of a plate bending motion is shown in Figure 4.2.

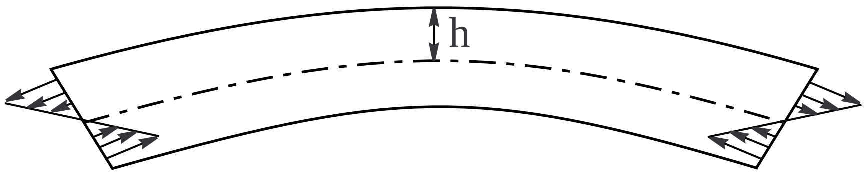

Figure 4.2: Plate bending.

First, let us scale the original variables as

$$x_i = \ell\xi_i \ , \quad x_3 = h\zeta_p = a\zeta_q \ , \quad \text{and} \quad t = \frac{\eta^{-1}\ell}{c_2}\tau \quad (\text{or } \frac{\eta^{-2}h}{c_2}\tau) \ , \tag{4.4}$$

and define the dimensionless quantities, expressing the nonlocal stresses s_{mn} in the dimensionless form as a sum of their classical and nonlocal constituents, p_{mn} and q_{mn}, respectively. We obtain dimensionless displacements

$$u_i = \eta\ell v_i \ , \ u_3 = \ell v_3 \ , \tag{4.5}$$

strains

$$e_{ii} = \eta\varepsilon_{ii}\ ,\ e_{ij} = \eta\varepsilon_{ij}\ ,\ e_{3i} = \eta^2\varepsilon_{3i}\ ,\ e_{33} = \eta\varepsilon_{33}\ , \tag{4.6}$$

and nonlocal stresses

$$\begin{aligned} s_{ii} &= \eta\mu(p_{ii} + q_{ii})\ ,\\ s_{ij} &= \eta\mu(p_{ij} + q_{ij})\ ,\\ s_{3i} &= \eta^2\mu(p_{3i} + \eta q_{3i})\ ,\\ s_{33} &= \eta^3\mu(p_{33} + \eta^2 q_{33})\ , \end{aligned} \tag{4.7}$$

where $i \neq j = 1, 2$, Einstein's summation convention is not employed here, μ is a Lamé constant, and the dimensionless quantities v, ε, p and q are assumed to be of the same asymptotic order. Note again that the novelty of the scaling (4.4) is that ζ_p is a transverse dimensionless variable normalised by the macroscale h, while ζ_q is its analogue normalised by the microscale a, see [24].

Now using (2.31), i.e.,

$$s_{mn}(\boldsymbol{x}) = \frac{1}{a\sqrt{\pi}} \int\limits_{-h}^{h} \exp\left[-\frac{(x_3' - x_3)^2}{a^2}\right] \sigma_{mn}(x_1, x_2, x_3') dx_3'\ , \tag{4.8}$$

and expressing the classical stresses σ_{mn} in terms of strains e_{mn} using (2.2), let us write down non-dimensional equations expressing nonlocal stresses (presented as sums of dimensionless p_{mn} and q_{mn}) in terms of dimensionless strains ε_{mn}. We are now able to obtain

$$
\begin{aligned}
p_{ij} + q_{ij} &= \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij} d\zeta_q' \,, \\
p_{ii} + q_{ii} &= \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{ii} + (\kappa^{-2} - 2)(\varepsilon_{jj} + \varepsilon_{33})\right) d\zeta_q' \,, \\
p_{3i} + \eta q_{3i} &= \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{3i} d\zeta_q' \,, \\
&\eta^2(p_{33} + \eta^2 q_{33}) \\
&= \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{33} + (\kappa^{-2} - 2)(\varepsilon_{ii} + \varepsilon_{jj})\right) d\zeta_q' \,,
\end{aligned}
\tag{4.9}
$$

We may now express the strains e_{mn} in terms of displacements u_n as in (2.3), but using dimensionless form. We have

$$
\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_i}\right) \,, \\
\varepsilon_{ii} &= \frac{\partial v_i}{\partial \xi_i} \,, \\
\eta^2 \varepsilon_{3i} &= \frac{1}{2}\left(\frac{\partial v_3}{\partial \xi_i} + \frac{\partial v_i}{\partial \zeta_p}\right) \,, \\
\eta^2 \varepsilon_{33} &= \frac{\partial v_3}{\partial \zeta_p} \,.
\end{aligned}
\tag{4.10}
$$

The nonlocal equations of motion (4.1) for stresses s_{3i} and s_{33} can be rewritten as

$$
\begin{aligned}
\frac{\partial p_{3i}}{\partial \zeta_p} + \frac{\partial q_{3i}}{\partial \zeta_q} &= -\frac{\partial (p_{ii} + q_{ii})}{\partial \xi_i} - \frac{\partial (p_{ij} + q_{ij})}{\partial \xi_j} + \eta^2 \frac{\partial^2 v_i}{\partial \tau^2} \,, \\
\frac{\partial p_{33}}{\partial \zeta_p} + \eta \frac{\partial q_{33}}{\partial \zeta_q} &= -\frac{\partial (p_{3i} + \eta q_{3i})}{\partial \xi_i} - \frac{\partial (p_{3j} + \eta q_{3j})}{\partial \xi_j} + \frac{\partial^2 v_3}{\partial \tau^2} \,.
\end{aligned}
\tag{4.11}
$$

The boundary conditions (4.2) then become

$$
p_{3i} + \eta q_{3i} = 0 \,, \; p_{33} + \eta^2 q_{33} = 0 \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm \eta^{-1}) \,. \tag{4.12}
$$

4.2 Asymptotic derivation of a 2D plate bending theory

Now we expand all the dimensionless quantities, i.e., the displacements v_n, non-local stress constituents p_{mn} and q_{mn}, and strains ε_{mn} as asymptotic series in terms of the small parameter $\eta = \frac{h}{\ell} = \frac{a}{h}$

$$\begin{pmatrix} v_n \\ \\ p_{mn} \\ q_{mn} \\ \\ \varepsilon_{mn} \end{pmatrix} = \begin{pmatrix} v_n^{(0)} \\ \\ p_{mn}^{(0)} \\ q_{mn}^{(0)} \\ \\ \varepsilon_{mn}^{(0)} \end{pmatrix} + \eta \begin{pmatrix} v_n^{(1)} \\ \\ p_{mn}^{(1)} \\ q_{mn}^{(1)} \\ \\ \varepsilon_{mn}^{(1)} \end{pmatrix} + \ldots \tag{4.13}$$

Substitution of these expansions into equations $(4.9)_{1,2,4}$, (4.10), (4.11), and the boundary conditions (4.12), results, at leading order, in the following set of equations

$$
\begin{aligned}
&\frac{\partial v_3^{(0)}}{\partial \zeta_p} = 0 \, , \\
&\frac{\partial v_i^{(0)}}{\partial \zeta_p} = -\frac{\partial v_3^{(0)}}{\partial \xi_i} \, , \\
&\varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^{(0)}}{\partial \xi_j} + \frac{\partial v_j^{(0)}}{\partial \xi_i}\right) \, , \\
&\varepsilon_{ii}^{(0)} = \frac{\partial v_i^{(0)}}{\partial \xi_i} \, , \\
&\varepsilon_{33}^{(0)} = -(1-2\kappa^2)(\varepsilon_{ii}^{(0)} + \varepsilon_{jj}^{(0)}) \, , \\
&p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij}^{(0)} d\zeta_q' \, , \\
&p_{ii}^{(0)} + q_{ii}^{(0)} \\
&\qquad = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{ii}^{(0)} + (\kappa^{-2}-2)(\varepsilon_{jj}^{(0)} + \varepsilon_{33}^{(0)})\right) d\zeta_q' \, , \\
&\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(0)} + q_{ii}^{(0)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(0)} + q_{ij}^{(0)})}{\partial \xi_j} \, , \\
&\frac{\partial p_{33}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{3i}^{(0)}}{\partial \xi_i} - \frac{\partial p_{3j}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_3^{(0)}}{\partial \tau^2} \, ,
\end{aligned}
\tag{4.14}
$$

and the boundary conditions

$$
p_{3i}^{(0)} = 0 \, , \; p_{33}^{(0)} = 0 \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm \eta^{-1}) \, . \tag{4.15}
$$

Upon integrating of the equations $(4.14)_{1,2}$ with respect to ζ_p, then using the obtained results in $(4.14)_{3-5}$, we arrive at

$$\begin{aligned} &v_3^{(0)} = w_3^{(0)}, \quad v_i^{(0)} = -\zeta_p \frac{\partial w_3^{(0)}}{\partial \xi_i} \,, \\ &\varepsilon_{ij}^{(0)} = -\zeta_p \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j}, \quad \varepsilon_{ii}^{(0)} = -\zeta_p \frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2}, \quad \varepsilon_{33}^{(0)} = (1-2\kappa^2)\zeta_p \Delta_\xi w_3^{(0)} \,, \end{aligned} \tag{4.16}$$

where $w_3^{(0)} = w_3^{(0)}(\xi_i, \xi_j, \tau)$ is an arbitrary function that does not depend on the transverse co-ordinate ζ_p (or ζ_q); and $\Delta_\xi = \frac{\partial^2}{\partial \xi_i^2} + \frac{\partial^2}{\partial \xi_j^2}$ (i.e., a 2D Laplacian with respect to dimensionless quantities ξ_i and ξ_j).

Equation $(4.14)_6$ then becomes,

$$p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2\eta}{\sqrt{\pi}} \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j} \int_{-\eta^{-1}}^{\eta^{-1}} \zeta_q' \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \,, \tag{4.17}$$

with $\zeta_p = \eta \zeta_q$. Next, on making the substitution $t = \zeta_q' - \zeta_q$ in equation (4.17), and then integrating by parts, at leading order we have

$$p_{ij}^{(0)} + q_{ij}^{(0)} = -\frac{2}{\sqrt{\pi}} \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j} \zeta_p \int_{-\eta^{-1}-\zeta_q}^{\eta^{-1}-\zeta_q} \exp\left[-t^2\right] dt \,. \tag{4.18}$$

We can rewrite the integral in this equation as

$$\begin{aligned} &\int_{-\eta^{-1}-\zeta_q}^{\eta^{-1}-\zeta_q} \exp\left[-t^2\right] dt \\ &\quad = \int_{-\infty}^{\infty} \exp\left[-t^2\right] dt - \int_{\eta^{-1}-\zeta_q}^{\infty} \exp\left[-t^2\right] dt - \int_{-\infty}^{-\eta^{-1}-\zeta_q} \exp\left[-t^2\right] dt \,, \end{aligned} \tag{4.19}$$

and noting that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ (e.g., see [54]), we now obtain

$$p_{ij}^{(0)} + q_{ij}^{(0)} = -\frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j} \zeta_p \left\{2 - \operatorname{erfc}(\eta^{-1} - \zeta_q) - \operatorname{erfc}(\eta^{-1} + \zeta_q)\right\} \,, \tag{4.20}$$

where $\operatorname{erfc}(x)$ is given by (3.29).

In what follows, we consider equation (4.20) and, as previously, separate the stress components p_{mn}, having polynomial variations across the thickness and coinciding with those in the classical plate theory, from the stress components q_{mn}. The latter correspond to boundary layers of width $O(a)$ (where a is the chosen internal microscale size) localised near each of the faces of the plate. Thus, we obtain

$$
\begin{aligned}
p_{ij}^{(0)} &= -2\zeta_p \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j} \ , \\
q_{ij}^{(0)}(\eta\zeta_q) &= \eta\zeta_q \frac{\partial^2 w_3^{(0)}}{\partial \xi_i \partial \xi_j} \left\{ \operatorname{erfc}\left(\eta^{-1} - \zeta_q\right) + \operatorname{erfc}\left(\eta^{-1} + \zeta_q\right) \right\} \ ,
\end{aligned}
\tag{4.21}
$$

reiterating that $\zeta_p = \eta\zeta_q \sim 1$.

Next, equation $(4.14)_7$ becomes

$$
\begin{aligned}
p_{ii}^{(0)} + q_{ii}^{(0)} = -\frac{\eta}{\sqrt{\pi}} & \left(4(1-\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + 2(1-2\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) \\
& \times \int_{-\eta^{-1}}^{\eta^{-1}} \zeta_q' \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \ .
\end{aligned}
\tag{4.22}
$$

Following a similar scheme to that used to derive equations (4.21) (yielding exactly the same integral as in (4.17)) and as before splitting the dimensionless stresses into $p_{ii}^{(0)}$ and $q_{ii}^{(0)}$, we have

$$
\begin{aligned}
p_{ii}^{(0)} &= -\zeta_p \left(4(1-\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + 2(1-2\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) \ , \\
q_{ii}^{(0)}(\eta\zeta_q) &= \frac{\eta\zeta_q}{2} \left(4(1-\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + 2(1-2\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) \\
&\qquad \times \left\{ \operatorname{erfc}\left(\eta^{-1} - \zeta_q\right) + \operatorname{erfc}\left(\eta^{-1} + \zeta_q\right) \right\} \ .
\end{aligned}
\tag{4.23}
$$

Now we separate the p and q stress components in the governing equation $(4.14)_8$. On integrating of the p part of $(4.14)_8$ given by the equation

$$
\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(0)}}{\partial \xi_j} \ , \tag{4.24}
$$

with respect to ζ_p, we obtain

$$
p_{3i}^{(0)} = 2\zeta_p^2 (1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} + C_0 \ , \tag{4.25}
$$

where $C_0 = C_0(\xi_1, \xi_2, \tau)$ is an arbitrary function independent of the transverse coordinate. If we now satisfy the corresponding boundary condition $(4.15)_1$, namely

$$p_{3i}^{(0)} = 0 \quad \text{at } \zeta_p = \pm 1 , \tag{4.26}$$

we can determine the arbitrary function C_0 to conclude that

$$p_{3i}^{(0)} = 2(\zeta_p^2 - 1)(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} . \tag{4.27}$$

Then, using this formula in $(4.14)_9$ and integrating with respect to ζ_p, we obtain

$$p_{33}^{(0)} = \zeta_p \left(2(1 - \frac{\zeta_p^2}{3})(1-\kappa^2)\Delta_\xi^2 w_3^{(0)} + \frac{\partial^2 w_3^{(0)}}{\partial \tau^2} \right) . \tag{4.28}$$

Finally, satisfying the boundary condition $(4.15)_2$, i.e.,

$$p_{33}^{(0)} = 0 \quad \text{at } \zeta_p = \pm 1 , \tag{4.29}$$

we arrive at

$$\frac{4}{3}(1-\kappa^2)\Delta_\xi^2 w_3^{(0)} + \frac{\partial^2 w_3^{(0)}}{\partial \tau^2} = 0 , \tag{4.30}$$

which in terms of the original dimensional variables yields the classical Kirchhoff equation of plate bending

$$D\Delta^2 u_3 + 2\rho h u_{3,tt} = 0 , \tag{4.31}$$

where $D = \frac{8\mu h^3(1-\kappa^2)}{3}$ (or $D = \frac{2Eh^3}{3(1-\nu^2)}$ expressed in terms of Young modulus E and Poisson ratio ν) is the conventional bending stiffness.

In what follows, we also need to calculate $q_{3i}^{(0)}$. Therefore, we now integrate the q part of $(4.14)_8$, i.e., the following equation

$$\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial q_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial q_{ij}^{(0)}}{\partial \xi_j} , \tag{4.32}$$

with respect to ζ_q. On substituting of $q_{ij}^{(0)}$ from (4.21) and $q_{ii}^{(0)}$ from (4.23) into the equation (4.32), we arrive at

$$\frac{\partial q_{3i}^{(0)}(\eta\zeta_q)}{\partial \zeta_q} = -2(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} \eta\zeta_q \left\{ \operatorname{erfc}(\eta^{-1} - \zeta_q) + \operatorname{erfc}(\eta^{-1} + \zeta_q) \right\} , \tag{4.33}$$

and on integrating it with respect to ζ_q we have

$$q_{3i}^{(0)}(\eta\zeta_q) = -2(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} \int_0^{\zeta_q} \eta\zeta_q' \left\{\mathrm{erfc}\left(\eta^{-1}-\zeta_q'\right) + \mathrm{erfc}\left(\eta^{-1}+\zeta_q'\right)\right\} d\zeta_q' \,, \quad (4.34)$$

which yields

$$\begin{aligned} q_{3i}^{(0)} &= -(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} \left\{\eta\left(\zeta_q^2-\eta^{-2}\right)\left[\mathrm{erfc}\left(\eta^{-1}-\zeta_q\right)+\mathrm{erfc}\left(\eta^{-1}+\zeta_q\right)\right]\right. \\ &\quad + \frac{4}{\sqrt{\pi}}\eta^{-1}\int_{\eta^{-1}}^{\infty} e^{-t^2}dt \qquad (4.35) \\ &\quad \left. + \frac{2}{\sqrt{\pi}}\left(\exp\left[-(\eta^{-1}-\zeta_q)^2\right]+\exp\left[-(\eta^{-1}+\zeta_q)^2\right]-2\exp\left[-\eta^{-2}\right]\right)\right. . \end{aligned}$$

Recalling that $\eta^{-1} = \frac{h}{a} \gg 1$, we can neglect exponentially small term $\int_{\eta^{-1}}^{\infty} e^{-t^2}dt \sim \int_{\infty}^{\infty} e^{-t^2}dt = 0$ in (4.35) to obtain the following expression

$$\begin{aligned} q_{3i}^{(0)} &= -(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} \left\{\eta\left(\zeta_q^2-\eta^{-2}\right)\left[\mathrm{erfc}\left(\eta^{-1}-\zeta_q\right)+\mathrm{erfc}\left(\eta^{-1}+\zeta_q\right)\right]\right. \\ &\quad \left. + \frac{2}{\sqrt{\pi}}\left(\exp\left[-\left(\eta^{-1}-\zeta_q\right)^2\right]+\exp\left[-\left(\eta^{-1}+\zeta_q\right)^2\right]\right)\right\} . \end{aligned} \qquad (4.36)$$

Note that the first term in curly braces in the equation (4.36), i.e.,

$$\eta\left(\zeta_q^2-\eta^{-2}\right)\left[\mathrm{erfc}\left(\eta^{-1}-\zeta_q\right)+\mathrm{erfc}\left(\eta^{-1}+\zeta_q\right)\right] ,$$

is of order unity. This can be easily verified by rearranging it as

$$\eta\left(\zeta_q-\eta^{-1}\right)\left(\zeta_q+\eta^{-1}\right)\left[\int_{\eta^{-1}-\zeta_q}^{\infty} e^{-t^2}dt + \int_{\eta^{-1}+\zeta_q}^{\infty} e^{-t^2}dt\right] , \qquad (4.37)$$

where the complementary error function is presented in an integral form. The first integral above takes its maximum value when $\eta^{-1}-\zeta_q \sim 1 \quad \Rightarrow \quad \zeta_q \sim \eta^{-1}-1$ and in this case expression (4.37) can be estimated as $C\left[C + \int_{C\eta^{-1}}^{\infty} e^{-t^2}dt\right]$, where $C \sim 1$ and the integral term is exponentially small as discussed after (4.35). Therefore, the considered expression is of order unity. Similarly, the second integral in (4.37) takes its maximum value when $\eta^{-1} + \zeta_q \sim 1 \quad \Rightarrow \quad \zeta_q \sim 1 - \eta^{-1}$ and the considered expression is again of order unity. Thus, (4.37) is indeed of order unity.

At the next order, the equation set analogous to (4.14) is given by

$$
\begin{aligned}
&\frac{\partial v_3^{(1)}}{\partial \zeta_p} = 0 \, , \\
&\frac{\partial v_i^{(1)}}{\partial \zeta_p} = -\frac{\partial v_3^{(1)}}{\partial \xi_i} \, , \\
&\varepsilon_{ij}^{(1)} = \frac{1}{2}\left(\frac{\partial v_i^{(1)}}{\partial \xi_j} + \frac{\partial v_j^{(1)}}{\partial \xi_i}\right) \, , \\
&\varepsilon_{ii}^{(1)} = \frac{\partial v_i^{(1)}}{\partial \xi_i} \, , \\
&\varepsilon_{33}^{(1)} = -(1-2\kappa^2)(\varepsilon_{ii}^{(1)} + \varepsilon_{jj}^{(1)}) \, , \\
&p_{ij}^{(1)} + q_{ij}^{(1)} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij}^{(1)} d\zeta_q' \, , \\
&p_{ii}^{(1)} + q_{ii}^{(1)} \\
&\qquad = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{ii}^{(1)} + (\kappa^{-2}-2)(\varepsilon_{jj}^{(1)} + \varepsilon_{33}^{(1)})\right) d\zeta_q' \, , \\
&p_{3i}^{(1)} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{3i}^{(1)} d\zeta_q' \, , \\
&\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(1)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(1)} + q_{ii}^{(1)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(1)} + q_{ij}^{(1)})}{\partial \xi_j} \, , \\
&\frac{\partial p_{33}^{(1)}}{\partial \zeta_p} + \frac{\partial q_{33}^{(0)}}{\partial \zeta_q} = -\frac{\partial (p_{3i}^{(1)} + q_{3i}^{(0)})}{\partial \xi_i} - \frac{\partial (p_{3j}^{(1)} + q_{3j}^{(0)})}{\partial \xi_j} + \frac{\partial^2 v_3^{(1)}}{\partial \tau^2} \, ,
\end{aligned}
\tag{4.38}
$$

and the boundary conditions are

$$
p_{3i}^{(1)} = -q_{3i}^{(0)} \, , \; p_{33}^{(1)} = 0 \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm\eta^{-1}) \, . \tag{4.39}
$$

On integrating $(4.38)_{1,2}$, using the same approach as at the leading order, we obtain

$$
\begin{aligned}
&v_3^{(1)} = w_3^{(1)}, \quad v_i^{(1)} = -\zeta_p \frac{\partial w_3^{(1)}}{\partial \xi_i} \,, \\
&\varepsilon_{ij}^{(1)} = -\zeta_p \frac{\partial^2 w_3^{(1)}}{\partial \xi_i \partial \xi_j}, \quad \varepsilon_{ii}^{(1)} = -\zeta_p \frac{\partial^2 w_3^{(1)}}{\partial \xi_i^2}, \quad \varepsilon_{33}^{(1)} = (1-2\kappa^2)\zeta_p \Delta_\xi w_3^{(1)} \,,
\end{aligned}
\tag{4.40}
$$

where $w_3^{(1)} = w_3^{(1)}(\xi_i, \xi_j, \tau)$ is an arbitrary function not dependent on ζ_p or ζ_q, and $\Delta_\xi = \frac{\partial^2}{\partial \xi_i^2} + \frac{\partial^2}{\partial \xi_j^2}$ is a 2D Laplacian in the dimensionless quantities ξ_i and ξ_j.

On making use of (4.40), equation $(4.38)_6$ becomes

$$
p_{ij}^{(1)} + q_{ij}^{(1)} = \frac{2\eta}{\sqrt{\pi}} \frac{\partial^2 w_3^{(1)}}{\partial \xi_i \partial \xi_j} \int_{-\eta^{-1}}^{\eta^{-1}} \zeta_q' \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \,, \tag{4.41}
$$

and a series of transformations similar to those for leading order, after separation of $p_{ij}^{(1)}$ and $q_{ij}^{(1)}$, leads to

$$
\begin{aligned}
&p_{ij}^{(1)} = -2\zeta_p \frac{\partial^2 w_3^{(1)}}{\partial \xi_i \partial \xi_j} \,, \\
&q_{ij}^{(1)}(\eta\zeta_q) = \eta\zeta_q \frac{\partial^2 w_3^{(1)}}{\partial \xi_i \partial \xi_j} \left\{ \operatorname{erfc}(\eta^{-1} - \zeta_q) + \operatorname{erfc}(\eta^{-1} + \zeta_q) \right\} \,,
\end{aligned}
\tag{4.42}
$$

where we recall that $\zeta_p = \eta\zeta_q \sim 1$. Next, equation $(4.38)_7$ takes the form

$$
\begin{aligned}
p_{ii}^{(1)} + q_{ii}^{(1)} = &-\frac{\eta}{\sqrt{\pi}} \left(4(1-\kappa^2) \frac{\partial^2 w_3^{(1)}}{\partial \xi_i^2} + 2(1-2\kappa^2) \frac{\partial^2 w_3^{(1)}}{\partial \xi_j^2} \right) \\
&\times \int_{-\eta^{-1}}^{\eta^{-1}} \zeta_q' \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \,.
\end{aligned}
\tag{4.43}
$$

After separation of $p_{ii}^{(1)}$ and $q_{ii}^{(1)}$, we have

$$
\begin{aligned}
&p_{ii}^{(1)} = -2\zeta_p \left(2(1-\kappa^2) \frac{\partial^2 w_3^{(1)}}{\partial \xi_i^2} + (1-2\kappa^2) \frac{\partial^2 w_3^{(1)}}{\partial \xi_j^2} \right) \,, \\
&q_{ii}^{(1)}(\eta\zeta_q) = \eta\zeta_q \left(2(1-\kappa^2) \frac{\partial^2 w_3^{(1)}}{\partial \xi_i^2} + (1-2\kappa^2) \frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) \\
&\qquad \times \left\{ \operatorname{erfc}(\eta^{-1} - \zeta_q) + \operatorname{erfc}(\eta^{-1} + \zeta_q) \right\} \,.
\end{aligned}
\tag{4.44}
$$

On integrating of the p part of $(4.38)_9$, namely the equation

$$\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(1)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(1)}}{\partial \xi_j} \tag{4.45}$$

with respect to ζ_p, we obtain

$$p_{3i}^{(1)} = \zeta_p^2 2(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(1)}}{\partial \xi_i} + C_1(\xi_1, \xi_2, \tau) \;, \tag{4.46}$$

where $C_1 = C_1(\xi_1, \xi_2, \tau)$ is an arbitrary function independent of the transverse coordinate. After satisfying the boundary condition $(4.39)_1$

$$p_{3i}^{(1)} = -q_{3i}^{(0)} \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm \eta^{-1}) \;, \tag{4.47}$$

we can find C_1 and conclude that

$$p_{3i}^{(1)} = 2(1-\kappa^2)\Delta_\xi \left[\frac{\partial w_3^{(1)}}{\partial \xi_i}(\zeta_p^2 - 1) + \frac{1}{\sqrt{\pi}} \frac{\partial w_3^{(0)}}{\partial \xi_i} \right] \;. \tag{4.48}$$

Using the expression above, we integrate $(4.38)_{10}$ with respect to ζ_p to obtain

$$p_{33}^{(1)} = \zeta_p \left(2(1-\kappa^2)\Delta_\xi^2 \left[(1 - \frac{\zeta_p^2}{3}) w_3^{(1)} + \frac{1}{\sqrt{\pi}} w_3^{(0)} \right] + \frac{\partial^2 w_3^{(1)}}{\partial \tau^2} \right) \;. \tag{4.49}$$

Finally, satisfying the boundary condition $(4.39)_2$

$$p_{33}^{(1)} = 0 \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm \eta^{-1}) \;, \tag{4.50}$$

we establish the equation

$$\frac{4}{3}(1-\kappa^2)\Delta_\xi^2 \left[w_3^{(1)} - \frac{3}{2\sqrt{\pi}} w_3^{(0)} \right] + \frac{\partial^2 w_3^{(1)}}{\partial \tau^2} = 0 \;. \tag{4.51}$$

Multiplying (4.51) by η and adding the resulting formula to the Kirchhoff equation (4.30) results in

$$\frac{4}{3}(1-\kappa^2) \left[1 - \frac{3\eta}{2\sqrt{\pi}} \right] \Delta_\xi^2 W_3 + \frac{\partial^2 W_3}{\partial \tau^2} = 0 \;, \tag{4.52}$$

where $W_3 = w_3^{(0)} + \eta w_3^{(1)}$ is the dimensionless transverse displacement.

In terms of the original variables, we finally have a plate bending equation that takes nonlocality into account, see [24]. This equation is given by

$$D'\Delta^2 u_3 + 2\rho h u_{3,tt} = 0 \ , \tag{4.53}$$

with $u_3 = \ell(w_3^{(0)} + \eta w_3^{(1)})$ and the refined plate bending stiffness D' is given by

$$D' = D\left(1 - \frac{3a}{2h\sqrt{\pi}}\right) \ , \tag{4.54}$$

where we recall that $D = \frac{8\mu h^3(1-\kappa^2)}{3}$ (or $D = \frac{2Eh^3}{3(1-\nu^2)}$) and note that a is an internal characteristic length (for example, lattice parameter or granular distance). Note that the nonlocal bending stiffness D' in equation (4.54), to within higher order terms in η, coincides with that in [123], where the traditional Kirchhoff hypotheses for thin plate theory were adapted. The observed softening effect has the same origin as the decrease of the surface wave speed addressed in Section 3.4

4.3 Numerical results

Here we illustrate the effect of the boundary layers localised near the plate faces by plotting the dimensionless classical and nonlocal stress components p_{mn} and q_{mn} versus the transverse co-ordinate ζ_p. First let us rewrite the p and q components corresponding to the stress s_{ij}, see (4.21), as

$$\begin{aligned} p_{ij}^{(0)} &= 2\frac{\partial^2 w_3^{(0)}}{\partial\xi_i\partial\xi_j}P_{ij} \ , \\ q_{ij}^{(0)} &= 2\frac{\partial^2 w_3^{(0)}}{\partial\xi_i\partial\xi_j}Q_{ij} \ , \end{aligned} \tag{4.55}$$

where

$$P_{ij} = -2\zeta_p \tag{4.56}$$

and

$$Q_{ij} = \eta\zeta_q\left\{\mathrm{erfc}\left(\eta^{-1} - \zeta_q\right) + \mathrm{erfc}\left(\eta^{-1} + \zeta_q\right)\right\} \ . \tag{4.57}$$

Recalling that $\zeta_q = \eta^{-1}\zeta_p$, the expression for Q_{ij} becomes

$$Q_{ij} = \zeta_p \left\{ \operatorname{erfc}\left(\eta^{-1}(1-\zeta_p)\right) + \operatorname{erfc}\left(\eta^{-1}(1+\zeta_p)\right) \right\} \ . \tag{4.58}$$

Similarly, in the case of the stress s_{ii}, see (4.23), we obtain

$$\begin{aligned} p_{ii}^{(0)} &= \left(2(1-\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + (1-2\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) P_{ii} \ , \\ q_{ii}^{(0)} &= \left(2(1-\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_i^2} + (1-2\kappa^2)\frac{\partial^2 w_3^{(0)}}{\partial \xi_j^2} \right) Q_{ii} \ , \end{aligned} \tag{4.59}$$

where

$$P_{ii} = -2\zeta_p \tag{4.60}$$

and

$$Q_{ii} = \eta\zeta_q \left\{ \operatorname{erfc}\left(\eta^{-1} - \zeta_q\right) + \operatorname{erfc}\left(\eta^{-1} + \zeta_q\right) \right\} \ . \tag{4.61}$$

Expression (4.61) after simplifying, based on the same logics as for derivation of Q_{ij} in (4.58), becomes

$$Q_{ii} = \zeta_p \left\{ \operatorname{erfc}\left(\eta^{-1}(1-\zeta_p)\right) + \operatorname{erfc}\left(\eta^{-1}(1+\zeta_p)\right) \right\} \ . \tag{4.62}$$

Below we define $Q' = Q_{ij} = Q_{ii}$. A graphs for Q' versus ζ_p is plotted in Figure 4.3.

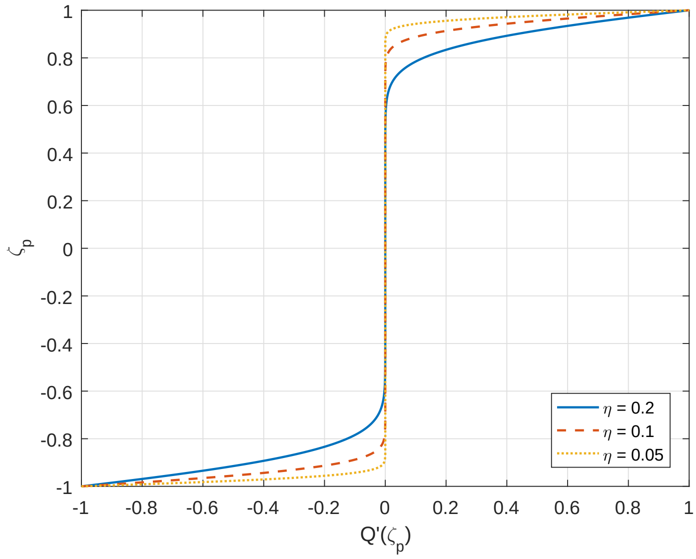

Figure 4.3: The nonlocal component of the stresses s_{ij} and s_{ii} $(i \neq j = 1, 2)$ for plate bending.

It is noted that $Q' \approx \pm 1$ on the plate faces $x_3 = \pm h$ $(\zeta_p = \pm 1)$. Let us now demonstrate the exponential type of attenuation of Q' with distance from one of the plate faces, for example, $x_3 = h$ $(\zeta_p = 1)$. As in the previous chapter, we substitute the complementary error function in Q' above by its asymptotic expansion (3.69) that works for large arguments, keeping the leading order term only and neglecting asymptotically small terms. We obtain

$$Q'_{asymp} = \zeta_p \frac{\exp\left[-(\eta^{-1}(1-\zeta_p))^2\right]}{\sqrt{\pi}(\eta^{-1}(1-\zeta_p))} \,. \tag{4.63}$$

See Figure 4.4 for comparison of approximate (exponential) and original asymptotic (complementary error function) solutions. This asymptotic expansion only works for large argument, which is shown in Figure 4.4 and similar figures in what follows.

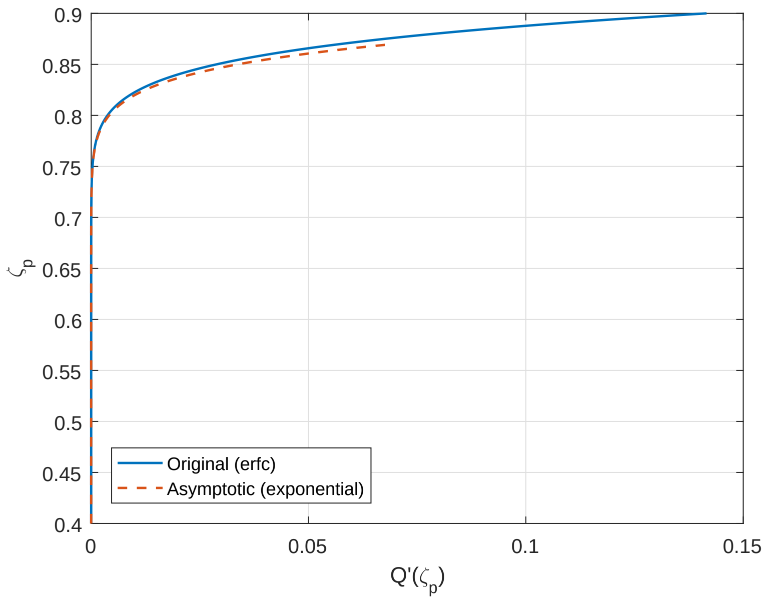

Figure 4.4: Attenuation of Q' in a plate bending problem; $\eta = 0.1$.

Now let us in the same fashion proceed with the p and q components of the stress s_{3i}, see (4.27) and (4.36), thus

$$p_{3i}^{(0)} = 2(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} P_{3i} \ ,$$
$$q_{3i}^{(0)} = 2(1-\kappa^2)\Delta_\xi \frac{\partial w_3^{(0)}}{\partial \xi_i} Q_{3i} \ , \tag{4.64}$$

where

$$P_{3i} = \zeta_p^2 - 1 \tag{4.65}$$

and

$$Q_{3i} = -\frac{1}{\sqrt{\pi}} \left\{ \eta \left(\zeta_q^2 - \eta^{-2}\right) \left[\int_{\eta^{-1}-\zeta_q}^{\infty} e^{-t^2} dt + \int_{-\infty}^{-\eta^{-1}-\zeta_q} e^{-t^2} dt \right] \right.$$
$$\left. + \left(\exp\left[-\left(\eta^{-1} - \zeta_q\right)^2 \right] + \exp\left[-\left(\eta^{-1} + \zeta_q\right)^2 \right] \right) \right\} \ . \tag{4.66}$$

After truncation of $O(\eta)$ terms, we obtain

$$Q_{3i} = \eta^{-1}(1-\zeta_p)\,\mathrm{erfc}\,(\eta^{-1}(1-\zeta_p)) + \eta^{-1}(1+\zeta_p)\,\mathrm{erfc}\,(\eta^{-1}(1+\zeta_p)) \\ -\frac{1}{\sqrt{\pi}}\left(\exp\left[-\left(\eta^{-1}(1-\zeta_p)\right)^2\right] + \exp\left[-\left(\eta^{-1}(1+\zeta_p)\right)^2\right]\right) . \tag{4.67}$$

The graph for Q_{3i} against ζ_p is plotted in Figure 4.5.

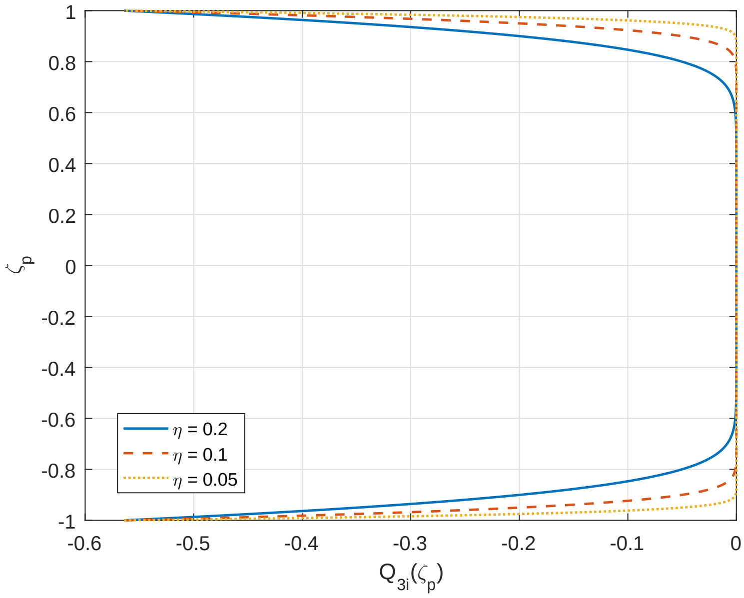

Figure 4.5: The nonlocal part of the stresseses s_{3i} $(i = 1, 2)$ for plate bending.

We note that $P_{3i} = 0$ and $Q_{3i} \approx -\frac{1}{\sqrt{\pi}}$ on the plate faces $x_3 = \pm h$ $(\zeta_p = \pm 1)$. Now we demonstrate the exponential type of attenuation of Q_{3i} with distance from one of the plate faces, for example, $x_3 = h$ $(\zeta_p = 1)$. Substituting the complementary error function in Q_{3i} above by its asymptotic expansion (3.69) for large argument, keeping the terms up to the first order and neglecting asymptotically small terms, we have

$$Q_{3i}^{asymp} = -\frac{\exp\left[-(\eta^{-1}(1-\zeta_p))^2\right]}{2\sqrt{\pi}(\eta^{-1}(1-\zeta_p))^2} , \tag{4.68}$$

see Figure 4.6.

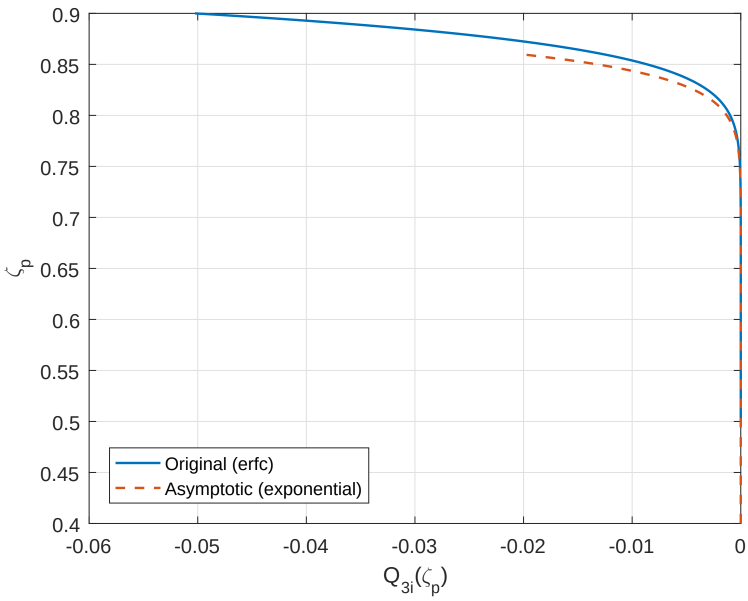

Figure 4.6: Attenuation of Q_{3i} for plate bending; $\eta = 0.1$.

We now calculate with p part of the stress s_{33}, see (4.28). First, from the Kirchhoff equation (4.30) we express the inertial term as

$$\frac{\partial^2 w_3^{(0)}}{\partial \tau^2} = -\frac{4}{3}(1-\kappa^2)\Delta_\xi^2 w_3^{(0)} \ . \tag{4.69}$$

On substituting of the latter into (4.28) we have

$$p_{33}^{(0)} = 2(1-\kappa^2)\Delta_\xi^2 w_3^{(0)} P_{33} \ , \tag{4.70}$$

where

$$P_{33} = \zeta_p \left[\left(1 - \frac{\zeta_p^2}{3}\right) - \frac{2}{3} \right] \ , \tag{4.71}$$

see P_{33} plotted in Figure 4.7.

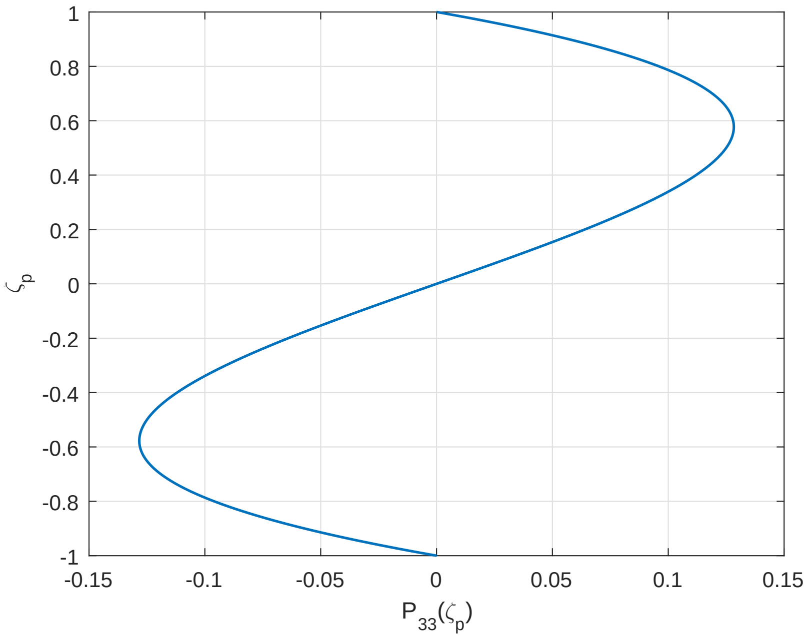

Figure 4.7: The classical part of the stress s_{33} for plate bending.

Note that $P_{33} = 0$ on the plate faces $x_3 = \pm h$ ($\zeta_p = \pm 1$).

Finally, we derive the refined dispersion relation that accounts for nonlocality and compare it with the classical Rayleigh-Lamb equation and its leading order asymptotic long-wave low-frequency approximation. The Rayleigh-Lamb dispersion relation as well as its asymptotic approximation are presented in Section 2.1.2. We obtain a nonlocal dispersion relation for a plane harmonic wave propagating with frequency ω and wave number k from the derived nonlocal plate bending equation (4.53). In order to do this, we adopt a travelling wave solution $U_3 = \exp\left[i(K\xi_1 - \Omega\tau)\right]$ in (4.53), having

$$\frac{4}{3}(1-\kappa^2)K^4\left(1 - \frac{3}{2}\frac{\eta}{\sqrt{\pi}}\right) - \Omega^2 = 0\,, \tag{4.72}$$

where κ is defined by (2.12). Obviously, at $\eta = 0$ this dispersion relation coincides with its classical, 'local' version (2.21) up to $O(\Omega)$ terms.

Numerical results are presented below and include the following curves: fundamental Rayleigh-Lamb antisymmetric modes calculated using the classical transcendental relation (2.20); ‘local’ asymptotic solution, i.e., (2.21) or, equivalently, (4.72) for $\eta = 0$; nonlocal asymptotic solutions (4.72) for $\eta = 0.1$ and $\eta = 0.2$, all plotted for $\nu = 0.3$ in Figure 4.8 and for $\nu = 0.45$ in Figure 4.9.

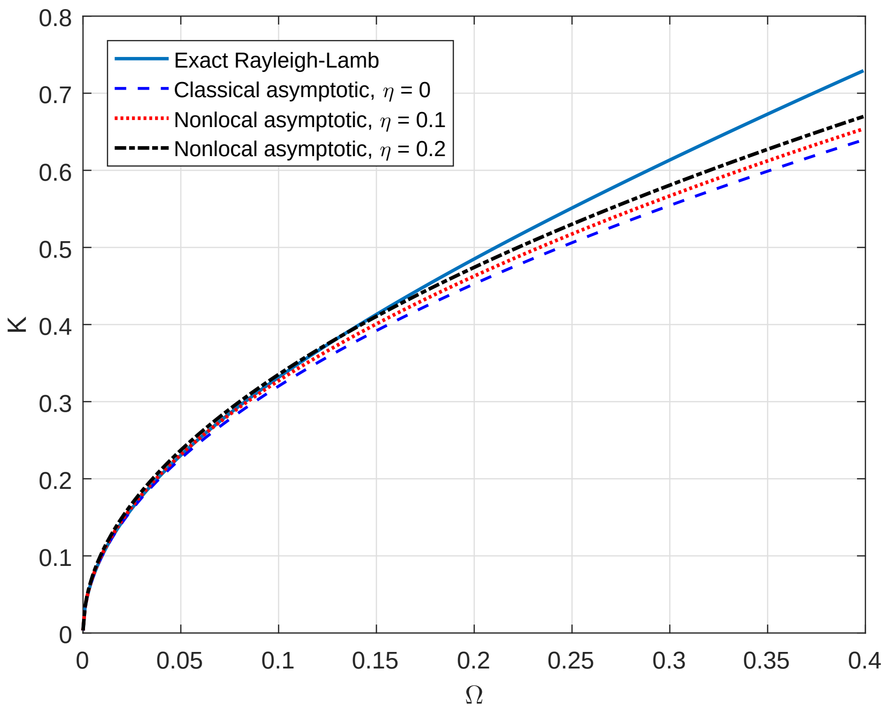

Figure 4.8: Dispersion of bending wave (antisymmetric mode); $\nu = 0.3$.

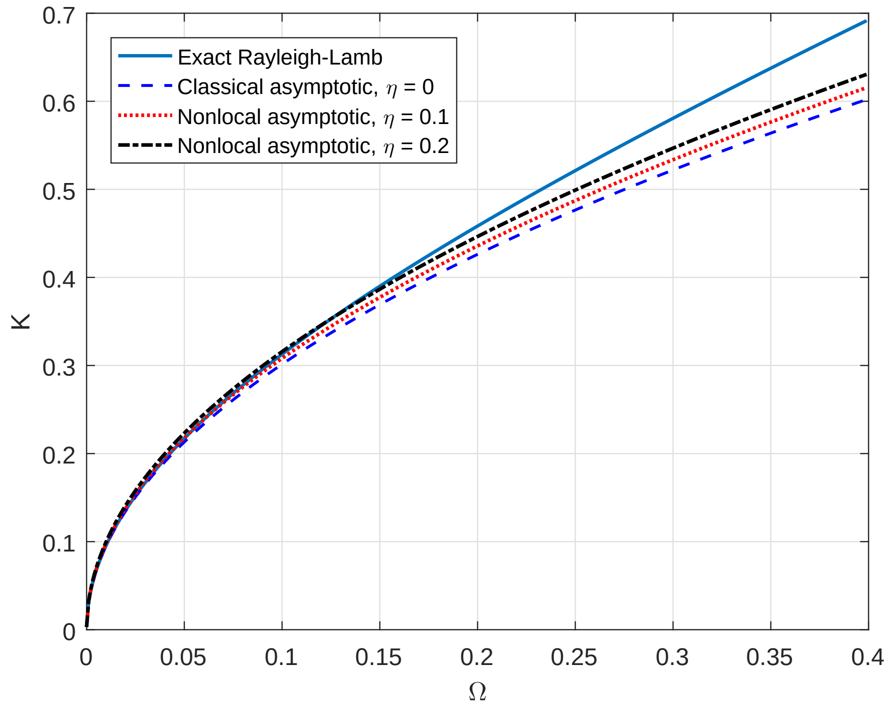

Figure 4.9: Dispersion of bending wave (antisymmetric mode); $\nu = 0.45$.

The curves plotted in these figures confirm that the nonlocal correction to the classical plate bending theory (see equation (4.53)) is meaningful only at relatively low frequencies. The point is that at higher frequencies nonlocal corrections become negligible in comparison to truncations within the classical plate theory. It is also remarkable that the curve corresponding to the nonlocal plate theory (for $\eta = 0.1$, 0.2 in Figures 4.8 and 4.9) intersect with that calculated from the Rayleigh-Lamb dispersion equation.

5 A nonlocal theory for plate extension

In this chapter, the classical plate extension equation is refined by introducing a correction to account for nonlocal effects arising from the presence of boundary layers near plate faces.

5.1 Problem statement and asymptotic scaling

Let us again consider an elastic plate of thickness $2h$ with traction-free faces, see Figure 4.1. As previously, x_3 is a transverse co-ordinate with the origin coinciding with the midplane of the plate, so that the faces of the plate are expressed by $x_3 = \pm h$. Similarly to the plate bending problem considered in the previous chapter, equations of motion in 3D nonlocal elasticity are given by

$$s_{mn,m} = \rho \frac{\partial^2 u_n}{\partial t^2} \ , \tag{5.1}$$

where $m, n = 1, 2, 3$. The constitutive relations for an isotropic material are given by (2.2) and (2.3). Note again that Einstein's summation convention over repeated indices is employed in the equations (5.1) and (2.2).

The boundary conditions imposed on the traction-free faces are

$$s_{3n} = 0 \quad \text{at } x_3 = \pm h \ . \tag{5.2}$$

As in Chapter 4, we assume that the half thickness of the plate, h, is much smaller than a typical wavelength ℓ and much greater then the internal microscale a, yielding $a \ll h \ll \ell$, see Figure 4.1. In addition, we specify a single small geometric parameter as in Chapter 4

$$\eta = \frac{a}{h} = \frac{h}{\ell} \ll 1 \ . \tag{5.3}$$

Here, we again express the nonlocal stresses s_{mn} as a sum of its classical counterpart denoted $p_{mn}(\zeta_p)$ and nonlocal additional component $q_{mn}(\zeta_q)$, see [24].

Next, the same asymptotic integration method as in Chapters 3 and 4 is used (e.g., see [50], [52], and [6]) for derivation of the refined plate extension equation that incorporates a correction to account for nonlocal behaviour of the plate. See Figure 5.1 for a schematic illustration of an extensional motion of the plate.

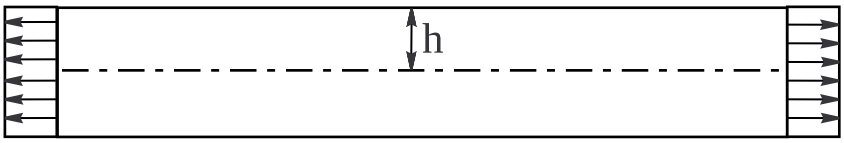

Figure 5.1: Plate extension.

Let us scale the original variables as

$$x_i = \xi_i \ell \ , \quad x_3 = h\zeta_p = a\zeta_q \ , \quad \text{and} \quad t = \frac{\ell}{c_2}\tau = \eta^{-1}\frac{h}{c_2}\tau \ , \tag{5.4}$$

and define the dimensionless displacements and strains as follows

$$u_i = \ell v_i \ , \ u_3 = \eta \ell v_3 \ , \tag{5.5}$$

$$e_{ii} = \varepsilon_{ii} \ , \ e_{ij} = \varepsilon_{ij} \ , \ e_{33} = \varepsilon_{33} \ , \tag{5.6}$$

with the nonlocal stresses

$$\begin{aligned} s_{ii} &= \mu(p_{ii} + q_{ii}) \ , \\ s_{ij} &= \mu(p_{ij} + q_{ij}) \ , \\ s_{3i} &= \eta\mu(p_{3i} + \eta q_{3i}) \ , \\ s_{33} &= \eta^2\mu(p_{33} + \eta^2 q_{33}) \ , \end{aligned} \tag{5.7}$$

and also assuming that

$$\frac{\partial u_i}{\partial x_3} = \eta\gamma_{i3} \ , \tag{5.8}$$

where $i \neq j = 1, 2$, Einstein's summation convention is not employed here, μ is a Lamé constant, dimensionless quantities v, ε, p and q are assumed to be of the same asymptotic order, and γ_{i3} is of order unity.

Using (2.31), i.e.,

$$s_{mn}(\boldsymbol{x}) = \frac{1}{a\sqrt{\pi}} \int\limits_{-h}^{h} \exp\left[-\frac{(x_3' - x_3)^2}{a^2}\right] \sigma_{mn}(x_1, x_2, x_3') dx_3' \ , \tag{5.9}$$

and expressing the classical stresses σ_{mn} in terms of strains e_{mn} as in (2.2), let us write down non-dimensional equations expressing nonlocal stresses (presented as sums of dimensionless p_{mn} and q_{mn}) in terms of strains ε_{mn}. We thus have

$$\begin{aligned}
&p_{ij} + q_{ij} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij} d\zeta_q' \ , \\
&p_{ii} + q_{ii} = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{ii} + (\kappa^{-2} - 2)(\varepsilon_{jj} + \varepsilon_{33})\right) d\zeta_q' \ , \\
&\eta^2(p_{33} + \eta^2 q_{33}) \\
&\qquad = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(\kappa^{-2}\varepsilon_{33} + (\kappa^{-2} - 2)(\varepsilon_{ii} + \varepsilon_{jj})\right) d\zeta_q' \ ,
\end{aligned} \tag{5.10}$$

with κ defined by (2.12), and on substituting ε_{33}, obtained from $(5.10)_3$ into $(5.10)_2$, we establish that

$$\begin{aligned}
&p_{ij} + q_{ij} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij} d\zeta_q' \ , \\
&p_{ii} + q_{ii} = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(4(1 - \kappa^2)\varepsilon_{ii} + 2(1 - 2\kappa^2)\varepsilon_{jj}\right) d\zeta_q' \\
&\qquad\qquad + \eta^2(1 - 2\kappa^2)(p_{33} + \eta^2 q_{33}) \ .
\end{aligned} \tag{5.11}$$

Then we may express the strains e_{mn} in terms of the displacements u_n, as in (2.3), which in dimensionless form yields

$$\begin{aligned}
&\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_i}\right) \ , \\
&\varepsilon_{ii} = \frac{\partial v_i}{\partial \xi_i} \ , \\
&\varepsilon_{33} = \frac{\partial v_3}{\partial \zeta_p} \ ,
\end{aligned} \tag{5.12}$$

with (5.8) becoming

$$\frac{\partial v_i}{\partial \zeta_p} = \eta^2 \gamma_{i3} \; . \tag{5.13}$$

The nonlocal equations of motion (5.1) for stresses s_{3i} and s_{33} can be written as

$$\begin{aligned} &\frac{\partial p_{3i}}{\partial \zeta_p} + \frac{\partial q_{3i}}{\partial \zeta_q} = -\frac{\partial (p_{ii} + q_{ii})}{\partial \xi_i} - \frac{\partial (p_{ij} + q_{ij})}{\partial \xi_j} + \frac{\partial^2 v_i}{\partial \tau^2} \; , \\ &\frac{\partial p_{33}}{\partial \zeta_p} + \eta \frac{\partial q_{33}}{\partial \zeta_q} = -\frac{\partial (p_{3i} + \eta q_{3i})}{\partial \xi_i} - \frac{\partial (p_{3j} + \eta q_{3j})}{\partial \xi_j} + \frac{\partial^2 v_3}{\partial \tau^2} \; . \end{aligned} \tag{5.14}$$

The boundary conditions (5.2) then become

$$p_{3i} + \eta q_{3i} = 0 \; , \; p_{33} + \eta^2 q_{33} = 0 \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm \eta^{-1}) \; . \tag{5.15}$$

5.2 Asymptotic derivation of a 2D plate extension theory

Let us expand all the dimensionless quantities v_n, ε_{mn}, and p_{mn} and q_{mn} as asymptotic series in terms of the small parameter $\eta = \frac{h}{\ell} = \frac{a}{h}$, obtaining

$$\begin{pmatrix} v_n \\ \\ p_{mn} \\ q_{mn} \\ \\ \varepsilon_{mn} \end{pmatrix} = \begin{pmatrix} v_n^{(0)} \\ \\ p_{mn}^{(0)} \\ q_{mn}^{(0)} \\ \\ \varepsilon_{mn}^{(0)} \end{pmatrix} + \eta \begin{pmatrix} v_n^{(1)} \\ \\ p_{mn}^{(1)} \\ q_{mn}^{(1)} \\ \\ \varepsilon_{mn}^{(1)} \end{pmatrix} + \ldots \tag{5.16}$$

Substitution of the expansions (5.16) into equations (5.11), (5.12), (5.13), (5.14), and boundary conditions (5.15) yields, at leading order, the following equations

$$
\begin{aligned}
&\frac{\partial v_i^{(0)}}{\partial \zeta_p} = 0 \, , \\
&\varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^{(0)}}{\partial \xi_j} + \frac{\partial v_j^{(0)}}{\partial \xi_i}\right) \, , \\
&\varepsilon_{ii}^{(0)} = \frac{\partial v_i^{(0)}}{\partial \xi_i} \, , \\
&p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij}^{(0)} d\zeta_q' \, , \\
&p_{ii}^{(0)} + q_{ii}^{(0)} \\
&\qquad = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(4(1-\kappa^2)\varepsilon_{ii}^{(0)} + 2(1-2\kappa^2)\varepsilon_{jj}^{(0)}\right) d\zeta_q' \, , \\
&\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(0)} + q_{ii}^{(0)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(0)} + q_{ij}^{(0)})}{\partial \xi_j} + \frac{\partial^2 v_i^{(0)}}{\partial \tau^2} \, , \\
&\frac{\partial p_{33}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{3i}^{(0)}}{\partial \xi_i} - \frac{\partial p_{3j}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_3^{(0)}}{\partial \tau^2} \, ,
\end{aligned}
\tag{5.17}
$$

accompanied by the boundary conditions

$$
p_{3i}^{(0)} = 0 \, , \; p_{33}^{(0)} = 0 \quad \text{at } \zeta_p = \pm 1 \; (\zeta_q = \pm \eta^{-1}) \, . \tag{5.18}
$$

On integrating the equation $(5.17)_1$ with respect to ζ_p, and then using the obtained result in $(5.17)_{2,3}$, we have

$$
\begin{aligned}
&v_i^{(0)} = w_i^{(0)} \, , \\
&\varepsilon_{ii}^{(0)} = \frac{\partial w_i^{(0)}}{\partial \xi_i}, \quad \varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i}\right) \, ,
\end{aligned}
\tag{5.19}
$$

where $w_i^{(0)} = w_i^{(0)}(\xi_i, \xi_j, \tau)$ is an arbitrary function that does not depend on the transverse co-ordinate (ζ_p or ζ_q); and $\Delta_\xi = \frac{\partial^2}{\partial \xi_i^2} + \frac{\partial^2}{\partial \xi_j^2}$.

Equation $(5.17)_4$ may now be shown to take the form

$$p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{1}{\sqrt{\pi}} \left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i} \right) \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \, . \tag{5.20}$$

Next, on making a substitution $t = \zeta_q' - \zeta_q$ in equation (5.20), at leading order we obtain

$$p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{1}{\sqrt{\pi}} \left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i} \right) \int_{-\eta^{-1}-\zeta_q}^{\eta^{-1}-\zeta_q} \exp\left[-t^2\right] dt \, . \tag{5.21}$$

Now we rewrite the integral in (5.21) as

$$\begin{aligned} &\int_{-\eta^{-1}-\zeta_q}^{\eta^{-1}-\zeta_q} \exp\left[-t^2\right] dt \\ &= \int_{-\infty}^{\infty} \exp\left[-t^2\right] dt - \int_{\eta^{-1}-\zeta_q}^{\infty} \exp\left[-t^2\right] dt - \int_{-\infty}^{-\eta^{-1}-\zeta_q} \exp\left[-t^2\right] dt \, , \end{aligned} \tag{5.22}$$

and making use of the result $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$, e.g., see [54], we readily have

$$p_{ij}^{(0)} + q_{ij}^{(0)} = \frac{1}{2} \left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i} \right) \left\{ 2 - \mathrm{erfc}\left(\eta^{-1} - \zeta_q\right) - \mathrm{erfc}\left(\eta^{-1} + \zeta_q\right) \right\} \, , \tag{5.23}$$

where, as previously, $\mathrm{erfc}(x)$ is given by (3.29).

Considering equation (5.23), we may write down separately the classical stress components $p_{ij}^{(0)}$ and the nonlocal stress components $q_{ij}^{(0)}$ (using the same methods as in the previous chapter), now resulting in

$$\begin{aligned} p_{ij}^{(0)} &= \frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i} \, , \\ q_{ij}^{(0)} &= -\frac{1}{2} \left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i} \right) \left\{ \mathrm{erfc}\left(\eta^{-1} - \zeta_q\right) + \mathrm{erfc}\left(\eta^{-1} + \zeta_q\right) \right\} \, . \end{aligned} \tag{5.24}$$

Next, equation $(5.17)_5$ becomes

$$\begin{aligned} p_{ii}^{(0)} + q_{ii}^{(0)} = \frac{1}{\sqrt{\pi}} &\left(4(1-\kappa^2) \frac{\partial w_i^{(0)}}{\partial \xi_i} + 2(1-2\kappa^2) \frac{\partial w_j^{(0)}}{\partial \xi_j} \right) \\ &\times \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \, . \end{aligned} \tag{5.25}$$

Then, using a similar scheme to that used to derive (5.24) (therefore, obtaining the same integral as in (5.20)) and splitting $p_{ii}^{(0)}$ and $q_{ii}^{(0)}$ as before, we arrive at

$$p_{ii}^{(0)} = 4(1-\kappa^2)\frac{\partial w_i^{(0)}}{\partial \xi_i} + 2(1-2\kappa^2)\frac{\partial w_j^{(0)}}{\partial \xi_j} \ ,$$

$$q_{ii}^{(0)} = -\left(2(1-\kappa^2)\frac{\partial w_i^{(0)}}{\partial \xi_i} + (1-2\kappa^2)\frac{\partial w_j^{(0)}}{\partial \xi_j}\right) \tag{5.26}$$

$$\times \left\{\operatorname{erfc}\left(\eta^{-1}-\zeta_q\right) + \operatorname{erfc}\left(\eta^{-1}+\zeta_q\right)\right\} \ .$$

We now separate p and q components in the governing equation $(5.17)_6$. The p part of $(5.17)_6$ becomes

$$\frac{\partial p_{3i}^{(0)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(0)}}{\partial \xi_j} + \frac{\partial^2 v_i^{(0)}}{\partial \tau^2} \ . \tag{5.27}$$

Note that the inertial term was assumed to appear in the p part as above. On substituting of $p_{ij}^{(0)}$ from (5.24) and $p_{ii}^{(0)}$ from (5.26) into (5.27) and on integrating the resulting equation with respect to ζ_p, we obtain

$$p_{3i}^{(0)} = -\zeta_p\left(4(1-\kappa^2)\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_i^{(0)}}{\partial \xi_j^2} + (3-4\kappa^2)\frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 w_i^{(0)}}{\partial \tau^2}\right) \ . \tag{5.28}$$

On satisfying the corresponding boundary condition $(5.18)_1$

$$p_{3i}^{(0)} = 0 \quad \text{at } \zeta_p = \pm 1 \ (\zeta_q = \eta^{-1}) \ , \tag{5.29}$$

we can write down the following equations

$$4(1-\kappa^2)\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_i^{(0)}}{\partial \xi_j^2} + (3-4\kappa^2)\frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 w_i^{(0)}}{\partial \tau^2} = 0 \ , \tag{5.30}$$

which in the original variables becomes the classical plate extension equation. It can be written in the vector form as

$$\frac{A}{2}\left((1-\nu)\Delta \boldsymbol{u} + (1+\nu)\,\mathrm{grad\,div}\,\boldsymbol{u}\right) - 2\rho h \boldsymbol{u}_{tt} = 0 \ , \tag{5.31}$$

where $\boldsymbol{u} = \ell(w_1^{(0)} + \eta w_1^{(1)},\ w_2^{(0)} + \eta w_2^{(1)})$ is a 2D displacement vector, Δ and grad div are 2D operators, and $A = \frac{2Eh}{1-\nu^2}$ is usually referred to as the so-called extensional stiffness.

In what follows, we also need to calculate $q_{3i}^{(0)}$. This can be done by integrating the q part of $(5.17)_6$, i.e., the following equation

$$\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = -\frac{\partial q_{ii}^{(0)}}{\partial \xi_i} - \frac{\partial q_{ij}^{(0)}}{\partial \xi_j}\ , \tag{5.32}$$

with respect to ζ_q. Note that the inertial term does not appear in the q part above as it was already included in the p part (5.27).

On substituting $q_{ij}^{(0)}$ from (5.24) and $q_{ii}^{(0)}$ from (5.26) into equation (5.32), we arrive at

$$\begin{aligned}\frac{\partial q_{3i}^{(0)}}{\partial \zeta_q} = \frac{1}{2}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \xi_j}\right)\right] \\ \times\left\{\operatorname{erfc}(\eta^{-1}-\zeta_q) + \operatorname{erfc}(\eta^{-1}+\zeta_q)\right\}\ .\end{aligned} \tag{5.33}$$

On now integrating with respect to ζ_q we have

$$\begin{aligned}q_{3i}^{(0)} = \frac{1}{2}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j}\right)\right] \\ \times\int_0^{\zeta_q}\left\{\operatorname{erfc}(\eta^{-1}-\zeta_q) + \operatorname{erfc}(\eta^{-1}+\zeta_q)\right\} d\zeta_q'\ ,\end{aligned} \tag{5.34}$$

which then becomes

$$\begin{aligned}q_{3i}^{(0)} = -\frac{1}{2}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j}\right)\right] \\ \times\Bigg\{\left(\eta^{-1}-\zeta_q\right)\operatorname{erfc}(\eta^{-1}-\zeta_q) - \left(\zeta_q+\eta^{-1}\right)\operatorname{erfc}(\eta^{-1}+\zeta_q) \\ +\frac{2\eta^{-1}}{\sqrt{\pi}}\left(\int_{-\infty}^{-\eta^{-1}} e^{-t^2}dt - \int_{\eta^{-1}}^{\infty} e^{-t^2}dt\right) \\ -\frac{1}{\sqrt{\pi}}\left(\exp[-(\eta^{-1}-\zeta_q)^2] - \exp[-(\eta^{-1}+\zeta_q)^2]\right)\Bigg\}\ .\end{aligned} \tag{5.35}$$

If we now neglect exponentially small terms, the following expression is obtained

$$
\begin{aligned}
q_{3i}^{(0)} = -\frac{1}{2}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j}\right)\right] \\
\times \Bigg\{ \left(\eta^{-1}-\zeta_q\right)\operatorname{erfc}\left(\eta^{-1}-\zeta_q\right) - \left(\zeta_q+\eta^{-1}\right)\operatorname{erfc}\left(\eta^{-1}+\zeta_q\right) \\
-\frac{1}{\sqrt{\pi}}\left(\exp\left[-(\eta^{-1}-\zeta_q)^2\right] - \exp\left[-(\eta^{-1}+\zeta_q)^2\right]\right)\Bigg\}\,.
\end{aligned}
\tag{5.36}
$$

Similar analysis of asymptotic orders in (5.36) (as in (4.36)) shows that the term

$$
\left(\eta^{-1}-\zeta_q\right)\operatorname{erfc}\left(\eta^{-1}-\zeta_q\right) - \left(\zeta_q+\eta^{-1}\right)\operatorname{erfc}\left(\eta^{-1}+\zeta_q\right)
$$

is of order unity. This fact can be verified by transforming the expression above into

$$
\left(\eta^{-1}-\zeta_q\right)\int_{\eta^{-1}-\zeta_q}^{\infty} e^{-t^2}dt - \left(\eta^{-1}+\zeta_q\right)\int_{\eta^{-1}+\zeta_q}^{\infty} e^{-t^2}dt\ ,
$$

indicating that both terms take their maximum value when $\eta^{-1} \pm \zeta_q \sim 1 \quad \Rightarrow \quad \zeta_q \sim \pm(1-\eta^{-1})$, therefore substituting $\zeta_q \sim \pm(1-\eta^{-1})$ into the expression under consideration, we conclude that it is indeed of order unity.

At the next order we have

$$
\begin{aligned}
&\frac{\partial v_i^{(1)}}{\partial \zeta_p} = 0 \ , \\
&\varepsilon_{ij}^{(0)} = \frac{1}{2}\left(\frac{\partial v_i^{(1)}}{\partial \xi_j} + \frac{\partial v_j^{(1)}}{\partial \xi_i}\right) \ , \\
&\varepsilon_{ii}^{(0)} = \frac{\partial v_i^{(1)}}{\partial \xi_i} \ , \\
&p_{ij}^{(1)} + q_{ij}^{(1)} = \frac{2}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \varepsilon_{ij}^{(1)} d\zeta_q' \ , \\
&p_{ii}^{(1)} + q_{ii}^{(1)} \\
&\qquad = \frac{1}{\sqrt{\pi}} \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] \left(4(1-\kappa^2)\varepsilon_{ii}^{(1)} + 2(1-2\kappa^2)\varepsilon_{jj}^{(1)}\right) d\zeta_q' \ , \\
&\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} + \frac{\partial q_{3i}^{(1)}}{\partial \zeta_q} = -\frac{\partial (p_{ii}^{(1)} + q_{ii}^{(1)})}{\partial \xi_i} - \frac{\partial (p_{ij}^{(1)} + q_{ij}^{(1)})}{\partial \xi_j} + \frac{\partial^2 v_i^{(1)}}{\partial \tau^2} \ , \\
&\frac{\partial p_{33}^{(1)}}{\partial \zeta_p} + \frac{\partial q_{33}^{(0)}}{\partial \zeta_q} = -\frac{\partial (p_{3i}^{(1)} + q_{3i}^{(0)})}{\partial \xi_i} - \frac{\partial (p_{3j}^{(1)} + q_{3j}^{(0)})}{\partial \xi_j} + \frac{\partial^2 v_3^{(1)}}{\partial \tau^2} \ ,
\end{aligned}
\tag{5.37}
$$

and the boundary conditions are

$$
p_{3i}^{(1)} = -q_{3i}^{(0)} \ , \ p_{33}^{(1)} = 0 \quad \text{at } \zeta_p = \pm 1 \ (\zeta_q = \pm \eta^{-1}) \ . \tag{5.38}
$$

On integrating the equations (5.37), again using the same approach as previously, we obtain

$$
\begin{aligned}
&v_i^{(1)} = w_i^{(1)}, \\
&\varepsilon_{ij}^{(1)} = \frac{1}{2}\left(\frac{\partial w_i^{(1)}}{\partial \xi_j} + \frac{\partial w_j^{(1)}}{\partial \xi_i}\right), \quad \varepsilon_{ii}^{(1)} = \frac{\partial w_i^{(1)}}{\partial \xi_i} \ ,
\end{aligned}
\tag{5.39}
$$

where $w_i^{(1)} = w_i^{(1)}(\xi_i, \xi_j, \tau)$ is an arbitrary function not dependent on the transverse co-ordinate ζ_p or ζ_q.

Equation $(5.37)_4$ takes the form

$$p_{ij}^{(1)} + q_{ij}^{(1)} = \frac{1}{\sqrt{\pi}} \left(\frac{\partial w_i^{(1)}}{\partial \xi_j} + \frac{\partial w_j^{(1)}}{\partial \xi_i} \right) \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \,, \tag{5.40}$$

and a series of transformations, analogous to those at leading order, after separation of $p_{ij}^{(1)}$ and $q_{ij}^{(1)}$, leads to

$$\begin{aligned} p_{ij}^{(1)} &= \frac{\partial w_i^{(1)}}{\partial \xi_j} + \frac{\partial w_j^{(1)}}{\partial \xi_i} \,, \\ q_{ij}^{(1)} &= -\frac{1}{2} \left(\frac{\partial w_i^{(1)}}{\partial \xi_j} + \frac{\partial w_j^{(1)}}{\partial \xi_i} \right) \left\{ \operatorname{erfc}(\eta^{-1} - \zeta_q) + \operatorname{erfc}(\eta^{-1} + \zeta_q) \right\} \,. \end{aligned} \tag{5.41}$$

Next, equation $(5.37)_5$ becomes

$$\begin{aligned} p_{ii}^{(1)} + q_{ii}^{(1)} = \frac{1}{\sqrt{\pi}} &\left(4(1-\kappa^2) \frac{\partial w_i^{(1)}}{\partial \xi_i} + 2(1-2\kappa^2) \frac{\partial w_j^{(1)}}{\partial \xi_j} \right) \\ &\times \int_{-\eta^{-1}}^{\eta^{-1}} \exp\left[-(\zeta_q' - \zeta_q)^2\right] d\zeta_q' \,. \end{aligned} \tag{5.42}$$

Similarly, after separation of $p_{ii}^{(1)}$ and $q_{ii}^{(1)}$, we have

$$\begin{aligned} p_{ii}^{(1)} &= 4(1-\kappa^2) \frac{\partial w_i^{(1)}}{\partial \xi_i} + 2(1-2\kappa^2) \frac{\partial w_j^{(1)}}{\partial \xi_j} \,, \\ q_{ii}^{(1)} &= -\left(2(1-\kappa^2) \frac{\partial w_i^{(1)}}{\partial \xi_i} + (1-2\kappa^2) \frac{\partial w_j^{(1)}}{\partial \xi_j} \right) \\ &\quad \times \left\{ \operatorname{erfc}(\eta^{-1} - \zeta_q) + \operatorname{erfc}(\eta^{-1} + \zeta_q) \right\} \,. \end{aligned} \tag{5.43}$$

Now, let us consider the p part of the governing equation $(5.37)_6$, namely

$$\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} = -\frac{\partial p_{ii}^{(1)}}{\partial \xi_i} - \frac{\partial p_{ij}^{(1)}}{\partial \xi_j} + \frac{\partial^2 v_i^{(1)}}{\partial \tau^2} \,, \tag{5.44}$$

and substitute $p_{ij}^{(1)}$ from (5.41) and $p_{ii}^{(1)}$ from (5.43) into (5.44), resulting in

$$\frac{\partial p_{3i}^{(1)}}{\partial \zeta_p} = -4(1-\kappa^2) \frac{\partial^2 w_i^{(1)}}{\partial \xi_i^2} - \frac{\partial^2 w_i^{(1)}}{\partial \xi_j^2} - (3-4\kappa^2) \frac{\partial^2 w_j^{(1)}}{\partial \xi_i \xi_j} + \frac{\partial^2 w_i^{(1)}}{\partial \tau^2} \,. \tag{5.45}$$

Integrating this equation with respect to ζ_p yields

$$p_{3i}^{(1)} = -\zeta_p \left(4(1-\kappa^2) \frac{\partial^2 w_i^{(1)}}{\partial \xi_i^2} + \frac{\partial^2 w_i^{(1)}}{\partial \xi_j^2} + (3-4\kappa^2) \frac{\partial^2 w_j^{(1)}}{\partial \xi_i \xi_j} - \frac{\partial^2 w_i^{(1)}}{\partial \tau^2} \right) , \tag{5.46}$$

and on satisfying of the corresponding boundary conditions $(5.38)_1$

$$p_{3i}^{(1)} = -q_{3i}^{(0)} \quad \text{at } \zeta_p = \pm 1 \ (\zeta_q = \pm \eta^{-1}) , \tag{5.47}$$

we arrive at the following equation

$$\begin{aligned} &\left[\Delta_\xi w_i^{(1)} + (3-4\kappa^2) \left(\frac{\partial^2 w_i^{(1)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(1)}}{\partial \xi_i \partial \xi_j} \right) - \frac{\partial^2 w_i^{(1)}}{\partial \tau^2} \right] \\ &\qquad - \frac{1}{2\sqrt{\pi}} \left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2) \left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j} \right) \right] = 0 . \end{aligned} \tag{5.48}$$

Multiplying (5.48) by η and adding the resulting formula to the classical plate extension (5.30) yields

$$\left[\Delta_\xi W_i + (3-4\kappa^2) \left(\frac{\partial^2 W_i}{\partial \xi_i^2} + \frac{\partial^2 W_j}{\partial \xi_i \partial \xi_j} \right) \right] \left(1 - \frac{\eta}{2\sqrt{\pi}} \right) - \frac{\partial^2 W_i}{\partial \tau^2} = 0 , \tag{5.49}$$

where $W_i = w_i^{(0)} + \eta w_i^{(1)}$. Expressing this equation in terms of the original variables, we finally arrive at the plate extension equation taking into account nonlocality, see [24]. In the vector form, it is given by

$$\frac{A'}{2} \left((1-\nu) \Delta \boldsymbol{u} + (1+\nu) \operatorname{grad} \operatorname{div} \boldsymbol{u} \right) - 2\rho h \boldsymbol{u}_{tt} = 0 , \tag{5.50}$$

with $\boldsymbol{u} = \ell(w_1^{(0)} + \eta w_1^{(1)}, \ w_2^{(0)} + \eta w_2^{(1)})$ a 2D displacement vector and the refined extensional stiffness of the plate taking the form

$$A' = A \left(1 - \frac{a}{2h\sqrt{\pi}} \right) , \tag{5.51}$$

where $A = \frac{2Eh}{1-\nu^2}$ is the conventional extensional stiffness as before and a is an internal characteristic length (for example, lattice parameter or granular distance). The non-local extensional stiffness A' in the equation (4.54), to within higher order terms in η, coincides with that in [123].

5.3 Numerical results

Similarly to the previous chapters, let us plot p_{mn} (classical) and q_{mn} (nonlocal) stress components versus the transverse coordinate ζ_p. We start with p and q components of stresses s_{ij} and s_{ii}. Note that the classical components p_{ij} and p_{ii} are uniform in the transverse co-ordinate, so we only plot the q components q_{ij} and q_{ii}. To this end, we present q_{ij}, see (5.24), in the following form

$$q_{ij}^{(0)} = \left(\frac{\partial w_i^{(0)}}{\partial \xi_j} + \frac{\partial w_j^{(0)}}{\partial \xi_i} \right) Q_{ij} \ , \tag{5.52}$$

where

$$Q_{ij} = -\frac{1}{2} \left\{ \mathrm{erfc} \left(\eta^{-1}(1 - \zeta_p) \right) + \mathrm{erfc} \left(\eta^{-1}(1 + \zeta_p) \right) \right\} \ . \tag{5.53}$$

After carrying out the same rearrangements for the q part corresponding to the stress s_{ii}, see (5.26), we have

$$q_{ii}^{(0)} = \left(4(1 - \kappa^2) \frac{\partial w_i^{(0)}}{\partial \xi_i} + 2(1 - 2\kappa^2) \frac{\partial w_j^{(0)}}{\partial \xi_j} \right) Q_{ii} \ , \tag{5.54}$$

where

$$Q_{ii} = -\frac{1}{2} \left\{ \mathrm{erfc} \left(\eta^{-1}(1 - \zeta_p) \right) + \mathrm{erfc} \left(\eta^{-1}(1 + \zeta_p) \right) \right\} \ . \tag{5.55}$$

We define $Q' = Q_{ij} = Q_{ii}$, see Figure 5.2.

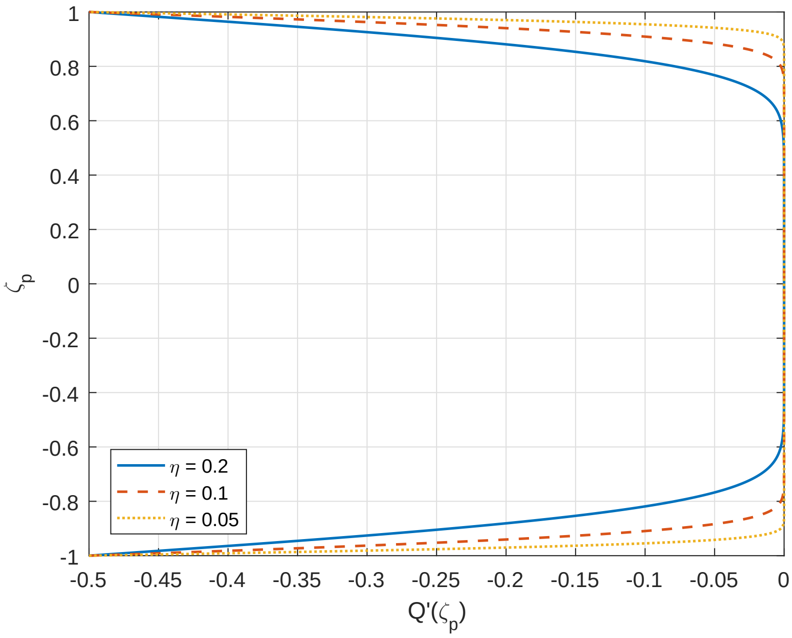

Figure 5.2: The nonlocal component of the stresses s_{ij} and s_{ii} ($i \neq j = 1, 2$) for plate extension.

Note that $Q' \approx -\frac{1}{2}$ on the plate faces $x_3 = \pm h$ ($\zeta_p = \pm 1$). Let us now show the exponential type of attenuation of Q' with distance from one of the plate faces, for example, $x_3 = h$ ($\zeta_p = 1$). As previously, we substitute the complementary error function in Q' by its asymptotic expansion (3.69) for large arguments, keeping only the leading order term and neglecting asymptotically small terms, thus obtaining

$$Q'_{asymp} = -\frac{\exp\left[-(\eta^{-1}(1-\zeta_p))^2\right]}{2\sqrt{\pi}\eta^{-1}(1-\zeta_p)}\,, \tag{5.56}$$

see Figure 5.3.

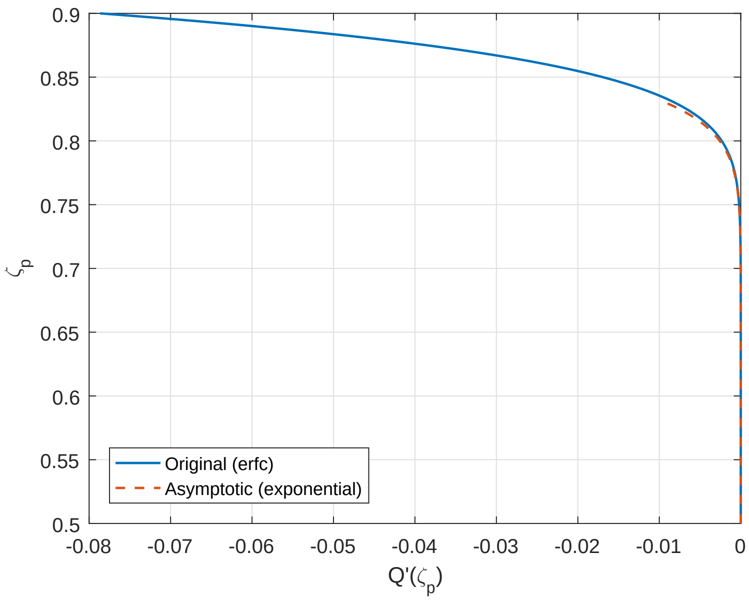

Figure 5.3: Attenuation of Q' for plate extension; $\eta = 0.1$.

Now let us in the same spirit proceed with the p and q components of the stress s_{3i}, see (5.28) and (5.36), from which

$$p_{3i}^{(0)} = \left(4(1-\kappa^2)\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_i^{(0)}}{\partial \xi_j^2} + (3-4\kappa^2)\frac{\partial^2 w_j^{(0)}}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 w_i^{(0)}}{\partial \tau^2}\right) P_{3i} \,,$$
$$q_{3i}^{(0)} = -\frac{1}{2}\left[\Delta_\xi w_i^{(0)} + (3-4\kappa^2)\left(\frac{\partial^2 w_i^{(0)}}{\partial \xi_i^2} + \frac{\partial^2 w_j^{(0)}}{\partial \xi_i \xi_j}\right)\right] Q_{3i} \,, \tag{5.57}$$

where

$$P_{3i} = -\zeta_p \tag{5.58}$$

and

$$Q_{3i} = \eta^{-1}(1-\zeta_p)\,\mathrm{erfc}\,(\eta^{-1}(1-\zeta_p)) - \eta^{-1}(1+\zeta_p)\,\mathrm{erfc}\,(\eta^{-1}(1+\zeta_p))$$
$$-\frac{1}{\sqrt{\pi}}\left(\exp\left[-(\eta^{-1}(1-\zeta_p))^2\right] - \exp\left[-(\eta^{-1}(1+\zeta_p))^2\right]\right) \,. \tag{5.59}$$

The graph for and Q_{3i} versus ζ_p is provided in Figure 5.4.

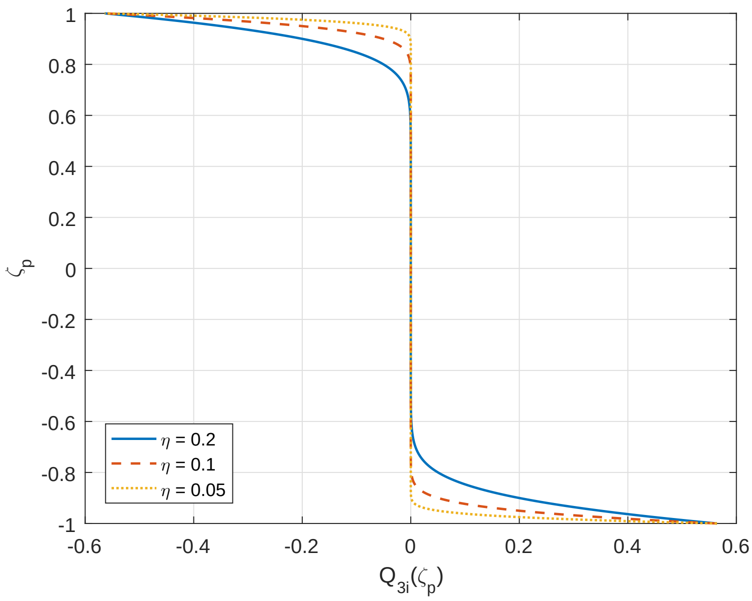

Figure 5.4: The nonlocal part of the stresseses s_{3i} $(i = 1, 2)$ for plate extension.

Note that $P_{3i} = \mp 1$ and $Q_{3i} \approx \pm \frac{1}{\sqrt{\pi}}$ on the plate faces $x_3 = \pm h$ $(\zeta_p = \pm 1)$. Let us now demonstrate the exponential type of attenuation of Q_{3i} with distance from one of the plate faces, for example, $x_3 = h$ $(\zeta_p = 1)$. Substituting the complementary error function in Q_{3i} above by its asymptotic expansion for large arguments using (3.69), keeping the terms up to the first order and neglecting asymptotically small terms, we have

$$Q_{3i}^{asymp} = -\frac{\exp\left[-(\eta^{-1}(1-\zeta_p))^2\right]}{2\sqrt{\pi}(\eta^{-1}(1-\zeta_p))^2} \, . \tag{5.60}$$

We remark that Q_{3i}^{asymp} in (5.60) coincides with that in (4.68) in the previous chapter, see Figure 4.6. Next, we derive the refined dispersion relation that accounts for nonlocality in the case of plate extension and compare it with the classical Rayleigh-Lamb equation and its leading order asymptotic long-wave low-frequency approximation. The Rayleigh-Lamb dispersion relation, as well as its asymptotic approximation, can be found in Section 2.1.2. We obtain a nonlocal dispersion relation from the derived nonlocal plate bending equation (5.50) by adopting a travelling wave solution

$U_3 = \exp\left[i(K\xi_1 - \Omega\tau)\right]$ in (5.50), obtaining

$$4(1-\kappa^2)K^2\left(1-\frac{\eta}{2\sqrt{\pi}}\right) - \Omega^2 = 0\ , \tag{5.61}$$

where $\kappa = \sqrt{\frac{1-2\nu}{2-2\nu}}$. As in the previous chapter, at $\eta = 0$ this dispersion relation coincides with its classical version (2.24) up to $O(\Omega)$ terms.

Numerical results below include the following curves: fundamental Rayleigh-Lamb symmetric modes calculated using the classical transcendental relation (2.23); 'local' asymptotic solution, i.e., (2.24) or, equivalently, (5.61) for $\eta = 0$; nonlocal asymptotic solutions (5.61) for $\eta = 0.1$ and $\eta = 0.2$. All of the graphs plotted for $\nu = 0.3$ in Figure 5.5 and its zoomed-in fragments in Figures 5.6-5.8 and, similarly, for $\nu = 0.45$ in Figure 5.9 and its zoomed-in fragments in Figures 5.10-5.12.

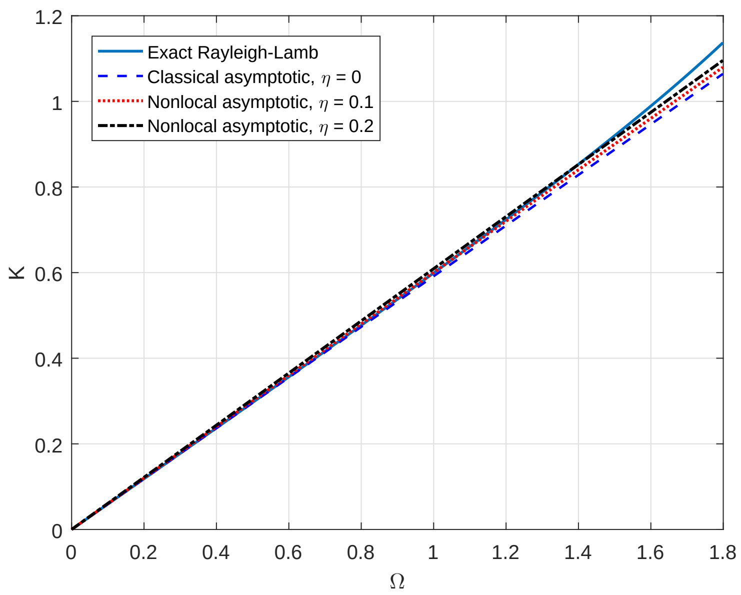

Figure 5.5: Dispersion of extensional wave (symmetric mode); $\nu = 0.3$.

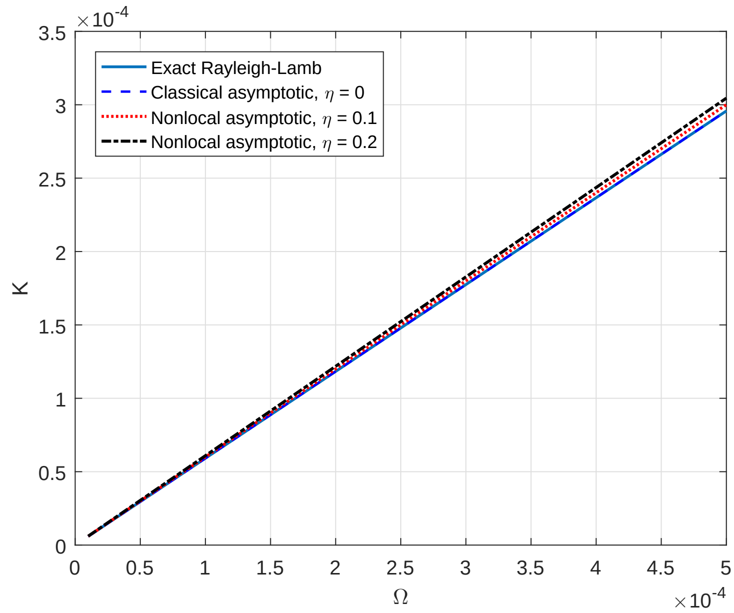

Figure 5.6: Zoomed-in, part 1 (origination) of Figure 5.5.

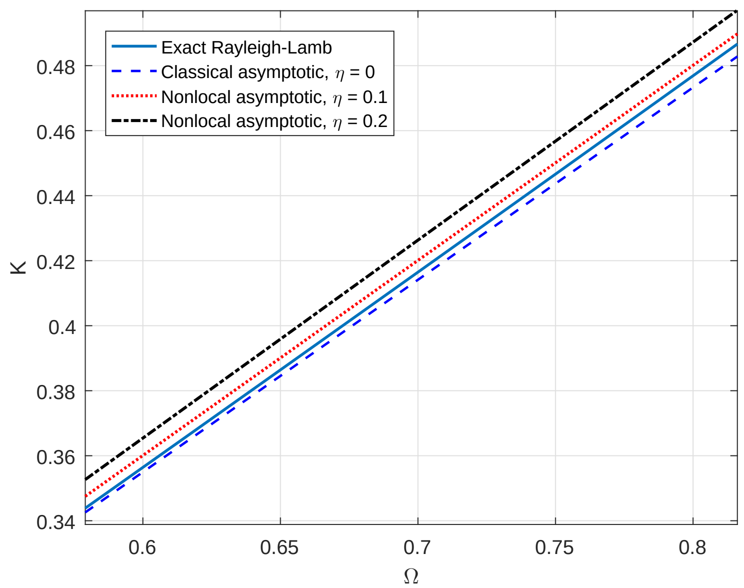

Figure 5.7: Zoomed-in, part 2 (diversion of asymptotic from exact) of Figure 5.5.

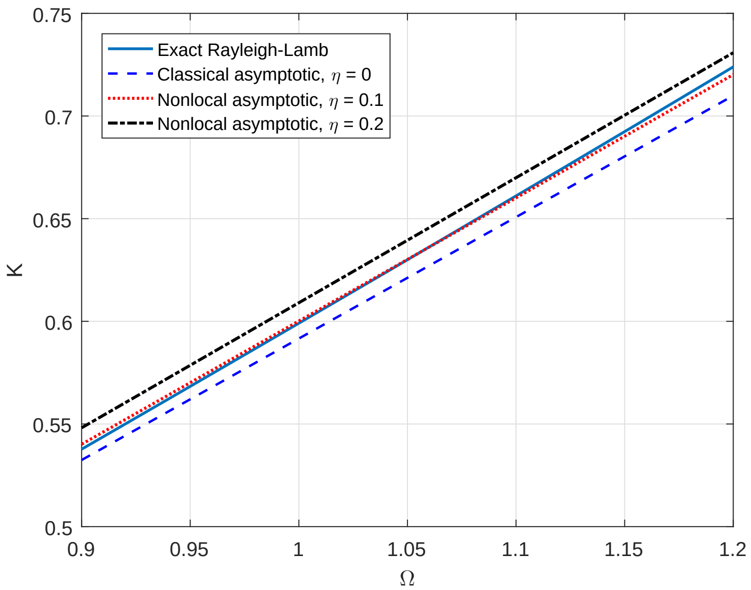

Figure 5.8: Zoomed-in, part 3 (intersection of nonlocal and exact) of Figure 5.5.

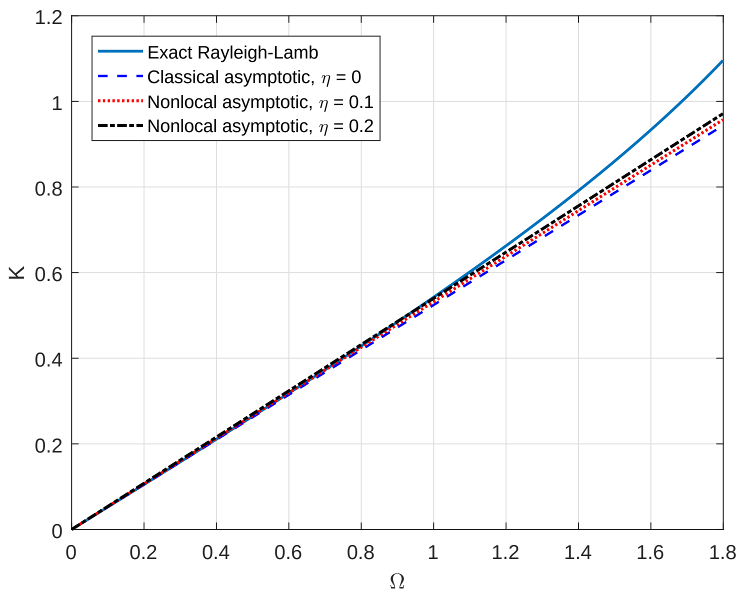

Figure 5.9: Dispersion of extensional wave (symmetric mode); $\nu = 0.45$.

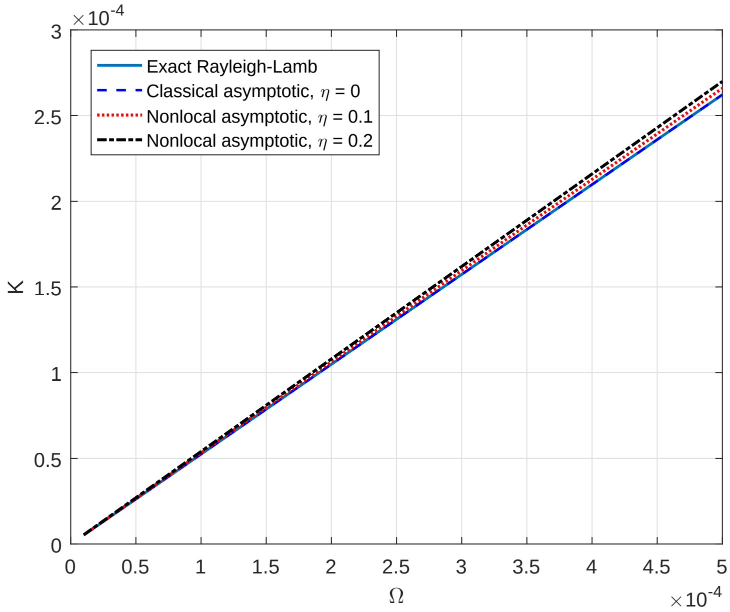

Figure 5.10: Zoomed-in, part 1 (origination) of Figure 5.9.

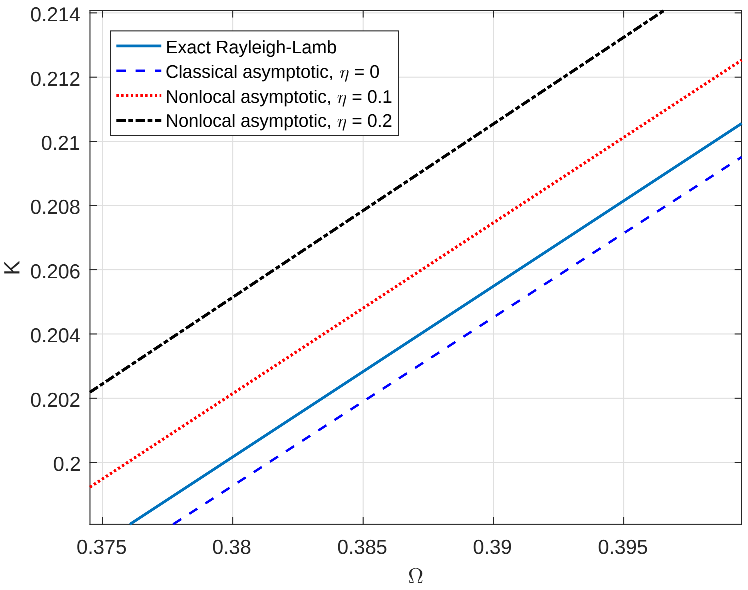

Figure 5.11: Zoomed-in, part 2 (diversion of asymptotic from exact) of Figure 5.9.

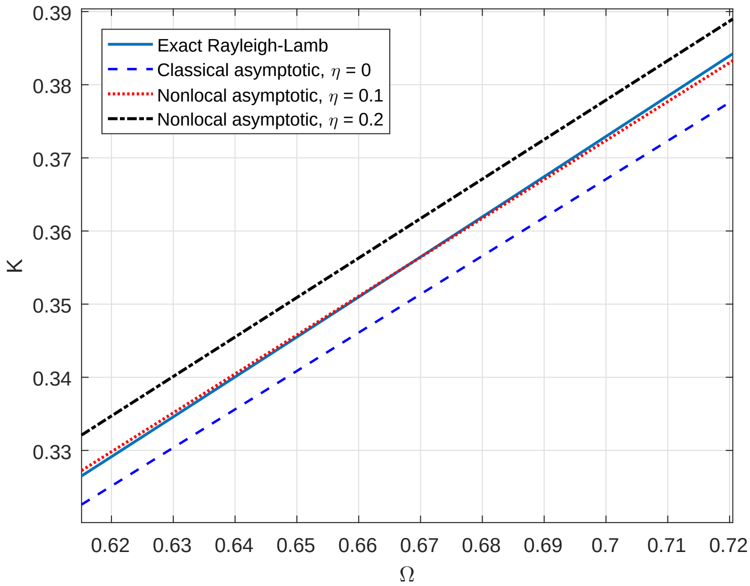

Figure 5.12: Zoomed-in, part 3 (intersection of nonlocal and exact) of Figure 5.9.

The graphs plotted in these figures confirm that the nonlocal correction to the classical plate extensional theory in (5.50) is meaningful only at relatively low frequencies. Again, as for plate bending, this is due to non-accuracy of the adapted leading-order long-wave low-frequency model. It is also of interest that the curve corresponding to the nonlocal plate theory (for $\eta = 0.1$, 0.2 in Figures 5.5 and 5.9) intersect with that calculated using the classical Rayleigh-Lamb dispersion equation, see Figures 5.8 and 5.12.

6 Conclusion

The classical theory of elasticity has been proven to work well for macroscale problems using continuum mechanics, when internal particle interactions are considered to be of contact type (zero range). However, when it is necessary to consider micro- and especially nano-scale, then long-range, cohesive intermolecular forces acting inside the body become significant. Therefore, in this case they should be accounted for, necessitating the introduction of nonlocal elasticity theory (e.g., see Eringen [41]). When an internal characteristic length is introduced and the classic stress tensors in the governing equations replaced by nonlocal stress tensors, which capture the long-range intermolecular forces by employing nonlocal elastic moduli by integrating over the whole volume, it becomes possible to account for long-range forces.

An asymptotic approach to solving nonlocal boundary value problems shows the importance of analysing near-surface behaviour. It has been demonstrated that the effect of a boundary layer near the surface of a half-space can be incorporated just by refining the boundary conditions in classical elasticity. In particular, the effective boundary conditions (3.119) involve an explicit correction to their classical counterparts, arising from taking into account nonlocal phenomena. Similar nonlocal corrections found in the literature (e.g, see [44]) happened to be much smaller than the correction obtained in this work, see remarks in Section 3.4 and 3.3. The linear elastodynamic equations, subject to the derived effective boundary conditions on the free surface of a homogeneous half-space, allow us to determine the interior stress and strain fields outside a narrow near-surface layer of the thickness satisfying the asymptotic inequality (3.152). As an example, an $O(\frac{a}{\ell})$ nonlocal correction to the Rayleigh surface wave speed was calculated, see (3.162) and Figure 3.8. In addition, a nonlocal correction was observed within the similar problem of Rayleigh wave on a surface under a moving load, see (3.173), and the critical Rayleigh wave speeds plotted in Figures 3.11 and 3.12. This correction exceeds the $O(\frac{a^2}{\ell^2})$ correction involved in the nonlocal equations of motion in Eringen [44], see [23] and [74].

The boundary layer near the surface of a half-space is expressed in terms of the complementary error function in relations (3.66) (or (3.68)) and (3.74) and its effect on the stress field is shown in Figures 3.3 and 3.5. We recall that the approximate nature of nonlocal models originates from truncation of homogenisation procedures, including asymptotic homogenisation in periodic structures, e.g., see [125], [100], which underlies the corresponding macroscale relations. Considering this case, the truncation error in the classical boundary conditions should be of the same order as the deviation from the uniform microscale variation of the sought for solution. This microscale variation is expected to be negligibly small comparing to the $O(\frac{a}{\ell})$ correction suggested in the thesis. For instance, it is $O(\frac{a^2}{\ell^2})$ for a range of periodic lattices, e.g., see [26]. This problem is certainly worth a deep analysis.

Another important result of the present work is that the effect of the boundary layers arising due to nonlocal interactions can be incorporated just by modifying the bending and extensional stiffness in the classical equations of plate motion. The nonlocal stiffness D' (for plate bending) and A' (for plate extension) are defined by formulae (4.54) and (5.51), respectively. The boundary layers, expressed in terms of complementary error function, are given by (4.58) (or (4.62)) and (4.5) and their effect on the stresses is demonstrated in Figures 4.3 and 4.5 for plate bending and expressed in (5.53) (or (5.55)) and (5.4) and shown in Figures 5.2 and 5.4 for plate extension, see [24].

The range of validity of equations (4.53) and (5.50) is actually not restricted to the assumed set up of a single small parameter, where $\frac{h}{\ell} \sim \frac{a}{h} \ll 1$. Consequently, the equations are also applicable for $\frac{h^2}{\ell^2} \ll \frac{a}{h} \ll 1$. Note that at $\frac{a}{h} \sim \frac{h^2}{\ell^2}$, the $O(\frac{h^2}{l^2})$ terms characteristic of the asymptotics for Timoshenko-Reissner theories must be kept, e.g., see [52] and [39].

Let us remark that the proposed approach also creates an opportunity for various generalisations and extensions. First of all, it is not restricted to use of exponential kernel (2.28) considered in this work. It might be expected that we would observe a similar nonlocal effect for a range of kernels involving a small microscale parameter and decaying at infinity. The approach can also be applied to the analysis of anisotropic media. In addition, the obtained results may be extended to non-locally elastic solids with a boundary of arbitrary shape. More general boundary conditions may also be considered such as fixed faces or sliding contact.

The general asymptotic scheme utilised in the thesis may potentially have numerous applications outside the field of nonlocal elasticity. For instance, it can be applied to analysis of solids with localised near-surface inhomogeneities such as functionally graded structures, e.g., see a review by Birman & Byrd [19]. This scheme could also be adapted for the long-wave dynamic analysis of vertically inhomogeneous foundations, e.g., see Muravskii [94] and references therein.

Analysis of the dynamic behaviour of elastic waveguides, such as beams and thin elastic shells subjected to the boundary conditions of the form (3.119) imposed on the free faces, would also be of considerable interest. This would seem to be a generalisation of the studied example for the Rayleigh surface wave. Another area of interest for further development could be in asymptotic justification of the nonlocal constitutive relations (2.26) in [44] near plate faces, for example, by homogenising the discrete lattice structure, e.g., see [106].

The discrete models accounting for nonlocal interactions may also support the existence of boundary layers near plate faces similar to the considered kernel normalised over a 3D domain by the expression (2.27). It is worth noting that an alternative approach in [123], based on normalising the nonlocal kernels over the plate thickness, does not predict boundary layers. Moreover, originating from nonlocal differential formulations (see also [102]), they can approximate slowly varying behaviour only.

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