ISOMORPHISMS IN SWITCHING CLASSES OF GRAPHS

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CONTENTS

ABSTRACT of the Thesis "Isomorphisms in Switching Classes of Graphs" by David Harries, University of Keele, 1977.

We introduce notation and terminology to investigate conditions on a permutation group G sufficient to ensure that G fixes a graph in any switching class of graphs that it stabilises. We show that cyclic groups, groups of odd order, groups of order $4k+2$ and all stabilisers of switching classes of graphs on an odd number of vertices have this property. In Chapter 5 we give a necessary and sufficient condition for a dihedral group to have this property.

In Chapter 6 we consider switching classes containing forests and graphs with a given girth $g \geqslant 5$. We give necessary and sufficient conditions for the stabilisers of all such switching classes to fix graphs in their classes.

Finally we give a brief account of the link between strong graphs and switching, and give an example of a class of switching classes with doubly transitive stabilisers.

CHAPTER 1. INTRODUCTION

J. H. Van Lint and J. J. Seidel [3] state the following problem: "Problem. Give, for all n, a survey of the equivalence classes, under complementation, of all n-graphs." We may rewrite their problem in the following way: Give a survey of all switching classes of graphs.

We will present an approach towards the solution of this problem.

We shall observe that a switching class may contain a graph such that all isomorphic copies of that graph lie outside its switching class. We call such graphs representatives of their switching classes. We shall also observe that there are switching classes that have no representative. We consider the problem of characterising switching classes that contain representatives and those that do not, from two points of view. The first approach concerns the stabiliser group of a switching class.

Let G be a permutation group stabilising the switching class $\mathcal{S}(\Gamma)$ of a graph Γ. Although every element of G occurs in the automorphism group of some graph in $\mathcal{S}(\Gamma)$, the group G does not necessarily fix a graph in the class. If it does, we say that G is exposable in $\mathcal{S}(\Gamma)$. We consider the following problem: what conditions on G are sufficient to ensure that G is exposable in all the switching classes that it stabilises? It is implicit in the work of Mallows and Sloane [4] that it is sufficient for G to be cyclic. Further known conditions are that G be of odd order, or of order $4k+2$. In Chapters 3 and 4 we present these results and others using our methods. Our main result, however, is in Chapter

5 where we give necessary and sufficient conditions on the permutation representation of a dihedral group G such that G is exposable in all switching classes that it stabilises.

Our second approach looks at some properties of graphs that are preserved under permutation of the vertices and that may or may not be preserved by switching. In Example 3.8 we give a very simple condition on the number of edges of a graph on an even number of vertices to ensure that such a graph is a representative. In Chapter 6 we study graphs with a given girth and give a complete analysis of switching classes containg graphs of girth $g \geqslant 5$ and forests. That is, we decide which switching classes, containing a graph with the above property, also contain a representative. Finally, in Chapter 7, we consider strong graphs. We give a brief sketch of the connection between strong graphs and switching.

Our aim is to present a general approach for studying isomorphisms in switching classes of graphs, which we apply to obtain the results mentioned.

CHAPTER 2. NOTATION AND PRELIMINARY RESULTS

We consider the collection $\mathcal{G}$ of labelled, undirected graphs on n vertices, without loops and without multiple edges. Let Γ be a graph in $\mathcal{G}$. We label its vertices 1, 2, ..., n, and call the set of labels $\Omega = \{1, 2, \ldots, n\}$.

A graph Γ in $\mathcal{G}$ is described by its <u>adjacency matrix</u> $A = A(\Gamma)$, whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} -1, & \text{if } i \text{ and } j \text{ are adjacent;} \\ 1, & \text{if } i \text{ and } j \text{ are not adjacent, } i \neq j; \\ 0, & \text{if } i = j. \end{cases}$$

Given a permutation π in Σ, the symmetric group on Ω, we define $\pi\Gamma$ to be the labelled graph, such that $\{\pi(i), \pi(j)\}$ is an edge in $\pi\Gamma$ if and only if $\{i, j\}$ is an edge in Γ. We say that Γ and $\pi\Gamma$ are isomorphic. The permutation π is a permutation of the vertices of Γ and can be represented by a permutation matrix $P = (p_{ij})$, where

$$p_{ij} = \begin{cases} 1, & \text{if } \pi(j) = i; \\ 0, & \text{otherwise.} \end{cases}$$

We note that according to the above definitions, PAP^{-1} is the adjacency matrix of $\pi\Gamma$, where A is the adjacency matrix of Γ.

An <u>automorphism</u> of a graph Γ is a permutation π of its vertices such that $\pi\Gamma = \Gamma$. The set of all automorphisms of Γ forms a group denoted by Aut Γ.

We next introduce an $n \times n$ diagonal matrix $S_i = (s_{k\ell i})$ called a <u>switch-matrix</u>, where

$$s_{k\ell i} = \begin{cases} -1, & \text{if } k = \ell = i; \\ 1, & \text{if } k = \ell, \text{ and } k \neq i; \\ 0, & \text{if } k \neq \ell. \end{cases}$$

Let Γ be a graph on n vertices with adjacency matrix A. Then $S_i A S_i^{-1}$ is the adjacency matrix of the graph obtained from Γ by deleting all the edges in Γ that are incident to the vertex i, and adding edges $\{i, j\}$ for all vertices j not adjacent to vertex i.

We call the above operation <u>switching Γ with respect to vertex i</u>, and denote the graph obtained from Γ by $s_i\Gamma$, where s_i is a <u>switch</u> associated with the switch-matrix S_i.

We now establish some simple results to demonstrate the partitioning of $\mathcal{G}$ into disjoint switching classes of graphs.

<u>Lemma 2.1</u>. <u>Switching is a commutative operation</u>.

<u>Proof</u>. This Lemma is established by observing that switch-matrices are diagonal.

A switch with respect to a set of vertices can now be defined unambiguously.

A switch s with respect to a set of vertices $i_1, \ldots, i_r$ is defined to be the composition of switches $s = s_{i_1} \cdots s_{i_r}$. We say that $i_1, \ldots, i_r$ is the support of s and write $\text{supp } s = \{i_1, \ldots, i_r\}$. Note that the switch-matrix S, that s is associated with, is the diagonal matrix $S_{i_1} \cdots S_{i_r}$.

<u>Lemma 2.2</u>. <u>Switching is an equivalence relation on $\mathcal{G}$</u>.

<u>Proof</u>. (i) We prove that Γ is switching equivalent to itself by noting that $s\Gamma = \Gamma$ if and only if $\text{supp } s = \phi$, the empty set, or $\text{supp } s = \Omega$.

(ii) Suppose that Γ is switching equivalent to Γ', then $S A(\Gamma) S^{-1} = A(\Gamma')$ for some switch-matrix S. Since $S^2 = I$, the identity matrix, we have $SA(\Gamma')S^{-1} = A(\Gamma)$. Hence Γ' is switching equivalent to Γ.

(iii) Suppose that Γ is switching equivalent to Γ', and that Γ' is switching equivalent to Γ''. Then

$SA(\Gamma)S = A(\Gamma')$ and $S^*A(\Gamma')S^* = A(\Gamma'')$

for some switch-matrices S and S^*. Consequently

$$S^*SA(\Gamma)SS^* = A(\Gamma'').$$

By (i), (ii) and (iii), switching is an equivalence relation on $\mathcal{G}$.

We have noted that a switch s fixes a graph Γ (that is, $s\Gamma = \Gamma$) if and only if supp $s = \phi$, or supp $s = \Omega$. When supp $s = \phi$, write $s = e$, and when supp $s = \Omega$, write $s = \hat{e}$. Since $s^2 = e$ for all switches s, we can state that $s\Gamma = s'\Gamma$ if and only if $s's = e$ or $s's = \hat{e}$. In the latter case write $s' = \hat{s}$, noting that $\hat{s} = s\hat{e}$, and supp $\hat{s} = \Omega \setminus$ supp s.

Lemma 2.3. *Switching partitions $\mathcal{G}$ into $2^{\frac{1}{2}(n-1)(n-2)}$ disjoint classes, called switching classes, each class containing $2^{(n-1)}$ graphs.*

Proof. By Lemma 2.2, switching partitions $\mathcal{G}$ into disjoint classes. There are 2^n different $n \times n$ diagonal matrices with diagonal entries 1 or -1. However, it is clear from our previous remarks that $SA(\Gamma)S = S'A(\Gamma)S'$, where S and S' are switch-matrices, if and only if either $S' = S$ or $S' = -S$. Therefore the switching class containing Γ contains $2^{(n-1)}$ different graphs. It is well known that $|\mathcal{G}| = 2^{\frac{1}{2}n(n-1)}$. From the above it follows that switching partitions $\mathcal{G}$ into $2^{1/2(n-1)(n-2)}$ disjoint switching classes.

We denote the switching class containing Γ, by $\mathcal{S}(\Gamma)$. The <u>stabiliser</u> of $\mathcal{S}(\Gamma)$ is the group Stab $\mathcal{S}(\Gamma)$ of all permutations in Σ that permute the members of $\mathcal{S}(\Gamma)$ among themselves; that is,

Stab $\mathcal{S}(\Gamma) = \{\pi \in \Sigma \mid \Gamma' \in \mathcal{S}(\Gamma) \Rightarrow \pi\Gamma' \in \mathcal{S}(\Gamma)\}$.

In order to study the stabiliser of a switching class further, we must first establish a relationship between switches with support in Ω and permutations in Σ.

<u>Lemma 2.4.</u> <u>Let P be the permutation matrix corresponding to $\pi \in \Sigma$, and let S be the switch-matrix corresponding to s with supp s $= \{i_1, \ldots, i_r\} \subseteq \Omega$. Then PSP^{-1} is a switch-matrix corresponding to the switch denoted ${}_\pi s = \pi s \pi^{-1}$, with supp ${}_\pi s = \{\pi(i_1), \ldots, \pi(i_r)\} \subseteq \Omega$.</u>

<u>Proof</u>. The proof is clear from the fact that S is the diagonal matrix with diagonal entries $(S)_{i_j i_j} = -1$, $1 \leq j \leq r$, and +1 elsewhere.

The following corollary concerns the manipulation of switches with permutations.

<u>Corollary 2.5. Let s be a switch with supp s $\subseteq \Omega$, and let $\pi \in \Sigma$ be expressed as a product of disjoint cycles.</u>

(i) <u>${}_\pi s = s$ if and only if supp s involves complete cycles of π.</u>

(ii) <u>${}_\pi s = \hat{s}$ if and only if π consists of even length cycles only, and supp s involves alternate symbols from each cycle of π.</u>

Proof. Immediate from Lemma 2.4.

Our next result shows that a necessary and sufficient condition for a permutation to belong to the stabiliser of $\mathcal{S}(\Gamma)$ is that it maps any one graph in $\mathcal{S}(\Gamma)$ to a graph in this class.

Lemma 2.6. $\pi\Gamma \in \mathcal{S}(\Gamma)$ if and only if $\pi \in$ Stab $\mathcal{S}(\Gamma)$.

Proof. Suppose that $\pi\Gamma \in \mathcal{S}(\Gamma)$. Then for some switch s, $\pi\Gamma = s\Gamma$. Now consider an arbitrary switch s′. Then by Lemma 2.4,

$$\pi(s'\Gamma) = \pi s'\pi^{-1}(\pi\Gamma) = {}_{\pi}s'(s\Gamma) = s^{*}\Gamma \in \mathcal{S}(\Gamma)$$

Therefore $\pi \in$ Stab $\mathcal{S}(\Gamma)$. The converse is true by definition.

We are now in a position to state the major problem studied in this thesis.

Under what 'conditions' does a switching class $\mathcal{S}(\Gamma)$ contain a graph Γ', such that Aut Γ' = Stab $\mathcal{S}(\Gamma)$? We shall approach this problem from two points of view.

Our first approach concerns the structure of permutation groups. Let G be a subgroup of Stab $\mathcal{S}(\Gamma)$. Two possibilities arise: either G is a subgroup of the automorphism group of some graph in $\mathcal{S}(\Gamma)$, or there is no graph in $\mathcal{S}(\Gamma)$ fixed by G.

Definition 2.7. Suppose that G is a subgroup of Stab $\mathcal{S}(\Gamma)$. We say that G is exposable in $\mathcal{S}(\Gamma)$ if there is a graph Γ' in $\mathcal{S}(\Gamma)$ such that $G \subseteq$ Aut Γ'. If G is a subgroup of

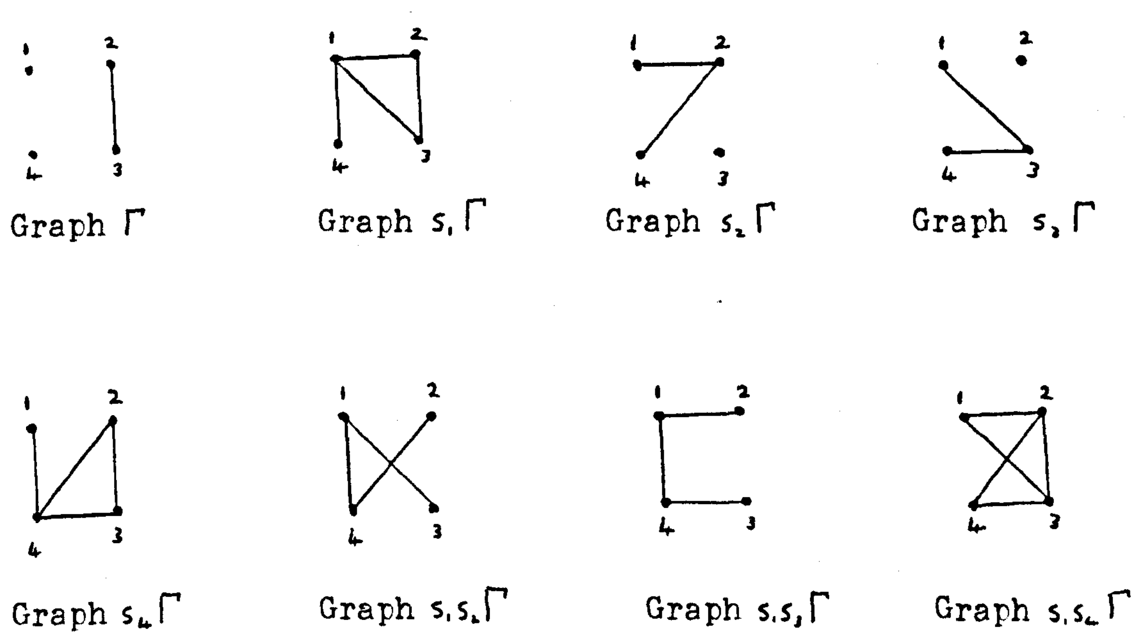
Stab $\mathcal{S}(\Gamma)$ but there is no graph in $\mathcal{S}(\Gamma)$ with the above property, then we say that G is hidden in $\mathcal{S}(\Gamma)$. A permutation group G is always exposable if it is exposable in every switching class that it stabilises.

Our second approach will concern the structure of graphs.

Definition 2.8. Let Γ' be a graph in $\mathcal{S}(\Gamma)$. We say that Γ' is a representative of $\mathcal{S}(\Gamma)$ if Aut Γ' = Stab $\mathcal{S}(\Gamma)$.

From these two definitions it is clear that Stab $\mathcal{S}(\Gamma)$ is exposable in $\mathcal{S}(\Gamma)$ if and only if $\mathcal{S}(\Gamma)$ contains a representative.

Example 2.9.



Graph Γ Graph $s_1\Gamma$ Graph $s_2\Gamma$ Graph $s_3\Gamma$

Graph $s_4\Gamma$ Graph $s_1s_2\Gamma$ Graph $s_1s_3\Gamma$ Graph $s_1s_4\Gamma$

Figure 1

Figure 1 illustrates a complete switching class of graphs on four vertices. There are eight graphs in the class, as Lemma 2.3 demands. Note that

$$s_2 s_3 s_4 \Gamma = s_1 \Gamma, \qquad s_3 s_4 \Gamma = s_1 s_2 \Gamma,$$

and so on. This is a consequence of the fact that $s_1 s_2 s_3 s_4 \Gamma = \Gamma$. It will be found that $\mathcal{G}$, the collection of graphs on four vertices, is the disjoint union of eight switching classes. Six of these classes contain graphs isomorphic to Γ, a further class contains the graph with no edges, called the empty graph on four vertices and denoted N_4, and the final class contains the complete graph, denoted K_4. The graph Γ is a representative of $\mathcal{S}(\Gamma)$, for if it were not, $\mathcal{S}(\Gamma)$ would contain an isomorphic copy of Γ, different to Γ. The graph $s_1 s_4 \Gamma$ is also a representative of $\mathcal{S}(\Gamma)$.

Consequently

Stab $\mathcal{S}(\Gamma) = \{(14)(2)(3),\ (1)(23)(4),\ (14)(23),\ 1\} = \mathrm{Aut}\,\Gamma$.

In Figure 1 the following pairs of graphs are isomorphic:

$$s_1 \Gamma \text{ and } s_4 \Gamma; \quad s_2 \Gamma \text{ and } s_3 \Gamma; \quad s_1 s_2 \Gamma \text{ and } s_1 s_3 \Gamma.$$

For example, let $\pi = (1)(23)(4)$.

Then we have the following algebraic verifications:

$$\pi(s_2 \Gamma) = s_3 \Gamma,$$

$$\Longleftrightarrow \quad \pi s_2 \pi^{-1}(\pi \Gamma) = s_3 \Gamma,$$

$$\Longleftrightarrow \quad {}_{\pi}s_2(\pi \Gamma) = s_3 \Gamma,$$

$$\Longleftrightarrow \quad \pi \Gamma = {}_{\pi}s_2\, s_3 \Gamma,$$

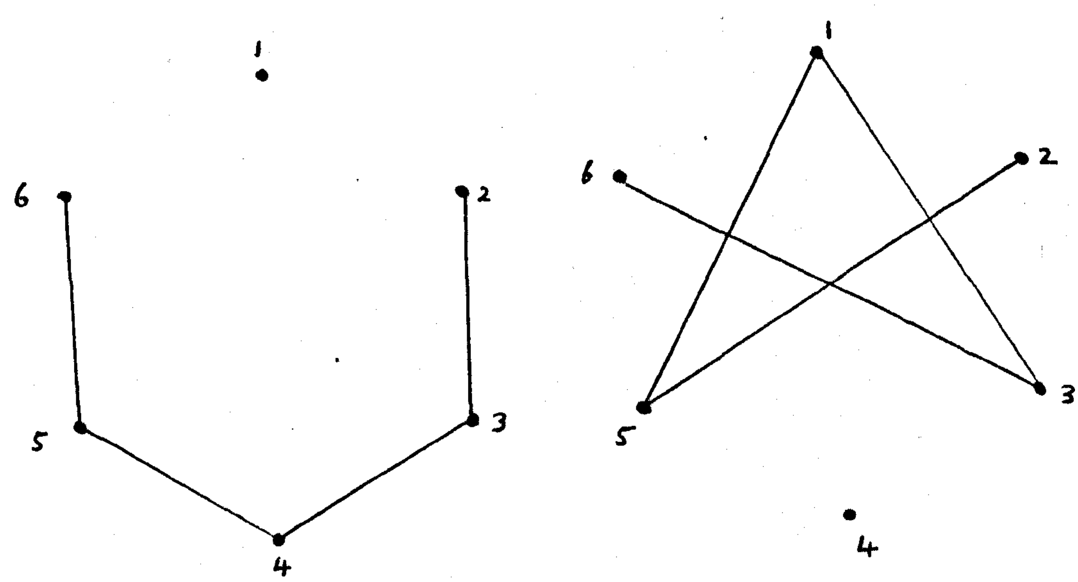
$$\Longleftrightarrow \quad \pi \Gamma = s_3 s_3 \Gamma,$$

$$\Longleftrightarrow \quad \pi \Gamma = \Gamma.$$

This is typical of the sort of calculations we do when considering isomorphisms in switching classes.

In order to illustrate the smallest example of a switching class that contains no representative, we must consider a switching class of graphs on six vertices.

Example 2.10.



Graph Γ Graph $s_1 s_5 \Gamma$

Figure 2

Figure 2 illustrates two isomorphic graphs, Γ and $s_1 s_5 \Gamma$, that lie in the same switching class. We shall now prove that the switching class $\mathcal{S}(\Gamma)$ has no representative.

An isomorphism mapping Γ to $s_1 s_5 \Gamma$ is $\pi = (14)(2)(35)(6)$. That is, $\pi\Gamma = s_1 s_5 \Gamma$. The automorphism group of Γ, is $\mathrm{Aut}\,\Gamma = \langle \sigma \rangle$, where $\sigma = (1)(26)(35)(4)$. From Definition 2.8, if $\mathcal{S}(\Gamma)$ contains a representative, Γ', then $\langle \pi, \sigma \rangle \subseteq \mathrm{Aut}\,\Gamma'$, since $\langle \pi, \sigma \rangle \subseteq \mathrm{Stab}\,\mathcal{S}(\Gamma)$.

Suppose that $\sigma(s\Gamma) = s\Gamma$ for some switch s. Then, since $\sigma\Gamma = \Gamma$, we have ${}_\sigma s = s$, or ${}_\sigma s = \hat{s}$. By Corollary 2.5(ii) ${}_\sigma s = \hat{s}$ is impossible since σ contains a 1-cycle. If ${}_\sigma s = s$ then either both 3 and 5 lie in the support of s or neither 3 or 5 lie in the support of s. Suppose that $\pi(s'\Gamma) = s'\Gamma$ for some switch s'. Then, since $\pi\Gamma = s_3 s_5 \Gamma$, we have ${}_\pi s' s' = s_3 s_5$ or ${}_\pi s' s' = s_1 s_2 s_4 s_6$. In the second case it will be observed that s' does not exist. In the first case exactly one of 3 and 5 lies in the support of s'. We have proved that for all choices of s and s' with the above properties, $s \neq s'$ and $\hat{s} \neq s'$. Consequently $\mathcal{S}(\Gamma)$ contains no representative, and the group $G = \langle \pi, \sigma \rangle$ is hidden in $\mathcal{S}(\Gamma)$.

The final result in this chapter shows that Example 2.10 is by no means unique.

Lemma 2.11. For all even values of n greater than four, there is a switching class of graphs on n vertices, that has no representative.

Proof. We give a construction for Γ, and show that $\mathcal{S}(\Gamma)$ contains no representative.

Let Γ_1 and Γ_2 be two isomorphic graphs each on n_1 vertices. The vertices of Γ_1 are labelled $1, \ldots, n_1$, where $n_1 \geqslant 1$. The vertices of Γ_2 are labelled $n_1+1, \ldots, 2n_1$, where $\delta(i) = n_1 + i$, $1 \leqslant i \leqslant n_1$, and δ is an isomorphism mapping Γ_1 to Γ_2. We now introduce four new vertices labelled $2n_1+1$, $2n_1+2$, $2n_1+3$ and $2n_1+4$, and construct a graph Γ_2' from Γ_2 by defining their adjacencies in the following way. Let vertices labelled $2n_1+1$ and $2n_1+2$ be adjacent to every vertex in Γ_2. Let $2n_1+3$ be adjacent to $2n_1+1$, and $2n_1+4$ be adjacent to $2n_1+2$. Let Γ, a graph on n vertices, be the disjoint union of Γ_1 and Γ_2'. (That is, Γ is the graph whose vertex-set is the vertex-set of Γ_1 and Γ_2', and whose edge-set is the edge-set of Γ_1 and Γ_2'.) We claim that $\mathcal{S}(\Gamma)$ has the required properties. Clearly n is even, and can take any even value greater than four by suitable choice of n_1, since $n = 2n_1+4$. We observe that $\langle \sigma \rangle \subseteq \mathrm{Aut}\,\Gamma$, where $\sigma = (2n_1+1\ 2n_1+2)(2n_1+3\ 2n_1+4)$. Furthermore, the graph $s\Gamma$ is isomorphic to Γ, where $s = s_{2n_1+1} s_{2n_1+2}$, under the isomorphism $\pi = \delta(2n_1+1\ 2n_1+2)$. That is, $\pi\Gamma = s\Gamma$. We show that $\langle \pi, \sigma \rangle \not\subseteq \mathrm{Aut}\,\Gamma'$ for any $\Gamma' \in \mathcal{S}(\Gamma)$.

By using Corollary 2.5 and noting that both σ and π contain 1-cycles, we see that whenever $\sigma(s'\Gamma) = s'\Gamma$, the support of s' contains either both of $2n_1+1$ and $2n_1+2$, or neither of them. Whenever $\pi(s''\Gamma) = s''\Gamma$, exactly one of $2n_1+1$ and $2n_1+2$ lies in the support of s''. Therefore $\langle \pi, \sigma \rangle$ is hidden in $\mathcal{S}(\Gamma)$.

The next three chapters approach the problem from the point of view of permutation group structure.

CHAPTER 3. SOME ALWAYS EXPOSABLE PERMUTATION GROUPS

In this chapter we give a necessary and sufficient condition on a subgroup G of the stabiliser of a switching class, for G to be exposable in the class. We use this result to show that certain permutation grpups are always exposable, and that certain switching classes contain a representative.

Theorem 3.1. A necessary and sufficient condition for a permutation group G to be exposable in a switching class $\mathcal{S}(\Gamma)$ is that G has an orbit on $\mathcal{S}(\Gamma)$ of odd length.

Proof. Suppose that $\Gamma_1, \ldots, \Gamma_r$ is an orbit of G on $\mathcal{S}(\Gamma)$, where r is odd. Let $s^{(i)}$ denote a switch such that

$$s^{(i)}\Gamma = \Gamma_i, \quad i = 1, \ldots r.$$

A permutation π in G permutes the graphs $\Gamma_1, \ldots, \Gamma_r$. Suppose that $\pi\Gamma = s\Gamma$. Then

$$\pi\Gamma_i = \pi s^{(i)}\Gamma = {}_{\pi}s^{(i)}s\Gamma = s^{(j)}\Gamma = \Gamma_j,$$

for some j = 1, ..., r. Put $s' = s^{(1)}\ldots s^{(r)}$. Since r is odd,

$${}_{\pi}s's'\Gamma = (s)^r\Gamma = s\Gamma = \pi\Gamma,$$

and hence $\pi(s'\Gamma) = s'\Gamma$. The choice of s' is independent from the choice of π in G, so G fixes $s'\Gamma$, and G is exposable in $\mathcal{S}(\Gamma)$. Conversely, if G is exposable in $\mathcal{S}(\Gamma)$, then G has an orbit on $\mathcal{S}(\Gamma)$ of length one.

The following lemma and corollary establish properties of switching classes that we will use in conjunction with Theorem 3.1.

Lemma 3.2. Every switching class contains a unique graph in which a given labelled vertex is adjacent to a given set of labelled vertices and not adjacent to their complement.

Proof. No two graphs in a switching class can have the same labelled vertex adjacent to the same set of labelled vertices, since a non-trivial switch on a graph changes the adjacencies of all vertices in that graph. Since a switching class contains $2^{(n-1)}$ graphs, all possible adjacencies for a given labelled vertex must occur.

Corollary 3.3. Every switching class contains a unique graph in which a given labelled vertex is isolated. (An isolated vertex has no adjacencies.)

Corollary 3.4. If a permutation group G has an orbit of odd length on Ω, then G is always exposable.

Proof. Let G stabilise $\mathcal{S}(\Gamma)$. Then there is a correspondence between an odd orbit $1, \ldots, r$ of G on Ω and an orbit of G on $\mathcal{S}(\Gamma)$ consisting of graphs in which vertices $1, \ldots, r$ are isolated. The number of graphs in $\mathcal{S}(\Gamma)$ that have isolated vertices with labels from the set $\{1, \ldots, r\}$ is a divisor of r, and therefore odd. These graphs form an orbit of G on $\mathcal{S}(\Gamma)$. By Theorem 3.1, G is exposable in $\mathcal{S}(\Gamma)$, and, since $\mathcal{S}(\Gamma)$ was an arbitrary switching class stabilised by G, G is always exposable.

Theorem 3.5. (Seidel [8]) Every switching class of graphs on an odd number of vertices contains a representative.

Proof. The stabiliser of a switching class of graphs on an odd number of vertices, of necessity, has an orbit of odd length on Ω since $|\Omega| = n$, which is odd. Therefore, when n is odd, every stabiliser is always exposable by Corollary 3.4. Hence the switching class contains a representative.

Note. Theorem 3.5 can be proved directly from the definition of switching. We proceed to do this, in order to emphasise a difference between switching classes of graphs on an odd number of vertices and an even number of vertices. On switching a graph on an even number of vertices with respect to one vertex, the parity of the valency of each vertex is changed. Clearly it follows that on switching such a graph with respect to an even number of vertices the parity of the valency of each vertex remains unchanged, and on switching the graph with respect to an odd number of vertices the parity of the valency of each vertex is changed. If Γ is a graph on n vertices where n is even, and k labelled vertices have even valency, then those k vertices are either all even valent or all odd valent in each graph in $\mathcal{S}(\Gamma)$, and the other (n-k) vertices are either all odd valent or all even valent respectively in each graph in $\mathcal{S}(\Gamma)$.

On switching a graph on an odd number of vertices with respect to one vertex, the parity of the valency of each vertex, except for the vertex switched, is changed. It follows that if Γ is a graph on n vertices where n is odd, and vertices $i_1, \ldots, i_k$ are all the even valent vertices of Γ, then no other graph in $\mathcal{S}(\Gamma)$ has the property that $i_1, \ldots, i_k$ are all its even valent vertices. This fact, together with the facts that $\mathcal{S}(\Gamma)$ contains $2^{(n-1)}$ graphs, and that no graph can have an odd number of odd valent vertices, tells us that any odd set of labelled vertices occur as all the even valent vertices of some graph in $\mathcal{S}(\Gamma)$. In particular there is a unique graph in $\mathcal{S}(\Gamma)$ in which all its vertices are even valent. This graph must be a representative of $\mathcal{S}(\Gamma)$. We have proved Theorem 3.5 and shown that every switching class of graphs on an odd number of vertices contains a 'natural' representative, an Euler graph.

The following corollary is a direct consequence of Theorem 3.1 and Corollary 3.4.

Corollary 3.6. A group of odd order is always exposable.

We shall be able to strengthen the above result using the results of the next chapter.

Corollary 3.7. Let G be a permutation group containing a subgroup H that is always exposable. If the index of H in G is odd, then G is always exposable.

Proof. Suppose that G stabilises $\mathcal{S}(\Gamma)$. Then there is a graph Γ' in $\mathcal{S}(\Gamma)$ which is fixed by H. The graph Γ' lies in an orbit of G on $\mathcal{S}(\Gamma)$ whose length divides the index of H in G. The length of this orbit is odd and so G is always exposable, by Theorem 3.1.

Corollary 3.4 takes us some way towards proving that all cyclic groups $G = \langle \pi \rangle$ are always exposable. It is certainly true that if π, expressed as a product of disjoint cycles, contains an odd length cycle, then G is always exposable. It remains to be proved that if π consists entirely of even length cycles then G is always exposable.

Theorem 3.1 suggests another method of deciding whether a switching class contains a representative or not. Choose a graph property known to be preserved by permutations of the vertices, and count the number of graphs in a switching class with that property. If the number of such graphs is odd, then the switching class contains a representative. Generally this approach appears to be rather difficult. We have made some progress, however, by choosing the property of girth (the length of the shortest circuit in a graph), where, for graphs with girth greater than four, either switching cannot preserve the girth or, in special cases, it is very easy to count the number of graphs in a switching class with a given girth. This analysis will be done in a later chapter.

We end this chapter with an example illustrating this approach.

Example 3.8. Let Γ be a graph on n vertices, where n is even. Let s be a switch. A necessary condition that Γ and $s\Gamma$ be isomorphic graphs is that both graphs have the same number of edges. The following operation on Γ is equivalent to switching Γ with respect to those vertices whose labels lie in the support of s. Partition the vertices of Γ into two sets, V_1 and V_2, such that the labels of V_1 are all the symbols in supp s and the labels of V_2 are all the symbols in supp $\hat{s}$. Preserve the edges incident to two vertices in V_1 and to two vertices in V_2; cancel all the edges incident to a vertex in V_1 and a vertex in V_2, and introduce an edge between each pair of non-adjacent vertices, one of which lies in V_1 and the other in V_2. Clearly, if Γ and $s\Gamma$ are isomorphic and $|\text{supp } s| = k$, it is necessary that the number of edges joining the two sets of vertices, V_1 and V_2 in Γ, be $\frac{1}{2}k(n-k)$. Note that, since n is even, then k is even. With these restrictions the minimum of $\frac{1}{2}k(n-k)$ is $(n-2)$. We have proved that if Γ has less than $(n-2)$ edges then it is a representative, since there is no switch s such that $s\Gamma$ has the same number of edges as Γ. Finally we note that the structure, and action of the stabiliser of $\mathcal{S}(\Gamma)$, on $\mathcal{S}(\Gamma)$, is identical with the structure, and action of the stabiliser of $\mathcal{S}(\Gamma^c)$ on $\mathcal{S}(\Gamma^c)$; where Γ^c is the graph with the same vertex-set as Γ and two vertices are adjacent in Γ^c if and only if they are not adjacent in Γ. The graph Γ^c is called the <u>complement</u> of Γ. Hence any graph on n vertices, where n is even, with greater than $\frac{1}{2}n(n-1)-(n-2)$ edges is also a representative.

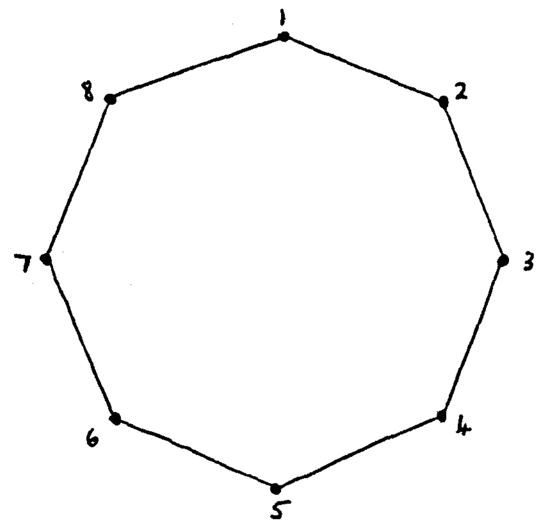
We now turn to cyclic permutation groups, and prove that they are always exposable.

CHAPTER 4. CYCLIC SUBGROUPS OF STABILISERS

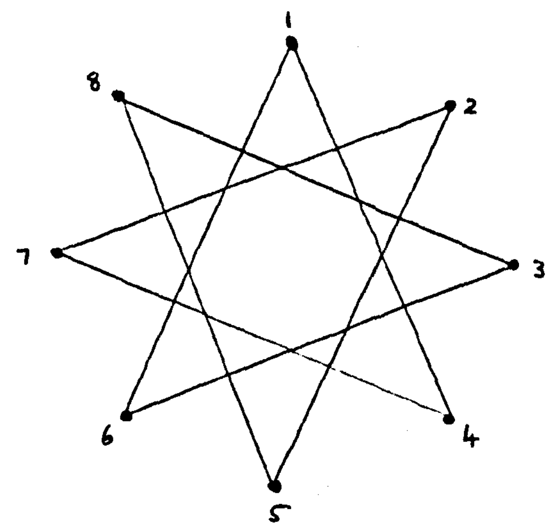
In order to make further progress on the question of which subgroups of stabilisers are exposable, we must return to the results of Chapter 2. We observe that the graphs in $\mathcal{G}$ are permuted by switches, by permutations in Σ and by compositions of these operations, which we call switch-permutations. Their totality forms a group W, where the law of composition of a switch and a permutation is given by Lemma 2.4. Every element $w \in W$ is uniquely expressible as a switch-permutation $w = s\pi$, where s is a switch, and $\pi \in \Sigma$. Lemma 2.6 tells us that associated with every graph Γ is a group Q of switch-permutations that fixes Γ. That is, $\pi\Gamma = s\Gamma$ if and only if $s\pi$ is a switch-permutation fixing Γ. We say that the permutation group $\{\pi \in \Sigma \mid s\pi \in Q$ for some switch $s\}$ is the group of permutations associated with Q. If Q is the largest subgroup of W fixing Γ then clearly Stab $\mathcal{S}(\Gamma)$ is the permutation group associated with Q. In this chapter and the next, we consider cyclic and dihedral groups associated with subgroups of W and establish conditions under which these groups are exposable. This problem resolves itself into two parts. It will be made clear shortly that not all subgroups of W fix graphs. We establish a necessary and sufficient condition on a subgroup of W in order that a graph exists fixed by it. Our second problem is to discover whether or not its associated permutation group is exposable.

We illustrate these problems in the following example.

Example 4.1.



Graph Γ.



Graph $s_1 s_3 s_5 s_7 \Gamma$.

Figure 3

Let s be the switch $s = s_1 s_3 s_5 s_7$, then $s\Gamma$ is the only graph in $\mathcal{S}(\Gamma)$ isomorphic to Γ, where Γ is the graph illustrated in Figure 3. Also, since $\mathrm{Aut}\,\Gamma = \langle \alpha, \beta \rangle$, where $\alpha = (1)(28)(37)(46)(5)$ and $\beta = (13)(2)(48)(57)(6)$, the stabiliser $\mathrm{Stab}\,\mathcal{S}(\Gamma) = \langle \alpha, \beta, \mu \rangle$, where $\mu\Gamma = s\Gamma$. Put $\mu = (15)(28)(3)(46)(7)$, and μ has the property that $\mu\Gamma = s\Gamma$. An analysis, similar to that of Example 2.10, of the action of μ and α on Γ, shows that $\mathcal{S}(\Gamma)$ has no representative. In other words the group $Q = \langle \alpha, s\mu \rangle$ fixes Γ, but the permutation group $\langle \alpha, \mu \rangle$ associated with Q is hidden in $\mathcal{S}(\Gamma)$. However, the group $Q' = \langle \beta, s\mu \rangle$ also fixes Γ but the permutation group $\langle \beta, \mu \rangle$, associated with Q' is exposable, since $\langle \beta, \mu \rangle \subseteq \mathrm{Aut}\,(s_4 s_8 \Gamma)$. Finally, the group $Q'' = \langle s_1 \alpha \rangle$ fixes no graph in $\mathcal{S}(\Gamma)$;

in fact Q'' fixes no graph in $\mathcal{G}$, the set of all graphs on eight vertices. This is clear from the observation that if the pair $\{1, 5\}$ is an edge or a non-edge in our putative graph, $s_1\alpha$ maps $\{1, 5\}$ to a non-edge or an edge respectively. Hence there is no graph fixed by Q''.

<u>Notation 4.2</u>. We introduce a convenient notation for the switch-permutations $w = s\pi$ of W. The permutation π is written as a product of disjoint cycles, and a bar is placed over each symbol that occurs in supp s. To illustrate, in Example 4.1

$$s_1 s_3 s_5 s_7 \mu = (\bar{1}\bar{5})(28)(\bar{3})(46)(\bar{7}).$$

We observe that the action of $s_1 s_3 s_5 s_7 \mu$ on all graphs is identical to that of

$$\hat{e}\, s_1 s_3 s_5 s_7 \mu = s_2 s_4 s_6 s_8 \mu = (15)(\bar{2}\bar{8})(3)(\bar{4}\bar{6})(7).$$

We now establish a criterion for the existence of a graph fixed by a subgroup Q of W.

<u>Theorem 4.3</u>. <u>A subgroup Q of W does not fix any graph in $\mathcal{G}$ if and only if some element of Q involves either a switch-transposition $(\bar{i}j)$ or switch-1-cycles $(\bar{i})(j)$</u>...

<u>Proof</u>. In view of the action of permutations and switches on graphs, a necessary and sufficient condition for a switch-permutation $s\pi$ to fix a graph Γ is as follows: for all $p, q \in \Omega$, $\{p, q\}$ and $\{\pi(p), \pi(q)\}$ are both edges or both non-edges of Γ if and only if supp s contains both or neither of $\pi(p)$ and $\pi(q)$. The construction of a graph fixed by Q will break down if and only if the stage is reached that an unordered pair $\{i, j\}$ represents both an

edge and a non-edge. This will arise if and only if Q contains a switch-permutation $s\pi$ such that supp s contains exactly one of i and j and either (1) $\pi(i) = j$, $\pi(j) = i$, or (2) $\pi(i) = i$, $\pi(j) = j$.

<u>Corollary 4.4. Let a group Q of switch-permutations fix a graph. If $s\pi$ and $s'\pi$ belong to Q then $s' = s$ or $s' = \hat{s}$</u>.

<u>Proof</u>. $s\pi \in Q$, $s'\pi \in Q \Rightarrow s\pi\pi^{-1}s' = ss' \in Q$.
By Theorem 4.3, $ss' = e$ or $ss' = \hat{e}$.

We note also that if Q fixes a graph then Q fixes exactly 2^{ν} graphs, where ν is the number of orbits of unordered pairs $\{i, j\}$, $i \neq j$, in $\Omega \times \Omega$ under the action of the permutation group associated with Q. It is clear from the proof of Theorem 4.3 that we assign in each orbit one pair to be an edge or a non-edge.

<u>Example 4.5</u>. Consider the subgroup Q of W generated by $w = (\bar{1}4\bar{5}8)(2\bar{7}6\bar{3}) = s_1 s_3 s_5 s_7 \mu$. We calculate $w^2 = (s_1 s_3 s_5 s_7)(s_4 s_2 s_8 s_6)\mu^2 = (\bar{1}\bar{5})(\bar{2}\bar{6})(\bar{3}\bar{7})(\bar{4}\bar{8})$.
By Theorem 4.3 there exists a graph on 8 vertices fixed by Q. It will be found that there are exactly 2^8 different graphs fixed by Q since $\langle\mu\rangle$ has eight orbits of unordered pairs on $\Omega \times \Omega$.

We now apply Theorem 4.3 to subgroups of W whose associated permutation groups are cyclic.

<u>Lemma 4.6. Consider the r-cycle $\sigma = (1 \ldots r)$ and the switch s, with supp $s \subseteq \{1, \ldots, r\}$. Then</u>

$$(s\sigma)^r = \begin{cases} (1)\ldots(r), & \text{if } |\text{supp } s| \text{ is even,} \\ (\bar{1})\ldots(\bar{r}), & \text{if } |\text{supp } s| \text{ is odd.} \end{cases}$$

Moreover, if $r = 2k$, then $(s\sigma)^k$ involves a switch-transposition $(\bar{i}j)$ if and only if $|\text{supp } s|$ is odd.

Proof. Application of the formula

$$(s\sigma)^m = s\,{}_{\sigma}s\,{}_{\sigma^2}s \ldots {}_{\sigma^{(m-1)}}s\,\sigma^m.$$

Definition 4.7. (i) A switch s is strongly compatible with a permutation π if supp s contains an even number of symbols from each cycle of π, where π is expressed as the product of disjoint cycles (including 1-cycles).

(ii) A switch s is compatible with π if either s or $\hat{s}$ is strongly compatible with π.

(iii) A switch s is weakly compatible with permutations π and μ if s is strongly compatible with π and compatible, but not strongly compatible, with μ, or if s is strongly compatible with μ and compatible, but not strongly compatible, with π.

Lemma 4.8. A switch s is compatible with a permutation π if and only if $s\pi$ fixes some graph.

Proof. By Lemma 4.6, a switch s is compatible with permutation π if and only if no power of $s\pi$ contains switch-1-cycles of the form $(\bar{i})(j)\ldots$, or switch-transpositions of the form $(\bar{i}j)$. By Theorem 4.3, this condition holds if and only if $s\pi$ fixes some graph.

Lemma 4.9. A switch s is compatible with a permutation π if and only if there exists a switch s' such that

$$s\pi = s'\pi s', \quad \text{or} \quad \hat{s}\pi = s'\pi s'.$$

Proof. If $s\pi = s'\pi s'$ then $s = s'_{\pi}s'$ and clearly s is compatible with π.

Conversely, suppose that s is compatible with π. We suppose, without loss of generality that supp s involves an even number of symbols from each cycle of π. Consider a particular cycle of π, which we write $\sigma = (1 \ldots r)$. If supp s $\cap$ supp σ is not empty then it is expressible in the form

$$\text{supp } s \cap \text{supp } \sigma = \{ i_1, \ldots, i_{2k} \},$$

where $1 \leqslant i_1 \leqslant \cdots \leqslant i_{2k} \leqslant r$. Let s′ be a switch such that

$$\text{supp } s' \cap \text{supp } \sigma = \{ i \mid i_{2q-1} \leqslant i < i_{2q},\ q = 1, \ldots, k \}$$

Then

$$\text{supp}_{\pi} s' \cap \text{supp } \sigma = \{ i \mid i_{2q-1} < i \leqslant i_{2q},\ q = 1, \ldots, k \},$$

and so $\text{supp } s'_{\pi}s' \cap \text{supp } \sigma = \{ i_1, \ldots, i_{2k} \}$.

We define s' to be the switch whose support is obtained by applying the above construction to all the cycles of π that have common symbols with supp s. Then $s'_{\pi}s' = s$ and $s\pi = s'\pi s'$. This completes the proof.

Theorem 4.10. (Mallows and Sloane [4]) A cyclic group is always exposable.

Proof. Suppose a cyclic group G stabilises $\mathcal{S}(\Gamma)$. If $G = \langle \pi \rangle$, then there is a switch s such that $s\pi$ fixes Γ. By Lemma 4.8, s is compatible with π. By Lemma 4.9, there is a switch s′ such that $s'\pi s'$ is equal to either $s\pi$ or $\bar{s}\pi$. But then $s'\pi s'\Gamma = \Gamma$, and hence $\pi(s'\Gamma) = s'\Gamma$. Thus π fixes the graph $s'\Gamma$, and G is exposable in $\mathcal{S}(\Gamma)$.

The following result strengthens Corollary 3.6.

Theorem 4.11. Let G be a permutation group of order $2^m(2k+1)$ containing an element of order 2^m. Then G is always exposable. In particular, all groups of order 4k+2 are always exposable.

Proof. Let π in G have order 2^m, and let $H = \langle \pi \rangle$. By Theorem 4.10, H is always exposable. H has index 2k+1 in G so, by Corollary 3.7, G is always exposable. It follows that all groups of order 4k+2 are always exposable, from the fact that all such groups contain an element of order two.

Theorem 4.10 allows us to derive an explicit formula for the number, t_n, of switching classes of graphs on n vertices, up to isomorphism. That is, we count the number of orbits of W on $\mathcal{G}$. In this calculation we shall find that the number t_n is a function of the number of cycles of a given length in each permutation of Σ, and so we state the following definition. See Robinson [5].

Definition 4.12. The cycle type of $\pi \in \Sigma$ is the ordered n-tuple $a(\pi) = (a_1, \ldots, a_n)$, where if π is expressed as the product of disjoint cycles, then π contains a_i cycles of length i, $1 \leq i \leq n$. Note that $\sum_{i=1}^{n} i\, a_i = n$.

We apply the following well known result of Burnside to our case. Let t be the number of orbits of a group G on a set X. Let F(g) be the set of fixed points of $g \in G$; $F(g) = \{x \in X \mid g(x) = x\}$. Then

$$t|G| = \sum_{g \in G} |F(g)|. \text{ See Biggs [1].}$$

We regard $\mathcal{G}$ as the disjoint union of switching classes. Σ permutes the switching classes, by Lemma 2.6. A 'fixed point' of π, in its action on the set of all switching classes, is a class $\mathcal{S}(\Gamma)$ such that $\pi \in \text{Stab } \mathcal{S}(\Gamma)$. Denote the set of all such classes by $F_{\mathcal{S}}(\pi)$.

<u>Theorem 4.13. <u>Mallows and Sloane [4]</u>.</u>

$$t_n = \sum_a \frac{2^{\nu(a) - \lambda(a)}}{\prod_i i^{a_i} a_i!},$$

<u>the sum being over all ordered n-tuples $a = (a_1, \ldots, a_n)$ such that $n = \sum_{i=1}^{n} ia_i$, where</u>

$$\nu(a) = \sum_{\substack{i,j \\ i<j}} a_i a_j (i,j) + \sum_i i\left(a_{2i} + a_{2i+1} + \binom{a_i}{2}\right),$$

$$\lambda(a) = \sum_i a_i - \operatorname{sgn}\left(\sum_i a_{2i+1}\right),$$

<u>$\operatorname{sgn}(x) = 0$, if $x = 0$, $\operatorname{sgn}(x) = 1$, if $x > 0$, and (i,j) is the highest common factor of i and j.</u>

<u>Proof</u>. By Burnside's Lemma

$$t_n \, n! = \sum_\pi |F_{\mathcal{S}}(\pi)|.$$

$|F_{\mathcal{S}}(\pi)|$ may be calculated in the following way. Let $a(\pi)$ be the cycle type of π, then π lies in the automorphism group of $2^{\nu(a(\pi))}$ graphs, since $\nu(a(\pi))$ is the number of orbits of π on $\Omega \times \Omega$. Whenever $\pi \in \text{Stab } \mathcal{S}(\Gamma)$ for some switching class $\mathcal{S}(\Gamma)$, π lies in the automorphism group of $2^{\lambda(a(\pi))}$ graphs in $\mathcal{S}(\Gamma)$, by Theorem 4.10 and Corollary 2.5. Therefore $|F_{\mathcal{S}}(\pi)| = 2^{(\nu(a(\pi)) - \lambda(a(\pi)))}$, and $t_n = \frac{1}{n!} \sum_\pi 2^{(\nu(a(\pi)) - \lambda(a(\pi)))}$.

There are $n!/\prod_i i^{a_i} a_i!$ permutations in Σ of cycle type $(a_1, \ldots, a_n)$. So

$$t_n = \frac{1}{n!} \sum_\pi 2^{\nu(a(\pi)) - \lambda(a(\pi))},$$

$$= \frac{1}{n!} \sum_a 2^{\nu(a) - \lambda(a)} \frac{n!}{\prod_i i^{a_i} a_i!},$$

$$= \sum_a \frac{2^{\nu(a) - \lambda(a)}}{\prod_i i^{a_i} a_i!} \quad \text{as required.}$$

Finally, in this chapter, we give a formula for the number of graphs up to isomorphism in a switching class. That is, we count the number of orbits of Stab $\mathcal{S}(\Gamma)$ on $\mathcal{S}(\Gamma)$.

Let $G \subseteq$ Stab $\mathcal{S}(\Gamma)$. By Theorem 4.10 and Burnside's Lemma, the number of orbits of G on $\mathcal{S}(\Gamma)$ is t_G, where

$$t_G = \frac{1}{|G|} \sum_{\pi \in G} 2^{\lambda(a(\pi))} .$$

Here we regard G as acting on $\mathcal{S}(\Gamma)$, and the 'fixed points' of elements in G are the graphs in $\mathcal{S}(\Gamma)$ fixed by those elements. We note that t_G depends only on the cycle structure of the elements of G. Furthermore, it will be observed that for all n there is a switching class of graphs on n vertices, denoted $\mathcal{S}(N_n)$, consisting of the representative N_n and all the labelled complete bipartite graphs on n vertices. Since Stab $\mathcal{S}(N_n) = \Sigma$, we have proved the following lemma:

<u>Lemma 4.14.</u> <u>The number of graphs, up to isomorphism, in $\mathcal{S}(\Gamma)$ is</u>

$$t_{\mathrm{Stab}\,\mathcal{S}(\Gamma)} = \frac{1}{|\mathrm{Stab}\,\mathcal{S}(\Gamma)|} \sum_{\pi \in \mathrm{Stab}\,\mathcal{S}(\Gamma)} 2^{\lambda(a(\pi))} .$$

<u>Furthermore, $t_{\mathrm{Stab}\,\mathcal{S}(\Gamma)}$ is the number of orbits of Stab $\mathcal{S}(\Gamma)$ on the switching class $\mathcal{S}(N_n)$.</u>

CHAPTER 5. DIHEDRAL SUBGROUPS OF STABILISERS

Our aim in this chapter is to classify the dihedral subgroups in Σ which are always exposable. We see from Example 2.10 that not all dihedral groups are always exposable. The dihedral group Stab $\mathcal{S}(\Gamma)$ is hidden in $\mathcal{S}(\Gamma)$, where Γ is the graph of Figure 2.

Now let D be an arbitrary dihedral subgroup of Σ. Then D is generated by two involutions, α and β. The following lemma applies as a special case.

Lemma 5.1. Suppose that a subgroup G of Σ is generated by two permutations π and μ. If G stabilises a switching class $\mathcal{S}(\Gamma)$ then G is associated with a group Q fixing a graph in $\mathcal{S}(\Gamma)$, such that Q is generated by switch-permutations $s\pi$ and $s\mu$ for some switch s.

Proof. By Theorem 4.10, there is a graph Γ' in $\mathcal{S}(\Gamma)$ which is fixed by $\mu^{-1}\pi$. So there is a switch s such that

$$\pi\Gamma' = \mu\Gamma' = s\Gamma',$$

and the switch-permutations $s\pi$ and $s\mu$ fix Γ'.

According to Lemma 5.1, in order to study the action of the dihedral group D on a switching class which it stabilises, we can equivalently study subgroups Q of W that fix a graph, where Q is generated by switch-permutations $s\alpha$ and $s\beta$. We next establish a criterion, depending on s, α and β for the existence of a graph fixed by $Q = \langle s\alpha, s\beta \rangle$.

Lemma 5.2. Let α and β be involutions. There exists a graph fixed by $Q = \langle s\alpha, s\beta \rangle$ if and only if s is compatible with both α and β.

Proof. If Q fixes a graph then s is compatible with α and β by Lemma 4.8. Conversely, suppose that s is compatible with α and β. Then, since α and β are involutions,

$$_{\alpha}s = {}_{\beta}s = s. \qquad (5.3)$$

We will show the existence of a graph fixed by Q by an application of Theorem 4.3. The elements of Q are of the form

$$(s\alpha)^k (s\beta s\alpha)^\ell = (s\alpha)^k (\beta\alpha)^\ell,$$

where $k = 0$ or 1, and $\ell = 0,1,2,\ldots$, the last expression being obtained on applying (5.3). We must show that the conditions of Theorem 4.3 for the non-existence of a graph do not arise. This is clear when $k = 0$. Consider next an element $w = s\alpha(\beta\alpha)^\ell$ of Q. If the permutation $\alpha(\beta\alpha)^\ell$ transposes two symbols then by (5.3) supp s contains both or neither of these symbols. Finally, suppose that $\alpha(\beta\alpha)^\ell$ fixes two symbols i and j. If $\ell = 2m$, put $(\beta\alpha)^m(i) = p$, $(\beta\alpha)^m(j) = q$. Then

$$\alpha(p) = \alpha(\beta\alpha)^m(i) = (\beta\alpha)^m(i) = p,$$

and similarly $\alpha(q) = q$. Since s is compatible with α, supp s contains either both or neither of p and q, and hence also both or neither of $i = (\alpha\beta)^m(p)$ and $j = (\alpha\beta)^m(q)$. If $\ell = 2m+1$ a similar argument applies to the elements $\alpha(\beta\alpha)^m(i)$ and $\alpha(\beta\alpha)^m(j)$ which are fixed by β, using the hypothesis that s is compatible with β.

Having established our existence criterion for a graph fixed by Q, our next problem is to discover the conditions under which its associated permutation group D is exposable.

Lemma 5.4. Let the switch s be strongly compatible with both the involutions α and β. Then there is a switch s' such that

$$s\alpha = s'\alpha s' \quad \text{and} \quad s\beta = s'\beta s'. \qquad (5.5)$$

Proof. By our hypothesis on s, its support supp s is a union of orbits $\Delta_1, \ldots, \Delta_t$ of $D = \langle \alpha, \beta \rangle$ on Ω. Choose from each orbit Δ_r a symbol i_r, $r = 1, \ldots, t$. Then the switch s' is defined by the support

$$\text{supp } s' = \{(\beta\alpha)^m(i_r),\ r = 1, \ldots, t,\ m = 0, 1, 2, \ldots\}.$$

We will show that s' satisfies relations (5.5), in other words, that $s = s'_\alpha s' = s'_\beta s'$. This follows from the observation that supp s' consists of precisely one symbol from each transposition in α and in β whose symbols lie in the support of s. For if this is not the case then for some i_r in supp s' and some integer m,

$$(\beta\alpha)^m(i_r) = \alpha(i_r) \text{ or } \beta(i_r).$$

In either case this leads to the conclusion that either α or β fixes a symbol in Δ_r. This contradicts that s is strongly compatible with α and with β.

Corollary 5.6. Suppose that the graph Γ is fixed by $Q = \langle s\alpha, s\beta \rangle$, where α and β are involutions. If s is strongly compatible with both α and β then there is a graph Γ' in $\mathcal{S}(\Gamma)$ which is fixed by the dihedral group $D = \langle \alpha, \beta \rangle$.

Proof. Apply Lemma 5.4, putting $\Gamma' = s'\Gamma$.

We must now consider the case where a switch s is weakly compatible with α and β. The following examples motivate our next lemma.

Examples 5.7. (i) Consider the switch-involutions $s\alpha = (1)(2)(\bar{3}\bar{4})(\bar{5}\bar{6})(\bar{7}\bar{8})$ and $\hat{s}\beta = (\bar{1}\bar{2})(35)(46)(7)(8)$. Here s is weakly compatible with α and β. There exists a switch s' such that $s\alpha = s'\alpha s'$ and $\hat{s}\beta = s'\beta s'$. (Choose for example $\operatorname{supp} s' = \{1, 3, 5, 7\}$ or $\{2, 4, 6, 7\}$.) By Lemma 5.2 there exists a graph Γ fixed by $s\alpha$ and by $s\beta$, and hence also by $\hat{s}\beta$. The graph $s'\Gamma$ is fixed by $D = \langle \alpha, \beta \rangle$.

(ii) Put $s\alpha = (\bar{1}\bar{2})(\bar{3}\bar{4})(\bar{5}\bar{6})(\bar{7}\bar{8})(\bar{9}\ \overline{10})(11)(12)$ and $\hat{s}\beta = (1)(3)(24)(5\ 10)(67)(89)(\overline{11}\ \overline{12})$.

Here again s is weakly compatible with α and β. There is no switch s' such that $s\alpha = s'\alpha s'$ and $\hat{s}\beta = s'\beta s'$. Again, by Lemma 5.2, there exists a graph Γ fixed by $s\alpha$ and by $s\beta$, but in this case there is no graph in $\mathcal{S}(\Gamma)$ fixed by $D = \langle \alpha, \beta \rangle$.

The essential difference between Examples 5.7(i) and (ii) lies in the length of the orbits of D on Ω, none of whose symbols is fixed by α or by β. In Example 5.7(i) the only such orbit is $\{3, 4, 5, 6\}$ and in Example 5.7(ii) the only such orbit is $\{5, 6, 7, 8, 9, 10\}$. As the next lemma shows, the length of these orbits is crucial to our analysis.

Lemma 5.8. Let D be a dihedral group generated by involutions α and β. Suppose that a switch s is weakly compatible with α and β. Then there is a switch s' such that $s\alpha = s'\alpha s'$ and $\hat{s}\beta = s'\beta s'$ if and only if every orbit of D on Ω none of whose symbols is fixed by α or by β, has length divisible by four.

Proof. Suppose first that there is a switch s' such that

$s\alpha = s'\alpha s'$ and $\hat{s}\beta = s'\beta s'$. Then $s = s'_\alpha s'$ and $\hat{s} = s'_\beta s'$. The hypothesis allows us to assume, without loss of generality, that s is strongly compatible with α but not with β, and $\hat{s}$ is strongly compatible with β but not with α. Let Δ be an orbit of D on Ω none of whose symbols is fixed by α or by β, and assume by way of contradiction that $|\Delta| = 2+4k$ for some integer k. Then α and β each contain the symbols of Δ in $1+2k$ transpositions. In the case that the support of s involves all the symbols of Δ, the support of s' must contain exactly one symbol from each of those transpositions that occur in α, which is $1+2k$ symbols in all from Δ. However, because $\hat{s} = s'_\beta s'$, the support of s' must contain an even number of symbols from Δ. In the case that the support of $\hat{s}$ involves all the symbols of Δ, by the same argument applied to β and α, the support of s' involves both an odd and an even number of symbols from Δ. Hence $|\Delta| = 4k$ for some integer k.

Conversely, we must prove that subject to the condition of Lemma 5.8 an appropriate switch s' is constructible. We partition the orbits of D on Ω into three classes.

(i) Orbits containing a symbol fixed by α;

(ii) Orbits containing a symbol fixed by β;

(iii) Orbits none of whose symbols is fixed by α or by β.

The classes are disjoint, for suppose an orbit Δ is common to class (i) and class (ii). Then it contains a symbol fixed by α and a symbol fixed by β, and this contradicts the hypothesis that s is weakly compatible with α and β. Clearly orbits cannot be common to classes (i) and (iii) or to classes (ii) and (iii). We will now give a construction for a suitable switch s' in the case that the length of all orbits in class (iii) are multiples of four.

First consider an orbit Δ in class (i). Then the symbols of Δ are involved in, say, k transpositions of β where $|\Delta| = 2k$, and α fixes at least two symbols of Δ. We claim that $\alpha\beta$ is a 2k-cycle; for if not, consider a symbol i in Δ fixed by α. Every element of D is expressible in the form $(\alpha\beta)^r$ or $(\alpha\beta)^r\alpha$ for some integer r. If $\alpha\beta$ were not a 2k-cycle then, since $(\alpha\beta)^r\alpha(i) = (\alpha\beta)^r(i)$, the group D would not act transitively on Δ. Choose the support of s' from Δ as k alternate symbols from the cycle $\alpha\beta$, so chosen as to include the symbol i. We calculate

$$\alpha(\alpha\beta)^r(i) = \alpha(\alpha\beta)^r\alpha(i) = (\beta\alpha)^r(i) = (\alpha\beta)^{2k-r}(i)$$

and

$$\beta(\alpha\beta)^r(i) = \beta(\alpha\beta)^r\alpha(i) = (\beta\alpha)^{r+1}(i) = (\alpha\beta)^{2k-r-1}(i).$$

From this we see that α fixes setwise the support of s' from Δ, and β maps this support to its complement in Δ.

By reversing the roles of α and β the choice of the support of s' from an orbit in class (ii) is considered similarly.

Finally, consider an orbit Δ in class (iii). Then $|\Delta|$ is even, $|\Delta| = 2k$, say. Choose an arbitrary symbol i in Δ. We will show that the sets $\{(\alpha\beta)^r(i) \mid r = 1, \ldots, k\}$ and $\{(\alpha\beta)^r\alpha(i) \mid r = 1, \ldots, k\}$ are disjoint and hence exhaust Δ. For if not, then there integers b and c such that $(\alpha\beta)^b(i) = (\alpha\beta)^c\alpha(i)$ giving $\alpha(\beta\alpha)^{c-b}(i) = i$. This implies that α or β fixes a symbol in Δ, depending on the parity of (c-b).

We have thus proved that $\alpha\beta$ is the product of two k-cycles. Choose the support of s' in Δ to consist of (a) alternate symbols including i in the cycle of $\alpha\beta$ that contains i, and (b) alternate symbols in the other cycle of $\alpha\beta$ not including the symbol $\alpha(i)$. (It is at this stage that we require k to be even and hence $|\Delta|$ to be a multiple of four.) Here we assume that the support of s involves all the symbols of Δ. The involution β fixes setwise the support of s' from Δ and α maps this support to its complement in Δ. In the case that the support of $\hat{s}$ involves all the symbols of Δ, choose the support of s' in Δ to consist of (a) alternate symbols including i in the cycle of $\alpha\beta$ that contains i, and (b) alternate symbols in the other cycle of $\alpha\beta$ not including the symbol $\beta(i)$. The involution α fixes setwise the support of s' from Δ and β maps this support to its complement in Δ, as required.

With s' chosen as above it is clear that $s'_{\alpha}s' = s$ and $s'_{\beta}s' = \hat{s}$, and the proof is complete.

Corollary 5.9. *Suppose that the graph Γ is fixed by $Q = \langle s\alpha, s\beta\rangle$, where α and β are involutions. If s is weakly compatible with α and β then the dihedral group $D = \langle\alpha, \beta\rangle$ is exposable in $\mathcal{S}(\Gamma)$ if and only if every orbit of D on Ω containing no symbol fixed by α or by β has length divisible by four.*

It is clear that a dihedral group $D = \langle\alpha, \beta\rangle$ can stabilise many switching classes. Provided that a switch s is chosen compatible with α and β, a switching class stabilised by D can be constructed by applying Theorem 4.3 to the group Q generated by the switch-permutations $s\alpha$ and s β. Our next result gives a necessary and sufficient condition on a dihedral group D in a permutation representation to be always exposable.

<u>Theorem 5.10</u>. <u>A dihedral group D, represented as a permutation group on Ω, and generated by involutions α and β, is always exposable if and only if at least one of the following three conditions is satisfied</u>.

(1) <u>At least one of α and β fixes no symbol in Ω</u>.

(2) <u>Some orbit of D contains a symbol fixed by α and a symbol fixed by β</u>.

(3) (i) <u>α and β both fix symbols</u>. (ii) <u>The orbits containing symbols fixed by α contain no symbols fixed by β</u>. (iii) <u>Every orbit of D, none of whose symbols is fixed by α or by β has length divisible by four</u>.

<u>Remark 5.11</u>. Conditions (1) and (2) are equivalent to the following condition: for any suitable switch s which is compatible with α and with β, at least one of s and $\hat{s}$ is strongly compatible with both α and β.

<u>Proof of Theorem 5.10</u>. If condition (1) or condition (2) holds then, by Remark 5.11 and Corollary 5.6, the dihedral D is always exposable. If condition (3) holds the result follows by Corollary 5.9.

Conversely if none of conditions (1) - (3) hold, then again by Corollary 5.9 there is a switching class stabilised by D in which D is hidden.

We can simplify Theorem 5.10 in the following way.

Let the dihedral group D be hidden in $\mathcal{S}(\Gamma)$. By Theorem 4.11 $|D|$ is divisible by four, $|D| = 4k$, say. Let $D = \langle \alpha, \beta \rangle$ where α and β are involutions. Consider the dihedral subgroup D' of D, where $D' = \langle (\alpha\beta)^r\alpha\, ,\ (\alpha\beta)^{(2t+r+1)}\alpha \rangle$, where r and t are any non-negative integers. We claim that D' is hidden in $\mathcal{S}(\Gamma)$. Let $\alpha' = (\alpha\beta)^r\alpha$ and $\beta' = (\alpha\beta)^{(2t+r+1)}\alpha$. None of conditions (1) - (3) of Theorem 5.10 holds for involutions α' and β' since none of these conditions hold for α and β. Hence D is hidden in $\mathcal{S}(\Gamma)$ implies that D' is hidden in $\mathcal{S}(\Gamma)$. Now let $2t+1$ be the largest odd divisor of 4k. That is, let $4k = (2t+1)2^{\ell}$, where ℓ is an integer, $\ell \geq 2$. Define D' as before with the above restriction on t. Then D' is a subgroup of D of order 2^{ℓ}. We have proved the following corollary to Theorem 5.10.

Corollary 5.12. Let D_{2k} be a dihedral group of order 4k, and let $4k = (2t+1)2^{\ell}$, where t is a non-negative integer, and ℓ is an integer greater than one. D_{2k} is hidden in the switching class $\mathcal{S}(\Gamma)$ if and only if all the subgroups $D_{2^{\ell-1}}$ of D_{2k} are hidden in $\mathcal{S}(\Gamma)$.

Corollary 5.13. A dihedral group $D_{2^{\ell-1}}$, represented as a permutation group on Ω , and generated by involutions α and β , can be hidden in a switching class if and only if all of the following conditions are satisfied.

(1) Involutions α and β both fix symbols and the orbits of $D_{2^{\ell-1}}$ on Ω containing symbols fixed by α are disjoint from the orbits of $D_{2^{\ell-1}}$ on Ω containing symbols fixed by β.

(2) Involutions α and β have at least one transposition in

<u>common</u>.

<u>Proof</u>. Condition (1) negates conditions (1) and (2) of Theorem 5.10. The proof is complete if we can show that condition (2) negates condition (3) (iii) of Theorem 5.10. We must show that there is at least one orbit of $D_{2^{\ell-1}}$, none of whose symbols is fixed by α or by β, that has an even length not divisible by four. Since the length of each orbit of $D_{2^{\ell-1}}$ on Ω divides 2^{ℓ}, condition (2) establishes the result.

We note that it may be possible to use the methods of this chapter and Chapter 4 to study groups of different structures in their action on switching classes. In particular, it should be possible to analyse all two generator groups using these methods. The problem with groups having three or more generators is that if such a group G is hidden in a switching class, every two generator subgroup of G may be exposable. It would be very interesting to know if a switching class existed, having no representative, and every pair of permutations on the stabiliser generating a subgroup exposable in the switching class.

CHAPTER 6. SWITCHING CLASSES CONTAINING FORESTS, AND GRAPHS OF GIRTH $g \geqslant 5$.

In this chapter we analyse all switching classes containing forests, and graphs of girth $g \geqslant 5$. A forest is a graph with no circuits.

Our analysis will reveal all graphs with the above property (a) that are representatives of their switching classes; (b) that lie in switching classes containing a representative; and (c) that lie in switching classes containing no representative. We include graphs on an odd number of vertices for completeness.

The chapter will be divided into the following sections: (1) forests, (2) graphs with finite girth $g \geqslant 7$, (3) graphs with girth 6, and (4) graphs with girth 5.

Before beginning our analysis we state the following definitions and lemmas, and introduce notation that will be used throughout the chapter.

Definition 6.1. The vertices of Γ are denoted by $V\Gamma$. Let V_i be a subset of $V\Gamma$, then the vertex-subgraph of Γ, denoted $\langle V_i \rangle_\Gamma$, is the graph consisting of the vertices V_i and all the edges of Γ that are incident in Γ only with vertices belonging to V_i.

A switch s on Γ with respect to a subset of $V\Gamma$, is equivalent to the following operation. Partition $V\Gamma$ into two sets, $V\Gamma = V_1 \cup V_2$, where supp s contains all the labels of V_1 and supp $\hat{s}$ contains all the labels of V_2. The vertex-subgraphs $\langle V_1 \rangle_\Gamma$ and $\langle V_2 \rangle_\Gamma$ are preserved.

All the edges of Γ, incident in Γ with a vertex from V_1 and a vertex from V_2, are cancelled. All non-adjacent pairs $\{a, b\}$, where $a \in V_1$, $b \in V_2$, become edges.

Let Γ be a graph with girth $g \geq 5$ or a forest. We analyse the switching class $\mathcal{S}(\Gamma)$ by considering the conditions under which $\mathcal{S}(\Gamma)$ contains a graph Γ' with the same girth as Γ, or, if Γ is a forest, that Γ' is a forest with the same number of components as Γ. Let $s\Gamma = \Gamma'$ and let $V\Gamma = V_1 \cup V_2$, where supp s contains all the labels of V_1 and supp $\hat{s}$ contains all the labels of V_2.

<u>Lemma 6.2</u>. <u>Using the above notation and definitions; if there is an edge $\{a_1, a_2\}$ in $\langle V_i \rangle_\Gamma$ then each vertex in $\langle V_j \rangle_\Gamma$ is adjacent to exactly one of the vertices a_1 and a_2, where $i, j = 1, 2$ and $i \neq j$</u>.

<u>Proof</u>. Immediate from the fact that Γ and Γ' contain no 3-circuits.

<u>Corollary 6.3</u>. <u>If there is an edge in $\langle V_i \rangle_\Gamma$, then there are no edges in $\langle V_j \rangle_\Gamma$, where $i, j = 1, 2$ and $i \neq j$</u>.

<u>Proof</u>. The graphs Γ and Γ' contain no 3-circuits and no 4-circuits.

<u>Corollary 6.4</u>. <u>No two edges of $\langle V_i \rangle_\Gamma$ are incident, where $i = 1$ or 2</u>.

<u>Lemma 6.5</u>. <u>If N_4 is a vertex-subgraph of Γ, then at least three vertices of N_4 are in V_i, where $i = 1$ or 2</u>.

<u>Proof</u>. The resultant graph, on switching N_4 with respect to any two of its vertices, is a 4-circuit.

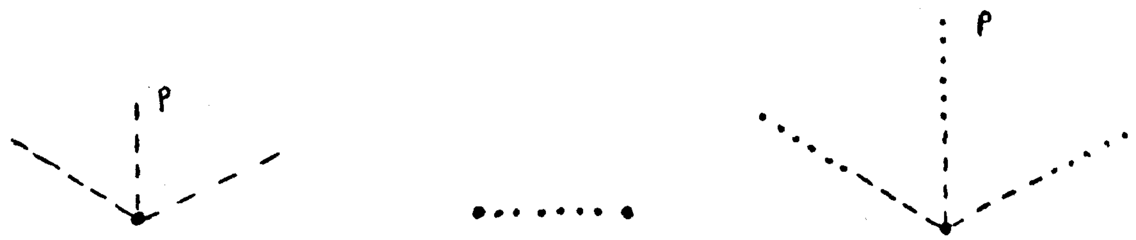
<u>Lemma 6.6</u>. <u>Let Γ and $s\Gamma$ be graphs with girth $g \geq 6$. If $\langle V_i \rangle_\Gamma$ has a non-adjacent pair of vertices $\{a_1, a_2\}$, and $\langle V_j \rangle_\Gamma$ has an edge $\{b_1, b_2\}$, then every other vertex in Γ not adjacent to any of $\{a_1, a_2, b_1, b_2\}$, lies in V_i, where $i, j = 1, 2$ and $i \neq j$.</u>

<u>Proof</u>. Let c be a vertex of Γ not adjacent to any of a_1, a_2, b_1 or b_2 and suppose that vertex c of $V\Gamma$, lies in V_j. Applying Lemma 6.2 to the vertex-subgraph $\langle\{a_1, a_2, b_1, b_2\}\rangle_\Gamma$, we see that either Γ contains the vertex subgraph N_4 consisting of the vertices $\{a_1, a_2, b_2, c\}$ say, contradicting Lemma 6.5, or, when $\{a_\ell, b_\ell\}$ are edges for $\ell = 1$ and 2, $s\Gamma$ contains a 5-circuit.

<u>Lemma 6.7</u>. <u>If Γ and $s\Gamma$ are graphs with girth $g \geq 7$, or forests with the same number of components, and $1 < |\text{supp } s| < n-1$, then there is either one edge or no edges in the graph $\langle V_1 \rangle_\Gamma \cup \langle V_2 \rangle_\Gamma$</u>

<u>Proof</u>. By Corollaries 6.3 and 6.4, we need only consider the possibility that there are at least two disjoint edges in $\langle V_i \rangle_\Gamma$ and no edges in $\langle V_j \rangle_\Gamma$, where $i, j = 1, 2, i \neq j$. Since V_j contains at least two vertices, application of Lemma 6.2 leads to a 4, 5, or 6-circuit in Γ.

In the following analysis the complete bipartite graph $K_{1,p}$ turns up frequently as a vertex-subgraph of significant graphs. We define a <u>p-claw</u> to be the graph $K_{1,p}$. We admit the possibility of a 0-claw, which is an isolated vertex. In diagrams we represent the p-claw as in Figure 4(a). If two vertices are joined by a dotted line, as in Figure 4(b),



(a) (b) (c)

Figure 4

then the presence or absence of an edge between these two vertices is irrelevant. The vertex whose distance is one from every other vertex in a p-claw is called the <u>centre</u> of the p-claw. Figure 4(c) represents a graph on $2p+1$ vertices. It contains a p-claw as a vertex-subgraph. Each of the non-central vertices of the p-claw is adjacent to at most one other vertex of the graph. Each of the remaining p vertices of the graph is adjacent to at most one non-central vertex of the p-claw. There are no other adjacencies in the graph.

(1) <u>Forests</u>.

We shall show that Figure 5 represents all the forests that are not representatives of their switching classes. The graph represented in Figure 5(a) is a tree on $p_1 + p_2 + 2$ vertices, when $\{1, 2\}$ is an edge, and a forest with two components, when $\{1, 2\}$ is not an edge. In Figure 5(a) $p_1 + p_2 \geqslant 1$, and in Figures 5(b) and 5(c), $p_1 + p_2 \geqslant 0$. Figure 5(d) represents a forest on $2p+1$ vertices, where $p \geqslant 1$.

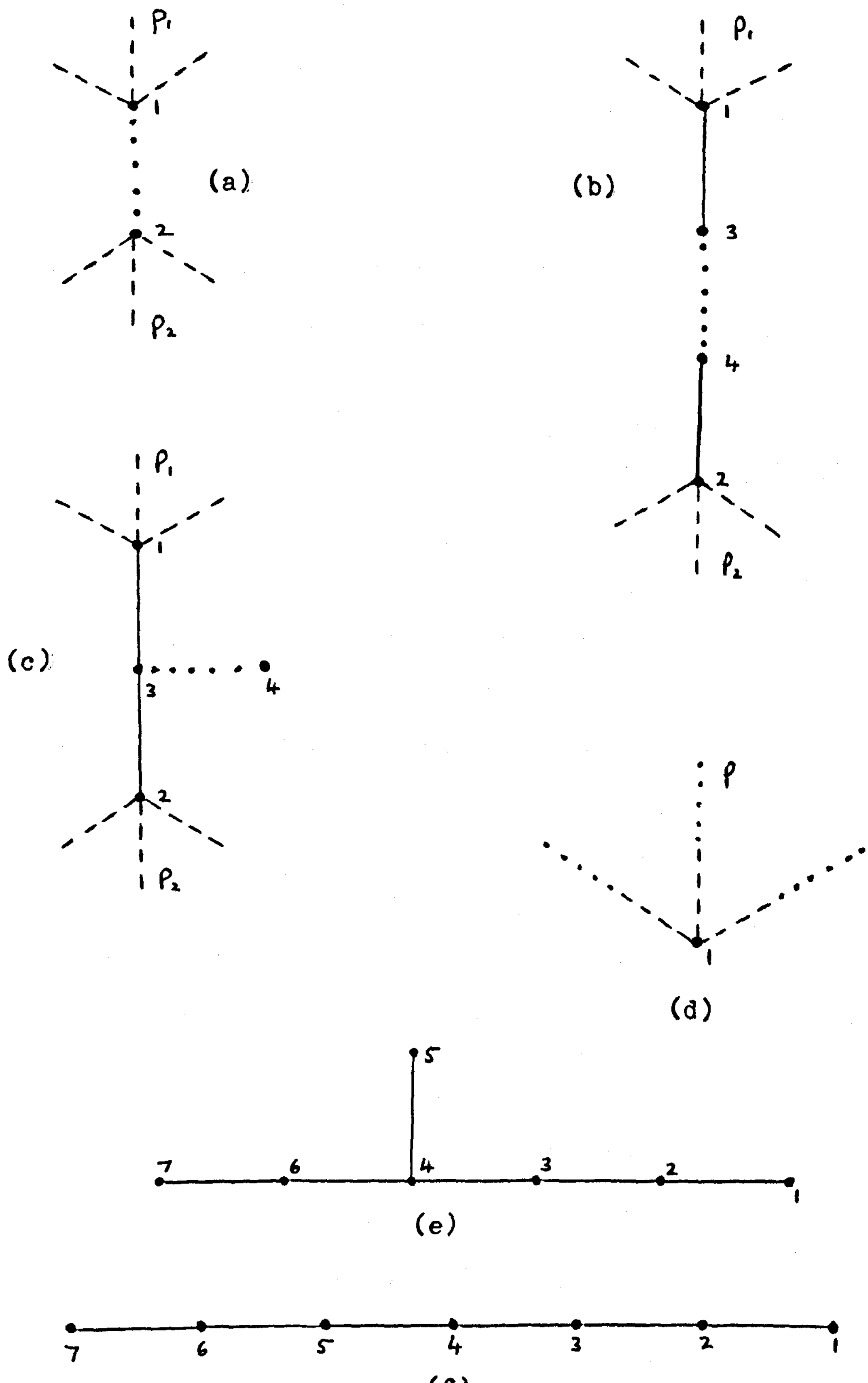


Figure 5

Theorem 6.8. A switching class contains distinct forests F and F', with the same number of components, if and only if F is isomorphic to a forest represented in Figure 5. Furthermore F is isomorphic to F'.

Proof. Let F be isomorphic to a forest in Figure 5. The following switches, with respect to vertices of the graphs of Figure 5, correspond to isomorphisms of those graphs: in Figures 5(a), 5(b) and 5(c), switch with respect to vertices labelled 1 and 2; in Figure 5(d) switch with respect to the vertex labelled 1; and in Figures 5(e) and 5(f), switch with respect to the vertices 1, 3, 5, and 7.

Conversely, suppose F and F' are two forests with the same number of components and lie in the same switching class. Then $sF = F'$ for some switch s. Let $|\text{supp } s| = k$. Since F and F' have the same number of edges, the number of edges in F with a vertex in $\langle V_1 \rangle_F$ and in $\langle V_2 \rangle_F$ is $\frac{1}{2}k(n-k)$, where $|VF| = n$. By Lemma 6.7, consideration of the following three cases will complete the proof:

(i) $1 < k < (n-1)$ and $\langle V_1 \rangle_F \cup \langle V_2 \rangle_F$ has one edge,

(ii) $1 < k < (n-1)$ and $\langle V_1 \rangle_F \cup \langle V_2 \rangle_F$ has no edges, and

(iii) $k = 1$.

Let F have c components.

(i) $1 < k < (n-1)$ and $\langle V_1 \rangle_F \cup \langle V_2 \rangle_F$ has one edge.

Since F has $n-c$ edges, we have

$$\tfrac{1}{2}k(n-k) = n-c-1.$$

For positive integers n and c and with the above restrictions on k, this equation has solutions $c = 1$ and $k = 2$ or $n-2$ only. Application of Lemma 6.2 yields the three trees of Figures 5(a), 5(b) and 5(c).

(ii) $1 < k < (n-1)$ and $\langle V_1 \rangle_F \cup \langle V_2 \rangle_F$ has no edges. In this case the applicable equation is

$$\tfrac{1}{2}k(n-k) = n-c.$$

This equation has solutions, $c = 1$, $n = 7$, and $k = 3$ or 4; or $c = 2$ and $k = 2$ or $n-2$.

On considering the eleven unlabelled trees on seven vertices, we find that the only ones with the appropriate properties are those shown in Figure 5(e) and 5(f). Turning to forests with two components and supposing without loss of generality that $k = 2$, it is straight forward to show that the only forests with the required properties are those of Figures 5(a) and 5(c), where the pairs of vertices $\{1, 2\}$ and $\{3, 4\}$ are non-edges respectively. It will be noted that when the pairs $\{1, 2\}$ and $\{3, 4\}$ of Figure 5(a) and 5(b), respectively, are non-edges then the graphs lie in the same family.

(iii) $k = 1$.

The vertex to be switched has valency $\tfrac{1}{2}(n-1)$. By application of Lemma 6.2 and Lemma 6.4, the $\tfrac{1}{2}(n-1)$ vertices not adjacent to the vertex to be switched, are either isolated or have valency one, and further application of these two lemmas yields the family of Figure 5(d).

We have now discovered all forests that are representatives of their switching classes; namely every forest not isomorphic to one in Figure 5. We consider in more detail the switching classes containing a forest in Figure 5 and determine which of these classes contain representatives. Since we know the answer to this question, quite generally, for graphs on an odd number of vertices, we restrict our analysis to forests on an even number of vertices.

<u>Theorem 6.9. A switching class contains a forest F and no representative if and only if F is isomorphic to a forest of Figure 5(c) with $p_1 = p_2$ and $|VF| \geq 6$.</u>

<u>Proof</u>. We prove first that if F is not isomorphic to a forest in Figure 5(c) with $p_1 = p_2$ and $|VF| \geqslant 6$, then the switching class containing F has a representative. By Theorem 6.8 and Theorem 3.5, we need only consider forests isomorphic to those of Figures 5(a), 5(b) and 5(c) where $p_1 + p_2$ is even.

(i) Let F be a forest of Figure 5(a) where the vertices of F corresponding to those labelled 1 and 2 in 5(a) are adjacent. Then sF is a representative of $\mathcal{S}(F)$, where s is a switch with respect to all the $p_1 + 1$ vertices of the p_1-claw of F, since sF has the unique property in $\mathcal{S}(F)$ that two of its vertices have valency $p_1 + p_2 + 1$. A similar argument holds when vertices 1 and 2 are not adjacent.

(ii) Let F be a forest of Figure 5(b), where the vertices of F corresponding to those labelled 3 and 4 in 5(b) are adjacent. Then sF is a representative of $\mathcal{S}(F)$, where s is a switch with respect to vertex 1 and the $p_1 + 1$ vertices adjacent to vertex 1, since sF has the unique property in $\mathcal{S}(F)$ that two of its vertices have valency $p_1 + p_2 + 3$.

(iii) Let F be a forest of Figure 5(c). We note first that if either p_1 or p_2 is zero and $\{3, 4\}$ is an edge then F is isomorphic to a tree of Figure 5(a). So we suppose that $p_1, p_2 > 0$ and $p_1 \neq p_2$, or if p_1 or p_2 is zero then $\{3, 4\}$ is a non-edge. In this case we have

$$s_1 s_2 F = (12)(34)F,$$

and no other switch on F is equivalent to an isomorphism of

F. Since $p_1 \neq p_2$, there is no automorphism of F interchanging 1 and 2, and so Stab $\mathcal{S}(F)$ = Aut(s,F). Therefore $s_1 F$ is a representative of $\mathcal{S}(F)$.

Conversely, let F be a forest of Figure 5(c) with $p_1 = p_2$ and $|VF| \geqslant 6$. Let σ be the involution (12)(34), and let π be the involution interchanging 1 and 2, interchanging all vertices in the two p_1-claws of F and fixing 3 and 4. Then $\sigma F = s_1 s_2 F$, and $\pi F = F$, and the dihedral group $D_2 = \langle \sigma, \sigma\pi \rangle$ is hidden in $\mathcal{S}(F)$ by Corollary 5.9, where $Q = \langle s_1 s_2 \sigma, s_1 s_2 \sigma\pi \rangle$. Therefore $\mathcal{S}(F)$ contains no representative.

Theorem 6.8 can be used to give examples of switching classes with trivial stabilisers. The tree T of Figure 6 has Aut T = $\{1\}$. By Theorem 6.8, T is a representative of $\mathcal{S}(T)$. Hence Stab $\mathcal{S}(T) = \{1\}$.

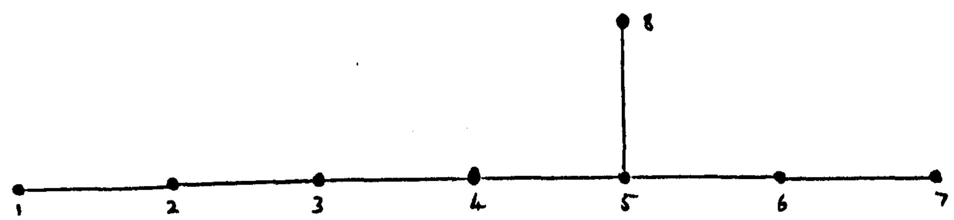


Figure 6

(2) Graphs of finite girth $g \geqslant 7$.

Theorem 6.10. Let Γ be a graph with finite girth $g \geqslant 7$. Then $\mathcal{S}(\Gamma)$ has no representative if and only if Γ is an 8-circuit.

Proof. Example 4.1 proves that if $\mathcal{S}(\Gamma)$ contains a representative then Γ is not an 8-circuit. Conversely, suppose that Γ is not an 8-circuit. We will prove that Γ is a representative. We suppose that Γ and $s\Gamma$ both have girth $g \geq 7$, and show a contradiction.

By Lemma 6.7, consideration of the following three cases will complete the proof: (i) $|\mathrm{supp}\ s| = 1$ or $n-1$; (ii) $2 \leq |\mathrm{supp}\ s| \leq n-2$ and $\langle V_1 \rangle_\Gamma \cup \langle V_2 \rangle_\Gamma$ has no edges; (iii) $2 \leq |\mathrm{supp}\ s| \leq n-2$ and $\langle V_1 \rangle_\Gamma \cup \langle V_2 \rangle_\Gamma$ has one edge.

(i) $|\mathrm{supp}\ s| = 1$ or $n-1$.

Since Γ has a circuit length $g \geq 7$, there are incident edges in one of the graphs $\langle V_1 \rangle_\Gamma$ or $\langle V_2 \rangle_\Gamma$, contradicting Corollary 6.4.

(ii) $2 \leq |\mathrm{supp}\ s| \leq n-2$ and there are no edges in $\langle V_1 \rangle_\Gamma \cup \langle V_2 \rangle_\Gamma$.

In this case Γ is bipartite and consequently there are no odd length circuits in Γ. Therefore Γ either has even girth $g \geq 10$ or girth 8 and $|V\Gamma| > 8$. In either case Γ will contain an N_4 vertex-subgraph, two vertices of which will be in $\langle V_1 \rangle_\Gamma$ and two in $\langle V_2 \rangle_\Gamma$, contradicting Lemma 6.5.

(iii) $2 \leq |\mathrm{supp}\ s| \leq n-2$ and there is one edge in $\langle V_1 \rangle_\Gamma \cup \langle V_2 \rangle_\Gamma$.

Application of Lemma 6.2, Lemma 6.5, and Lemma 6.6 yields a contradiction in this case.

(3) <u>Graphs of girth 6</u>.

In Figure 7(a), p_1, p_2 and p_3 are any non-negative integers.

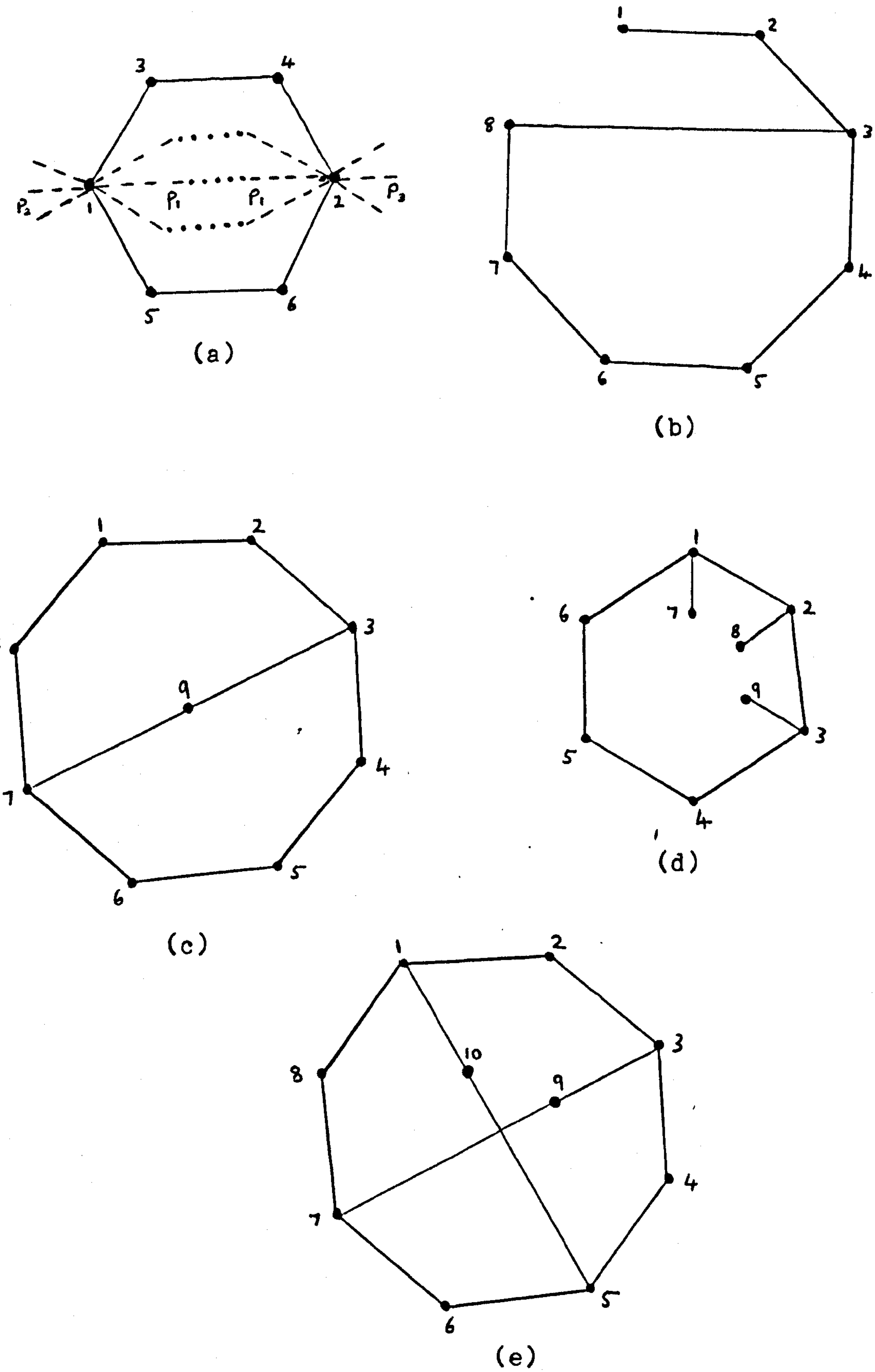


Figure 7

<u>Theorem 6.11. A switching class contains distinct isomorphic graphs, Γ and Γ', of girth 6 if and only if Γ is isomorphic to a graph of Figure 7.</u>

<u>Proof</u>. Suppose Γ is isomorphic to a graph of Figure 7. The following switches with respect to the vertices of the graphs of Figure 7 correspond to isomorphisms of those graphs: switch vertices 1 and 2 in the graph of Figure 7(a); switch vertices 1, 2 and 3 in the graph of Figure 7(d), and switch vertices 1, 3, 5 and 7 in the graphs of Figure 7(b), 7(c) and 7(e).

Conversely, let Γ and Γ', two distinct isomorphic graphs of girth 6, lie in the same switching class. Let $s\Gamma = \Gamma'$ and $|\text{supp } s| = k$. By Corollary 6.3 and Corollary 6.4, consideration of the following two cases will complete the proof. (i) $\langle V_1\rangle_\Gamma \cup \langle V_2\rangle_\Gamma$ has no edges, and (ii) $\langle V_2\rangle_\Gamma$ is the union of disjoint edges and $\langle V_1\rangle_\Gamma$ has no edges.

(i) The graph $\langle V_1\rangle_\Gamma \cup \langle V_2\rangle_\Gamma$ has no edges implies that Γ is bipartite and, since Γ has girth 6, $3 \leqslant k \leqslant n-3$. In the case that $k = 3$, application of Lemma 6.5 leads to a graph isomorphic to the graph of Figure 7(d). In the case that $k = 4$ and $n \geqslant 8$, application of Lemma 6.5 yields graphs isomorphic to those of Figures 7(b), 7(c) and 7(e). The case that $5 \leqslant k \leqslant n-5$ leads to a contradiction, as application of Lemma 6.5 gives a graph containing a 4-circuit.

(ii) The graph $\langle V_2\rangle_\Gamma$ is the union of disjoint edges and $\langle V_1\rangle_\Gamma$ has no edges implies that $k = |V_1| = |\text{supp } s| = 2$, since, by Lemmas 6.5 and 6.6, if $k \geqslant 3$ then Γ contains a 4-circuit or a 5-circuit. It follows immediately that Γ is isomorphic to the graph of Figure 7(a).

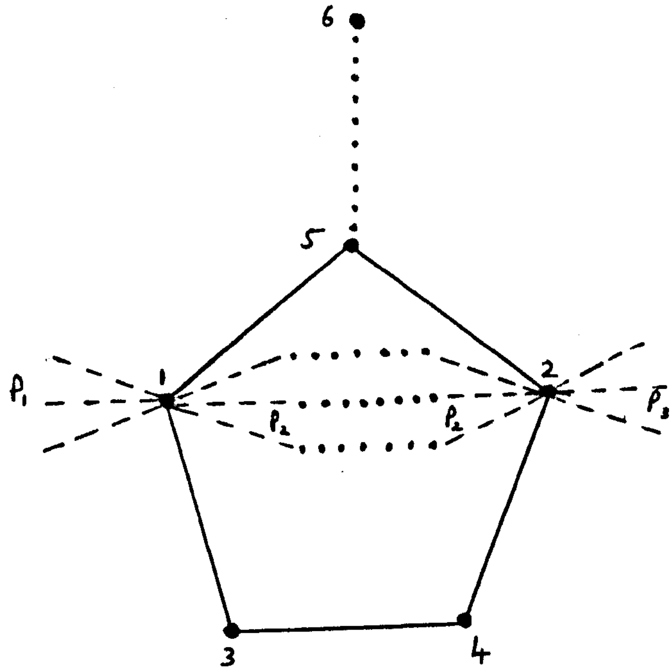
Theorem 6.12. <u>Let Γ be a graph with girth 6. Then $\mathcal{S}(\Gamma)$ has no representative if and only if Γ is isomorphic to the graph of Figure 7(b) or 7(e).</u>

<u>Proof</u>. Let Γ be a graph isomorphic to the graph of Figure 7(b). Switching with respect to the vertices 1, 3, 5 and 7 of Figure 7(b) corresponds to an isomorphism $\sigma = (13)(26)(57)$ of the graph, and the involution $\pi = (48)(57)$ is an automorphism of this graph. The group $\langle\sigma , \sigma\pi\rangle$ is hidden in its switching class. Similarly the group $\langle\sigma', \sigma'\pi'\rangle$, where $\sigma' = (26)(48)(9\ 10)$ and $\pi' = (13)(48)(57)(9\ 10)$, is hidden in the switching class of the graph of Figure 7(e).

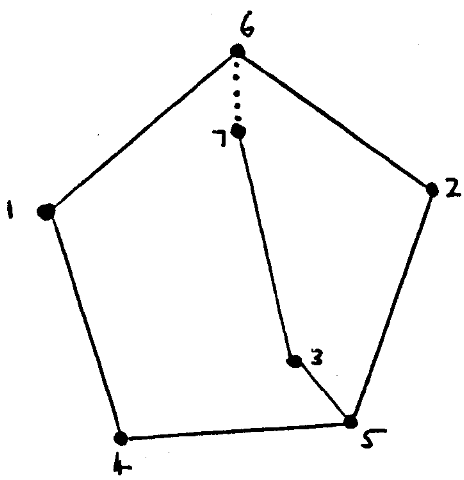
Conversely, we show that if Γ is a graph isomorphic to the graph of Figure 7(a), then $\mathcal{S}(\Gamma)$ contains a representative. Let at least one of the integers p_1, p_2 and p_3 be positive. Let s be a switch with respect to the vertices in Γ corresponding to the vertex in Figure 7(a) labelled 1, and all the vertices adjacent to vertex 1. Then $s\Gamma$ is a representative of $\mathcal{S}(\Gamma)$ since $s\Gamma$ is the unique graph in its switching class that has two vertices with valency n-1. Now let $p_1 = p_2 = p_3 = 0$, then Γ is a 6-circuit and $\mathcal{S}(\Gamma)$ has the representative $K_2 \cup K_2 \cup K_2$.

By Theorem 6.11 all other graphs of girth 6 are representatives, except those of Figure 7(c) and 7(d), which have an odd number of vertices.

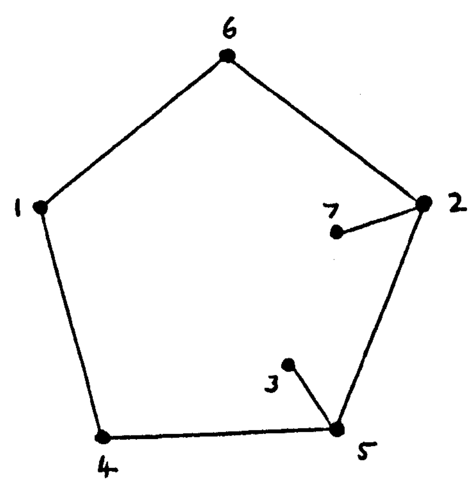
(4) Graphs of girth 5.



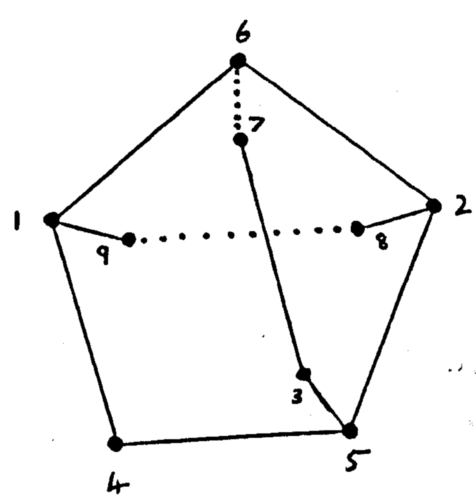
(a)



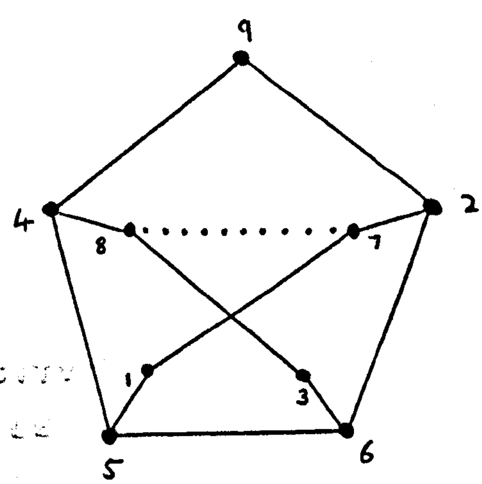
(b)



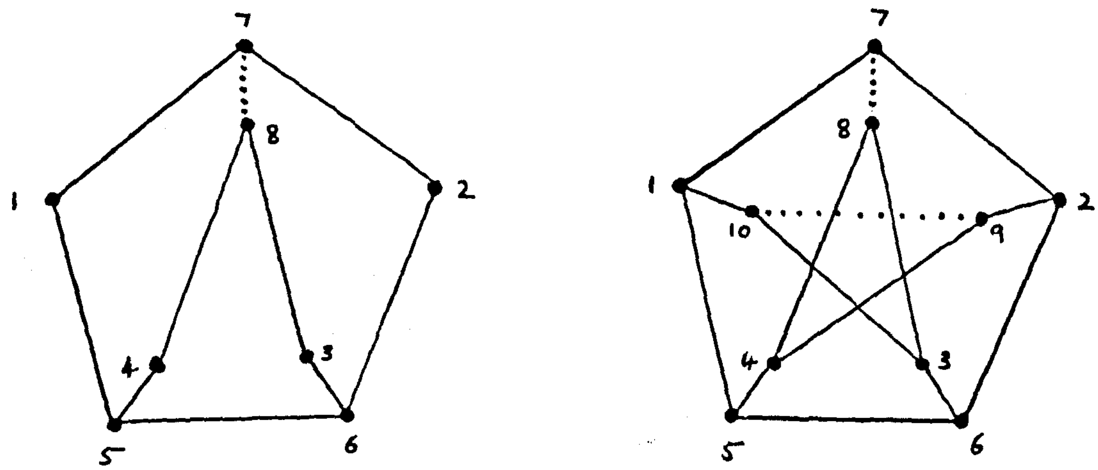
(c)



(d)



(e)



(f) (g)

Figure 8

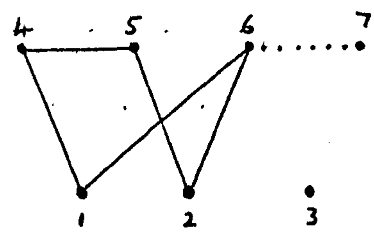
In Figure 8(a), p_1, p_2 and p_3 are any non-negative integers.

Theorem 6.13. A switching class contains distinct isomorphic graphs, Γ and Γ', of girth 5 if and only if Γ is isomorphic to a graph of Figure 8.

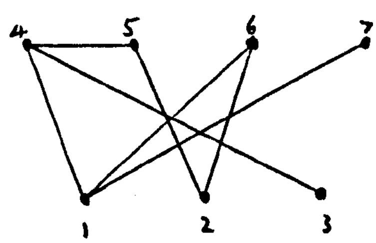
Proof. Suppose Γ is isomorphic to a graph of Figure 8. The following switches with respect to vertices of the graphs of Figure 8 correspond to isomorphisms of those graphs. Switch vertices 1 and 2 in the graph of Figure 8(a); switch vertices 1, 2 and 3 in the graphs of Figures 8(b), 8(c) and 8(d), and switch vertices 1, 2, 3 and 4 in the graphs of Figures 8(e), 8(f) and 8(g).

Conversely, let Γ and Γ' be distinct isomorphic graphs of girth 5 in the same switching class. Let $s\Gamma = \Gamma'$ and $|\text{supp } s| = k$. We prove that Figure 8 gives all graphs with this property by considering all possible values of k. Clearly $1 < k < n-1$. When $k = 2$ Lemma 6.2 and Lemma 6.5 give the graphs of Figure 8(a). We give a detailed proof of the case k = 3.

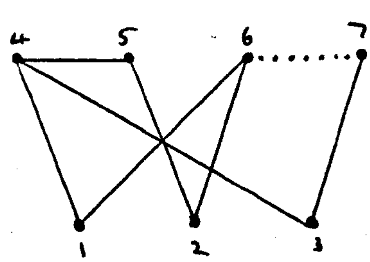
Let supp s = $\{1, 2, 3\}$. Since Γ and Γ' have the same number of edges, $|V\Gamma|$ is odd; also Γ contains a 5-circuit so $|V\Gamma| \geq 7$. We begin by displaying minimum graphs on seven vertices with the above properties, and use Lemmas 6.2 and 6.5 to extend these graphs, by adding edges and vertices, to graphs with the required properties.



We observe that, by Lemma 6.2, vertex 3 is adjacent to vertex 4 or vertex 5. Without loss of generality, let $\{3, 4\}$ be an edge. We further note that the set $\{1, 3, 5, 7\}$ forms the vertex set of a N_4 subgraph of our graph, in contradiction to the conditions of Lemma 6.5. Since $\{1, 5\}$ and $\{3, 5\}$ cannot be edges the following graphs result. The pair $\{1, 7\}$ is an edge implies that $\{6, 7\}$ is not an edge, and the graph



is isomorphic to the graph of Figure 8(c). The pair $\{3, 7\}$ is an edge yields the graphs



, isomorphic to the graphs of Figure 8(b). Retaining k = 3 and extending to nine vertices leads to graphs isomorphic to the graphs of Figure 8(d) Extension beyond nine vertices, retaining k = 3, is impossible using Lemma 6.5 and that Γ has girth 5.

Similarly, putting $k = 4$ and $n \geqslant 8$ leads to graphs isomorphic to those of Figures 8(e), 8(f), 8(g) and a graph isomorphic to the graph of Figure 8(a) where $p_1 = p_3 = 0$, $p_2 = 1$ and $\{5, 6\}$ is an edge.

Using Lemma 6.5 we see that $5 \leqslant k \leqslant n-5$ contradicts the girth restriction on Γ. Hence Figure 8 displays all graphs of girth 5 with the required property.

Theorem 6.14. Let Γ be a graph of girth 5. Then $\mathcal{S}(\Gamma)$ has no representative if and only if Γ is isomorphic to a graph of Figure 8(f), 8(g) or of Figure 8(a) with $p_1 = p_3$.

Proof. The proof is similar to the proof of Theorem 6.12. We note that the graphs of Figures 8(b), 8(c), 8(d) and 8(e) have an odd number of vertices, and that the switching class of a graph of Figure 8(a) has a representative whenever $p_1 \neq p_3$.

We have concluded our analysis of switching classes containing graphs of girth 5. The difficulty of extending the analysis to girth 4 graphs by the methods of this Chapter is that none of the lemmas and corollaries at the beginning of this Chapter are applicable to the case, except Lemma 6.2.

Turning our attention to switching classes with no representative, we conclude this Chapter with the following observation and Example. We observe that if Γ is a graph of girth $g \geqslant 5$ or a forest and $\mathcal{S}(\Gamma)$ has no representative, then Stab $\mathcal{S}(\Gamma)$ has a dihedral subgroup isomorphic to D_2 that is hidden in $\mathcal{S}(\Gamma)$. However, Corollary 5.13 suggests that there may be switching classes with no representative but with all the D_2 subgroups of their stabilisers exposable. Clearly all the graphs in such a switching class would have girth 4 or girth 3. Example 6.15 demonstrates that this possibility is a reality.

Example 6.15.

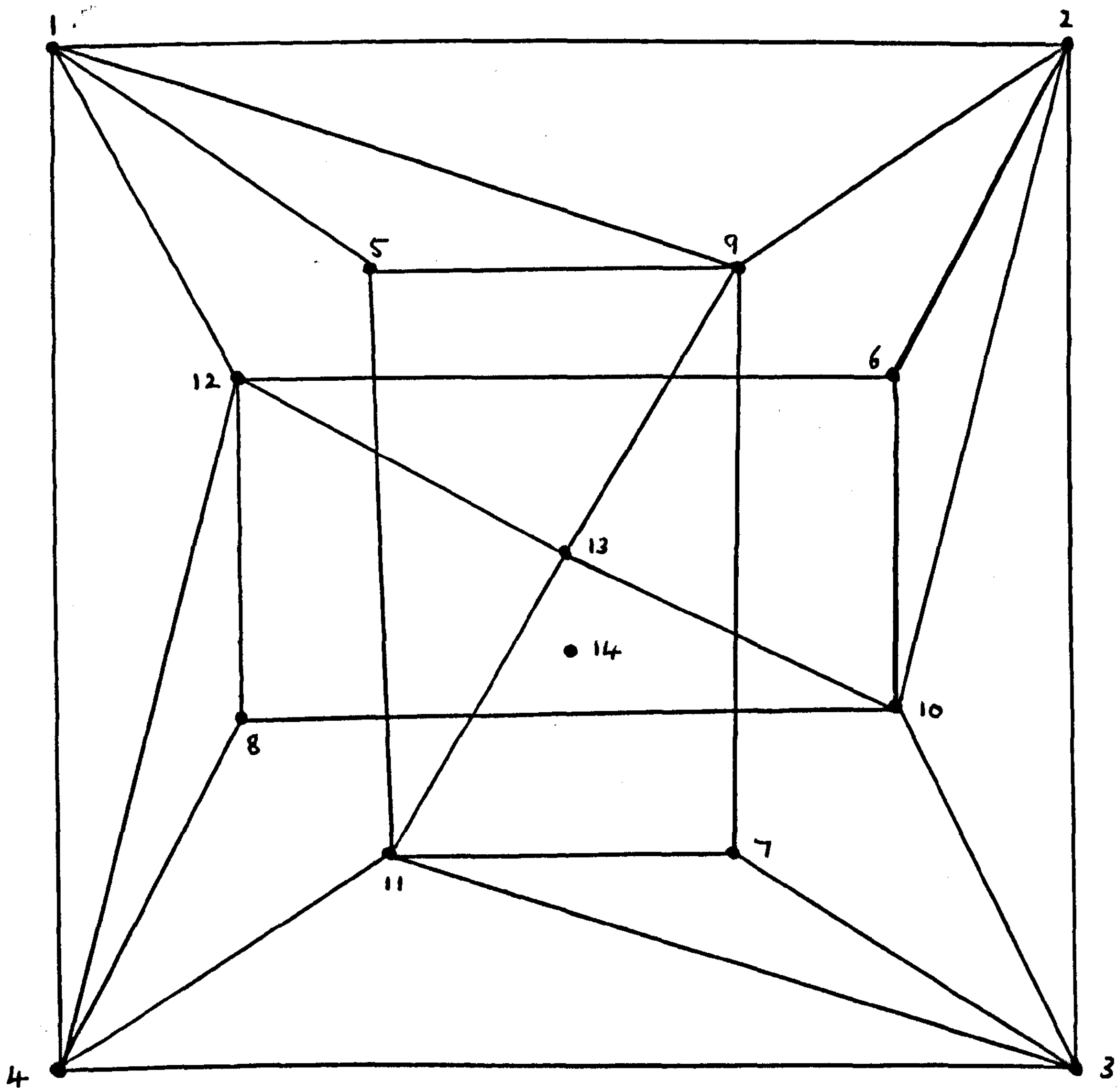


Figure 9

Let Γ be the graph on 14 vertices represented in Figure 9. It is easy to verify that switch-permutations $s\alpha$ and $s\beta$ fix Γ, where $s\alpha = (12)(34)(56)(78)(\bar{9}\ \bar{11})(\bar{10})(\bar{12})(13\ 14)$ and $s\beta = (1)(3)(24)(5)(7)(68)(\bar{9}\ \bar{10})(\bar{11}\ \bar{12})(13\ 14)$. We observe that $\mathcal{S}(\Gamma)$ has no representative, since $D = \langle \alpha, \beta \rangle$ is hidden in $\mathcal{S}(\Gamma)$. We also observe that Aut $\Gamma = \langle \alpha\beta \rangle$. Furthermore s is the only switch on Γ corresponding to an isomorphism of Γ. We prove this by noting that vertex 14 is isolated. Any switch corresponding to an isomorphism of Γ must not only isolate a vertex, but must have a support of even order, and vertex 13 is the only vertex in Γ with a positive even valency. Therefore Stab $\mathcal{S}(\Gamma) = \langle \alpha, \beta \rangle \cong D_4$, and all proper subgroups of Stab $\mathcal{S}(\Gamma)$ are exposable in $\mathcal{S}(\Gamma)$.

CHAPTER 7. SWITCHING CLASSES CONTAINING STRONG GRAPHS

Seidel [6] [7] [8] and Cameron and Van Lint [2] have extensively studied 'strong graphs'. In this final chapter we briefly mention some basic aspects of their work that are relevant to our study. We give a necessary condition for a switching class to contain a 'strong graph' and no representative and we give examples of switching classes of 'strong graphs' that contain no representative. We begin by introducing our notation and defining the property of strength.

Notation 7.1. Let Γ be a graph with an edge $\{i_1, j_1\}$ and a non-edge $\{i_2, j_2\}$. Let $u_1(i_1, j_1)$ be the number of vertices of Γ, not including j_1, adjacent to i_1 and not adjacent to j_1; let $u_2(i_1, j_1)$ be the number of vertices of Γ, not including i_1, adjacent to j_1 and not adjacent to i_1; let $v_1(i_2, j_2)$ be the number of vertices of Γ adjacent to i_2 and not adjacent to j_2, and let $v_2(i_2, j_2)$ be the number of vertices of Γ adjacent to j_2 and not adjacent to i_2.

We are interested in graphs with the property that $u_1(i_1, j_1) + u_2(i_1, j_1)$ and $v_1(i_2, j_2) + v_2(i_2, j_2)$ are the same numbers for all vertices i_1 and j_1 adjacent, and i_2 and j_2 non-adjacent, respectively.

Definition 7.2. A graph Γ is strong if there exist integers u and v such that for all edges $\{i_1, j_1\}$ of Γ, $u_1(i_1, j_1) + u_2(i_1, j_1) = u$, and for all non-edges $\{i_2, j_2\}$ of Γ, $v_1(i_2, j_2) + v_2(i_2, j_2) = v$. If Γ is complete

<u>then we define $v = 0$, and if Γ is empty we define $u = 0$</u>.

<u>Example 7.3</u>. (i) The complete bipartite graph $K_{a,b}$ is strong with $u = a+b-2$ and $v = 0$.
(ii) The graph of Figure 8(a), with $p_1 = p_2 = p_3 = 0$ and $\{5, 6\}$ a non-edge, is strong with $u = v = 2$.
(iii) The graph of Figure 8(g), where $\{7, 8\}$ and $\{9, 10\}$ are edges, known as <u>Petersen's graph</u>, is strong, with $u = v = 4$.
(iv) The graph $K_2 \cup K_2 \cup K_2$ is strong, with $u = 0$ and $v = 2$.

The next theorem gives a necessary condition on the parameters of a strong graph for its switching class to have no representative.

<u>Theorem 7.4</u>. <u>Let Γ be a strong graph on n vertices with parameters u and v.</u>
(i) <u>Every graph in $\mathcal{S}(\Gamma)$ is strong with parameters u and v if and only if $u + v = n-2$.</u>
(ii) <u>If $u + v \neq n-2$ then Γ is the only strong graph in $\mathcal{S}(\Gamma)$.</u>

<u>Proof</u>. (i) Let Γ and $s\Gamma$ be strong graphs with parameters u and v. Let i and j be vertices of Γ such that $\{i, j\}$ is an edge of Γ and a non-edge of $s\Gamma$, and let $u_1(i,j)_\Gamma + u_2(i,j)_\Gamma$ be associated with Γ and $v_1(i,j)_{s\Gamma} + v_2(i,j)_{s\Gamma}$ be associated with $s\Gamma$. Then

$$u_1(i, j)_\Gamma + u_2(i, j)_\Gamma = n-2-(v_1(i, j)_{s\Gamma} + v_2(i, j)_{s\Gamma}). \qquad ①$$

In the case that $s\Gamma$ has all the edges of Γ, we choose $\{i, j\}$ to be a non-edge in Γ and an edge in $s\Gamma$ and a similar equation results. Hence we have $u + v = n-2$, as required.

Conversely, we suppose that Γ has parameters u and v where $u + v = n-2$. From the fact that if $\{i, j\}$ is an edge (non-edge) of Γ and $s\Gamma$ then the number $u_1(i, j) + u_2(i, j)$ $(v_1(i, j) + v_2(i, j))$ is the same in both Γ and $s\Gamma$, and from equation 1, it follows immediately that all graphs in $\mathcal{S}(\Gamma)$ are strong with parameters u and v.

(ii) It is clear from the proof of part (i) that if Γ and $s\Gamma$ are strong then it is necessary for the parameters of Γ and $s\Gamma$ to be the same. Hence if $u + v \neq n-2$ then Γ is the only strong graph in $\mathcal{S}(\Gamma)$.

We note that Theorem 7.4 does not give sufficient conditions on a strong graph for its switching class to have no representative. Trivial counterexamples are provided by the switching classes $\mathcal{S}(K_n)$ and $\mathcal{S}(N_n)$. J. J. Seidel [9] has found non-trivial examples of switching classes of strong graphs containing a representative. Graphs with such properties exist on 26 and 30 vertices.

The graph of Example 7.3(i), $K_{a,b}$, lies in $\mathcal{S}(N_{a+b})$. The switching classes containing the graphs of Example 7.3(ii) and 7.3(iii) have no representatives, by Theorem 6.14, and the graph of Example 7.3(iv) is a representative by Theorem 7.4(ii) (and by Theorem 6.8).

We turn now to consider switching classes with doubly transitive stabilisers. We first characterise a switching class of strong graphs in the following lemma.

Lemma 7.5. <u>Let Γ be a graph on n vertices not in $\mathcal{S}(N_n)$.</u> <u>The graph Γ is a strong graph with parameters u and v, where $u + v = n-2$, if and only if $\mathcal{S}(\Gamma)$ contains n distinct graphs of the form $\Gamma' \cup N_1$, where Γ' is a regular graph of valency v on n-1 vertices</u>.

Proof. Suppose that Γ is a strong graph not in $\mathcal{S}(N_n)$ with parameters u and v where $u + v = n-2$. Then, by Theorem 7.4, every graph in $\mathcal{S}(\Gamma)$ is strong with the same parameters. In particular, the graph with a given vertex i isolated is strong. In this graph $v_1(i, j) + v_2(i, j) = v$ for all vertices $j \neq i$. Since $v_1(i, j) = 0$, we have $v_2(i, j) = v$ for all vertices $j \neq i$ and Γ is a regular graph of valency v. If $v = 0$ then $u = n-2$ and $\Gamma \in \mathcal{S}(N_n)$, contradicting our hypothesis, hence $\mathcal{S}(\Gamma)$ contains n distinct graphs of the form $\Gamma' \cup N_1$.

Conversely, suppose that a switching class $\mathcal{S}(\Gamma)$ contains n distinct graphs of the form $\Gamma' \cup N_1$, where Γ' is a regular graph on n-1 vertices of valency v. Since the graph in which vertex i is isolated is unique in $\mathcal{S}(\Gamma)$, for all vertices i, $v_1(i, j) + v_2(i, j) = v$ for all pairs (i, j), $i \neq j$. The result follows immediately.

Lemma 7.5 shows that associated with every switching class of strong graphs on n vertices there are strong, regular graphs on n-1 vertices with valency v and with parameters u and v, where $u + v = (n-1)-1$. A strong, regular graph is called a <u>strongly regular graph</u>. An example of this is the 5-circuit associated with the graph of Example 7.3(ii).

We can now prove the following theorem about doubly transitive stabilisers of switching classes.

<u>Theorem 7.6</u>. <u>Let G be a subgroup of the stabiliser of a switching class $\mathcal{S}(\Gamma)$. If G is doubly transitive on Ω then $\mathcal{S}(\Gamma)$ consists of strong graphs only</u>.

<u>Proof</u>. Let G_i denote the subgroup of G that fixes $i \in \Omega$. Since G is doubly transitive on Ω, G_i is transitive on $\Omega \setminus \{i\}$, for all $i \in \Omega$. Now G_i is a subgroup of the automorphism group of the graph Γ_i in $\mathcal{S}(\Gamma)$ in which the vertex labelled i is isolated. Therefore either Γ_i is the graph N_n, where $|V\Gamma| = n$, or Γ_i is the union of a regular graph on n-1 vertices of valency $v > 0$, and an isolated vertex labelled i. Since this argument follows for each $i \in \Omega$, by Lemma 7.5, $\mathcal{S}(\Gamma)$ consists of strong graphs only.

If we consider switching classes that do not contain K_n or N_n, then we can make a further comment on the structure of doubly transitive stabilisers.

<u>Lemma 7.7</u>. <u>The stabiliser of a switching class $\mathcal{S}(\Gamma)$ is 3-transitive on Ω if and only if $\mathcal{S}(\Gamma)$ is the switching class $\mathcal{S}(K_n)$ or $\mathcal{S}(N_n)$</u>.

<u>Proof</u>. Clearly Stab $\mathcal{S}(K_n)$ = Stab $\mathcal{S}(N_n)$ = Σ, which is 3-transitive on Ω.

Conversely, suppose that Stab $\mathcal{S}(\Gamma)$ is 3-transitive on Ω. Then (Stab $\mathcal{S}(\Gamma)$ $)_i$ is doubly transitive on $\Omega \setminus \{i\}$, and either Γ_i is the graph N_n or Γ_i is the graph $K_{n-1} \cup N_1$. Hence $\mathcal{S}(\Gamma)$ is the switching class $\mathcal{S}(K_n)$ or $\mathcal{S}(N_n)$.

We illustrate Theorem 7.6 with the following example.

Example 7.8.

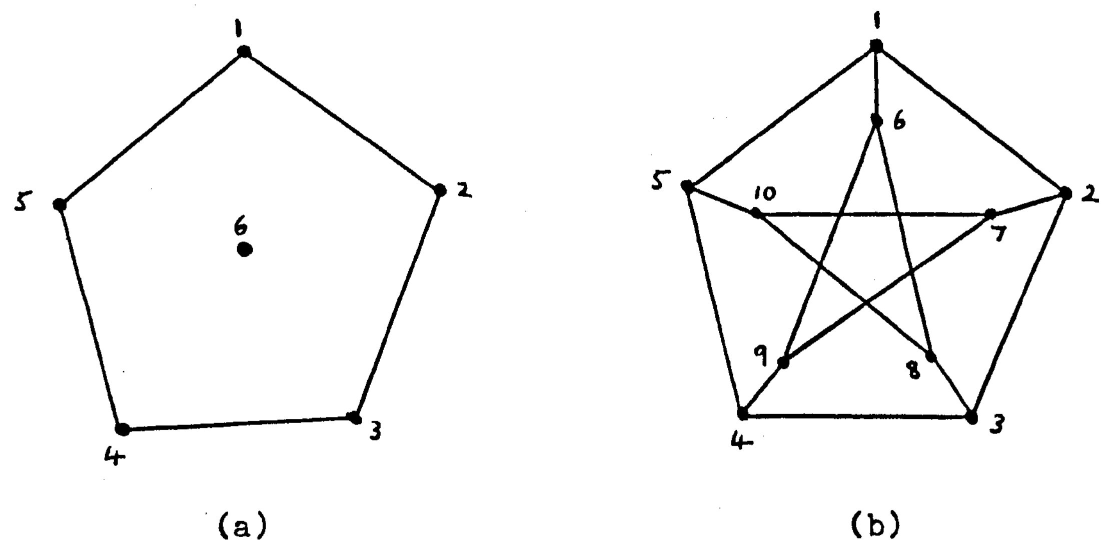


Figure 10

The graphs in Figure 10(a) and 10(b) are the graphs mentioned in Examples 7.3(ii) and 7.3(iii), respectively.

The stabiliser of the switching class of the graph of Figure 10(a) is isomorphic to A_5, the group of all even permutations of 5 symbols. A_5 is doubly transitive in its representation on 6 symbols. The stabiliser of the switching class containing Petersen's graph is isomorphic to the symmetric group on 6 symbols, which is doubly transitive in its representation on 10 symbols.

The switching classes of the graphs of Example 7.8 are particular examples of a general class of switching classes that can be constructed in the following way. Let n-1 be a prime power with $n \equiv 2 \pmod 4$. The graph Γ' has a vertex set consisting of the elements of GF(n-1), with two vertices adjacent if and only if their difference is a non-zero square. The graph Γ' is strongly regular on n-1 vertices with parameters $u = v = \frac{1}{2}(n-2)$ and valency $\frac{1}{2}(n-2)$. A graph constructed in the above way is called a <u>Paley graph</u>. Let Γ be the graph $\Gamma' \cup N_1$. The stabiliser Stab $\mathcal{S}(\Gamma)$ is doubly transitive on Ω, where $|\Omega| = n$. For a proof of this see, for example, Seidel [7], page 507.

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