Keele UNIVERSITY

CONTRIBUTIONS TO THE THEORY OF LINEAR TOPOLOGICAL SPACES

Thesis submitted for the Degree of Doctor of Philosophy

at the University of Keele

by

SUNDAY OSARUMWENSE IYAHEN B.Sc.

Department of Mathematics,
University of Keele.

June, 1967.

ACKNOWLEDGEMENTS

I wish to thank

Professor A.P. Robertson, my supervisor, for his help in carrying out the work reported in this thesis and for his constant encouragement and friendly disposition,

Mr. I. Tweddle and Dr. J.W. Baker for useful discussions and correspondence,

Professor A. Olubummo who first stimulated my interest in Functional analysis,

Professors R.A. Rankin and I.N. Sneddon and the members of staff of and research students in the Departments of Mathematics at the Universities of Glasgow and Keele for their friendliness,

my parents, relations and friends for their constant encouragement,

Miss K.B. Davies for her excellent typing of the thesis,

and the Government of the Republic of Nigeria for financial support.

Chapter 1 and parts of Chapter 2 form an introduction to the thesis and contain a brief resume of known theory. In sections 2.4 and 2.6 and from Chapter 3 onwards, the work reported in this thesis is claimed as original, except where it is explicitly stated otherwise. This work has not been submitted for a Higher degree of this or any other University before.

ABSTRACT

This thesis is mainly concerned with linear topological spaces in which local convexity is not assumed. In particular it contains a study of the closed graph and open mapping theorems in this context, together with results analogous to the Banach-Steinhaus theorem. Many of the techniques and notions used to study these important theorems in locally convex spaces are no longer effective for general linear topological spaces and much of this thesis is taken up with the development of alternative methods and definitions.

The first of these is the notion of a *-inductive limit of linear topological spaces. This plays much the same part in the theory of general linear topological spaces as an inductive limit does for locally convex spaces, and natural analogues are proved for most of the known results on inductive limits. After this has been introduced, it is shown that the *-inductive limit topology of a sequence of locally convex spaces is locally convex.

Then a study is made of ultrabarrelled spaces, which replace barrelled spaces in certain theorems when local convexity is not assumed. Also ultrabornological and quasi-ultrabarrelled spaces are defined and studied. Any *-inductive limit of members of one of these classes has the same property. In particular, any *-inductive limit of complete metric linear spaces has the three properties. However, an uncountable direct sum of Banach spaces has none of these properties and none of these properties passes on to closed linear subspaces. Ultrabarrelled spaces are characterised in terms of closed linear maps

into complete metric linear spaces and similar characterisations are given for ultrabornological and quasi-ultrabarrelled spaces in terms of bounded and closed bounded linear maps, respectively. These notions find application in the study of two-norm spaces.

The next section of the thesis looks at semiconvex spaces, spaces in which there is a neighbourhood base of the origin consisting of semiconvex sets. For these, there can be defined a type of inductive limit topology which is in some respects intermediate between that of the ordinary inductive limit of locally convex spaces and *-inductive limit of general linear topological spaces. Such is called a **-inductive limit topology. Similarly there are spaces (called hyperbarrelled spaces) fitting naturally between barrelled spaces and ultrabarrelled spaces, with analogues for bornological and quasi-barrelled spaces. A thorough study is made of these, in which results rather similar to those already found for ultrabarrelled spaces are obtained. For example, hyperbarrelled spaces are characterised in terms of closed linear maps into complete separated locally bounded spaces. It is also shown that any product of separated hyperbarrelled spaces is hyperbarrelled.

Finally, the problem of characterising the sorts of spaces that can be range spaces in various forms of the closed graph theorem is considered. Various general classes $D_r(\mathcal{A}_1;\mathcal{A}, T)$ and $D(\mathcal{A}_1;\mathcal{A}, T)$ of linear topological spaces are defined, generalising in a natural way the B_r-complete and B-complete spaces. These are used to find extensions of the known closed graph and open mapping theorems. The notions are also meaningful for commutative topological groups and, for these, analogues of the known theorems are proved.

CONTENTS

CHAPTER I

INTRODUCTION

Let T be a set of linear maps from a locally convex linear topological space E to another. If E is barrelled then, T is equicontinuous if it is pointwise bounded and each of its members is continuous. This result is often referred to as the Banach-Steinhaus theorem. There are other well known conditions which ensure that T is equicontinuous. One of these requires that E be bornological and T uniformly bounded on bounded sets. Another requires that E be quasi-barrelled and T be a set of continuous linear maps which is uniformly bounded on bounded sets. These results contribute immensely to the importance of barrelled, bornological and quasi-barrelled spaces. One of our main objectives in this thesis is to study classes of linear topological spaces which can be used to replace these in situations where local convexity is not assumed. We also wish to generalise the notions of B-completeness and B_r-completeness, with a view to extending known closed graph and open mapping theorems.

The idea of a *-inductive limit of linear topological spaces is presented in Chapter 3. This plays much the same part in the theory of general linear topological spaces as inductive limit does for locally convex linear topological spaces, and natural analogues are proved for most of the known results on inductive limits. In addition the useful result is established that the *-inductive limit topology of a sequence of locally convex linear topological spaces is locally convex.

W. Robertson, in (37) introduced the notion of an ultrabarrelled space Such spaces effectively replace barrelled spaces in certain important results when local convexity is not assumed. One example of such a phenomenon is in the Banach-Steinhaus theorem ((37),Theorem 5). Another is in the closed graph and open mapping theorems ((37), Proposition 15). In Chapter 4, ultrabarrelled spaces are further studied, and ultrabornological and quasi-ultrabarrelled spaces are defined and studied. These respectively bear the same relationship to bornological and quasi-barrelled spaces as ultrabarrelled spaces do to barrelled ones. It is proved that a *-inductive limit of members of one of the three classes has the same property. In particular, every *-inductive limit of complete metric linear spaces has all three properties. But an uncountable direct sum of Banach spaces has none of these properties and none of these properties passes on to closed linear subspaces. It is shown that a linear topological space E is ultrabarrelled if and only if every closed linear map from E into any complete metric linear space is continuous. Similar characterisations are given for ultrabornological and quasi-ultrabarrelled spaces in terms of bounded and closed bounded linear maps respectively. The last section of this chapter deals with two-norm spaces, where these notions find applications.

Husain, in (14) introduced the classes of countably barrelled and countably quasi-barrelled spaces, classes which respectively include barrelled and quasi-barrelled spaces and for which analogues of the Banach-Steinhaus theorem hold for sequences of mappings into locally convex spaces. In Chapter 5, there is a short section containing counter examples on countably barrelled and countably quasi-barrelled spaces. Also, $\aleph$-ultrabarrelled and $\aleph$-quasi-ultrabarrelled spaces are defined and some results which hold for them are indicated. These spaces are slight generalisations of those which respectively replace countably barrelled and countably quasi-barrelled spaces when considering general linear topological spaces. In the rest of the chapter, a study is made of hyperbarrelled, hyperbornological, quasi-hyperbarrelled, $\aleph$-hyperbarrelled and $\aleph$-quasi-hyperbarrelled spaces. The first three are the spaces which respectively replace barrelled, bornological and quasi-barrelled spaces in the theory of semiconvex spaces. The last two are generalisations of what replace countably barrelled and countably quasi-barrelled spaces in a similar situation. A useful tool in this study is the notion of a **-inductive limit of semiconvex spaces, a concept which is in many ways intermediate between that of an inductive limit of locally convex linear topological spaces and a *-inductive limit of linear topological spaces. Most of the results obtained are similar to those already proved for ultrabarrelled ultrabornological and quasi-ultrabarrelled spaces. One example is that a **-inductive limit of members of one of these classes has the same property. Another is that a semiconvex space E is hyperbarrelled if and only if every closed linear map from E into any complete separated locally bounded space is continuous; with similar characterisations for hyperbornological and quasi-hyperbarrelled spaces in terms of bounded and closed bounded linear maps respectively. It is shown that every countable product of separated hyperbornological spaces is of the same sort, and that any separated product of members of one of the other classes belongs to the class.

Chapter 6 is concerned with the problem of characterising the sorts of spaces that can be range spaces in various forms of the closed graph theorem. Various general classes $D_r(\mathcal{A}_1;\mathcal{A}, T)$ and $D(\mathcal{A}_1;\mathcal{A}, T)$ of linear topological spaces are defined, generalising in a natural way the B_r-complete and B-complete spaces. These are used to describe extensions of the known closed graph and open mapping theorems. The notions of $D_r(\mathcal{A}_1;\mathcal{A}, T)$-spaces and $D(\mathcal{A}_1;\mathcal{A}, T)$-spaces are meaningful for commutative topological groups, and for these, analogues of the known theorems are proved.

Some of the basic information needed and notation used in the rest of the thesis are in Chapter 2. Apart from the material in sections 2.4 and 2.6, most of what is in this chapter can be found in the current literature on linear topological spaces and topological groups.

CHAPTER 2

GENERAL THEORY

2.1 Linear topological spaces

Our linear spaces shall be over the field K of real or complex numbers and it shall be assumed that K has its usual topology. Any topology on a linear space such that addition and scalar multiplication are each continuous simultaneously in both variables, is called linear. A linear space E (over K) on which is defined a linear topology u is called a linear topological space (over K) and denoted by (E,u). Linear topological spaces over the reals were first studied by Von Neumann (28) and Kolmgoroff (21). The definition given here is equivalent to theirs. We shall denote a linear topological space (over K) by l.t.s.

Let E be an l.t.s. If $\mathcal{U}$ is a base of neighbourhoods at zero (the origin) in E then, for any x in E, the family of sets (x + U), as U runs through $\mathcal{U}$ is a base of neighbourhoods at x. Thus a linear topology is completely determined by a base of neighbourhoods of the origin. As a result of this, a linear map from an l.t.s. E into an l.t.s. is continuous (open) at some point of E if and only if it has the property at the origin. Let f be a map of a topological space G into another H. The map f is said to be nearly open if for every neighbourhood U of any point x in G, the closure of f(U) in H is a neighbourhood of f(x) in H. And f is said to be nearly continuous if for any x in G and every neighbourhood V of f(x) in H, the closure of $f^{-1}(V)$ in G is a neighbourhood of x in G. If G, H are linear topological spaces and f a linear map, f is nearly open if and only if for every neighbourhood U of the origin in G, the closure of f(U) in H is a neighbourhood of the origin in H. And f is nearly continuous if and only if for every neighbourhood V of the origin in H, the closure of $f^{-1}(V)$ in G is a neighbourhood of the origin in G. We shall henceforth use the terms "neighbourhood" and "base of neighbourhoods" in an l.t.s. to respectively denote "neighbourhood of the origin" and "base of neighbourhoods of the origin". And in considering the notions of continuity, openness, near continuity and near openness of a linear map from an l.t.s. to another, we shall limit our consideration to behaviour at the origin. There is clearly no loss of generality in doing so.

A subset B of a linear space is called balanced if for every x in B, λx is in B for all λ in K with $|\lambda| \leq 1$. And B is said to be absorbent if for any x in E, there exists a positive number α such that x is in λB for all λ in K with $|\lambda| \geq \alpha$. In an l.t.s., there is a base of neighbourhoods $\mathcal{U}$ say, made up of balanced absorbent sets U such that $U_1 + U_1 \subseteq U$ for some U_1 in $\mathcal{U}$. Conversely, if E is a linear space then a filter base $\mathcal{U}$ of balanced absorbent subsets is a base of neighbourhoods for a linear topology on E if for every U in $\mathcal{U}$, there is a U_1 in $\mathcal{U}$ with $U_1 + U_1 \subseteq U$. In particular since an l.t.s. is regular, any l.t.s. has a base $\mathcal{U}$ of closed balanced absorbent neighbourhoods such that for any U in $\mathcal{U}$, there is U_1 in $\mathcal{U}$ with $U_1 + U_1 \subseteq U$.

A subset A of a linear space is said to absorb a subset B, if for some positive number α, $B \subseteq \lambda A$ for all λ in K with $|\lambda| \geq \alpha$. A subset of an l.t.s. which is absorbed by every neighbourhood is said to be bounded. A subset of an l.t.s. which absorbs bounded sets is called bornivorous. A subset B of an l.t.s. E is bounded if and only if for any sequence (x_n) of points of B and any sequence (λ_n) of positive real numbers converging to zero, the sequence $(\lambda_n x_n)$ converges to the origin in E. A linear map from one l.t.s. to another is called bounded if it maps bounded sets to bounded sets. A linear map f from an l.t.s. E to another l.t.s. F is said to be sequentially continuous if for every sequence (x_n) converging to some x in E, $(f(x_n))$ converges to f(x) in F. Thus a sequentially continuous linear map from an l.t.s. to another is bounded. In particular a continuous linear map from an l.t.s. to another is bounded.

We say that a topological space is separated if it satisfies Hausdorff's separation axiom. An l.t.s. is separated if and only if the intersection of members of a base of neighbourhoods is the origin. An l.t.s. E need not be complete, but can be embedded uniquely as a dense subspace of a complete l.t.s. $E^\wedge$ ((18), Chapter 2, 7.10), called the completion of E. If $\mathcal{U}$ is a base of neighbourhoods for the topology of E then, the closures in $E^\wedge$ of members of $\mathcal{U}$ is a base of neighbourhoods for the topology of $E^\wedge$. An l.t.s. E is said to be sequentially complete if every Cauchy sequence in E converges. The space E is said to be quasi-complete if every closed bounded subset of E is complete. Since each Cauchy sequence in an l.t.s. is bounded, any quasi-complete l.t.s. is sequentially complete.

A subset B of a linear space is called convex if for any points x,y in B, $\lambda x + (1 - \lambda)y$ is in B for all real λ between zero and one. A balanced convex set in a linear space is said to be absolutely convex. An l.t.s. is called a locally convex l.t.s. if it has a base of convex neighbourhoods. We shall henceforth refer to a locally convex l.t.s. as a locally convex space and call its topology a locally convex topology or more shortly a convex topology. A closed absolutely convex absorbent subset of an l.t.s. is called a barrel. A locally convex space has a base of neighbourhoods consisting of barrels. The difference between a locally convex space and a non-locally convex l.t.s. is of basic importance in the study of these spaces. If E is a linear space, any linear map from E to K is called a linear functional on E, and the linear space of all linear functionals on E is called the algebraic dual of E and denoted by E^*. For an l.t.s. E, the linear subspace of E^* consisting of continuous linear functionals on E is called the dual of E and denoted by E'. A linear subspace F of E^* such that for each non-zero x_o in E, there is f in F for which $f(x_o) \neq 0$ is said to separate the points of E. By the Hahn-Banach extension theorem (see (36), Chapter 2, Theorem 3, Corollary), if E is a separated locally convex space then, E' separates the points of E.

If a separated l.t.s. E is not locally convex, E' may or may not separate the points of E, for example, see Day (6), Walters (40).

Let (E,u) be an l.t.s. with dual E'. The absolute convex u-neighbourhoods form a base of neighbourhoods for the finest convex topology on E coarser than u. This topology is called the convex topology derived from u and denoted by u^{oo}. The dual of (E,u^{oo}) is E' and (E,u^{oo}) is separated if and only if E' separates the points of E. If E is an infinite dimensional linear space and F a linear subspace of E^* which separates the points of E, various separated linear topologies can be defined on E with F (= E') as dual. As shown by Mackey (25), there is a finest as well as coarsest convex topology on E with E' as dual. The coarsest one is called the weak topology on E with E' as dual, and denoted by $\sigma(E,E')$ while the finest is called the Mackey topology with E' as dual, and denoted by $\tau(E,E')$. Since for any linear topology u on E, $(E,u)' = (E,u^{oo})'$, $\sigma(E,E')$ is the coarsest linear topology on E for which every x' in E' is continuous. The space E may be identified with a linear subspace of E'^* such that with this identification, E separates the points of E'. Thus we have the topologies $\sigma(E',E)$ and $\tau(E',E)$ on E'.

Let E be a separated l.t.s. with dual E' separating the points of E. If A is a subset of E, the subset of E' consisting of continuous linear functionals on E not exceeding unity in absolute value on A is called the polar of A (in E') and denoted by A^o. The set A^o is absolutely convex. Furthermore, it is absorbent if A is $\sigma(E,E')$-bounded.

Thus if $\mathcal{A}$ is a set of $\sigma(E,E')$-bounded subsets of E, there is a coarsest convex topology $v(\mathcal{A})$ say, on E' for which the polars of all members of $\mathcal{A}$ are neighbourhoods. A base of neighbourhoods for this topology is the scalar multiples of finite intersections of polars of members of $\mathcal{A}$. The topology $\sigma(E',E)$ is $v(\mathcal{A})$, when $\mathcal{A}$ is the set of all finite subsets of E, while the topology $\tau(E',E)$ is $v(\mathcal{A})$ when $\mathcal{A}$ is the set of all absolutely convex $\sigma(E,E')$-compact subsets of E. If $\mathcal{A}$ is the set of all $\sigma(E,E')$-bounded subsets of E, then $v(\mathcal{A})$ is finer than $\tau(E',E)$. It is then called the strong topology on E' and denoted by $\beta(E',E)$. The topologies $\sigma(E,E')$, $\tau(E,E')$ and $\beta(E,E')$ are similarly described in terms of polars of sets of $\sigma(E',E)$-bounded subsets of E' Clearly, a weak topology is independent of any topology on the dual space.

If E is a separated locally convex space, it often happens that a problem in E has an equivalent formulation in E' (under some suitable topology) which may be more easily tackled. Since for a separated non-locally convex space E, E' may consist of only the zero functional, this duality theory is not available for non-locally convex spaces. This partially accounts for the fact that comparatively little is known about linear topological spaces which we do not assume locally convex.

Throughout this thesis, the Hebrew alphabet $\aleph$ shall denote an arbitrary cardinal number and $\aleph_0$ shall denote the cardinal number of a countable set.

2.2 Upper bound and lower bound topologies for linear spaces

Let E be a linear space and suppose that for each α in an index set Γ, λ_α is a linear topology on E with $\mathcal{U}_\alpha$ as a base of neighbourhoods. Denote by $\mathcal{U}$ the family of sets

$$(U = \bigcap_{\alpha\epsilon\phi} U_\alpha : U_\alpha \epsilon \mathcal{U}_\alpha)$$

as ϕ varies over all finite subsets of Γ. Then $\mathcal{U}$ is a base of neighbourhoods for a linear topology λ say, on E which is the coarsest linear topology on E finer than λ_α for all α in Γ. The topology λ is called the upper bound of the set (λ_α). It is easy to see that λ is convex if each λ_α is. Also, a linear map from an l.t.s. F into (E,λ) is continuous if and only if it is continuous from F into (E,λ_α) for each α in Γ. The finest linear topology on a linear space E is the upper bound of all linear topologies on E. We shall denote this topology by s. The finest convex topology $(\tau(E,E^*))$ on E is the upper bound of all convex topologies on E. The topologies s and $\tau(E,E^*)$ coincide if E is at most countably dimensional ((19), Theorem 3.1).

Suppose that F is a linear space and that for each α in an index set Γ, t_α is a linear map of F into an l.t.s. $(E_\alpha,\lambda_\alpha)$. If $\mathcal{U}_\alpha$ is a base of neighbourhoods for λ_α then the inverse images by t_α of members of $\mathcal{U}_\alpha$ form a base of neighbourhoods for a linear topology on F, denoted by $t_\alpha^{-1}(\lambda_\alpha)$. The upper bound v of the set $(t_\alpha^{-1}(\lambda_\alpha) : \alpha\epsilon\Gamma)$ is the coarsest linear topology on F for which each t_α is continuous. If $\bigcap_{\alpha\epsilon\Gamma} t_\alpha^{-1}(0)$ is the origin in F then, (F,v) is called the projective limit of $((E_\alpha;\lambda_\alpha) : \alpha\epsilon\Gamma)$ by $(t_\alpha : \alpha\epsilon\Gamma)$. In this case, (F,v) is locally convex (separated) if each $(E_\alpha, \lambda_\alpha)$ is. Also a linear map g from an l.t.s. G into (F,v) is continuous if and only if $t_\alpha \circ g$ is continuous for each α in Γ.

The upper bound of a set of linear topologies is an example of projective limit topologies. Other examples are the induced topology on a linear subspace of an l.t.s., and the product topology for linear topological spaces. Any topological product contains topological copies of its factors which are closed in the product space if each of its factors is separated; (this follows easily from (18), Chapter 2, 5.9). A product of complete (sequentially complete, quasi-complete) linear topological spaces is of the same sort. By Theorem 1 of (20), any separated l.t.s. is a subspace of a product of metric linear spaces. As an easy consequence of this, any complete separated l.t.s. is a closed subspace of a product of complete metric linear spaces which, by (4), Page 4, Exc. 7, is of the second category (in itself).

Let $(\eta_\alpha : \alpha \varepsilon \Psi)$ be a set of linear topologies on a linear space E. The set Φ of all linear topologies on E which are each coarser than η_α for all α in Ψ is not empty, since it contains the trivial topology. The upper bound v say, of Φ is in Φ. It is the finest linear topology on E coarser than each η_α. The topology v is called the lower bound of $(\eta_\alpha : \alpha \varepsilon \Psi)$. If each η_α is a convex topology then the upper bound u say, of all convex topologies on E which are each coarser than every η_α, is the finest convex topology on E coarser than η_α for arbitrary α. The topology u is an example of what is called an inductive limit topology for locally convex spaces.

For each γ in an index set Φ , let E_γ be a locally convex space and u_γ a linear map of E_γ into a linear space E spanned by $\bigcup_{\gamma\varepsilon\Phi} u_\gamma(E_\gamma)$. The finest convex topology w say, on E for which each u_γ is continuous is known as the inductive limit topology on E of $(E_\gamma : \gamma\varepsilon\Phi)$ by $(u_\gamma : \gamma\varepsilon\Phi)$. We also say that (E,w) is the inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$. A linear map t from (E,w) into a locally convex space is continuous if and only if each $t o u_\gamma$ is continuous. A set T of linear maps from (E,w) into a locally convex space is equicontinuous if and only if each set $T o u_\gamma$ is equicontinuous.

Quotient and direct sum topologies for locally convex spaces are inductive limit topologies.

If (E,w) is the inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$ such that E is the union of the subspaces $u_\gamma(E_\gamma)$, then (E,w) is called the generalized strict inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$. In particular, if Φ is countable (say Φ is the set of positive integers), (E_i) a sequence of strictly increasing linear subspaces of E and the topology of E_i coincides with that induced by E_{i+1}, (E,w) is called the strict inductive limit of (E_i). In this case, if each E_i is a Fréchet space then, (E,w) is called an L.F. space, (see (7)). (For a discussion of projective and inductive limits of locally convex spaces see, for example, (36), Chapters 5 and 7).

2.3 Barrelled, bornological and quasi-barrelled spaces

A set T of linear maps from an l.t.s. E to another F is said to be pointwise bounded if T(x) is bounded in F for each x in E; T is said to be uniformly bounded on bounded sets if T(B) is bounded in F for every bounded subset B of E, and T is called equicontinuous if for every neighbourhood V in F, there exists a neighbourhood U in E such that $T(U) \subseteq V$.

A locally convex space is called barrelled if every barrel is a neighbourhood.

A separated locally convex space E is barrelled if and only if every pointwise bounded set of continuous linear maps from E into any locally convex space is equicontinuous. By a result in (26), this is equivalent to the condition that every closed linear map from E into any Banach space be continuous.

The class of barrelled spaces is quite extensive. Every locally convex space of the second category is barrelled. Inductive limits, separated products as well as completions, of barrelled spaces are barrelled. A hyperplane in a barrelled space is barrelled, though a closed linear subspace of a barrelled space need not be barrelled. In fact, by Theorem 1.1 of (22), any separated locally convex space is a closed linear subspace of some barrelled space. If there is a continuous nearly open linear map from a barrelled space into a locally convex space F, then F is barrelled.

In the study of the closed graph and open mapping theorems, it proves useful that any linear map from a barrelled space into a locally convex space is nearly continuous and that a linear map from a locally convex space onto a barrelled space is nearly open (see (33), 4.8).

A locally convex space E is called bornological if every bounded linear map from E into any locally convex space is continuous.

A locally convex space is bornological if and only if every absolutely convex bornivorous subset is a neighbourhood. A separated locally convex space E is bornological if and only if a set of linear maps from E into any locally convex space is equicontinuous if it is uniformly bounded on bounded sets. By a result in (26), this is equivalent to the condition that every bounded linear map from E into any Banach space is continuous.

Any metrizable locally convex space is bornological. If for some index set Φ, K^{Φ} is bornological, so is a product $X_{\gamma\varepsilon\Phi} E_{\gamma}$ of separated bornological spaces ((4), Page 15, Exc. 18(b)). Whether or not an arbitrary product of separated bornological spaces is bornological depends on the existence of an Ulam measure (see (18), Chapter 5, 19.9) . The property of being bornological is inherited by hyperplanes but not by arbitrary closed linear subspaces. Any inductive limit of bornological spaces is of the same sort. In particular, any inductive limit of normed linear spaces is bornological. Conversely, any separated bornological space E is an inductive limit of some normed linear spaces. If E is also sequentially complete, then it is an inductive limit of Banach spaces. By using an example in Page 155 of (22) we see that a barrelled space need not be bornological. Also a bornological space need not be barrelled, since a countably dimensional normed linear space is not barrelled. However, the completion of a bornological space is barrelled.

A locally convex space is called quasi-barrelled if every bornivorous barrel is a neighbourhood.

A separated locally convex space E is quasi-barrelled if and only if every set of continuous linear maps from E into any locally convex space, which is uniformly bounded on bounded sets, is equicontinuous. By a result in (26), a separated locally convex space E is quasi-barrelled if and only if every closed bounded linear map from E into any Banach space is continuous.

A bornological or barrelled space is quasi-barrelled, but a quasi-barrelled space need neither be barrelled nor bornological ((27), Page 816). Inductive limits and separated products of quasi-barrelled spaces are quasi-barrelled. As in the case of barrelled and bornological spaces, a closed linear subspace of a quasi-barrelled space need not be of the same sort. If there is a continuous nearly open linear map from a quasi-barrelled space into a locally convex space F, then F is quasi-barrelled. Any sequentially complete quasi-barrelled space is barrelled.

2.4 Suprabarrels and ultrabarrels

Let B be a balanced subset of an l.t.s. E. If there exists a sequence (B_n) of balanced absorbent (bornivorous) subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for all positive integers n, we say that B is a suprabarrel (bornivorous suprabarrel) in E. If in addition B is closed, we call it an ultrabarrel (a bornivorous ultrabarrel) in E.

In either of the cases considered above, we say that (B_n) is a <u>defining sequence</u> for B.

Clearly if B is an absolutely convex absorbent (bornivorous) subset of an l.t.s. then it is a suprabarrel (bornivorous suprabarrel) with $(1/2^n B)$ as a defining sequence. However, a suprabarrel (bornivorous suprabarrel) need not be convex and need not have a defining sequence of convex sets. For, if E is a complete non-locally convex locally bounded l.t.s., then its closed unit ball B is a bornivorous suprabarrel with $(\lambda_n B)$ as a defining sequence for some sequence (λ_n) of positive real numbers. But B is not convex and no member of a defining sequence for B can be convex. Similarly, every barrel (bornivorous barrel) in an l.t.s. is an ultrabarrel (a bornivorous ultrabarrel); but an ultrabarrel (a bornivorous ultrabarrel) need not be convex and need not have a defining sequence of convex sets.

In the situation of an l.t.s. where local convexity is not assumed, suprabarrels play parts often associated with absolutely convex absorbent sets in locally convex spaces. We give two instances:

(1) If B is a suprabarrel in a linear space E with (B_n) as a defining sequence, then the family of sets (B_n) is a base of neighbourhoods for a linear topology on E.

(2) Every balanced neighbourhood in an l.t.s. is a (bornivorous) suprabarrel and a base of neighbourhoods for the finest linear topology s on a linear space E is the family of all suprabarrels in E.

The following are easily verified. The closure of a suprabarrel (bornivorous suprabarrel) in an l.t.s. is an ultrabarrel (a bornivorous ultrabarrel); and an ultrabarrel (a bornivorous ultrabarrel) has a defining sequence of closed sets. Thus in referring to a defining sequence (B_n) for an ultrabarrel, it shall always be assumed that each B_n is closed. Let t be a linear map from an l.t.s. E into another, F. Then $t^{-1}(B)$ is a suprabarrel (bornivorous suprabarrel) in E if B is a suprabarrel in F (B is a bornivorous suprabarrel in F and t is bounded). In particular if t is continuous, $t^{-1}(B)$ is an ultrabarrel (bornivorous ultrabarrel) in E if B is an ultrabarrel (a bornivorous ultrabarrel) in F. If t maps E onto F then, for any suprabarrel C in E, t(C) is a suprabarrel in F.

The notion of a suprabarrel makes sense in any linear space (it does not depend on any topology on the space). As such, we may refer to suprabarrels in a linear space.

2.5 Topological groups

Let u be a topology on a group E for which the group operation (generally denoted by product) is continuous simultaneously in both variables. If group inversion is also continuous, then u is called a group topology on E. A group E on which is defined a group topology u is called a topological group and denoted by (E,u).

Let E be a topological group. If $\mathcal{U}$ is a base of neighbourhoods of the identity then, the family of sets $(xU : U \in \mathcal{U})$ (equivalently $(Ux : U \in \mathcal{U})$) is a base of neighbourhoods at x for every x in E.

Thus a group homomorphism from a topological group to another is continuous (open) if and only if it is continuous (open) at the identity. Also, a group homomorphism f from a topological group E to a topological group F is nearly open if and only if for every neighbourhood U of the identity in E, the closure of f(U) in F is a neighbourhood of the identity in F. And f is nearly continuous if and only if the closure of $f^{-1}(V)$ in E is a neighbourhood of the identity in E for every neighbourhood V of the identity in F. We shall henceforth use the terms "neighbourhoods" and "base of neighbourhoods" in a topological group to respectively denote "neighbourhood of the identity" and "base of neighbourhoods of the identity". In considering the notions of continuity, openness, near continuity and near openness, we shall limit our consideration to behaviour at the identity, as there is no loss of generality in doing so.

A subset B of a group is called symmetric if x^{-1} is in B for every point x in B. In a topological group E there exists a base $\mathcal{U}$ of symmetric neighbourhoods such that (i) for every U in $\mathcal{U}$, there is U_1 in $\mathcal{U}$ with $U_1U_1 \subseteq U$, (ii) if $U \in \mathcal{U}$ and $x \in U$, then there is U_1 in $\mathcal{U}$ such that $xU_1 \subseteq U$, and (iii) for every U in $\mathcal{U}$ and point x in E, there is U_1 in $\mathcal{U}$ such that $xU_1x^{-1} \subseteq U$. Conversely, if E is a group and $\mathcal{U}$ is a filter base of symmetric subsets of E satisfying conditions (i), (ii) and (iii), then there is a group topology on E with $\mathcal{U}$ as a base of neighbourhoods. Since a topological group is regular, any topological group has a base of closed symmetric neighbourhoods satisfying conditions (i), (ii) and (iii). A topological group is separated if and only if the members of a base of neighbourhoods intersect at the identity.

A topological space is said to be Lindelöf if every open cover has a countable subcover. Let E be a topological group which is of the second category in itself and, F a Lindelöf topological group. Then, by ((16), page 213), any group homomorphism from E into F is nearly continuous and, any group homomorphism from F onto E is nearly open.

A subset A of a topological space E is said to satisfy the condition of Baire if there is an open set U such that the complement of A with respect to U and the complement of U with respect to A are each of the first category in E. Since each closed set is the union of its interior and boundary it follows that every closed subset of a topological space satisfies the condition of Baire. Let t be a group homomorphism from a topological group E to another. If G is a subgroup of the second category in E, and G satisfies the condition of Baire, then, by Theorem 2 of (31), t is continuous on E if it is continuous on G.

The definition of inductive limits of topological groups used in this thesis, is due to Varopoulos (39). Let F be a group. For each α in an index set Φ, let t_α be a group homomorphism from a topological group E_α into F. If u is the finest group topology on F for which each t_α is continuous then (F,u) is called the inductive limit of $(E_\alpha : \alpha\varepsilon\Phi)$ by $(t_\alpha : \alpha\varepsilon\Phi)$. The space (F,u) will also be called the inductive limit of $(E_\alpha; t_\alpha : \alpha\varepsilon\Phi)$. A group homomorphism t from (F,u) into a topological group is continuous if and only if each $t o t_\alpha$ is continuous. If (F,u) is the inductive limit of $(E_\alpha; t_\alpha : \alpha\varepsilon\Phi)$ such that F is the union of

$(t_\alpha(E_\alpha) : \alpha\epsilon\Phi)$, we shall call (F,u) the generalized strict inductive limit of $(E_\alpha; t_\alpha: \alpha\epsilon\Phi)$.

Throughout our groups shall be assumed commutative.

2.6 The graph of a map

The graph G of a map t from a set E into a set F is the subset of $E \times F$ consisting of all points $((x,t(x)):x \epsilon E)$. If E and F are topological spaces, t is said to be closed if G is closed in $E \times F$ under its product topology. The map t is closed if and only if for every net $(x_\alpha : \alpha\epsilon\Psi)$ converging to some x_o in E such that $(t(x_\alpha) : \alpha\epsilon\Psi)$ converges to y_o in F, $y_o = t(x_o)$.

Let t be a continuous map from a topological space E into another, F. If F is separated then t is closed. For, if (x,y) in $E \times F$ is not in the graph G of t then there exist disjoint neighbourhoods U of $f(x)$ and V of y. The set $(t^{-1}(U), V)$ is a neighbourhood of (x,y) not meeting G. For a group homomorphism f from a topological group E_1 into a topological group F_1 to be closed, it is necessary that F_1 be separated. For, suppose that F_1 is not separated, that e, e_1 are the identities of E_1, F_1 respectively and that y is in the closure of e_1 in F, but $y \neq e_1$. Then (e,y) is not in the graph of f, but it is in its closure in $E_1 \times F_1$. Thus f is not closed.

The following two lemmas are easily proved.

Lemma 2.6.1.

Let t be a map from a set E into a set F and u a map of a set H into E. Then the graph of the map $t\circ u$ from H into F is the inverse image of the graph of t by the map $(x,y) \to (u(x),y)$ of $H \times F$ into $E \times F$.

Corollary. Let E, F, H be topological spaces. If t is a closed map from E into F and u a continuous map from H into E. Then the map $t \circ u$ of H into F is closed.

Lemma 2.6.2.

Let t, f be group homomorphisms from a topological group E into another, F. If t is continuous and f is closed, then the group homomorphisms h and g from E into F are closed, where $h(x) = t(x)f(x)$ and $g(x) = t(x)f(x^{-1})$ for all x in E.

Let E, F be topological groups and s a group homomorphism of F into E. The filter condition is said to hold if whenever Φ is a Cauchy filter base in F such that $s(\Phi)$ is convergent to a point of s(F), Φ is necessarily convergent to a point of F. The inverse filter condition is said to hold if whenever Φ is a convergent filter base in F and $s(\Phi)$ is Cauchy, $s(\Phi)$ is necessarily convergent to a point of s(F). (See (37), Pages 243 and 253).

Lemma 2.6.3.(a).

Let F be a topological group and E_1 a subgroup of a separated topological group E. Let t be a group homomorphism from E_1 onto F with graph G. If the filter condition holds then, the closure of G in E x F is contained in E_1 x F. In particular, G is closed in E x F if it is closed in E_1 x F, provided that the filter condition holds.

Lemma 2.6.3.(b).

Let F be a topological group and E_1 a subgroup of a separated topological group E. Let s be a group homomorphism of F into E_1 with graph G. If the inverse filter condition holds then the closure of G in $F \times E$ is contained in $F \times E_1$. In particular, G is closed in $F \times E$ if it is closed in $F \times E_1$, provided that the inverse filter condition holds.

Proof:

(a) Let (x,y) be in the closure of G in $E \times F$. Let $\mathcal{U}$ be a base of neighbourhoods at x for the topology of E and $\mathcal{V}$ a base of neighbourhoods at y for the topology of F. Then $(U,V) \cap G \neq \emptyset$ for each $U \in \mathcal{U}$ and $V \in \mathcal{V}$. $\therefore$ $U \cap t^{-1}(V) \neq \emptyset$ for all $U \in \mathcal{U}$ and V in $\mathcal{V}$. We note that $U \cap t^{-1}(V)) \subseteq E_1$. Let Φ be the filter base generated by $(U \cap t^{-1}(V))$ as U, V run through $\mathcal{U}$, $\mathcal{V}$ respectively. Then clearly Φ converges to x in E and $t(\Phi)$ converges to y in $F = t(E_1)$. Since Φ converges in E, Φ is Cauchy in E_1. Now since the filter condition holds, we see that x is in E_1.

(b) Let (x,y) be in the closure of G in $F \times E$. Let $\mathcal{U}$ be a base of neighbourhoods at x for the topology of F and $\mathcal{V}$ a base of neighbourhoods at y for the topology of E. Then $(U,V) \cap G \neq \emptyset$ for each $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Therefore $U \cap s^{-1}(V) \neq \emptyset$ for each $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Let Φ be the filter base generated by $U \cap s^{-1}(V)$ as U, V run through $\mathcal{U}$, $\mathcal{V}$ respectively. Then Φ converges to x and the filter on E generated by $s(\Phi)$ converges to y. Therefore $s(\Phi)$ is Cauchy in E_1 and by the inverse filter condition, y is in s(F).

Let E_o, F be topological groups. Then, any group homomorphism t of E_o into F can be represented in the form $t = f o k_o$, where k_o is the canonical map of E_o onto $E_o/t^{-1}(e_1)$ (e_1 is the identity of F), and f is a group isomorphism of $E_o/t^{-1}(e_1)$ into F. We refer to f as the induced map of t. The induced map f is continuous, open, nearly continuous, nearly open if and only if t has the same property. Suppose that E_o is a subgroup of a topological group E and that the topology of E_o coincides with that induced by E. Since $t^{-1}(e_1) \subseteq E_o$, the topology of $E_o/t^{-1}(e_1)$ is that induced by $E/t^{-1}(e_1)$, and k_o is the restriction to E_o of the canonical map of E onto $E/t^{-1}(e_1)$.

Lemma 2.6.4.

Let E_o be a subgroup of a separated topological group E and suppose that for some topological group F (with identity e_1), t is a group homomorphism of E_o into F, with induced map f. If the graph of t is closed in E x F, then $E/t^{-1}(e_1)$ is separated. The graph of t is closed in E x F if and only if the graph of f is closed in $E/t^{-1}(e_1) \times F$.

Proof: To show that $E/t^{-1}(e_1)$ is separated, we prove that $t^{-1}(e_1)$ is closed in E. Let x be in the closure of $t^{-1}(e_1)$ in E. For every neighbourhood U in E and V in F, there exists x_1 in $t^{-1}(e_1)$ such that x is in x_1U and $t(x_1)$ (= e_1) is in V. Therefore, (x, e_1) is in $(x_1U, t(x_1)V)$ for every neighbourhood U in E and V in G. Since the graph of t is closed in E x F, it follows that (x , e_1) is in the graph of t and thus x is in $t^{-1}(e_1)$. Therefore, $t^{-1}(e_1)$ is closed in E.

We observe that the graph G of t is the inverse image of the graph G_1 of f by the continuous map $(x,y) \to (k(x),y)$ of E x F into $E/t^{-1}(e_1) \times F$ (k is the canonical map of E onto $E/t^{-1}(e_1)$). Therefore, G is closed if G_1 is.

Now suppose that G is closed. We show that G_1 must also be closed. Let (y,z) be in $E/t^{-1}(e_1) \times F$ but not in G_1. Clearly if $x \in k^{-1}(y)$, $(x,z) \notin G$. Since G is closed in E x F, for some neighbourhoods U in E and V in F, $(xU, zV) \cap G = \emptyset$, and this implies that $(yk(U), zV) \cap G_1 = \emptyset$. This gives the result.

Lemma 2.6.5.

Let (E,u), (F,v) be topological groups and t a group homomorphism of (E,u) into (F,v). Denote by $\mathcal{U}$, $\mathcal{V}$, bases of symmetric neighbourhoods for the topologies u, v respectively. Let w be the group topology on F with the family of sets $(t(U)V : U \in \mathcal{U}, V \in \mathcal{V})$ as a base of neighbourhoods. Then,

(a) the map t is closed if and only if w is separated.

(b) the map t is nearly continuous if and only if the identity map from (F,v) onto (F,w) is nearly open.

Proof:

(a) We note that v is finer than w and that t is a continuous map from (E,u) into (F,w).

If (F,w) is separated then, the graph of t is closed in (E,u) x (F,w). The graph of t must then be closed in (E,u) x (F,v), since v is finer than w.

Suppose that the graph G of t is closed in (E,u) x (F,v). Let z be a point of F different from the identity. If e is the identity in E then (e,z) is not in G. Since G is closed in (E,u) x (F,v), there exists U in $\mathcal{U}$ and V in $\mathcal{V}$ such that $(U,zV) \cap G = \emptyset$. Thus z is not in $t(U)V^{-1} = t(U)V$, and w is separated.

(b) The u-closure of any set B in E shall be denoted by cl(B).

Suppose that t is nearly continuous from (E,u) into (F,v). Let V be in $\mathcal{V}$ and let V_1 be in $\mathcal{V}$ such that $V_1V_1 \subseteq V$. Then for any V_α in $\mathcal{V}$ and U_β in $\mathcal{U}$,

$$\begin{aligned} VV_\alpha t(U_\beta) &\supseteq (V_1 t(U_\beta))V_1V_\alpha \\ &\supseteq t(t^{-1}(V_1)U_\beta)V_1V_\alpha \\ &\supseteq t(cl(t^{-1}(V_1)))V_1V_\alpha \\ &\supseteq t(cl(t^{-1}(V_1)))V_1 \ . \end{aligned}$$

This is true for all V_α in $\mathcal{V}$ and U_β in $\mathcal{U}$. Therefore the w-closure of V contains $t(cl(t^{-1}(V_1)))V_1$, and since t is nearly continuous, it follows that the identity map from (F,v) onto (F,w) is nearly open.

Now suppose that the identity map i say, from (F,v) onto (F,w) is nearly open. Let V be in $\mathcal{V}$. We show that $cl(t^{-1}(V))$ is a u-neighbourhood.

Let V_1 be a member of $\mathcal{V}$ such that $V_1V_1V_1 \subset V$. As i is nearly open, there exist V_2 in $\mathcal{V}$ and U in $\mathcal{U}$ such that

$$\{V_2 t(U) \subseteq V_1 t(U_\alpha)V_\beta, \text{ for all } U_\alpha \varepsilon \mathcal{U}, V_\beta \varepsilon \mathcal{V} \} \ . \qquad (*)$$

We note that we can choose V_2 such that $V_2 \subseteq V_1$, and we do so.

From (*),

$$t(U) \subseteq V_1 V_2^{-1} V_\beta t(U_\alpha) \text{ for all } U_\alpha \text{ in } \mathcal{U} \text{ and } V_\beta \text{ in } \mathcal{V} \ .$$

Thus $t(U) \subseteq Vt(U_\alpha)$ for all U_α in $\mathcal{U}$.

i.e. $U \subseteq t^{-1}(Vt(U_\alpha)) \subseteq t^{-1}(V)U_\alpha$ for all U_α in $\mathcal{U}$.

Hence $U \subseteq cl(t^{-1}(V))$ and t is nearly continuous.

We observe that in the above lemma, if (E,u), (F,v) are linear topological spaces and t a linear map, then (F,w) is an l.t.s. In this case (a) coincides with Lemma 3 of (17). We also note that if in addition u,v are convex topologies, so is w.

Let E,F be linear spaces and G a linear subspace of $E \times F$. Then, G is the graph of a linear map from a subspace of E into F if and only if $y = o$ whenever $(o,y) \in G$.

Let E be a normed linear space which is not barrelled. By applying a result of Mahowald (26), we see that there exists a closed linear map, t say, from E into some Banach space F such that t is not continuous. By using the same result again, we deduce that the graph of any linear extension of t to $E^\wedge$ cannot be closed in $E^\wedge \times F$. Thus, a closed linear map from a dense subspace of an l.t.s. into a complete l.t.s. may not have a closed linear extension to the whole space. However, we have the following result.

Lemma 2.6.6.

Let E_o be a linear subspace of an l.t.s. E of finite co-dimension. Then, any closed linear map from E_o into an l.t.s. F has a closed linear extension from all of E into F.

Proof: Let t be a linear map from E_o into F with graph G closed in $E_o \times F$. Let G_1 denote the closure of G in $E \times F$. If $(o,y) \in G_1$, then $(o,y) \in G_1 \cap (E_o \times F) = G$, since G is closed in $E_o \times F$. Therefore, $y = o$. Thus G_1 is the graph of a linear extension t_1 say, of t.

There exists some linear extension t_2 of t_1 mapping E into F. For, let E_1 be the domain of t_1 and suppose that E_1 is of co-dimension N with respect to E. Let $(x_n : 1 \leq n \leq N)$ be points of E not in E_1, such that x_n is not in the linear subspace of E spanned by E_1 and $(x_i : 1 \leq i \leq n - 1)$ for n with $2 \leq n \leq N$. Any x in E may be represented in the form $x_o + \Sigma_{1 \leq n \leq N} \lambda_n x_n$ for some x_o in E_1 and scalars $(\lambda_n : 1 \leq n \leq N)$. The map f from E into F defined as follows is a linear extension of t_1 : $f(x) = f(x_o + \Sigma_{1 \leq n \leq N} \lambda_n x_n) = t_1(x_o)$. The result now follows from the observation that the graph G_2 of any linear extension of t_1 can be represented in the form $G_2 = G_1 + F$, where F is a finite dimensional linear subspace of E x F.

CHAPTER 3

*-INDUCTIVE LIMITS OF LINEAR TOPOLOGICAL SPACES

3.1 Definition and general properties

Let E be a linear space, and suppose that, for each γ in an index set Γ, E_γ is an l.t.s. and u_γ is a linear map of E_γ into E. The set Φ of all linear topologies on E for which each u_γ is continuous, is not empty, since it contains the trivial topology. The upper bound ζ say of the members of Φ is in Φ; it is the finest linear topology on E for which all the u_γ are continuous.

Definition 3.1.1

Suppose that E, E_γ and u_γ are as above and that in addition the union of u_γ (E_γ) spans E. Then the topology ζ on E defined above is called the *-inductive limit topology on E induced by $(E_\gamma; u_\gamma: \gamma\varepsilon\Gamma)$. We shall say that (E,ζ) is the *-inductive limit of the spaces (E_γ) by the mappings u_γ, or more shortly that (E,ζ) is the *-inductive limit of $(E_\gamma; u_\gamma: \gamma\varepsilon\Gamma)$.

With the notation above, the following are easily verified.

(a) A base of neighbourhoods for the topology ζ is the family of all suprabarrels U in E such that for each U_n in a defining sequence for U, $u_\gamma^{-1}(U_n)$ is a neighbourhood in E_γ for each γ in Γ.

(b) If each E_γ is the *-inductive limit of $(E_{\gamma,\alpha}; u_{\gamma,\alpha}: \alpha\varepsilon\Psi)$, then (E,ζ) is the *-inductive limit of $(E_{\gamma,\alpha}; u_\gamma o u_{\gamma,\alpha}: \gamma\varepsilon\Gamma, \alpha\varepsilon\Psi)$.

(c) A linear map t of (E,ζ) into an l.t.s. F is continuous if and only if $t o u_\gamma$ is a continuous map of E_γ into F for each γ in Γ.

((a), (b) and (c) are still true even if, in Definition 3.1.1., the union of $u_\gamma(E_\gamma)$ does not span E).

(d) A set T of linear maps of (E, ζ) into an l.t.s. F, is equicontinuous if and only if $T \circ u_\gamma$ is an equicontinuous set of linear maps of E_γ into F for each γ in Γ.

<u>Examples of *-inductive limits of linear topological spaces</u>

(1) Let E be an l.t.s., E_o a linear subspace of E and k_1 the canonical map of E onto E/E_o. Then E/E_o under its quotient topology is the *-inductive limit of $(E;k_1)$.

(2) Let E be a linear space and $\{\lambda_\alpha, \alpha\in\Phi\}$ a set of linear topologies on E. Suppose that, for each α in Φ, i_α is the identity map of (E,λ_α) into E. If λ is the lower bound of the λ_α, then (E,λ) is the *-inductive limit of $[(E,\lambda_\alpha);\ i_\alpha:\ \alpha\in\Phi.]$.

(3) Let E be a linear space over the complex field. Then E is linearly isomorphic to $\Sigma_{\alpha\in\Phi} C_\alpha$ for some index set Φ (C_α(=K) is a copy of the complex field for each α in Φ). Under its finest linear topology, s, E is the *-inductive limit of (C_α) by the injection maps. Since (E,s) is not necessarily locally convex ((19), Theorem 3.1) it follows that a *-inductive limit of locally convex spaces need not be locally convex.

<u>Proposition 3.1.1.</u>

Let (F,τ) be the *-inductive limit of $[(E_\gamma,\tau_\gamma);u_\gamma:\ \gamma\in\Phi]$. Then $(F,\ \tau^{oo})$ is the inductive limit of $[(E_\gamma,\tau_\gamma^{oo});\ u_\gamma:\gamma\in\Phi]$.

<u>Proof</u>: Since τ^{oo} is coarser than τ, u_γ is a continuous linear map of (E_γ,τ_γ) into (F,τ^{oo}) for each γ in Φ. Therefore each u_γ is a continuous

linear map of $(E_\gamma, \tau_\gamma^{oo})$ into (F, τ^{oo}). Hence τ^{oo} is coarser than the inductive limit topology λ on F induced by $[(E_\gamma, \tau_\gamma^{oo});\ u_\gamma: \gamma \varepsilon \Phi]$. Since λ is necessarily locally convex and coarser than τ, it must be identical with τ^{oo}.

<u>Proposition 3.1.2.</u>

Let (E, ζ) be the $*$-inductive limit of $(E_\gamma;\ u_\gamma: \gamma \varepsilon \Gamma)$. For each γ in Γ, let V_γ be a balanced neighbourhood of the origin in E_γ and let

$$U = \bigcup_\phi \Sigma_{\gamma \varepsilon \phi}\, u_\gamma(V_\gamma), \ \ldots\ldots\ldots *$$

the union being taken over all finite subsets ϕ of Γ. Then U is a neighbourhood of the origin in (E, ζ).

If Γ is countable, then as V_γ runs through a base of balanced neighbourhoods of the origin in E_γ, the above sets form a base of neighbourhoods of the origin for (E, ζ).

<u>Proof</u>: To prove that U is a ζ-neighbourhood, it is sufficient to construct a defining sequence (U_n) say, for U such that for each n, $u_\gamma^{-1}(U_n)$ is a neighbourhood in E_γ for each γ in Γ. Let $\mathcal{V}_\gamma$ be a base of balanced neighbourhoods in E_γ. Then there is a sequence (V_γ^n) of neighbourhoods from $\mathcal{V}_\gamma$, with $V_\gamma^1 + V_\gamma^1 \subseteq V_\gamma$ and $V_\gamma^{n+1} + V_\gamma^{n+1} \subseteq V_\gamma^n$ for all $n \geq 1$. Let

$$U_n = \bigcup_\phi \Sigma_{\gamma \varepsilon \phi}\, u_\gamma(V_\gamma^n)$$

the union being over all finite subsets ϕ of Γ. Then clearly $U_1 + U_1 \subseteq U$, $U_{n+1} + U_{n+1} \subseteq U_n$ for all $n \geq 1$ and each U_n is balanced. Also each $u_\gamma(V_\gamma^n)$ is absorbent in $u_\gamma(E_\gamma)$ and the union of the latter sets spans E; hence, U_n is absorbent. Thus U is a suprabarrel in E with (U_n) as a defining sequence. Clearly, for each n, $u_\gamma^{-1}(U_n)$ is a neighbourhood in E_γ for each γ in Γ.

To prove the remaining part, we notice that, as V_γ runs through $\mathcal{V}_\gamma$, the sets (*) satisfy the condition to form a base of neighbourhoods for a linear topology η say on E which is clearly coarser than ζ. But if W_o is a ζ-neighbourhood, there are ζ-neighbourhoods W_n with $W_{n+1} + W_{n+1} \subseteq W_n$ for $n \geq 0$. If $\{\gamma(n)\}$ is an enumeration of Γ, there are balanced neighbourhoods $V_{\gamma(n)}$ in $\mathcal{V}_{\gamma(n)}$ with $u_{\gamma(n)}(V_{\gamma(n)}) \subseteq W_n$ and then

$$U = \bigcup_{n \geq 1} \Sigma_{1 \leq r \leq n} u_{\gamma(r)} (V_{\gamma(r)})$$

is an η-neighbourhood contained in W_o.

From the proposition above it follows that the *-inductive limit topology of a sequence of locally convex spaces is convex and therefore coincides with the inductive limit topology of the spaces, by Proposition 3.1.1. Also if (E,ζ) is the *-inductive limit of a sequence (E_i) of linear topological spaces by linear maps (u_i), it follows from Proposition 3.1.2. and Proposition 2(a) of section 2 of (39) that ζ is the finest group topology on E for which each u_i is continuous. However, ζ is not necessarily the finest topology on E for which each u_i is continuous, even if each E_i is a Banach space (see (8) pages 98-99).

3.2 *-direct sums

If for each γ in an index set Γ, E_γ is an l.t.s., we shall call the linear space $\Sigma_{\gamma\epsilon\Gamma} E_\gamma$ under the *-inductive limit topology of (E_γ) by the injection maps, the *-direct sum of (E_γ).

The proofs of Propositions 3.2.1, 3.2.2 and 3.2.3. are easy and will be omitted.

Proposition 3.2.1.

If E is the *-inductive limit of (E_γ: $\gamma\varepsilon\Gamma$) then E is topologically isomorphic to a quotient of the *-direct sum of (E_γ: $\gamma\varepsilon\Gamma$).

Proposition 3.2.2.

Let (E_γ: $\gamma\varepsilon\Gamma$) be a family of linear topological spaces. The dual of the *-direct sum of (E_γ: $\gamma\varepsilon\Gamma$) is the product $F = X_\gamma E_\gamma'$. If E_γ' separates the points of E_γ, then F separates the points of the direct sum of (E_γ : $\gamma\varepsilon\Gamma$). The dual of the product space $X_\gamma E_\gamma$ is the direct sum $G = \Sigma_\gamma E_\gamma'$. If E_γ' separates the points of E_γ, then G separates the points of the product space $X_\gamma E_\gamma$.

Proposition 3.2.3.

Let (E,ζ) be the *-direct sum of a family (E_γ: $\gamma\varepsilon\Gamma$) of linear topological spaces. Then ζ is finer than the product topology η, say, on E. If ϕ is a finite subset of Γ, then ζ and η coincide on $\Sigma_{\gamma\varepsilon\phi}E_\gamma$. The topology ζ induces the original one on each E_γ.

Corollary 1. The *-direct sum topology on a finite direct sum is identical with the product topology.

Corollary 2. The *-direct sum of a family (E_γ: $\gamma\varepsilon\Gamma$) of linear topological spaces is separated if and only if each E_γ is separated.

Remarks

(1) Suppose that the index set Γ be represented as the disjoint union of two subsets Φ and Ψ. If E, F, G are the *-direct sums of linear topological spaces (E_γ: $\gamma\varepsilon\Gamma$), (E_γ: $\gamma\varepsilon\Phi$), (E_γ: $\gamma\varepsilon\Psi$) respectively, then E, being the *-direct sum of F and G is in fact F x G, by Corollary 1 of Proposition 3.2.3. Thus for any subset Φ of an index set Γ, the

*-direct sum of (E_γ: $\gamma\varepsilon\Phi$) is a quotient of the *-direct sum of (E_γ: $\gamma\varepsilon\Gamma$). If E_γ is separated for each γ in Γ, it then follows by Corollary 2 of Proposition 3.2.3. that $\Sigma_{\gamma\varepsilon\Phi} E_\gamma$ is closed in the *-direct sum of (E_γ: $\gamma\varepsilon\Gamma$). By a similar argument, one can show that if (E_γ: $\gamma\varepsilon\Gamma$) is a family of locally convex spaces, then for any $\Phi \subseteq \Gamma$, the direct sum of (E_γ: $\gamma\varepsilon\Phi$) is a quotient of the direct sum of (E_γ: $\gamma\varepsilon\Gamma$).

(2) Suppose that, for each γ in an index set Γ, the l.t.s. F_γ is the *-direct sum of G_γ and H_γ. If E, F, G are the *-direct sums of (F_γ: $\gamma\varepsilon\Gamma$), (G_γ: $\gamma\varepsilon\Gamma$), (H_γ: $\gamma\varepsilon\Gamma$) respectively, then E, being the *-direct sum of F and G is in fact F x G by Corollary 1 of Proposition 3.2.3. Thus F (and G) is a quotient of E. By a similar argument one can show that, if (F_γ: $\gamma\varepsilon\Gamma$) is a family of locally convex spaces and, for each γ in Γ, F_γ is the topological direct sum of G_γ and H_γ, then the direct sums $\Sigma_{\gamma\varepsilon\Gamma} G_\gamma$ and $\Sigma_{\gamma\varepsilon\Gamma} H_\gamma$ are quotients of the direct sum $\Sigma_{\gamma\varepsilon\Gamma} F_\gamma$.

Let B be a subset of a linear space and $\lambda \not\geq 0$ a real number. In (29), B is said to be λ-convex if $\lambda(B + B) \subseteq B$, and B is said to be semiconvex if it is β-convex for some real number $\beta \not\geq 0$. However, for $\eta \geq 0$ we shall say that B is <u>η-convex</u> if $B + B \subseteq \eta B$, and call a subset C of a linear space a <u>semiconvex</u> set if it is ξ-convex (in this sense) for some $\xi \geq 0$.

<u>Definition 3.2.1</u>

We say that an l.t.s. is <u>almost convex</u> if it contains a fundamental system of bounded sets which are closed, balanced and semiconvex.

Clearly every locally convex space is almost convex and so is any locally bounded l.t.s. If E is a locally bounded l.t.s., any product of copies of E is almost convex.

Proposition 3.2.4.

Let $[(E_\gamma, \xi_\gamma) : \gamma \epsilon \Gamma]$ be a family of separated linear topological spaces. Then a subset A of the *-direct sum of $[(E_\gamma, \xi_\gamma) : \gamma \epsilon \Gamma]$ is bounded (precompact) if and only if it is contained in a finite sum of subsets of the E_γ which are ξ_γ-bounded (precompact).

Proof: If A is contained in a finite sum of subsets of the E_γ which are ξ_γ-bounded (precompact) then clearly A is bounded (precompact).

Now suppose that A is bounded (precompact). Let p_γ be the projection from the *-direct sum onto (E_γ, ξ_γ). Each $p_\gamma(A)$ is bounded (precompact). We show that $p_\gamma(A) = \{0\}$ except for finitely many γ, and this will give the result since $A \subseteq \Sigma_{\gamma\epsilon\Gamma}\, p_\gamma(A)$.

Suppose not. Then there exists a sequence $(\gamma(n))$ from Γ and a sequence of points x_n such that $x_n \neq 0$ and x_n is in $p_{\gamma(n)}(A)$. Since each E_γ is separated, there exists for each n a balanced $\xi_{\gamma(n)}$-neighbourhood $U_{\gamma(n)}$ of the origin such that x_n is not in n $U_{\gamma(n)}$.

Let $U = \bigcup_{j=1}^{\infty} \Sigma_{n=1}^{j} U_{\gamma(n)}$. Then by Proposition 3.1.2, U is a neighbourhood of the origin in the *-direct sum topology of $(E_{\gamma(n)})$. Clearly $A \not\subseteq nU$ for any n and, since the *-direct sum of $(E_\gamma : \gamma\epsilon\Gamma)$ induces the *-direct sum topology on $\Sigma_{n=1}^{\infty} E_{\gamma(n)}$ (see Remark 1 after Proposition 3.2.3), A cannot be bounded under the *-direct sum topology of $(E_\gamma : \gamma\epsilon\Gamma)$ (and thus cannot be precompact under the same topology).

Corollary 1 The *-direct sum of infinitely many separated linear topological spaces is never metrizable.

Proof: By Remark 1 after Proposition 3.2.3, it is sufficient to prove the assertion when the index set is countable. So, let E be the *-direct sum of a sequence (E_i) of separated linear topological spaces. Suppose that E is metrizable and let (U_n) be a decreasing sequence of sets forming a local base. Since $F_n = \Sigma_{1 \leq i \leq n} E_i$ is a proper subspace of E, there exists a sequence (x_n) of points such that x_n is in U_n but x_n is not in F_n. By the proposition, (x_n) is not bounded but since x_n is in U_n and (U_n) is decreasing, (x_n) must be bounded. This contradiction gives the result.

Corollary 2 A *-direct sum of separated almost convex linear topological spaces is almost convex.

Proof: This follows immediately from the proposition.

Corollary 3 A *-direct sum of separated sequentially (quasi-) complete linear topological spaces is sequentially (quasi-) complete.

3.3 Strict *-inductive limits

If E is the *-inductive limit of $(E_\gamma; u_\gamma: \gamma \varepsilon \Phi)$, such that E is the union of the subspaces $u_\gamma(E_\gamma)$, then we say that E is the generalized strict *-inductive limit of $(E_\gamma; u_\gamma: \gamma \varepsilon \Phi)$. In particular if Φ is countable (say Φ is the set of positive integers), (E_i) is a sequence of strictly increasing linear subspaces of E, and the topology of E_i coincides with that induced by E_{i+1}, we say that E is the strict *-inductive limit of (E_i).

Remarks

(1) If E is the strict *-inductive limit of (E_i), then by using an inductive argument we see that the topology of E_i coincides with that induced by E_{i+n} for any positive integer, n.

(2) Also it is not difficult to show that if E is the strict *-inductive limit of (E_i), then for any sub-sequence $(i(r))$ of the positive integers, E is the strict *-inductive limit of $(E_{i(r)})$.

Examples of strict *-inductive limits

(1) Any strict inductive limit of a sequence of locally convex spaces.

(2) If E is the *-direct sum of a sequence (E_i) of linear topological spaces, then E is the strict *-inductive limit of (F_j) where F_j is the *-direct sum of $(E_i)_{1 \leq i \leq j}$.

Proposition 3.3.1

Let (E,ξ) be the strict *-inductive limit of (E_n,ξ_n). Then ξ coincides with ξ_n on E_n.

Proof: It is sufficient to show that ξ induces on E_n a topology finer than ξ_n. Let W_n be a ξ_n-neighbourhood of the origin and suppose that U_n is a ξ_n-neighbourhood of the origin such that

$$U_n + U_n + \dots + U_n \text{ (n + 1 terms)} \subseteq W_n \quad . \qquad (1)$$

There exists a balanced ξ_{n+1}-neighbourhood U_{n+1} such that

$$(U_{n+1} + U_{n+1}) \cap E_n \subseteq U_n \quad , \qquad (2)$$

Similarly there exists a balanced ξ_{n+2}-neighbourhood U_{n+2} such that

$$(U_{n+2} + U_{n+2}) \cap E_{n+1} \subseteq U_{n+1} \quad ,$$

From this and (2) it follows that

$$(U_{n+2} + U_{n+2} + U_{n+1}) \cap E_n \subseteq U_n \quad .$$

For if z is in $(U_{n+2} + U_{n+2} + U_{n+1}) \cap E_n$, then z is in E_n and $z = z_1 + z_2$, where z_1 is in U_{n+1} and z_2 is in $U_{n+2} + U_{n+2}$. Since $E_n \subsetneqq E_{n+1} \subsetneqq E_{n+2}$, z_2 must be in E_{n+1} and therefore

$$(U_{n+2} + U_{n+2} + U_{n+1}) \cap E_n \subseteq$$

$$((U_{n+2} + U_{n+2}) \cap E_{n+1} + U_{n+1}) \cap E_n \subseteq U_n \quad .$$

Similarly we can find balanced ξ_{n+j}-neighbourhoods U_{n+j} such that for any positive integer r,

$$(U_{n+1} + U_{n+2} + \dots + U_{n+r} + U_{n+r}) \cap E_n \subseteq U_n \quad .$$

Therefore , $\wedge$ by (1) $\wedge$

$$(\bigcup_{r \geq 1} U_n + U_n + \dots + U_n (\text{n terms}) + \Sigma_{1 \leq j \leq r} U_{n+j}) \cap E_n \subseteq W_n \; .$$

Thus, by Proposition 3.1.2, ξ coincides with ξ_n on E_n, since $U_n \cap E_i$ is a balanced ξ_i-neighbourhood for $1 \leq i \leq n$.

<u>Corollary 1</u>. Let (E,ξ) be the strict $*$-inductive limit of a sequence $((E_n,\xi_n))$ of separated linear topological spaces. Then (E,ξ) is separated.

<u>Proof</u>: Let $x \neq 0$ be in E. For some n, x is in E_n and, since (E_n, ξ_n) is separated, there exists a ξ_n-neighbourhood U_n of the origin not containing x. By the proposition, there is a ξ-neighbourhood U of the origin such that $U \cap E_n \subseteq U_n$. Clearly x is not in U and therefore (E,ξ) is separated.

Corollary 2: If (E,ξ) is the strict *-inductive limit of a sequence $((E_n,\xi_n))$ of linear topological spaces such that, for each n, E_n is closed in (E_{n+1}, ξ_{n+1}), then E_n is closed in (E,ξ).

Proof: Let x be in E and suppose that x is not in E_N for some positive integer N. Clearly x is in some E_{N+m}. Since E_N is closed in E_{N+m}, there exists a ξ_{N+m}-neighbourhood U_{N+m} such that $(x + U_{N+m}) \cap E_N = \emptyset$. By the proposition, there exists a ξ-neighbourhood U such that $U \cap E_{N+m} \subseteq U_{N+m}$. Then $(x + U) \cap E_N = \emptyset$, from which the result follows.

Proposition 3.3.2

Let (E,ξ) be the strict *-inductive limit of a sequence (E_n) of linear topological spaces. Suppose that (E,ξ) is a topological subspace of an l.t.s. F and that, for each n, E_n is closed in F, then E is closed in F.

Proof: For any subset A of F, cl(A) shall denote the closure of A in F. Let x be in cl(E) and suppose that x is not in E_n for any n. Then for each n, there exists a balanced ξ-neighbourhood of the origin W_n such that $(x + \mathrm{cl}(W_n)) \cap E_n = \emptyset$ and (W_n) may be so chosen that

$$W_{n+1} + W_{n+1} + W_{n+1} \subseteq W_n \quad . \tag{1}$$

Let $U_n = W_n \cap E_n$ and $U = \bigcup_{n \geq 1}(\Sigma_{1 \leq i \leq n} U_i)$. Then U is a ξ-neighbourhood by Proposition 3.1.2 and thus $x + \mathrm{cl}(U)$ meets E_N for some positive integer N. Now, if z_1 is in U, then z_1 is in $\Sigma_{1 \leq i \leq k} U_i$ for some positive integer k (which we may choose greater than N). Therefore z_1 is in $\Sigma_{1 \leq i \leq N} U_i + \Sigma_{N+1 \leq i \leq k} U_i$ and this is contained in $E_N + \Sigma_{N+1 \leq i \leq k} W_i$. Thus z_1 is in $E_N + W_{N+1} + W_{N+2} + \dots + W_{k-1} + W_{k-1}$,

and using (1) successively we see that z_1 is in $E_N + W_{N+1} + W_{N+1}$.
Therefore $U \subseteq E_N + W_{N+1} + W_{N+1}$ and thus

$$\mathrm{cl}(U) \subseteq E_N + W_{N+1} + W_{N+1} + \mathrm{cl}(W_{N+1})$$

$$\subseteq E_N + \mathrm{cl}(W_N) ;$$

since $x + \mathrm{cl}(U)$ meets E_N, so does $x + \mathrm{cl}(W_N)$.

This is impossible. Thus $\mathrm{cl}(E) = E$.

Corollary: Any strict *-inductive limit of a sequence of complete separated linear topological spaces is complete.

Proof: This follows from the proposition on choosing F to be the completion of (E, ξ).

We note in passing that the methods of proofs of Propositions 3.3.1 and 3.3.2 can be used to get similar results for commutative topological groups (not necessarily locally compact); cf. (39), Section 3, Propositions ~~3~~ 4 and 5.

Proposition 3.3.3

If E is the strict *-inductive limit of a sequence (E_n) of separated linear topological spaces such that, for each n, E_n is closed in E_{n+1}, then E is contained as a dense topological subspace of the strict *-inductive limit of a sequence of complete separated linear topological spaces.

Proof: If E is not complete, let $\hat{E}$ be its completion. For any subset A of $\hat{E}$, let $\mathrm{cl}(A)$ denote the closure of A in $\hat{E}$. Let F be the linear space $\bigcup_{n \geq 1} \mathrm{cl}(E_n)$. The space F is dense in $\hat{E}$ since F contains E.

Also $cl(E_n)$ is a proper subspace of $cl(E_{n+1})$, since each E_n is a closed proper subspace of E_{n+1}. If η is the strict *-inductive limit topology on F of $(cl(E_n))$, then (F,η) is complete, by the corollary of Proposition 3.3.2. Since the identity map from E_n into (F,η) is continuous for each n, η induces on E a topology coarser than its original topology, ξ, say. Also, since the identity map from $cl(E_n)$ into $E^\wedge$ is continuous for each n, η must be finer than the topology induced on F by $E^\wedge$, and this implies that η induces a topology on E finer than ξ. Thus E is a topological subspace of (F,η). The space (F, η) is complete and therefore the closure of E in (F,η) coincides with $E^\wedge$.

Hence $E^\wedge = (F,\eta)$, since $E \subseteq F \subseteq E^\wedge$. This completes the proof.

Corollary. If E is the strict *-inductive limit of a sequence (E_n) of separated linear topological spaces such that, for each n, E_n is closed in E_{n+1}, then E is not metrizable.

Proof: By Corollary 2 of Proposition 3.3.1, each E_n is closed in E. If E is complete and metrizable, then E is of the second category and $E = \bigcup_{n \geq 1} E_n$. This is impossible. If E is not complete, but metrizable, $E^\wedge$ is a complete metrizable l.t.s. which is the strict *-inductive limit of complete separated linear topological spaces, by the proposition. This is not possible as shown above.

CHAPTER 4

ULTRABARRELLED, ULTRABORNOLOGICAL AND QUASI-ULTRABARRELLED SPACES

4.1 Ultrabarrelled spaces

W. Robertson, in Theorem 4 of (37), proved that a locally convex space (F,η) is barrelled if and only if any convex topology on F with a base of η-closed neighbourhoods is necessarily coarser than η. She then proceeded to call an l.t.s. (E,τ) ultrabarrelled if any linear topology on E with a base of τ-closed neighbourhoods is necessarily coarser than τ. Thus any locally convex ultrabarrelled space is barrelled. In (37), the following results are also proved. Every l.t.s. of the second category is ultrabarrelled, but an ultrabarrelled space need not be of the second category. Let E be an ultrabarrelled space. Then any quotient of E by a linear subspace is ultrabarrelled and if F is a separated l.t.s. such that $E \subseteq F \subseteq E^\wedge$, then F is ultrabarrelled. Furthermore, when E is separated, any closed linear map from E into a complete metric linear space F is continuous and any continuous linear map from F onto E is open. And for E, an analogue of the Banach-Steinhaus theorem holds in the following form: Every pointwise bounded set of continuous linear maps from E into any l.t.s. is equicontinuous. It is also shown in (37) that if (E,u) is any ultrabarrelled space, then (E,u^{oo}) is barrelled, though it may not be ultrabarrelled.

Theorem 4.1.1.

An l.t.s. is ultrabarrelled if and only if every ultrabarrel is a neighbourhood of the origin.

Proof: Suppose that (E,η) is ultrabarrelled. Let B be an ultrabarrel in (E,η) with a defining sequence (B_n) of η-closed sets. Then the family of η-closed sets (B_n) is a base of neighbourhoods for a linear topology on E, which must be coarser than η since (E,η) is ultrabarrelled. Therefore B is an η-neighbourhood. Suppose that every ultrabarrel in an l.t.s. (E,η) is a neighbourhood. Let ξ be a linear topology on E with a base $\mathcal{U}$ of neighbourhoods consisting of balanced η-closed sets. Every U in $\mathcal{U}$, being then an ultrabarrel in (E,η) is an η-neighbourhood. This implies that ξ is coarser than η and thus (E,η) is ultrabarrelled.

Corollary 1. Every linear map of an l.t.s. onto an ultrabarrelled space is nearly open, and every linear map of an ultrabarrelled space into an l.t.s. is nearly continuous.

Proof: Let t be a linear map of an l.t.s. (E,u) onto an ultrabarrelled space (F,v). If U is a balanced u-neighbourhood, then U is a suprabarrel in (E,u) and thus $t(U)$ is a suprabarrel in (F,v). Therefore the v-closure of $t(U)$ is an ultrabarrel in (F,v) and, by the theorem, this set is a v-neighbourhood. Thus t is nearly open. By a similar argument, one can prove that a linear map of an ultrabarrelled space into an l.t.s. is nearly continuous.

Let (E,u) be a locally convex space. If there exists a continuous linear nearly open map of (E,u) onto an l.t.s. F say, then F is locally convex. It therefore follows from Corollary 1 that a non-locally convex ultrabarrelled topology on E can not be coarser than u. Since the finest linear topology on a countably dimensional linear space is locally convex, any ultrabarrelled topology on a countably dimensional linear space is necessarily locally convex.

Corollary 2. Let (E,u) be an ultrabarrelled space and (F,v) an l.t.s. If f is a continuous linear nearly open map of (E,u) into (F,v), then (F,v) is ultrabarrelled.

Proof: If B is a v-ultrabarrel, then $f^{-1}(B)$ is a u-ultrabarrel, since f is continuous. By the theorem, $f^{-1}(B)$ is a u-neighbourhood of the origin, and since f is nearly open, the v-closure of $f(f^{-1}(B))$ is a v-neighbourhood. And since the v-closure of $f(f^{-1}(B))$ is contained in B, B must be a v-neighbourhood, and, by the theorem (F,v) is ultrabarrelled.

An immediate consequence of the above is that a quotient by a linear subspace of an ultrabarrelled space is ultrabarrelled - a result due to W. Robertson ((37), Proposition 13). Also, since any linear space E under its finest linear topology s is clearly ultrabarrelled, by combining Corollaries 1 and 2, we get the following result.

Corollary 3 An l.t.s. (E,u) is ultrabarrelled if and only if the identity map from (E,s) onto (E,u) is nearly open.

Let (E,u) be the sequence space $l^{\frac{1}{2}}$. As shown in page 256 of (37), (E,u^{oo}) is a barrelled normed space which is not ultrabarrelled. Therefore, by Corollary 3, the identity map from (E,s) onto (E,u^{oo}) is not nearly open and thus the identity map from (E,u^{oo}) onto (E,s) is not nearly continuous. Since the identity map from $(E, \tau(E,E^*))$ onto (E,u^{oo}) is continuous and nearly open, it follows by Corollary 2 that $(E,\tau(E,E^*))$ is not ultrabarrelled.

Suppose that, for each γ in an uncountable index set Φ, E_γ is a (non-trivial) separated locally convex space. Each E_γ can be expressed as the direct sum of a closed hyperplane and a copy $K_\gamma(=K)$ of the scalar field; then by Remark 2 after Proposition 3.2.3, $\Sigma_{\gamma\varepsilon\Phi} K_\gamma$ is a quotient of $\Sigma_{\gamma\varepsilon\Phi} E_\gamma$. Now Φ is uncountable and therefore, for some subset Ψ of Φ, the space $E = l^{\frac{1}{2}}$ considered above, with its finest locally convex topology $\tau(E,E^*)$, can be expressed as a direct sum $\Sigma_{\gamma\varepsilon\Psi} K_\gamma$ of copies of the scalar field. Thus $(E, \tau(E,E^*))$ is a non-ultrabarrelled quotient of $\Sigma_{\gamma\varepsilon\Phi} E_\gamma$ and so the latter space can not be ultrabarrelled by Proposition 13 of (37). We note this for further reference.

<u>Corollary 4</u>. An uncountable direct sum of (non trivial) separated locally convex spaces cannot be ultrabarrelled.

If T is a pointwise bounded set of linear maps from an l.t.s. E into an l.t.s. F, then for any bornivorous suprabarrel B in F, $\bigcap_{t\varepsilon T} t^{-1}(B)$ is a suprabarrel in E. In particular if B is a bornivorous ultrabarrel and each t in T is continuous, then $\bigcap_{t\varepsilon T} t^{-1}(B)$ is an ultrabarrel in E. In this case, it follows from Theorem 4.1.1. that if E is ultrabarrelled, then T is equicontinuous. This is W. Robertson's analogue of the Banach-Steinhaus theorem ((37), Theorem 5).

Let (E,v) be an l.t.s. If B is a suprabarrel in E with (B_n) as a defining sequence, then B together with (B_n) is a base of neighbourhoods for a linear topology on E. If N is the linear subspace $\bigcap_{n \geq 1} B_n$ of E, and k_1 the canonical map of E onto E/N, it is not difficult to show that the linear topology w say, on E/N with base $(k_1(B_n))$ is metrizable. Let (E/N,w)^ denote the completion of (E/N,w) .

Lemma 4.1.1. With the notation introduced above,

(a) if B is an ultrabarrel, then the map k_1 of (E,v) into (E/N,w)^ is closed.

(b) if B is a bornivorous ultrabarrel, then the map k_1 of (E,v) into (E/N,w)^ is closed and bounded.

Proof:

(a) Since B is an ultrabarrel, we may assume by a remark in Section 2.4 that each B_n is v-closed. Let G denote the graph of k_1 and G_1 its closure in (E,v) x (E/N,w)^ . If A is contained in (E/N,w)^ , we denote the closure of A in (E/N,w)^ by cl(A). We show that if (x,y) is in G_1, then $y - k_1(x)$ is in $cl(k_1(B_n))$ for arbitrary n, and this will prove (a). Suppose that (x,y) is in G_1. Since E/N is dense in (E/N,w)^, then, for each n, there exists some z in E such that:

$$k_1(z) \in y - k_1(x) + cl(k_1(B_{n+4})) \quad . \qquad (*)$$

Since (x,y) is in G_1, then, for each v-neighbourhood V,

$$(x + V,\ y + cl(k_1(B_{n+4}))) \cap G \neq \emptyset \quad .$$

Therefore $y - k_1(x)$ is in $k_1(V) + cl(k_1(B_{n+4}))$, and using this in (*) we find that

$$k_1(z) \varepsilon\ k_1(V) + cl(k_1(B_{n+3})) \quad .$$

Thus, $k_1(z)$ is in $k_1(V) + k_1(B_{n+3}) + k_1(B_{n+3})$, which is contained in $k_1(V) + k_1(B_{n+2})$. This implies that $z \varepsilon V + B_{n+2} + N$ and thus $z \varepsilon V + B_{n+1}$. As this is true for all v-neighbourhoods V and B_{n+1} is v-closed, it follows that $z \varepsilon B_{n+1}$. Now, using this in (*), we find that

$$y - k_1(x) \varepsilon\ cl(k_1(B_n)) \quad .$$

(b) That k_1 is closed follows from (a). Let u be the linear topology on E with base (B_n). Since each B_n is v-bornivorous, the identity map from (E,v) onto (E,u) is bounded. Also, the map k_1 from (E,u) into (E/N,w)^ is bounded, being continuous. Therefore k_1 is a bounded linear map from (E,v) into (E/N,w)^.

Let E be an l.t.s. and N the intersection of the neighbourhoods of the origin in E. If t is a closed linear map from E into an l.t.s. F, then t(N) is the origin in F, and so t can be expressed in the form $t = t_1 o k_1$ where k_1 is the canonical map of E onto E/N and t_1 maps E/N into F. It is then easy to see that t_1 is also closed. If in addition t is bounded, so is t_1; also t is continuous if and only if t_1 is continuous.

Theorem 4.1.2.

An l.t.s. (E,v) is ultrabarrelled if and only if every closed linear map from (E,v) into any complete metric linear space is continuous.

Proof: If (E,v) is ultrabarrelled, then by the remark just above and Propositions 13 and 15(ii) of (37), every closed linear map from (E,v) into any complete metric linear space is continuous. Now suppose that every closed linear map from (E,v) into any complete metric linear space is continuous. Let B be a v-ultrabarrel with a defining sequence (B_n) of v-closed sets. Let N be the linear subspace $\bigcap_{n \geq 1} B_n$ of E and denote by w the metrizable linear topology on E/N with a neighbourhood base $(k_1(B_n))$, where k_1 is the canonical map of E into E/N. If $(E/N,w)^\wedge$ denotes the completion of (E/N,w) then, by Lemma 4.1.1., the graph of the map k_1 is closed in $(E,v) \times (E/N,w)^\wedge$. By the hypothesis therefore, k_1 is a continuous map from (E,v) into $(E/N,w)^\wedge$. Thus, $k_1^{-1}(k_1(B_1)) = B_1 + N$ is a v-neighbourhood, and since $B_1 + N$ is contained in B, B must be a v-neighbourhood. Thus by Theorem 4.1.1, (E,v) is ultrabarrelled.

Corollary 1. Any *-inductive limit of ultrabarrelled spaces is ultrabarrelled.

Proof: Let E be the *-inductive limit of ultrabarrelled spaces $(E_\gamma)_{\gamma\epsilon\Phi}$ by linear maps $(u_\gamma)_{\gamma\epsilon\Phi}$, and t a closed linear map from E into a complete metric linear space F. Since each u_γ is continuous, $t o u_\gamma$ is a closed linear map from E_γ into F for each γ in Φ. By the theorem, $t o u_\gamma$ is continuous. As this is true for all γ in Φ, t must be continuous and again, by the theorem, E is ultrabarrelled.

Corollary 2. Any countable inductive limit of locally convex ultrabarrelled spaces is ultrabarrelled. In particular any countable inductive limit of Fréchet spaces is ultrabarrelled.

Proof: That any countable inductive limit of locally convex ultrabarrelled spaces is ultrabarrelled follows from Corollary 1 and the remark after Proposition 3.1.2. The last part follows from this since, by the corollary of Proposition 12 of (37), any Fréchet space is ultrabarrelled.

Corollary 3. Let $(E_\gamma : \gamma \varepsilon \Phi)$ be a family of separated locally convex ultrabarrelled spaces. Then $E = \Sigma_{\gamma \varepsilon \Phi} E_\gamma$ is ultrabarrelled if and only if Φ is countable.

Proof: If Φ is countable, then by Corollary 2, E is ultrabarrelled. If Φ is uncountable, then by Corollary 4 of Theorem 4.1.1., E is not ultrabarrelled.

We note that, in Corollary 3, we cannot replace "direct sum" by "inductive limit". For if G is the direct sum of countably many copies of the scalar field and Φ is uncountable, then G, being a quotient of E, is an inductive limit of $(E_\gamma : \gamma \varepsilon \Phi)$.

Since every complete separated l.t.s. is a closed subspace of an l.t.s. of the second category - a product of complete metric linear spaces, it follows that a closed linear subspace of an ultrabarrelled space need not be ultrabarrelled, for an uncountably dimensional linear space under its finest locally convex topology is not ultrabarrelled, but is complete and separated. However, we have the following result.

Corollary 4. Any subspace of finite co-dimension in an ultrabarrelled space is ultrabarrelled.

Proof: Let E be an ultrabarrelled space, E_o a subspace of E of finite co-dimension and suppose that t is a closed linear map of E_o into F, a complete metric linear space.

By Lemma 2.6.6, there exists a closed linear extension from E into F which by the theorem must be continuous. Therefore t is continuous and again by the theorem, E_o is ultrabarrelled.

Finally we observe that by the method of proof of Proposition 14 of (37) one can show that if E_o, E_1, E are linear spaces, v a linear topology on E such that $E_o \subseteq E_1 \subseteq E$ and E_o is v-dense, then E_1, under the v-induced topology is ultrabarrelled if E_o, under the v-induced topology is ultrabarrelled.

4.2 Ultrabornological spaces

We call an l.t.s. E ultrabornological if every bounded linear map from E into any l.t.s. is continuous.

Clearly every metrizable l.t.s. is ultrabornological. Since a countably dimensional metrizable non-locally convex space is not ultrabarrelled (by the remark after Corollary 1 of Theorem 4.1.1), it follows that an ultrabornological space need not be ultrabarrelled.

If (E,u) is ultrabornological, then (E,u^{oo}) is bornological. For, let t be a bounded linear map of (E,u^{oo}) into a locally convex space F. Then t, being a bounded linear map from (E,u) into F, is continuous, and since F is locally convex it follows that t is continuous from (E,u^{oo}) into F. Thus (E,u^{oo}) is bornological. In particular, a locally convex ultrabornological space is bornological. However, a bornological space need not be ultrabornological. For, let E be a linear space of uncountable dimension. Then by Theorem 3.1 of (19), its finest linear topology s is strictly finer than $\tau(E,E^*)$, so that the identity map of $(E, \tau(E,E^*))$ into (E,s) is not continuous. But it is bounded, since every $\tau(E,E^*)$ - bounded subset is contained in a finite dimensional subspace of E. The l.t.s. $(E,\tau(E,E^*))$ is therefore not ultrabornological, though it is bornological.

Let E be an incomplete separated inductive limit of a sequence of Banach spaces (for an example of such a space, see (22), pages 437 and 438), and suppose that x is a point of $E^\wedge$ (the completion of E) not in E. By a result of Komura ((22), page 155), the linear subspace E_1 of $E^\wedge$ spanned by E and x is not bornological. Clearly E_1 is not ultrabornological. But, by Corollary 2 of Theorem 4.1.2 and Proposition 14 of (37), E_1 is ultrabarrelled. Thus an ultrabarrelled space need not be ultrabornological.

It is also a direct consequence of the definition that, if E is any ultrabornological space, then a linear map from E into an l.t.s. is continuous if and only if it is sequentially continuous.

Theorem 4.2.1.

Any $*$-inductive limit of ultrabornological spaces is ultrabornological.

Proof: Let E be the $*$-inductive limit of $(E_\gamma; u_\gamma: \gamma \in \Phi)$ where each E_γ is ultrabornological and let t be a bounded linear map from E into an l.t.s. F. Since u_γ is bounded, being continuous, it follows that $t \circ u_\gamma$ is a bounded linear map from E_γ into F. Since E_γ is ultrabornological, $t \circ u_\gamma$ is continuous. As this is true for all γ in Φ, it follows that t is continuous and therefore E is ultrabornological.

Corollary. Any countable inductive limit of locally convex ultrabornological spaces is ultrabornological.

It follows from the above corollary and Komura's example of a non-bornological barrelled space in ((22), page 155) that a non-ultrabornological space may contain a dense ultrabornological space.

Suppose that, for each γ in an uncountable index set Φ, E_γ is a (non-trivial) separated locally convex space and K_γ is a copy of the scalar field. As pointed out after the definition of an ultrabornological space above, $G = \Sigma_{\gamma \varepsilon \Phi} K_\gamma$ is not ultrabornological. Since G is a quotient of $\Sigma_{\gamma \varepsilon \Phi} E_\gamma$, it follows by Theorem 4.2.1 that an uncountable direct sum of separated (non-trivial) locally convex spaces cannot be ultrabornological.

Lemma 4.2.1. Let (E,v) be an l.t.s. Then there exists a finest linear topology u say, on E with the same bounded sets. The space (E,v) is ultrabornological if and only if v = u, and this is so if and only if every bornivorous suprabarrel in (E,v) is a v-neighbourhood.

Proof: If V_α is any bornivorous suprabarrel in (E,v), then V_α together with a defining sequence forms a base of neighbourhoods for a linear topology v_α say on E. Let u be the upper bound of (v_α) as V_α runs through all the bornivorous suprabarrels in (E,v). Clearly u is finer than v and u,v have the same bounded sets. The linear topology u is the finest one having the same bounded sets as v. For, suppose that W is a balanced neighbourhood of the origin in a linear topology u_1 on E having the same bounded sets as v, then W is a v-bornivorous suprabarrel and therefore u_1 is coarser than u. Next, (E,u) is ultrabornological; for if t is a bounded linear map from (E,u) into an l.t.s. F, then $t^{-1}(V)$ is a u-bornivorous suprabarrel for any balanced neighbourhood V in F. Since u and v have the same bounded sets, $t^{-1}(V)$ is a v-bornivorous suprabarrel and therefore $t^{-1}(V)$ is a u-neighbourhood and t is a continuous map from (E,u) into F. Thus (E,u) is ultrabornological. That u = v if and only if (E,v) is ultrabornological follows easily from here. The remaining part follows from the above construction of u.

With the notation used in the above lemma, we call u the <u>ultrabornological topology associated with v.</u>

Let T be a set of linear maps from an l.t.s. E into an l.t.s. F such that T is uniformly bounded on bounded sets. Also, let V be a balanced neighbourhood in F. Then V is a bornivorous suprabarrel in F, so that $\bigcap_{t\varepsilon T} t^{-1}(V)$ is a bornivorous suprabarrel in E. If E is ultrabornological, it follows from Lemma 4.2.1. that T is equicontinuous. We have thus proved the following result.

<u>Proposition 4.2.1.</u>

A set of linear maps from an ultrabornological space into an l.t.s. is equicontinuous provided that it is uniformly bounded on bounded sets.

<u>Theorem 4.2.2.</u>

An l.t.s. (E,v) is ultrabornological if and only if every bounded linear map from (E,v) into any complete metric linear space is continuous.

Proof: Suppose that every bounded linear map from (E,v) into any complete metric linear space is continuous. Let u be the ultrabornological topology associated with v. We show that u is coarser than v and it will follow by Lemma 4.2.1 that (E,v) is ultrabornological. Let U_o be a balanced u-neighbourhood in E and let (U_n) be a sequence of balanced u-neighbourhoods such that $U_{n+1} + U_{n+1} \subseteq U_n$ for all $n \geq 0$. Then U_o is a (bornivorous) suprabarrel in (E,v) with (U_n) as a defining sequence. Let N be the linear space $\bigcap_{n \geq 1} U_n$ and let w denote the metrizable linear topology on E/N with a local base $(k_1(U_n))$, where k_1 is the canonical map of E onto E/N. Let $(E/N,w)\hat{}$ be the completion of $(E/N,w)$. Clearly the map k_1 is continuous from (E,u) into $(E/N,w)\hat{}$, and therefore it is bounded from (E,v) into $(E/N,w)\hat{}$. By the hypothesis therefore, k_1 is a continuous map from (E,v) into $(E/N,w)\hat{}$. Hence $k_1^{-1}(K_1(U_1)) = U_1 + N$ is a v-neighbourhood of the origin and, since $U_1 + N \subseteq U$, U must be a v-neighbourhood. Thus u is coarser than v.

Theorem 4.2.3.

Let (E,u) be a separated almost convex ultrabornological space. Then (E,u) is a $*$-inductive limit of separated locally bounded spaces. If (E,u) is sequentially complete then, it is a $*$-inductive limit of complete separated locally bounded spaces and is therefore ultrabarrelled.

Proof: Let A be a balanced (closed) semiconvex bounded subset of (E,u), and let E_A be the linear subspace spanned by A. Then it is easy to see that E_A can be given a locally bounded metrizable topology v_A (with local base the sets $\frac{1}{n} A$ for $n = 1, 2, \ldots$) and that v_A is finer than the topology induced on E_A by u. As A runs through the balanced semiconvex (u-closed) u-bounded subsets of E, $\bigcup E_A$ spans E; let v be the *-inductive limit topology on E defined by the E_A and the injection maps into E. Then v is finer than u and, by Theorem 4.2.1, (E,v) is ultrabornological. Any u-bounded set A' in E is also v-bounded. For, since (E,u) is almost convex, there is a (u-closed) balanced u-bounded semiconvex set A containing A'. Then A' is v_A-bounded and so v-bounded. Thus the identity map of (E,u) into (E,v) is bounded and, since (E,u) is ultrabornological it is continuous. Hence v is identical with u.

If (E,u) is sequentially complete, each (E_A, v_A) is complete. For, any v_A-Cauchy sequence (x_n) is also u-Cauchy; it therefore converges in (E,u) to a say. Now for each $\Sigma > o$, $\exists\, n_o(\Sigma)$ such that $x_m - x_n \varepsilon\ \Sigma A$ for all m, n $\geq n_o(\Sigma)$ and so, on letting $n \to \infty$, $x_m - a\ \varepsilon\ \Sigma A$ for $m \geq n_o(\Sigma)$ (since we took the sets A u-closed). Thus $a\ \varepsilon\ E_A$ and $x_n \to a$ in (E_A, v_A) Hence, the metrizable space (E_A, v_A) is complete, and so (E,u) is a *-inductive limit of complete separated locally bounded spaces. By Corollary 1 of Theorem 4.1.2 it is therefore ultrabarrelled.

4.3 Quasi-ultrabarrelled spaces

We say that an l.t.s. E is quasi-ultrabarrelled if every bornivorous ultrabarrel in E is a neighbourhood of the origin. From Theorem 4.1.1., we deduce that every ultrabarrelled space is quasi-ultrabarrelled, and from Lemma 4.2.1, we deduce that every ultra-bornological space is quasi-ultrabarrelled.

If (E,u) is quasi-ultrabarrelled, then (E,u^{oo}) is quasi-barrelled. For, if B is a u^{oo}-bornivorous barrel in E, B is a u-bornivorous ultrabarrel and is therefore a u-neighbourhood of the origin. The set B must then be a u^{oo}-neighbourhood and thus (E,u^{oo}) is quasi-barrelled. In particular any locally convex quasi-ultrabarrelled space is quasi-barrelled.

If (E,v) is a quasi-ultrabarrelled space and u is the ultrabornological topology associated with v, then it is easy to see that the identity map from (E,u) onto (E,v) is nearly open. By an application of Lemma 4.2.1, one can show that if (E,v) is an l.t.s. such that the identity map from (E,u) (u is the ultrabornological topology associated with v) onto (E,v) is nearly open, then (E,v) is quasi-ultrabarrelled. Since for any linear space E, the finest linear topology s on E is the ultrabornological topology associated with $\tau(E,E^*)$, it follows from Corollaries 2 and 4 of Theorem 4.1.1 that if E has uncountable dimension then $(E, \tau(E,E^*))$ is not quasi-ultrabarrelled.

If in the argument preceding Proposition 4.2.1, each t in T is continuous and V is closed, we see that a set of continuous linear maps from a quasi-ultrabarrelled space into an l.t.s. is equicontinuous provided that it is uniformly bounded on bounded sets.

Theorem 4.3.1.

An l.t.s. (E,v) is quasi-ultrabarrelled if and only if every closed bounded linear map from (E,v) into any complete metric linear space is continuous.

Proof: Let (E,v) be quasi-ultrabarrelled and suppose that t is a closed bounded linear map from (E,v) into a complete metric linear space F. Since each quotient of (E,v) by a linear subspace is also quasi-ultrabarrelled, to prove that t is continuous, we may assume by the argument preceding Theorem 4.1.2, that (E,v) is separated. Since t is bounded, the v-closure of $t^{-1}(U)$ is a bornivorous ultrabarrel in (E,v) for every balanced neighbourhood U in F. And since (E,v) is quasi-ultrabarrelled, it follows that t is nearly continuous. Therefore, t is continuous, by ((16), Page 213).

The proof of the other part is as in Theorem 4.1.2, with the difference that we take B to be a bornivorous ultrabarrel, and this ensures that the map k_1 of (E,v) into (E/N,w)^ is bounded.

Corollaries 1 - 3 of Theorem 4.1.2 are true with "quasi-ultrabarrelled" replacing "ultrabarrelled". Only easy modifications of the methods of proof are needed in this case. In particular, if E_1, E_2 are quasi-ultrabarrelled so is their *-direct sum. If we now take E_1 to be an ultrabarrelled space which is not ultrabornological and E_2 as an ultrabornological space which is not ultrabarrelled, then the *-direct sum of E_1 and E_2 is quasi-ultrabarrelled, but is neither ultrabarrelled nor ultrabornological. Also, a closed linear subspace of an ultrabarrelled ultrabornological space need not be quasi-ultrabarrelled. For, in Problem D(b), Page 195 of (18), G* is a countable direct sum of Frechet spaces with a closed linear subspace H^o which is not quasi-ultrabarrelled since it is not quasi-barrelled.

Lemma 4.3.1. In a separated l.t.s., an ultrabarrel absorbs every balanced sequentially complete semiconvex bounded set.

Proof: Let (E,u) be a separated l.t.s., B an ultrabarrel in (E,u) and M a balanced sequentially complete semiconvex bounded subset of (E,u). We may without loss of generality assume that M spans E (for otherwise we may consider the subspace E_1 of E spanned by M and the ultrabarrel $B \cap E_1$ in E_1 under the u-induced topology). The locally bounded topology v say, on E with M as unit ball is finer than u; and using the fact that M is sequentially complete, one can show that (E,v) is complete. Therefore (E,v) is ultrabarrelled. Since B is a u-ultrabarrel and v is finer than u, B is a v-neighbourhood, being a v-ultrabarrel. As M is v-bounded, B absorbs M.

Theorem 4.3.2.

A separated sequentially complete almost convex quasi-ultrabarrelled space is ultrabarrelled.

Proof: Let (E,u) be such a space. Let B be an ultrabarrel in (E,u). Every bounded set in (E,u) is contained in a balanced closed semiconvex bounded set A and A is sequentially complete since E is. Hence by Lemma 4.3.1, B absorbs A and it follows easily from here that every ultrabarrel in (E,u) is a bornivorous ultrabarrel. Therefore (E,u) is ultrabarrelled by Theorem 4.1.1.

4.4. A generalization of two norm spaces

The two-norm spaces introduced by Alexiewicz in (1), (2) have been extensively studied by several authors, for example Wiweger (42) and Persson (30).

The idea of two-norm convergence (sometimes referred to as γ-convergence) is as follows. Let E be a linear space on which are defined two invariant metrics $d_1(\cdot,\cdot)$, $d_2(\cdot,\cdot)$ compatible with the linear space structure of E. A sequence (x_n) in E is said to be convergent to x_o in E in the two-norm sense if the supremum of $d_1(o,x_n)$ is finite and $d_2(x_o,x_n)$ tends to zero as n tends to infinity. A linear map f of E into an l.t.s. F is said to be continuous in the two-norm sense if whenever a sequence (x_n) converges to x_o in E in the two-norm sense, $f(x_n)$ converges to $f(x_o)$ in F. A two-norm space is a linear space provided with two metrics of the form of $d_1(\cdot,\cdot)$, $d_2(\cdot,\cdot)$ above.

Wiweger, in (42), constructed a topology which generates two-norm convergence. Persson, in (30), extended the theory of two-norm spaces to the situation of a linear space E provided with two convex topologies u,v such that every v-bounded set is u-bounded, and constructed a topology which generates the two-norm convergence. Wiweger's topology in (42) is also well defined in this case.

In this section, we point out how some of Persson's results in (30) carry over to certain classes of not necessarily locally convex linear topological spaces.

Let $(M_\alpha : \alpha\varepsilon\Phi)$ be a family of subsets of an l.t.s. (E,v) and let (M_α,v) denote M_α under the v-induced topology. The upper bound u say, of all linear topologies on E for which each identity map i_α from (M_α,v) into E is continuous is the finest linear topology on E such that each i_α is continuous. The topology u is finer than v, u coincides with v on each M_α and u is the finest linear topology on E which coincides with v on each M_α.

Let E be a linear space and u,v linear topologies on E. If every v-bounded subset of E is u-bounded, we call (E;u,v) a bitopological space. We denote by u(v), the finest linear topology on E coinciding with u on v-bounded sets, and say that u(v) is the mixed topology on E defined by u and v.

Clearly u is coarser than u(v) and if w is the ultrabornological topology associated with v, then u(v) = u(w). Also, since (E,w) is ultrabornological, u is coarser than w.

The proofs of Proposition 4.4.1, its corollary and Proposition 4.4.2 below are similar to those of Proposition 1.1, Corollary 1.1 and Proposition 1.2 of (30).

Proposition 4.4.1.

If (E;u,v) is a bitopological space, then every v-bounded set is u(v)-bounded.

Corollary. If (E;u,v) is a bitopological space and w is the ultrabornological topology associated with v, then u is coarser than u(v) and u(v) is coarser than w.

Proposition 4.4.2.

Let (E;u,v) be a bitopological space. Then, a linear map from (E,u(v)) into an l.t.s. is continuous if and only if its restriction to every v-bounded subset of E is continuous. Moreover, among the linear topologies on E which are identical with u on v-bounded sets, u(v) is the only one with this property.

We say that a bitopological space $(E;u,v)$ is normal if (E,v) is almost convex and has a base of balanced u-closed neighbourhoods.

Clearly if $(E;u,v)$ is a bitopological space, the topology $u(v)$ is determined by a fundamental system of v-bounded sets. Therefore, if $(E;u,v)$ is in particular normal we can limit our consideration to a fundamental system of balanced semiconvex v-bounded sets. In this case, the following lemma is useful.

Lemma 4.4.1. Let t be a linear map from an l.t.s. E into an l.t.s. F. If M is a balanced semiconvex subset of E, then t is uniformly continuous on M if and only if it is continuous on M at the origin.

Proof: Suppose that t is continuous on M at the origin. Let V be a neighbourhood of the origin in F. For some positive integer n, $M + M \subseteq nM$. Let V_n be a balanced neighbourhood of the origin in F such that

$$V_n + V_n + \dots + V_n \text{ (n terms)} \subseteq V.$$

Since t is continuous on M at the origin, there exists a balanced neighbourhood U of the origin in E such that $t(U \cap M)$ is contained in V_n. Let a be any point of M. If $x \in (a + U) \cap M$, then $x - a$ is in $U \cap (M - M)$ and this is contained in $n(U \cap M)$. Therefore $t(x)$ is in $t(a) + nt(U \cap M)$ and this is contained in $t(a) + V$. Therefore t is uniformly continuous on M.

From the above lemma and Proposition 4.4.1, we deduce that if $(E;u,v)$ is a normal bitopological space, then a base of neighbourhoods for the mixed topology $u(v)$ is the family $\mathcal{U}$ of all v-bornivorous suprabarrels in E such that for any balanced semiconvex v-bounded set in B and any U in $\mathcal{U}$, $U_n \cap B$ is a neighbourhood of the origin in (B,u) for each U_n in a defining sequence for U. Using this, we now show that every sequence (x_n) converging to zero in (E,u(v)) is v-bounded.

If (x_n) is a sequence on E which is not v-bounded, then there exists a balanced u-closed neighbourhood V^o in (E,v) and a subsequence $(x_{k(n)})$ of (x_n) such that for all positive integers n, $x_{k(n)} \notin nV^o$. Since nV^o is u-closed for each n, there exists a sequence $(U_{n(o)})$ of balanced u-neighbourhoods such that $x_{k(n)}$ is not in $(nV^o + U_{n(o)})$ for any n. Let $(V^m; m = 1,2\ldots)$ be a sequence of u-closed balanced v-neighbourhoods such that $V^{m+1} + V^{m+1} \subseteq V^m$ for all $m \geq 0$, and for each n, let $(U_{n(m)}: m = 1,2..)$ be a sequence of balanced u-neighbourhoods such that $U_{n(m+1)} + U_{n(m+1)} \subseteq U_{n(m)}$ for all $m \geq 0$.

Let

$$W^m = \bigcap_{n \geq 1} (nV^m + U_{n(m)}) .$$

It is not difficult to show that W^o is a v-bornivorous suprabarrel in E with (W^m) as a defining sequence. Since B is v-bounded, for any m, $B \subseteq nV^m$ for some n. Using this, one shows that for any m, $W^m \cap B$ is a neighbourhood of the origin in (B,u), and thus W^o is a u(v)-neighbourhood in E. And since W^o does not contain $x_{k(n)}$ for all positive integers n, (x_n) is not convergent to zero in (E,u(v)).

Let (E;u,v) be a normal bitopological space. If B is any u(v)-bounded subset of E, let (x_n) be any sequence of points of B and (λ_n) any sequence of positive scalars converging to zero. Since B is u(v)-bounded, $(\lambda_n^{\frac{1}{2}} x_n)$ converges to zero in (E,u(v)) and by the argument above, $(\lambda_n^{\frac{1}{2}} x_n)$ is v-bounded. Thus $(\lambda_n x_n)$ converges to zero in (E,v) and B is v-bounded. From this and Proposition 4.4.1, we deduce the following analogue of Theorem 1.1 of (30).

Theorem 4.4.1.

If (E;u,v) is a normal bitological space then a subset of E is v-bounded if and only if it is u(v)-bounded.

Corollary. Let (E;u,v) be a normal bitopological space. Then, a sequence in E is u(v)-convergent to x if and only if it is v-bounded and u-convergent to x.

As in Corollary 1.3 of (30), the above corollary shows the connection between the mixed topology and the notion of γ-convergence introduced by Alexiewicz in (2).

Let (E;u,v) be a normal bitopological space such that (E,v) is ultrabornological. If u = v, we deduce immediately from the corollary of Proposition 4.4.1 that v is the finest linear topology on E inducing the same topology on the bounded subsets of (E,v). In particular if (E,v) is locally bounded, v is the finest linear topology on E inducing the same topology on the unit ball of (E,v). If u ≠ v, then clearly (E,u) is not ultrabarrelled. If u and v are not identical on v-bounded sets, we have the following analogue of Proposition 1.3 of (30).

Proposition 4.4.3.

Let (E;u,v) be a normal bitopological space such that (E,v) is ultrabornological and v is not identical with u on v-bounded sets. Then (E,u(v)) is not quasi-ultrabarrelled.

We now give some examples of normal bitopological spaces.

1. Let E be a linear space. Let u be the topology $\sigma(E,E^*)$ ($\tau(E,E^*)$) and v the topology $\tau(E,E^*)$ ($\sigma(E,E^*)$). In either case, $(E;u,v)$ is a normal bitopological space and the finest linear topology s on E is the mixed topology determined by u and v. Since s is not necessarily locally convex, we see that the mixed topology determined by two locally convex topologies need not be locally convex.

2. Let E be the direct sum of a family of locally convex spaces. If v, u are respectively the direct sum and product topologies on E, then $(E;u,v)$ is a normal bitopological space.

3. Let F be a separated locally bounded space and suppose that E is the algebraic direct sum of countably many copies of F. If v,u are respectively the $*$-direct sum and product topologies on E, then $(E;u,v)$ is a normal bitopological space. For, since v is finer than u, every v-bounded set is u-bounded. Also, by Corollary 2 of Propoisition 3.2.4, (E,v) is almost convex. We now show that the topology v has a base of balanced u-closed neighbourhoods of the origin. Let B be the unit ball in F. Then $B + B \subseteq \beta B$ for some $\beta \geq 0$. Take one such β. A typical v-neighbourhood is of the form

$$U = \bigcup_{n \geq 1}(\Sigma_{1 \leq i \leq n} j_i (\alpha_i B))$$

where (α_i) is a sequence of positive real numbers and j_i is the injection map of F_i (=F) into E for each i. If x is in the u-closure of U, then for some positive integer n, $p_i(x) = 0$ for all integers i greater than n, where p_i is the projection of E onto F_i (=F) for each i. Let W be

$\bigcap_{1 \leq i \leq n} p_i^{-1}(\alpha_i B)$. Then W is a u-neighbourhood of the origin in E and since x is in the u-closure of U, there exists y in U such that x-y is in W. Since $y \in U$,

$$y \in \Sigma_{1 \leq i \leq m} j_i (\alpha_i B) \text{ for some integer } m \,. \quad (*)$$

Since $x-y \in W$,

$$\Sigma_{1 \leq i \leq n} p_i(x-y) \in \Sigma_{1 \leq i \leq n} (\alpha_i B) \,. \qquad (**)$$

Now, $\Sigma_{i \geq n+1} p_i(x-y) = -\Sigma_{i \geq n+1} p_i (y)$ is in $\Sigma_{n+1 \leq i \leq m} (\alpha_i B)$ and using (**), it follows that x-y is in $\Sigma_{1 \leq i \leq m} j_i (\alpha_i B)$. Therefore by (*), x is in $\beta(\Sigma_{1 \leq i \leq m} j_i (\alpha_i B))$, and thus $x \in \beta U$ and v has a base of u-closed neighbourhoods.

CHAPTER 5

SEMICONVEX SPACES

5.1 Semiconvex spaces: general

Let E be a linear space. As defined in section 3.2, if λ is a non-negative real number, a subset A of E is called λ-convex if $A + A \subseteq \lambda A$. A subset B of E is called a semiconvex subset if it is μ-convex for some $\mu \geq 0$.

A linear topology u on a linear space E with a base of balanced semiconvex neighbourhoods of the origin is called a semiconvex topology and (E,u) is known as a semiconvex space.

Any locally convex or locally bounded space is a semiconvex space and so is any product of semiconvex spaces. It is easy to see from the proof of Theorem 1 of (20) that any separated semiconvex space is topologically isomorphic to a subspace of a product of separated locally bounded spaces. Thus a separated l.t.s. is a semiconvex space if and only if it is a subspace of a product of separated locally bounded spaces.

Since the upper bound of any set of semiconvex topologies on a linear space is semiconvex, there exists a finest semiconvex topology on any linear space E. This shall be denoted by sc. A base of neighbourhoods for sc is the family of all balanced semiconvex absorbent subsets of E. Clearly $\tau(E,E^*)$ is coarser than sc and sc is coarser than the finest linear topology s on E; the three topologies coincide when the dimension of E is countable. When the dimension of E is uncountable

$\tau(E,E^*)$, sc and s are distinct. For, let G be the sequence space $l^{\frac{1}{2}}$ and H the space of all measurable functions on the closed interval (0,1) with the metrizable topology corresponding to convergence in measure. Since G and H have the same dimension ($2^{\aleph_0}$), they are algebraically isomorphic and we may identify them. That $\tau(G,G^*)$ is strictly coarser than sc follows from the fact that the topology of $l^{\frac{1}{2}}$ cannot be coarser than $\tau(G,G^*)$. Also, as H is not a semiconvex space ((43), Page 239), and any continuous nearly open linear image of a semiconvex space is obviously a semiconvex space, we deduce that the topology of H cannot be coarser than sc and thus sc is strictly coarser than s. If E is any uncountably dimensional linear space, G may be identified with a linear subspace of E, and from this, the assertion follows.

According to Simons ((44), Page 170), a function f on a linear space E into the non-negative reals is called an r-pseudometric ($o < r \leq 1$) if (i) there exists x in E such that $f(x) \neq 0$,

(ii) $f(x+y) \leq f(x) + f(y)$ for all x,y in E

and (iii) $f(\lambda x) = |\lambda|^r f(x)$ for each x in E and λ in the scalar field. The function f is said to give the topology u to E if the sets $(f^{-1}(o,{}^1/n) : n = 1,2,\ldots)$ form a base of u-neighbourhoods of the origin. From Theorem 1 of (44), we see that for every balanced semiconvex absorbent proper subset B of E there exists an r-pseudometric which gives the locally bounded topology on E with $({}^1/n\ B : n = 1,2, \ldots)$ as a base of neighbourhoods of the origin. And furthermore, if for each r in $o < r \leq 1$, $(f_\alpha : \alpha\epsilon\Phi_r)$ is the set of all r-pseudometrics on a linear space E and $(f_\alpha : \alpha\epsilon\Phi)$ is the union over r of $(f_\alpha : \alpha\epsilon\Phi_r)$ then, a base of neighbourhoods for sc is the family of sets $\{f_\alpha^{-1}(o,{}^1/n) : n = 1,2,\ldots, \alpha\varepsilon\Phi\}$. Now, using $(f_\alpha : \alpha\varepsilon\Phi)$ instead of invariant pseudometrics "q" in Problem E, Page 124 of (18), one can easily show that for any linear space E, (E,sc) is complete.

Simons's notion of an upper bound space in (44) clearly coincides with that of a semiconvex space. In Theorem 6 of (44), it is proved that if (E,u) is an upper bound space and B is a subset of E such that f(B) is bounded for each u-continuous invariant pseudometric f on E, then B is u-bounded. Simons asked whether this property characterises upper bound spaces. This is not so. For, if E is an uncountably dimensional linear space, (E,s) is not an upper bound space since s is strictly finer than sc. Let B be a subset of E such that for each s-continuous pseudometric f on E, f(B) is bounded. Clearly, for each sc-continuous pseudometric f on E f(B) is bounded and thus B is sc-bounded by Theorem 6 of (44). Since s and sc have the same bounded sets, B is s-bounded.

Let E be a linear space, and suppose that, for each γ in an index set Γ, E_γ is a semiconvex space and u_γ is a linear map of E_γ into E such that the union of the subspaces $u_\gamma(E_\gamma)$ spans E. The upper bound ζ say, of all semiconvex topologies on E for which each u_γ is continuous, is the finest semiconvex topology on E for which each u_γ is continuous. We shall call ζ the <u>**-inductive limit topology</u> on E induced by $(E_\gamma; u_\gamma : \gamma\varepsilon\Gamma)$ and say that (E,ζ) is the <u>**-inductive limit</u> of (E_γ) by (u_γ) or of $(E_\gamma; u_\gamma : \gamma\varepsilon\Gamma)$.

With the notation above, a base of neighbourhoods for the topology ζ is the family $\mathcal{U}$ of all balanced semiconvex subsets of E such that for every U in $\mathcal{U}$, $u_\gamma^{-1}(U)$ is a neighbourhood in E_γ for each γ in Γ. Also a linear map t from (E,ζ) into a semiconvex space is continuous if and only if $t \circ u_\gamma$ is continuous for each γ in Γ. If Γ is countable and for a fixed $\lambda > 0$ each E_γ has a base of balanced λ-convex neighbourhoods of the origin, then by Proposition 3.1.2, (E,ζ) is the *-inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Gamma)$.

However, since for an uncountably dimensional linear space E, the topologies $\tau(E,E^*)$, sc and s are distinct and $(E,\tau(E,E^*))$, (E,sc), (E,s) are respectively the inductive limit, **-inductive limit and *-inductive limit of some $(K_\gamma; u_\gamma : \gamma\varepsilon\Gamma)$ (each K_γ is a copy of the scalar field K), we see that a **-inductive limit of locally convex spaces need not be locally convex and that a *-inductive limit of locally convex spaces need not be a semiconvex space.

As in Chapter 3, one can define the notions of **-direct sum, generalized strict **-inductive limits and strict **-inductive limits of semiconvex spaces. It follows from Proposition 3.1.2 that a *-inductive limit of finitely many semiconvex spaces is semiconvex. In particular, the **-direct sum topology on a finite direct sum of semiconvex spaces is identical with the product topology.

5.2 Hyperbarrelled, hyperbornological and quasi-hyperbarrelled spaces

We call a semiconvex space hyperbarrelled (quasi-hyperbarrelled) if every closed balanced semiconvex absorbent (bornivorous) subset is a neighbourhood of the origin.

We say that a semiconvex space E is hyperbornological if every bounded linear map from E into any semiconvex space is continuous.

Clearly, if (E,u) is ultrabarrelled (ultrabornological, quasi-ultrabarrelled) and u^o is the finest semiconvex topology on E coarser than u, then (E,u^o) is hyperbarrelled (hyperbornological, quasi-hyperbarrelled), and if (F,v) is hyperbarrelled (hyperbornological, quasi-hyperbarrelled) then (F,v^{oo}) is barrelled (bornological, quasi-barrelled). In particular, every semiconvex ultrabarrelled (ultra-bornological, quasi-ultrabarrelled) space is hyperbarrelled (hyperborno-logical, quasi-hyperbarrelled) and every locally convex hyperbarrelled (hyperbornological, quasi-hyperbarrelled) space is barrelled (bornological, quasi-barrelled).

It is not difficult to show that a semiconvex space is hyper-bornological if and only if every balanced semiconvex bornivorous subset is a neighbourhood of the origin (in fact, the method used in Lemma 4.2.1 can easily be adapted). Let T be a set of linear maps from an l.t.s. E into an l.t.s. F. If B is a balanced semiconvex bornivorous subset of F and T is pointwise bounded (uniformly bounded on bounded sets), then $\bigcap_{t \in T} t^{-1}(B)$ is a balanced semiconvex absorbent (bornivorous) subset of E. If B is closed and each t in T is continuous, then $\bigcap_{t\varepsilon T} t^{-1}(B)$ is closed. The following result can be easily deduced from these observations.

Theorem 5.2.1.

Let T be a set of linear maps from a semiconvex space E into a semiconvex space F.

(a) If E is hyperbarrelled and T is pointwise bounded with each t in T continuous, then T is equicontinuous.

(b) If E is hyperbornological and T is uniformly bounded on bounded sets, then T is equicontinuous.

(c) If E is quasi-hyperbarrelled and T is uniformly bounded on bounded sets with each t in T continuous, then T is equicontinuous.

Theorem 5.2.2.

Any **-inductive limit of hyperbarrelled (hyperbornolgoical, quasi-hyperbarrelled) spaces is of the same sort.

Proof: Let (E,v) be the **-inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$, where each E_γ is hyperbarrelled. Let B be a v-closed balanced semiconvex absorbent subset of E. For each γ in Φ, $u_\gamma^{-1}(B)$ is a closed balanced semiconvex absorbent subset of E_γ; and is thus a neighbourhood, since E_γ is hyperbarrelled. As this is true for all γ in Φ, B must be a v-neighbourhood in E and thus (E,v) is hyperbarrelled.

If (E,v) is the **-inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$, where each E_γ is hyperbornological (quasi-hyperbarrelled), then by using a similar argument to the one above and choosing B to be a balanced semiconvex bornivorous (closed balanced semiconvex bornivorous) subset of (E,v), we see that (E,v) must be hyperbornological (quasi-hyperbarrelled).

Corollary 1. Every quotient by a linear subspace of a hyperbarrelled (hyperbornological, quasi-hyperbarrelled) space is of the same sort.

Corollary 2. Every product of finitely many hyperbarrelled (hyperbornological, quasi-hyperbarrelled) spaces is of the same sort.

Corollary 3. Every countable inductive limit of locally convex hyperbarrelled (hyperbornological, quasi-hyperbarrelled) spaces is of the same sort. In particular, every countable inductive limit of Fréchet spaces has all three properties.

From Corollary 3 above and Problem D(b), Page 195 of (18), we see that a closed linear subspace of a hyperbarrelled hyperbornological space need not be quasi-hyperbarrelled.

Corollary 4. Every **-inductive limit of complete separated locally bounded spaces is hyperbarrelled, hyperbornological and quasi-hyperbarrelled.

Since a countably dimensional normed linear space is not barrelled, it follows that a hyperbornological space need not be hyperbarrelled. Let E be an incomplete separated inductive limit of a sequence of Banach spaces and let x be a point of the completion $E^\wedge$ of E, which is not in E. As pointed out in Section 4.2, the linear subspace E_1 of $E^\wedge$ spanned by E and x is ultrabarrelled, and thus E_1 is hyperbarrelled. But it is not hyperbornological, since it is not bornological ((22), Page 155).

It is easy to see that every hyperbarrelled or hyperbornological space is quasi-hyperbarrelled. If F is a hyperbarrelled space which is not hyperbornological and G is a hyperbornological space which is not hyperbarrelled, then by Corollary 2 of Theorem 5.2.2, F x G is a quasi-hyperbarrelled space, which in view of Corollary 1 of Theorem 5.2.2 is neither hyperbarrelled nor hyperbornological.

A linear (bounded linear) map from a hyperbarrelled (quasi-hyperbarrelled) space into a semiconvex space is nearly continuous. Since by Corollary 1 of Theorem 5.2.2, every quotient of a hyperbarrelled (quasi-hyperbarrelled) space is of the same sort, we deduce from Page 213 of (16) and the argument preceding Theorem 4.1.2 that every closed (closed bounded) linear map from a hyperbarrelled (quasi-hyperbarrelled) space into a complete metrizable semiconvex space is continuous. In particular, every closed (closed bounded) linear map from a hyperbarrelled (quasi-hyperbarrelled) space into a complete separated locally bounded space is continuous. If B is a closed balanced semiconvex absorbent subset of a semiconvex space (E,v) (say B is λ-convex, $\lambda > 0$), and N is the intersection of $(\frac{1}{\lambda^n} B: \ n = 1,2,\ldots)$ the locally bounded topology w on the quotient space E/N with a base $(\frac{1}{\lambda^n} k_1(B): \ n = 1,2,\ldots)$ is separated (k_1 is the quotient map of E onto E/N), and by Lemma 4.1.1, the graph of k_1 is closed in $(E,v) \times (E/N, w)\hat{}$, k_1 being also bounded if B is bornivorous. Therefore, if every closed (closed bounded) linear map from (E,v) into any complete separated locally bounded space is continuous, B must be a v-neighbourhood, thus implying that (E,v) is hyperbarrelled (quasi-hyperbarrelled).

One can also show by modifying the proof of Theorem 4.2.2 that if every bounded linear map from a semiconvex space E into any complete separated locally bounded space is continuous, then E is hyperbornological.
We note these for further reference.

Theorem 5.2.3.

A semiconvex space E is hyperbarrelled (hyperbornological, quasi-hyperbarrelled) if and only if every closed (bounded, closed bounded) linear map from E into any complete separated locally bounded space is continuous.

Lemma 5.2.1. If f is a closed linear map from a product $X_{\gamma\varepsilon\Phi} E_\gamma$ of separated linear topological spaces into a complete separated locally bounded space F, then there is a finite subset Φ_o of Φ such that the restriction of f to $X_{\gamma\varepsilon\Phi/\Phi_o} E_\gamma$ is the zero map.

Proof: It is sufficient to prove that for some finite subset Φ_o of Φ, if $\gamma\varepsilon\Phi_1 = \Phi/\Phi_o$, then the restriction of f to E_γ is zero. For, if this is so, then $\Sigma_{\gamma\varepsilon\Phi_1} E_\gamma \subseteq f_1^{-1}(0)$, where f_1 is the restriction of f to $X_{\gamma\varepsilon\Phi_1} E_\gamma$. Since $\Sigma_{\gamma\varepsilon\Phi_1} E_\gamma$ is dense in $X_{\gamma\varepsilon\Phi_1} E_\gamma$ and $f_1^{-1}(0)$ is closed in $X_{\gamma\varepsilon\Phi_1} E_\gamma$ (because the graph of f_1 is closed in $X_{\gamma\varepsilon\Phi_1} E_\gamma \times F$), $f_1^{-1}(0) = X_{\gamma\varepsilon\Phi_1} E_\gamma$.

Let q be an r-pseudometric $(o < r \leq 1)$ which gives the topology of F. If there is no finite subset Φ_o of Φ such that for every γ in Φ/Φ_o, f is the zero map on E_γ, then, for some sequence $(\gamma_i : i = 1,2,...)$ of distinct members of Φ, there exist points x_{γ_i} such that x_{γ_i} is in E_{γ_i} and $q(f(x_{\gamma_i})) = i$. Clearly $(f(x_{\gamma_i}))$ is not bounded in F.

Now, $X_{\gamma\varepsilon\Phi} E_\gamma$ induces the product topology on the linear subspace $G = X_{i \geqslant 1} (K_{\gamma_i} x_{\gamma_i})$ (K_{γ_i} is a copy of the scalar field for each i). The restriction of f to the Frechet space G has a graph closed in G x F and is therefore continuous by Banach's closed graph theorem. This then implies that $(f(x_{\gamma_i}))$ is bounded in F. From this contradiction the result follows.

Since by Corollary 2 of Theorem 5.2.2, any finite product of hyperbarrelled (quasi-hyperbarrelled) spaces is of the same sort, the following result is immediate on using Lemma 5.2.1 and Theorem 5.2.3.

Theorem 5.2.4.

Any separated product of hyperbarrelled (quasi-hyperbarrelled) spaces is hyperbarrelled (quasi-hyperbarrelled).

By similarly using Lemma 5.2.1 and Mahowald's results in (26), one gets an alternative proof to the well known result that a separated product of barrelled (quasi-barrelled) spaces is of the same sort.

As any separated semiconvex space is a subspace of a product of separated locally bounded spaces which can be assumed complete, it follows that every separated semiconvex space is a subspace of some separated hyperbarrelled space. A hyperplane in a hyperbarrelled space is hyperbarrelled (this can be proved in a fashion similar to Corollary 4 of Theorem 4.1.2). The proof of the following result uses these observations and a method due to Komura ((22), Theorem 1.1).

Theorem 5.2.5.

Any separated semiconvex space is a closed subspace of some separated hyperbarrelled space.

Proof: Let E be a separated semiconvex space. If F is a separated hyperbarrelled space containing E, let $(e_\alpha : \alpha \varepsilon \Phi)$ be a Hamel basis for an algebraic supplement of E in F. For each α in Φ, let F_α be the linear subspace of F spanned by E and $(e_\lambda : \lambda \varepsilon \Phi, \lambda \neq \alpha)$. Clearly $E = \bigcap_{\alpha \varepsilon \Phi} F_\alpha$; and since each F_α is a hyperplane in F, each F_α is hyperbarrelled. It is easy to see that with the embedding map f of E into $X_{\alpha \varepsilon \Phi} F_\alpha$ specified by $f(x) = (x_\alpha)$, where $x_\alpha = x$ for all α in Φ, E becomes a linear topological subspace of $X_{\alpha \varepsilon \Phi} F_\alpha$. Since by Theorem 5.2.4., $X_{\alpha \varepsilon \Phi} F_\alpha$ is hyperbarrelled, all that remains is to prove that f(E) is closed in $X_{\alpha \varepsilon \Phi} F_\alpha$. If $((x_\alpha : \alpha \varepsilon \Phi)_\gamma : \gamma \varepsilon \Psi)$ is a net in f(E) which converges to $(y_\alpha : \alpha \varepsilon \Phi)$ in $X_{\alpha \varepsilon \Phi} F_\alpha$ then, for any α (say α_o), the net $(x_{\alpha_o \gamma} : \gamma \varepsilon \Psi)$ converges to y_{α_o}. As $x_\alpha = x_{\alpha_o}$ for all $\alpha \varepsilon \Phi, y_\alpha = y_{\alpha_o}$ for all $\alpha \varepsilon \Phi$. Thus $y_{\alpha_o} \varepsilon \bigcap_{\alpha \varepsilon \Phi} F_\alpha = E$, and $(y_\alpha) \varepsilon f(E)$. Therefore f(E) is closed in $X_{\alpha \varepsilon \Phi} F_\alpha$.

Let $(E_\gamma : \gamma \varepsilon \Phi)$ be a family of separated bornological spaces. If K^Φ is bornological so is $X_{\gamma \varepsilon \Phi} E_\gamma$. Bourbaki ((14),Page 15, exercise 18b) has a proof which uses the result that a separated locally convex space E is bornological if every bounded linear map of E into any Banach space is continuous. By using Theorem 5.2.3 in place of this, we see that the above stated result (on products of bornological spaces) holds with "bornological" replaced by "hyperbornological". In particular, a countable product of separated hyperbornological spaces is hyperbornological.

Since an uncountably dimensional linear space under its finest linear topology is not a semiconvex space, we see that an almost convex space need not be a semiconvex space. Theorems 5.2.6 and 5.2.7 below are results on almost convex semiconvex spaces which are analogues of Theorems 4.2.3 and 4.3.2. The proofs are omitted, being respectively similar to those of Theorems 4.2.3. and 4.3.2.

Theorem 5.2.6.

Let E be a separated almost convex hyperbornological space. Then E is a **-inductive limit of separated locally bounded spaces. If E is sequentially complete, then it is a **-inductive limit of complete separated locally bounded spaces and is therefore hyperbarrelled.

Theorem 5.2.7.

A separated sequentially complete almost convex quasi-hyperbarrelled space is hyperbarrelled.

5.3 Countably barrelled and countably quasi-barrelled spaces

Husain, in (14) called a separated locally convex space E with dual E' countably barrelled (countably quasi-barrelled) if every $\sigma(E',E)$-bounded ($\beta(E',E)$-bounded) subset of E' which is the countable union of equicontinuous sets is itself equicontinuous. He proved in Theorem 1 (Theorem 2) of (14) that a separated locally convex space E is countably barrelled (countably quasi-barrelled) if and only if every barrel (bornivorous barrel) which is the countable intersection of closed absolutely convex neighbourhoods in E is itself a neighbourhood. He also showed (Corollary 6) that a sequence of continuous linear maps from a countably barrelled (countably quasi-barrelled) space into a locally convex space is equicontinuous provided that it is pointwise bounded (uniformly bounded on bounded sets).

It is trivially true that every barrelled (quasi-barrelled) space is countably barrelled (countably quasi-barrelled) and a countably barrelled space is countably quasi-barrelled. In this section we give examples to show that (i) a countably barrelled space need not be barrelled (or even quasi-barrelled) and (ii) a countably quasi-barrelled space need not be countably barrelled. A third example shows that the property of being countably barrelled (countably quasi-barrelled) does not pass to closed linear subspaces.

(i) Let E be the strong dual of a metrizable locally convex space. Then, by ((8), Pages 71 and 88), E need not be quasi-barrelled. But E is countably barrelled, being countably quasi-barrelled and complete ((14), Propositions 1 and 4).

(ii) Denote by c the Banach space of all convergent sequences $x = (x_1, x_2, \ldots)$ with the supremum norm, by c_o the closed linear subspace of c consisting of sequences converging to zero and by ϕ the linear subspace consisting of all sequences containing only a finite number of non-zero entries. For each n, let f_n be the linear functional on ϕ defined by the equation $f_n(x) = n\, x_n$. As pointed out by Weston ((41), Page 1), (f_n) is a pointwise bounded sequence of continuous linear functionals on ϕ (under the norm topology induced from c) which is not equicontinuous. Thus by Corollary 6 of (14), ϕ is not countably barrelled, though it is countably quasi-barrelled, being bornological.

(iii) Since any separated locally convex space is a closed linear subspace of some barrelled space, to show that a closed linear subspace of a countably barrelled (countably quasi-barrelled) space need not be of the same sort, it is sufficient to give an example of a separated locally convex space which is not countably quasi-barrelled. Let (E,u) be c_o with the supremum norm topology u and v be the associated weak topology on c_o. For each n, let g_n be the linear map from (E,v) into (E,u) defined as follows: $g_n(x) = (x_1, x_2, \ldots, x_n, o, o, \ldots)$. Then (g_n) is a sequence of continuous linear maps from (E,v) into (E,u) such that for each x in E, $g_n(x)$ converges to x in (E,u). Moreover, (g_n) is uniformly bounded on bounded sets, for if B is the unit ball in (E,u), the union over n of $g_n(B)$ is contained in B. But (g_n) is not equicontinuous since v is strictly coarser than u. Therefore by Corollary 6 of (14), (E,v) is not countably quasi-barrelled.

5.4 $\aleph$-hyperbarrelled and $\aleph$-quasi-hyperbarrelled spaces

We say that a semiconvex space is $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled) if every closed balanced semiconvex absorbent (bornivorous) subset of the form

$$\left(\bigcap_{\gamma\in\Phi} U_\gamma : \text{for some } \lambda > 0, \text{ each } U_\gamma \text{ is a closed balanced } \lambda\text{-convex neighbourhood and the cardinality of } \Phi \text{ is } \aleph\right)$$

is a neighbourhood of the origin.

Clearly every hyperbarrelled (quasi-hyperbarrelled, $\aleph$-hyperbarrelled) space is $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled, $\aleph$-quasi-hyperbarrelled) for any $\aleph$ and if $\aleph_1$ is a cardinal number less than $\aleph_2$, then every $\aleph_2$-hyperbarrelled ($\aleph_2$-quasi-hyperbarrelled) space is $\aleph_1$-hyperbarrelled ($\aleph_1$-quasi-hyperbarrelled). By using Theorem 1 (Theorem 2) of (14) we see that every locally convex $\aleph_0$-hyperbarrelled ($\aleph_0$-quasi-hyperbarrelled) space is countably barrelled (countably quasi-barrelled). It therefore follows by Example (ii) of section 5.3 that for each $\aleph \geq \aleph_0$, an $\aleph$-quasi-hyperbarrelled space need not be $\aleph$-hyperbarrelled. Also, by using Example (iii) of section 5.3 and Theorem 5.2.5, we see that a closed linear subspace of an $\aleph$-hyperbarrelled ($\aleph$ -quasi-hyperbarrelled) space need not be of the same sort.

If $(f_\gamma : \gamma \varepsilon \Phi)$ is a set of continuous linear maps from an l.t.s. (E,u) into an l.t.s. F, then for every closed balanced semiconvex neighbourhood V (say V is λ-convex, $\lambda > 0$) in F each $f_\gamma^{-1}(V)$ is a closed balanced λ-convex neighbourhood. If $(f_\gamma : \gamma \varepsilon \Phi)$ is pointwise bounded (uniformly bounded on bounded sets), then $\bigcap_{\gamma \varepsilon \Phi} f_\gamma^{-1}(V)$ is absorbent (bornivorous). Using these, one can prove the following result.

<u>Theorem 5.4.1.</u>

Let Φ be a set of cardinality $\aleph$ and let $(f_\gamma : \gamma \varepsilon \Phi)$ be a set of continuous linear maps from a semiconvex space E into a semiconvex space. If E is $\aleph$ -hyperbarrelled ($\aleph$ -quasi-hyperbarrelled) then $(f_\gamma : \gamma \varepsilon \Phi)$ is equicontinuous, provided that it is pointwise bounded (uniformly bounded on bounded sets).

The following Corollary follows from the above theorem and Weston's main result in (41).

<u>Corollary</u>. Let (f_n) be a pointwise convergent sequence of continuous linear maps from an $\aleph_0$-hyperbarrelled space E into a separated semiconvex space F, and let f be the limit mapping i.e. $f(x) = \lim_{n \to \infty} f_n(x)$ for all x in E. Then f is continuous. If F is sequentially complete, then $(f_n(x))$ is necessarily convergent everywhere if it is bounded for each x in E and convergent through-out a set which is everywhere dense in E.

Let (E,u) be the sequence space $l^{\frac{1}{2}}$. For each $x = (x_1, x_2, \ldots)$ in E, let $t_n(x)$ be $(x_1, x_2, \ldots, x_n, o, o, \ldots)$. Then, as pointed out in Page 256 of (37) (t_n) is a sequence of continuous linear maps from (E,u^{oo}) into (E,u) such that for each x in E, $t_n(x)$ converges to x in (E,u). As the identity map from (E,u^{oo}) into (E,u) is not continuous, it follows from the corollary above that (E,u^{oo}) is not $\aleph_0$-hyper-barrelled, though it is countably barrelled, being barrelled. Also each t_n is continuous from $(E, \tau(E,E^*))$ into (E,u) and $t_n(x)$ converges to x in (E,u) for each x in E. Moreover, (t_n) is uniformly bounded on the $\tau(E,E^*)$-bounded subsets of E. For, any $\tau(E,E^*)$-bounded subset B of E is contained in some finite dimensional subspace E_o say, of E. The restrictions of (f_n) to E_o must be equicontinuous and thus $\bigcup_{n \geq 1} t_n(B)$ is u-bounded. As the identity map from $(E,\tau(E,E^*))$ onto (E,u) is not continuous, it follows from Theorem 5.4.1 that $(E, \tau(E,E^*))$ is not $\aleph_0$-quasi-hyperbarrelled. This implies that an uncountable direct sum of Banach spaces is not $\aleph_0$-quasi-hyperbarrelled and thus an inductive limit of Banach spaces need not be $\aleph_0$-quasi-hyperbarrelled. However, we have the following result.

Theorem 5.4.2.

Any **-inductive limit of $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled) spaces is of the same sort.

Proof: Let (F,v) be the **-inductive limit of $(E_\alpha;\ u_\alpha : \alpha\varepsilon\Psi)$, where each E_α is $\aleph$-hyperbarrelled. If

$V = (\bigcap_{\gamma\varepsilon\Phi} V_\gamma$: for some $\lambda > 0$, each V_γ is a closed balanced λ-convex v-neighbourhood and the cardinality of Φ is $\aleph$)

is absorbent, then for each α in Ψ,

$$u_\alpha^{-1}(V) = \bigcap_{\gamma\varepsilon\Phi} u_\alpha^{-1}(V_\gamma)$$

is absorbent and for each γ in Φ, each $u_\alpha^{-1}(V_\gamma)$ is a closed balanced λ-convex neighbourhood in E_α. Since E_α is $\aleph$-hyperbarrelled, $u_\alpha^{-1}(V)$ is a neighbourhood in E_α. As this is true for all α in Ψ, V is a v-neighbourhood and (F,v) is $\aleph$-hyperbarrelled.

Similarly, any **-inductive limit of $\aleph$-quasi-hyperbarrelled spaces is $\aleph$-quasi-hyperbarrelled.

Corollary 1. Any countable inductive limit of locally convex $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled) spaces is of the same sort.

Corollary 2. Any product of finitely many $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled) spaces is of the same sort.

Lemma 5.4.1. If B is a closed balanced semiconvex absorbent subset of a product $E = X_{\gamma\varepsilon\Phi} E_\gamma$ of semiconvex spaces, then there exists a finite subset Φ_o of Φ such that $X_{\gamma\varepsilon\Phi/\Phi_o} E_\gamma \subseteq B$.

Proof: Let F_γ denote E_γ under its finest semiconvex topology. By Theorem 5.2.4, the space $(F,v) = X_{\gamma\varepsilon\Phi} F_\gamma$ is hyperbarrelled. Since v is finer than the topology of E, B is v-closed and must therefore be a v-neighbourhood of the origin. From this, the result follows.

In the above Lemma, E is the **-direct sum of $X_{\gamma\varepsilon\Phi_o} E_\gamma$ and $X_{\gamma\varepsilon\Phi/\Phi_o} E_\gamma$, and therefore B is a neighbourhood of the origin in E if and only if $B \cap X_{\gamma\varepsilon\Phi_o} E_\gamma$ is a neighbourhood of the origin in $X_{\gamma\varepsilon\Phi_o} E_\gamma$. If each E_γ is $\aleph$-hyperbarrelled, so is $X_{\gamma\varepsilon\Phi_o} E_\gamma$ by Corollary 2 of Theorem 5.4.2. In this case, if B is of the form

$(\bigcap_{\alpha\varepsilon\Psi} U_\alpha$: for some $\lambda > 0$, each U_α is a closed balanced λ-convex neighbourhood and the cardinality of Ψ is $\aleph$),

$B \cap X_{\gamma\varepsilon\Phi_o} E_\gamma$ is a neighbourhood of the origin in $X_{\gamma\varepsilon\Phi_o} E_\gamma$ and B is thus a neighbourhood of the origin in E. Similarly if each E_γ is $\aleph$-quasi-hyperbarrelled and B is a bornivorous subset of E of the form

$(\bigcap_{\alpha\varepsilon\Psi} U_\alpha$: for some $\lambda > 0$, each U_α is a closed balanced λ-convex neighbourhood and the cardinality of Ψ is $\aleph$),

then it is a neighbourhood in E. We have thus proved the following result.

Theorem 5.4.3.

Any product of $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled) spaces is of the same sort.

By an argument similar to the one above, one can prove that a product of countably barrelled (countably quasi-barrelled) spaces is countably barrelled (countably quasi-barrelled).

The following is a generalization of 22.9 of (18). The argument used here is a slight modification of that in (18).

Lemma 5.4.2. If λ is a strictly positive real number then, in the strong dual of any metrizable locally convex space, every bornivorous set which is the intersection of a sequence of balanced λ-convex neighbourhoods is itself a neighbourhood.

Proof: Let E be a metrizable locally convex space with dual E´. We shall suppose that E´ has its strong topology $\beta(E',E)$.

Let

$$U = (\bigcap_{n \geq 1} U_n : \text{ each } U_n \text{ is a balanced } \lambda\text{-convex neighbourhood in } E')$$

be a bornivorous set in E´. To prove that U is neighbourhood in E´, it is sufficient to show that there exists a subset of U which is $\sigma(E',E)$-closed, absolutely convex and absorbent, since such a set is a neighbourhood in E´, being the polar in E´ of a $\sigma(E,E')$-bounded subset of E.

By 22.3 of (18), there exists in E´ a fundamental sequence (B_n) of bounded sets such that each B_n is absolutely convex and $\sigma(E',E)$-compact

Now U is a λ-convex bornivorous suprabarrel in E´ with $(^1/\lambda^n\ U)$ as a defining sequence. For each n, there are:

(i) a positive number t_n such that $t_n B_n \subseteq {}^1/\lambda^{n+1}\ U$, and

(ii) a $\sigma(E',E)$-closed absolutely convex neighbourhood W_n in E' such that $W_n \subseteq {}^1/\lambda\ U_n$.

The convex envelope G_n of $\bigcup_{1 \leq i \leq n} t_i B_i$ is a $\sigma(E',E)$-compact absolutely convex subset of ${}^1/\lambda\ U$. If $V_n = G_n + W_n$, then V_n is a $\sigma(E',E)$-closed absolutely convex neighbourhood in E' and $V_n \subseteq U_n$. Let

$$V = \bigcap (V_n : n = 1, 2, \ldots)$$

Then V is a $\sigma(E',E)$-closed absolutely convex subset of U which is absorbent in E' since it absorbs each B_n.

By an application of Lemma 4.3.1, one can easily show that any separated sequentially complete almost convex $\aleph$-quasi-hyperbarrelled space is $\aleph$-hyperbarrelled. It therefore follows from Lemma 5.4.2 that the strong dual of any metrizable locally convex space is $\aleph_o$-hyperbarrelled. And since by ((8), Pages 71 and 88) the strong dual of a metrizable locally convex space need not be quasi-barrelled, we conclude that an $\aleph_o$-hyperbarrelled space need not be quasi-hyperbarrelled.

Let E be an l.t.s., and let

$$U = \left(\bigcap_{\gamma \varepsilon \Phi} U_\gamma : \text{each } U_\gamma \text{ is a closed balanced neighbourhood and the cardinality of } \Phi \text{ is } \aleph\right),$$

be an ultrabarrel (a bornivorous ultrabarrel) in E with a defining sequence (U^n) such that each U^n is of the same form as U. If every such U is a neighbourhood, we say that E is <u>$\aleph$-ultrabarrelled</u> (<u>$\aleph$-quasi-ultrabarrelled</u>).

It is easy to see that the $\aleph$-ultrabarrelled ($\aleph$-quasi-ultrabarrelled) spaces bear a relationship to the ultrabarrelled (quasi-ultrabarrelled) spaces similar to that between $\aleph$-hyperbarrelled ($\aleph$-quasi-hyperbarrelled) spaces and hyperbarrelled (quasi-hyper-barrelled) spaces. It is not difficult to show that Theorem 5.4.1 and its corollary are true with "semiconvex space","hyperbarrelled" respectively replaced by "l.t.s." and "ultrabarrelled". Also, Theorem 5.4.2 and its corollaries are true with "**-inductive limit" and "hyperbarrelled" respectively replaced by "*-inductive limit" and "ultrabarrelled".

CHAPTER 6

D($\mathcal{A}_1;\mathcal{A}$, T) - SPACES

6.1 Completeness and the closed graph theorem

Let E,F be complete metric linear spaces. Banach, in (3) proved that any closed linear map of E into F is continuous and that any continuous linear map of E onto F is open. The first assertion is the classical closed graph theorem and the second, the open mapping theorem. These two results do not hold for arbitrary complete separated linear topological spaces, as the following example shows. Let (E,u) be an infinite dimensional complete metric linear space. Then, since the finest linear topology s on E is strictly finer than u, the identity map of (E,s) onto (E,u) is continuous but not open and this implies that its inverse is closed but not continuous. Yet (E,s), (E,u) are complete separated ultrabarrelled ultrabornological spaces. Thus the validity of the two theorems requires some unusually strong hypotheses. Various mathematicians, including Pták (32), (33), A.P. Robertson and W. Robertson (35), Kelley (17), Raikov (34) and Husain (9), (10), (11), (12) have studied the closed graph and open mapping theorems, especially for locally convex spaces.

Pták, in (32), (33) introduced the notions of B-completeness and B_r-completeness in locally convex spaces. Precisely, a separated locally convex space E is said to be a B-complete (B_r-complete) locally convex space if every continuous (continuous (1 - 1)) linear nearly open map of E onto any separated locally convex space is open. As shown by Kelley ((17), Theorem 2), for a separated locally convex space E the following are equivalent:

(1) The space E is a B-complete locally convex space.

(2) Every closed linear nearly open map of E onto any separated locally convex space is open.

(3) Every closed linear nearly continuous map from any separated locally convex space into any quotient of E by a closed linear subspace is continuous.

Also, for a separated locally convex space E, it follows easily from ((33), 3.6 and 3.8) that the following are equivalent:

(1) The space E is a B_r-complete locally convex space.

(2) Every closed linear (1 - 1) nearly open map of E onto any separated locally convex space is open.

(3) Every closed linear nearly continuous map from any separated locally convex space into E is continuous.

These two sets of results show the very close link between the closed graph theorem, the open mapping theorem, B-complete and B_r-complete locally convex spaces.

Pták in ((33), 5.6, 3.3, 4.1) showed that if E is a separated locally convex space with dual E′, then E is complete (B_r-complete, B-complete) if and only if every hyperplane in E′(every $\sigma(E', E)$-dense linear subspace of E′, every linear subspace of E′) having a $\sigma(E', E)$-closed intersection with the polar of each neighbourhood in E is $\sigma(E', E)$-closed. Separated locally convex spaces E with the property that every linear subspace of E' having a $\sigma(E',E)$-closed intersection with the polar of each neighbourhood in E is $\sigma(E',E)$-closed, were studied by Collins (5), who called them fully complete spaces. Hypercomplete spaces (of Kelley (17)) are the separated locally convex spaces having this property, with "linear subspaces of duals" replaced by "absolutely convex subsets of duals". Singer, in (38) called a separated locally convex space E strictly hypercomplete if every convex subset of E' having a $\sigma(E',E)$-closed intersection with the polar of each neighbourhood in E is $\sigma(E',E)$-closed. Thus every strictly hypercomplete space is hypercomplete, every hypercomplete space is a B-complete locally convex space, every B-complete locally convex space is a B_r-complete locally convex space and every B_r-complete locally convex space is complete.

Strictly hypercomplete spaces are those separated locally convex spaces for which the Krein-Smullian theorem (see (24)) holds. Every Frechet space is strictly hypercomplete (7). It is not difficult to show that the dual of a Frechet space is strictly hypercomplete for all convex topologies between the topology of compact convergence and the Mackey topology (in fact, the proof of a similar result for B-complete locally convex spaces, given in page 123 of (36) carries over easily). Thus a countable direct sum of reflexive Banach spaces is strictly hypercomplete. However, a hypercomplete space need not be strictly hypercomplete. For by((18), Page 178, example H) a product of reals need not be strictly hypercomplete, but is hypercomplete.

Since a complete S-space is strictly hypercomplete (for the definition of an S-space, see (13)) by Theorem 3 of (13), this example also proves false a conjecture of Husain's in page 258 of (13) that complete S-spaces and hypercomplete spaces are the same thing. By an easy modification of a method due to Collins ((15), Theorem 15) (Pták((33), 4.4)), one can prove that closed linear subspaces (quotients by closed linear subspaces) of strictly hypercomplete spaces are strictly hypercomplete.

Raikov, in (34) called an l.t.s. E a B-complete l.t.s. if every continuous linear nearly open map t of E onto any separated l.t.s. is open. He observed that for a locally convex space E, this definition is equivalent to that of Pták in (32), but that in addition every complete metric linear space is a B-complete l.t.s. He also pointed out that every closed linear nearly continuous map from any separated l.t.s. into a B-complete l.t.s. is continuous.

If in Raikov's definition above, we restrict "t" to be a (1 - 1) map we say that E is a <u>B_r-complete l.t.s.</u> Just as in the case of a B-complete l.t.s., a separated locally convex space is a B_r-complete l.t.s. if and only if it is a B_r-complete locally convex space. Analogously, one may define <u>B-complete</u> and <u>B_r-complete semiconvex spaces</u> as well as <u>B-complete</u> and <u>B_r-complete topological groups.</u> A separated semiconvex space is a B-complete (B_r-complete) semiconvex space if and only if it is a B-complete (B_r-complete) l.t.s. Examples of B-complete topological groups are complete metrizable or locally compact groups ((16), Page 213).

We now describe briefly an attempt to generalise the notions of B-completeness and B_r-completeness in locally convex spaces. Let $\mathcal{G}$ denote the class of all separated locally convex spaces with the property that every continuous linear (nearly open) map from any member of $\mathcal{G}$ onto any separated barrelled space is open. By Theorem 3(i) of (35), $\mathcal{G}$ includes all B-complete locally convex spaces. Husain and Mahowald, in (15) observed that a member of $\mathcal{G}$ need not be complete and hence need not be a B-complete locally convex space. This led Husain to study what he called B(σ)- and $B_r(\sigma)$-spaces in a number of papers including (9), (10), (11), (12). While B-complete and B_r-complete locally convex spaces derive a considerable part of their importance from their usefulness in proving closed graph and open mapping theorems, B(σ)- and $B_r(\sigma)$-spaces have not so far met with any appreciable success in this direction.

In this chapter we consider problems of the following type. If $\mathcal{A}_1$ is a class of separated linear topological spaces, find a necessary and sufficient condition for an l.t.s. F to have the property that every closed linear nearly continuous map from any member of $\mathcal{A}_1$ into F (any quotient of F by a closed linear subspace) is continuous. We give answers for some important classes $\mathcal{A}_1$ and use them to describe extensions to well known closed graph and open mapping theorems. As some of our methods work for topological groups, our subject is treated in this context in Sections 6.2 and 6.3, while in Sections 6.4, 6.5 and 6.6, we restrict our consideration to linear topological spaces. Throughout, all our topological spaces shall be assumed separated.

6.2 <u>The case when $\mathcal{A}_1$ is nearly full in $\mathcal{A}$</u>

Let $\mathcal{A}$ be a class of topological groups. For every E, F in $\mathcal{A}$, let T(E,F) denote a given set of group homomorphisms from E into F, and take T to be the union of T(E,F) as E, F vary over $\mathcal{A}$. We say that <u>($\mathcal{A}$,T) is admissible</u> if the following conditions are satisfied.

(1) For any E in $\mathcal{A}$, if E_o in $\mathcal{A}$ is a closed subgroup of E, then E/E_o is in $\mathcal{A}$ and the quotient map of E onto E/E_o is in $T(E,E/E_o)$.

(2) For E, F_o, F in $\mathcal{A}$ such that F_o is a subgroup of F, if t is in T(E,F) then, under their induced topologies the subgroups $t^{-1}(e_1)$, $t^{-1}(F_o)$ of E are in $\mathcal{A}$ (e_1 is the identity of F), the induced map of t is in $T(E/t^{-1}(e_1),F)$ (if $t^{-1}(e_1)$ is closed in E), and the restriction of t to $t^{-1}(F_o)$ is in both $T(t^{-1}(F_o),F_o)$ and $T(t^{-1}(F_o),F)$.

(3) For E, F, G in $\mathcal{A}$, if t_1 is in T(E,F) and t_2 is in T(F,G) then, the map $t_2 o t_1$ is in T(E,G), the subgroup $t_1(E)$ of F is in $\mathcal{A}$ and t_1 is in $T(E,t_1(E))$.

(4) For E_o, E, F in $\mathcal{A}$ such that E_o is a subgroup of E, if t in $T(E_o,F)$ is a (1 - 1) onto map then, t^{-1} is in T(F,E).

(5) If (E,u), (F,v) are in $\mathcal{A}$ and t is in T((E,u),(F,v)) then, the space (F,w) (if separated) is in $\mathcal{A}$, where the topology w has a base of neighbourhoods consisting of sets $(t(U)V : U \in \mathcal{U}, V \in \mathcal{V})$ ($\mathcal{U}$, $\mathcal{V}$ are respectively bases of symmetric neighbourhoods for the topologies u,v) and the identity map is in T((F,v),(F,w)).

If ($\mathcal{A}$,T) is admissible and E is in $\mathcal{A}$, in referring to a <u>subspace</u> E_o of E, it shall be assumed that E_o is in $\mathcal{A}$.

Let ($\mathcal{A}$,T) be admissible and suppose that $\mathcal{A}_1$, $\mathcal{A}_2$ are subclasses of $\mathcal{A}$. We say that $\mathcal{A}_1$ is <u>nearly full</u> in $\mathcal{A}$ with respect to T and $\mathcal{A}_2$ if, with the notation of (5) above, $(E,u) \varepsilon \mathcal{A}_1$, $(F,v) \varepsilon \mathcal{A}_2 \Rightarrow (F,w) \varepsilon \mathcal{A}_1$ whenever w is a separated topology.

Let $\mathcal{A}$ be the class of all topological groups (linear topological spaces, semiconvex spaces, locally convex spaces). For any E, F in $\mathcal{A}$, let T(E,F) be the set of all group homomorphisms (linear maps, linear maps, linear maps) from E into F. Clearly $(\mathcal{A},T)$ is admissible, and $\mathcal{A}$ is nearly full in $\mathcal{A}$ with respect to T and $\mathcal{A}_2$, for any subclass $\mathcal{A}_2$ of $\mathcal{A}$. If $\mathcal{A}_3$ is the class of all second category topological groups (ultrabarrelled spaces, hyperbarrelled spaces, barrelled spaces) and $\mathcal{A}_4$ the class of all Lindelöf topological groups which are either locally compact or complete metrizable (complete metric linear spaces, semiconvex complete metric linear spaces, Frechet spaces) then, $\mathcal{A}_3$ is nearly full in $\mathcal{A}$ with respect to T and $\mathcal{A}_4$. To prove this, one uses Lemma 2.6.5 and the fact that in any of the cases considered, every t in T(E,F) is nearly continuous if $E \varepsilon \mathcal{A}_3$ and $F \varepsilon \mathcal{A}_4$.

Let $(\mathcal{A},T)$ be admissible and let $\mathcal{A}_1$ be a subclass of $\mathcal{A}$. We say that E in $\mathcal{A}$ is a <u>$D(\mathcal{A}_1; \mathcal{A}, T)$-space</u> if for every E_1 in $\mathcal{A}_1$, any onto map t in $T(E,E_1)$ is open provided that it is nearly open and its graph is closed in $E \times E_1$. The class of all $D(\mathcal{A}_1; \mathcal{A}, T)$-spaces shall be denoted by $D(\mathcal{A}_1; \mathcal{A}, T)$. If in the definition of a $D(\mathcal{A}_1; \mathcal{A}, T)$-space the condition is only assumed satisfied by (1 - 1) maps "t", then we call E a <u>$D_r(\mathcal{A}_1; \mathcal{A}, T)$-space</u>, and correspondingly we have the class <u>$D_r(\mathcal{A}_1; \mathcal{A}, T)$</u>.

Clearly if $(\mathcal{A}, T)$ is admissible then $D(\mathcal{A}_1; \mathcal{A}, T) \subseteq D_r(\mathcal{A}_1; \mathcal{A}, T)$ for every subclass $\mathcal{A}_1$ of $\mathcal{A}$. If $\mathcal{A}_2, \mathcal{A}_3$ are subclasses of $\mathcal{A}$ such that $\mathcal{A}_2 \subseteq \mathcal{A}_3$, it is easy to see that $D_r(\mathcal{A}_3; \mathcal{A}, T) \subseteq D_r(\mathcal{A}_2; \mathcal{A}, T)$ and $D(\mathcal{A}_3; \mathcal{A}, T) \subseteq D(\mathcal{A}_2; \mathcal{A}, T)$. Also, if E is a $D(\mathcal{A}_1; \mathcal{A}, T)$-space then, for any closed subspace E_o of E, E/E_o is a $D(\mathcal{A}_1; \mathcal{A}, T)$-space. For, let F be in $\mathcal{A}_1$ and t in $T(E/E_o, F)$ be a closed nearly open onto map. If k_1 is the quotient map of E onto E/E_o, then the map $tok_1 \in T(E,F)$ since $(\mathcal{A}, T)$ is admissible, and is closed and nearly open. Therefore tok_1 is open and this implies that t is open. Thus E/E_o is a $D(\mathcal{A}_1; \mathcal{A}, T)$- space. It is also not difficult to show that if E is a $D_r(\mathcal{A}_1; \mathcal{A}, T)$-space then it is a $D(\mathcal{A}_1; \mathcal{A}, T)$-space if and only if E/E_o is a $D_r(\mathcal{A}_1; \mathcal{A}, T)$-space for every closed subspace E_o of E.

<u>Theorem 6.2.1.</u>

Let $(\mathcal{A}, T)$ be admissible and suppose that $\mathcal{A}_1, \mathcal{A}_2$ are subclasses of $\mathcal{A}$ such that $\mathcal{A}_1$ is nearly full in $\mathcal{A}$ with respect to T and $\mathcal{A}_2$. Then, for a space E in $\mathcal{A}_2$, the following are equivalent:

(1) For every F in $\mathcal{A}_1$, any continuous (1 - 1) nearly open onto map in T(E,F) is open.

(2) For every F in $\mathcal{A}_1$, any closed nearly continuous map in T(F,E) is continuous.

(3) The space E is a $D_r(\mathcal{A}_1; \mathcal{A}, T)$-space.

<u>Proof:</u>

(1) => (2): Let F be in $\mathcal{A}_1$ and let t in T(F,E) be a closed nearly continuous map. Denote by $\mathcal{U}, \mathcal{V}$ bases of symmetric neighbourhoods for

the topologies of F, E respectively, and let w be the topology on E with a neighbourhood base $(t(U)V : U\in\mathcal{U}, V\in\mathcal{V})$. Since t is closed and nearly continuous, it follows by Lemma 2.6.5 that (E,w) is separated and that the (continuous) identity map i say, from E onto (E,w) is nearly open. As $\mathcal{A}_1$ is nearly full in $\mathcal{A}$ with respect to T and $\mathcal{A}_2$, $(E,w)\in\mathcal{A}_1$; and because $(\mathcal{A},T)$ is admissible, $i\in T(E,(E,w))$. Therefore by (1), w coincides with the original topology of E. And since t is continuous from F into (E,w), (2) follows.

(2) => (3) Let F be in $\mathcal{A}_1$ and suppose that f in T(E,F) is a closed nearly open (1 - 1) onto map. Then $f^{-1}\in T(F,E)$ because $(\mathcal{A},T)$ is admissible, and f^{-1} is closed and nearly continuous. By (2) f^{-1} is continuous and thus f is open. This proves (3).

(3) => (1) obvious.

By using Theorem 6.2.1 and the fact that a quotient by a closed subspace of a $D(\mathcal{A}_1;\mathcal{A},T)$-space is also a $D(\mathcal{A}_1;\mathcal{A},T)$-space, one can prove the following result.

Theorem 6.2.2.

Let $(\mathcal{A},T)$ be admissible. Suppose that $\mathcal{A}_1, \mathcal{A}_2$ are sub-classes of $\mathcal{A}$ such that each quotient by a closed subspace of every member of $\mathcal{A}_2$ is also in $\mathcal{A}_2$ and that $\mathcal{A}_1$ is nearly full in $\mathcal{A}$ with respect to T and $\mathcal{A}_2$. Then, for a space E in $\mathcal{A}_2$ the following are equivalent:

(1) For every F in $\mathcal{A}_1$, any continuous nearly open onto map in T(E,F) is open.

(2) If E_1 is a quotient of E by a closed subspace and F is in $\mathcal{A}_1$ then, any closed nearly continuous map in $T(F,E_1)$ is continuous.

(3) The space E is a $D(\mathcal{A}_1;\mathcal{A},T)$-space.

Let $\mathcal{A}$ be the class of all topological groups (linear topological spaces, semiconvex spaces, locally convex spaces). If $E,F \in \mathcal{A}$, let $T(E,F)$ be the set of all group homomorphisms(linear maps, linear maps, linear maps) from E into F. It follows from Theorem 6.2.1 that the $D(\mathcal{A};\mathcal{A},T)$-spaces are the B_r-complete topological groups (linear topological spaces, semiconvex spaces, locally convex spaces). It similarly follows from Theorem 6.2.2 that the $D(\mathcal{A};\mathcal{A},T)$-spaces are the B-complete topological groups (linear topological spaces, semiconvex spaces, locally convex spaces).

With $\mathcal{A}$, T, $\mathcal{A}_1$, $\mathcal{A}_2$, E as in Theorem 6.2.2 (Theorem 6.2.1) let E_o be a subspace of E and suppose that the space F is in $\mathcal{A}_1$. It follows by an application of Lemma 2.6.3.(a) that a closed nearly open (closed nearly open (1 - 1)) onto map in $T(E_o,F)$ is open provided that the filter condition holds. In either of the two cases, it also follows by an application of Lemma 2.6.3.(b) that a closed nearly continuous map t in $T(F,E_o)$ is continuous provided that the inverse filter condition holds and $t \in T(F,E)$.

6.3 Inductive classes

Let $(\mathcal{A},T)$ be admissible. Let F be in $\mathcal{A}$ and suppose that for each γ in an index set Ψ, $E_\gamma \in \mathcal{A}$, $u_\gamma \in T(E_\gamma,F)$ and each u_γ is continuous. We say that $(\mathcal{A},T)$ is an inductive class if for each choice of F, Ψ, E_γ, u_γ as above, there is a topology w on F simultaneously satisfying the following conditions:

(1) $(F,w) \in \mathcal{A}$ and each $u_\gamma \in T(E_\gamma,(F,w))$.

(2) The topology w is the finest one on F satisfying (1) for which each u_γ is continuous.

(3) If $G \in \mathcal{A}$ then, any f in T((F,w), G) is continuous if and only if each $f \circ u_\gamma$ is continuous.

(4) If F_o is a closed subspace of (F,w) and k_1 is the canonical map of F onto F/F_o then, the quotient topology of $(F,w)/F_o$ is the finest topology on F/F_o for which $(F,w)/F_o$ is in $\mathcal{A}$ and such that (a) each $k_1 \circ u_\gamma$ is continuous and (b) for any H in $\mathcal{A}$, any f in $T((F,w)/F_o, H)$ is continuous if and only if each $f \circ k_1 \circ u_\gamma$ is continuous.

If conditions (1) → (4) are satisfied, we call (F,w) the <u>$(\mathcal{A}$,T)-inductive limit</u> of $(E_\gamma;\ u_\gamma : \gamma \in \Psi)$. If in particular F is the union over γ of $u_\gamma(E_\gamma)$, we say that (F,w) is the <u>generalized strict $(\mathcal{A}$,T)-inductive limit of $(E_\gamma;\ u_\gamma : \gamma \in \Psi)$</u>.

If E is the $(\mathcal{A}$,T)-inductive limit of $(E_\gamma;\ u_\gamma : \gamma \in \Psi)$ then, by condition (4) above, any quotient of E by a closed subspace is the $(\mathcal{A}$,T)-inductive limit of $(E_\gamma;\ k_1 \circ u_\gamma : \gamma \in \Psi)$, where k_1 is the canonical map of E onto the quotient space.

Let $(\mathcal{A}$,T) be admissible. We call a complete topological space E in $\mathcal{A}$ <u>extracomplete</u> if every quotient by a closed subspace of E is complete. Clearly any quotient by a closed subspace of an extracomplete space is extracomplete.

Let $(\mathcal{A},T)$ be admissible. Let $\mathcal{A}_1$ be the class of all members of $\mathcal{A}$ each of which is of the second category in itself, and $\mathcal{A}_2$ a subclass of $D_r(\mathcal{A};\mathcal{A},T)(D(\mathcal{A};\mathcal{A},T))$ consisting of spaces which are complete (extracomplete) [and which includes quotients of members]. If for every E_1 in $\mathcal{A}_1$ and E_2 in $\mathcal{A}_2$ any t in $T(E_1,E_2)$ is nearly continuous, we call the ordered pair <u>$(\mathcal{A}_1,\mathcal{A}_2)$</u> a <u>$\mu_{12}(\mathcal{A},T)$-pair</u> <u>$(\lambda_{12}(\mathcal{A},T)$-pair)</u>. We shall generally shorten $\mu_{12}(\mathcal{A},T)$pair$(\lambda_{12}(\mathcal{A},T)$pair) to μ_{12}pair (λ_{12}pair). Clearly every λ_{12} pair is also a μ_{12} pair.

Let $\mathcal{A}$ be the class of all topological groups (linear topological spaces, semiconvex spaces, locally convex spaces). If $E,F\varepsilon\mathcal{A}$, let $T(E,F)$ be the set of all group homomorphisms (linear maps, linear maps, linear maps) from E into F. By using the notion of inductive limits of topological groups defined in (39) (*-inductive limits of linear topological spaces, **-inductive limits of semiconvex spaces, inductive limits of locally convex spaces) we see that $(\mathcal{A},T)$ is an inductive class. Let $\mathcal{A}_1$ be the class of all members of $\mathcal{A}$ each of which is of the second category in itself. If $\mathcal{A}_2$ is the class of all Lindelöf topological groups which are either complete metrizable or locally compact (the class of all B-complete linear topological spaces, the class of all B-complete semiconvex spaces, the class of all B-complete locally convex spaces), then $(\mathcal{A}_1,\mathcal{A}_2)$ is a λ_{12} pair.

<u>Theorem 6.3.1</u>.

Let $(\mathcal{A},T)$ be an inductive class and $(\mathcal{A}_1,\mathcal{A}_2)$ a λ_{12} pair. If E is the $(\mathcal{A},T)$-inductive limit of $(E_\gamma\,;\,u_\gamma : \gamma\varepsilon\Phi)$ and F the generalized strict $(\mathcal{A},T)$-inductive limit of $(F_n; v_n : n = 1,2...)$, where each

$E_\gamma \in \mathcal{A}_1$ and each $F_n \in \mathcal{A}_2$ then,

(a) any closed map t in T(E,F) is continuous, and

(b) any closed onto map g in T(F,E) is open.

Proof:

(a) For each γ in Φ, $tou_\gamma \in T(E_\gamma,F)$ and is closed. Moreover, t is continuous if and only if each tou_γ is continuous. It is therefore sufficient to prove (a) assuming that $E \in \mathcal{A}_1$ i.e. that E is of the second category in itself. We now make this assumption. By a similar argument, one can show that it is sufficient to consider v_n as a (1 - 1) map. We may then identify F_n with the subspace $v_n(F_n)$ of F and thus consider F as the generalized strict $(\mathcal{A},T)$-inductive limit of $(F_n; i_n : n = 1,2, \ldots)$, where i_n is the identity map of F_n into F. With this identification, it is clear that for each n the topology of F_n is finer than that induced by F.

Now,

$$E = t^{-1}(\bigcup_{n \geq 1} (F_n)) = \bigcup_{n \geq 1} t^{-1}(F_n) \quad .$$

Since E is of the second category in itself, there exists some positive integer N such that $H = t^{-1}(F_N)$ is of the second category in E. As $(\mathcal{A},T)$ is admissible, H is in $\mathcal{A}_1$ and the restriction t_o of t to H is in $T(H,F_N)$. Moreover, the graph of t_o is closed in H x F_N, and since $(\mathcal{A}_1,\mathcal{A}_2)$ is a λ_{12} pair t_o is nearly continuous. Therefore, by Theorem 6.2.2, t_o is a continuous map from H into F_N. If t_1 is a continuous group homomorphism from the closure H_1 of H in E into F_N extending t_o, t_1 is also continuous from H_1 into F. By Lemma 2.6.2, the group homomorphism f from H_1 into F defined as follows, is closed .

$$f(x) = t_1(x)\ (t(x))^{-1} = t_1(x)\ t(x^{-1}).$$

Therefore, by Lemma 2.6.4, $f^{-1}(e_1)$ is closed in H_1 (e_1 is the identity of F) and since $f^{-1}(e_1) \supseteq H$, it follows that $f^{-1}(e_1) = H_1$ and that t, t_1 coincide on H_1. Thus t is a continuous map from H_1 into F.

Clearly H_1 is of the second category in E, and being closed it satisfies the condition of Baire. Therefore by a remark in section 2.5 t is a continuous map from E into F.

(b) If h is the induced map of g then, the graph of h^{-1} is closed in $E \times F/g^{-1}(e)$, (e is the identity of E). Also $h^{-1} \varepsilon T(E,F/g^{-1}(e))$ since $(\mathcal{A},T)$ is admissible; and $F/g^{-1}(e)$ is the generalized strict $(\mathcal{A},T)$-inductive limit of $(F_n; k_1 o\ v_n : n = 1,2 \ldots)$ (k_1 is the canonical map of F onto $F/g^{-1}(e)$) since $(\mathcal{A},T)$ is an inductive class. Therefore, by (a) h^{-1} is continuous and thus g is open.

By a method similar to that used above, but this time applying Theorem 6.2.1 instead of Theorem 6.2.2, one can prove the following result.

Theorem 6.3.2.

Let $(\mathcal{A},T)$ be an inductive class and $(\mathcal{A}_1,\mathcal{A}_2)$ a μ_{12} pair. If E is the $(\mathcal{A},T)$-inductive limit of $(E_\gamma; u_\gamma : \gamma \varepsilon \Phi)$ and F the generalized strict $(\mathcal{A},T)$-inductive limit of $(F_n; v_n : n = 1,2, \ldots)$, where each $E_\gamma \varepsilon \mathcal{A}_1$, each $F_n \varepsilon \mathcal{A}_2$ and v_n is a (1 - 1) map then,

(a) any closed map in T(E,F) is continuous, and

(b) any closed (1 - 1) onto map in T(F,E) is open.

Let $\mathcal{A}$ be the class of all locally convex spaces and $\mathcal{A}_1$ the class of all locally convex spaces of the second category. For E, F in $\mathcal{A}$, let T(E,F) be the set of all linear maps from E into F. If $\mathcal{A}_2$ is the class of all B-complete locally convex spaces then, Theorem 6.3.1 provides slightly strengthened forms of Theorems 2 and 3(ii) of (35). As is well known, this result implies that every closed linear map from a sequentially complete bornological space into an L.F. space is continuous.

Now consider the case when $\mathcal{A}$ is the class of all semiconvex spaces and $\mathcal{A}_1$ the class of all semiconvex spaces of the second category. For E, F in $\mathcal{A}$, let T(E,F) be the set of all linear maps from E into F. If $\mathcal{A}_2$ is the class of all B-complete semiconvex spaces then, Theorem 6.3.1 shows that every closed linear map from any **-inductive limit G of semiconvex spaces of the second category into a generalized strict **-inductive limit H of a sequence of B-complete semiconvex spaces is continuous and that every closed linear map of H onto G is open. By Theorem 5.2.6, it follows that every closed linear map from a sequentially complete separated almost convex hyperbornological space G into a generalized strict **-inductive limit H of a sequence of complete metrizable semiconvex spaces is continuous and that every closed linear map of H onto G is open. These remarks also hold on respectively replacing "semiconvex space", "hyperbornological" and "**-inductive limit" by "l.t.s.", "ultrabornological" and "*-inductive limit ".

We note however that a closed linear map from a separated inductive limit of Banach spaces into a B-complete semiconvex space need not be continuous. For, if (E,u) is the sequence space $l^{\frac{1}{2}}$, the identity map i from (E,u^{oo}) to (E,u) is closed and thus the graph of i is closed in $(E,\tau(E,E^*)) \times (E,u)$. But $\tau(E,E^*)$ is not finer than u.

6.4 General properties

As from now, we shall only be interested in three situations where $(\mathcal{A},T)$ is admissible. These shall be referred to as the admissible case (1), admissible case (2) and admissible case (3).

For the admissible case (1) $\mathcal{A}$ is the class of all linear topological spaces and for every E,F in $\mathcal{A}$, T(E,F) is the set of all linear maps from E into F. If we need to be specific $(\mathcal{A},T)$, $D(\mathcal{A}_1;\mathcal{A},T)$ and $D_r(\mathcal{A}_1;\mathcal{A},T)$ shall in this case be respectively denoted by $(\mathcal{A},T)_1$, $D(\mathcal{A}_1;(\mathcal{A},T)_1)$ and $D_r(\mathcal{A}_1;(\mathcal{A},T)_1)$.
The definitions of admissible case (2) and admissible case (3) are similar, only that we replace "linear topological spaces" by "semiconvex spaces" and "locally convex spaces" respectively.
The notations are also similar, only that we replace the suffix "1" by "2" and "3" respectively.

In any of these situations, $(\mathcal{A},T)$ is an inductive class. When we say that F is the $(\mathcal{A},T)$-inductive limit of $(E_\gamma ; u_\gamma : \gamma\varepsilon\Psi)$, it shall be assumed that the union of $(u_\gamma(E_\gamma))$ spans F. If $(\mathcal{A},T)$ is admissible, and $\mathcal{A}_1$ is a subclass of $\mathcal{A}$, $\mathcal{A}_1^*$ shall denote the class of all spaces each of which is the $(\mathcal{A},T)$-inductive limit of some $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$, where each $E_\gamma \varepsilon \mathcal{A}_1$.

In considering $D(\mathcal{A}_1;(\mathcal{A},T)_1)$ and $D_r(\mathcal{A}_1;(\mathcal{A},T)_1)$, $\mathcal{A}_1$ shall be a class of ultrabarrelled spaces. In the other two situations, $\mathcal{A}_1$ shall respectively be a class of hyperbarrelled and barrelled spaces. With such a choice, if $E \in \mathcal{A}$ and $F \in \mathcal{A}_1$, then, any onto map in T(E,F) is nearly open. As a result of this our definitions of $D(\mathcal{A}_1;\mathcal{A},T)$- and $D_r(\mathcal{A}_1;\mathcal{A},T)$-spaces now take the following form.

Let $(\mathcal{A},T)$ be admissible and suppose that $\mathcal{A}_1$ is a subclass of $\mathcal{A}$ (chosen as stated above). A space E in $\mathcal{A}$ is a <u>$D(\mathcal{A}_1;\mathcal{A},T)$-</u> <u>$(D_r(\mathcal{A}_1;\mathcal{A},T)$-)space</u> if for each E_1 in $\mathcal{A}_1$, every closed linear (closed linear (1 - 1)) map of E onto E_1 is open.

If $\mathcal{A}_1$ is a class of locally convex spaces then, according to Husain (9), a locally convex space E is a $B(\mathcal{A}_1)$-$(B_r(\mathcal{A}_1)$-)space if for each F in $\mathcal{A}_1$, every continuous (continuous (1 - 1)) linear nearly open map of E onto F is open. If $\mathcal{A}_1$ is a class of barrelled spaces then, every $D(\mathcal{A}_1;(\mathcal{A},T)_3)$-$(D_r(\mathcal{A}_1;(\mathcal{A},T)_3)$-)space is clearly a $B(\mathcal{A}_1)$-$(B_r(\mathcal{A}_1)$-)space. However, in this case I do not know of any $B(\mathcal{A}_1)$-$(B_r(\mathcal{A}_1)$-)space which is not a $D(\mathcal{A}_1;(\mathcal{A},T)_3)$-$(D_r(\mathcal{A}_1;(\mathcal{A},T)_3)$-) space.

For the remaining part of this chapter, the letters $\mathcal{C}$, $\mathcal{C}_u$, $\mathcal{C}_h$, $\mathcal{B}_n$, $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{B}_{11}$, $\mathfrak{F}$, $\mathfrak{F}_1$, $\mathfrak{F}_{11}$, $\mathcal{L}$, $\mathcal{N}$, $\mathcal{N}_1$, $\mathcal{N}_{11}$, $\mathfrak{F}_\delta$ shall respectively stand for the classes of all barrelled, ultrabarrelled, hyperbarrelled, Banach, second category locally convex, second category linear topological, second category semiconvex, Frechet, complete metric linear, semiconvex complete metric linear, complete separated locally bounded, sequentially complete bornological, sequentially complete almost convex ultrabornological, sequentially complete almost convex hyperbornological and finite dimensional linear topological spaces.

Proposition 6.4.1.

(a) Let $(\mathcal{A},T)$ be admissible and let $\mathcal{A}_1$ be a subclass of $\mathcal{A}$. Suppose that E, E_1 are in $\mathcal{A}$. If there is a continuous (continuous (1 - 1)) linear map of E onto E_1 then, E_1 is a $D(\mathcal{A}_1;\mathcal{A},T)$-($D_r(\mathcal{A}_1;\mathcal{A},T)$-) space if E is.

(b) Let $\mathcal{A}_1$ be a class of barrelled spaces. If u, v are convex topologies on a linear space E yielding the same dual then, (E,u) is a $D(\mathcal{A}_1; (\mathcal{A},T)_3)$-($D_r(\mathcal{A}_1; (\mathcal{A},T)_3)$-) space if and only if (E,v) is.

Proof:

(a) Suppose that E is a $D(\mathcal{A}_1;\mathcal{A},T)$-space and that h is a continuous linear map of E onto E_1. If f is a closed linear map of E_1 onto some H in $\mathcal{A}_1$ then, the map foh of E onto H is closed and therefore open. Since for any subset Q of E_1, $f(Q) = foh(h^{-1}(Q))$, we have that f is open and thus E_1 is a $D(\mathcal{A}_1;\mathcal{A},T)$-space. Similarly E_1 is a $D_r(\mathcal{A}_1;\mathcal{A},T)$-space if E is, provided that there is a continuous (1 - 1) linear map of E onto E_1.

(b) Let $(E,u)' = (E,v)' = E'$. By (a) it is sufficient to prove that $(E,\tau(E,E'))$ is a $D(\mathcal{A}_1;(\mathcal{A},T)_3)$-($D_r(\mathcal{A}_1;(\mathcal{A},T)_3)$-) space if $(E, \sigma(E,E')$ is.

Suppose that $(E,\sigma(E,E'))$ is a $D(\mathcal{A}_1;(\mathcal{A},T)_3)$-space, and let h be a closed linear map from $(E,\tau(E,E'))$ onto some H in $\mathcal{A}_1$. The graph of h is also closed in $(E,\sigma(E,E')) \times H$, since the graph is a linear subspace of $E \times H$ and the locally convex spaces $(E,\sigma(E,E')) \times H$ and $(E,\tau(E,E')) \times H$ have the same dual. Therefore h is an open map of $(E,\sigma(E,E'))$ onto H. Now, $(E,\sigma(E,E'))/h^{-1}(o)$, $(E,\tau(E,E'))/h^{-1}(o)$ have the weak and Mackey topologies respectively with the same dual. If f is the induced map of h then, since f^{-1} is a continuous linear map of H into $(E,\sigma(E,E'))/h^{-1}(o)$ and H has a Mackey topology, f^{-1} is a continuous map from H into $(E,\tau(E,E'))/h^{-1}(o)$. Therefore h is an open map of $(E,\tau(E,E'))$ onto H and thus $(E,\tau(E,E'))$ is a $D(\mathcal{A}_1;(\mathcal{A},T)_3)$-space. Similarly (E,u) is a $D_r(\mathcal{A}_1;(\mathcal{A},T)_3)$-space if and only if (E,v) is.

Let (E,u) be a metrizable l.t.s. with dual E' separating the points of E. By an application of ((35), page 9), we see that the space $(E',\tau\{E',(E,u^{oo})^\wedge\})$ is a B-complete locally convex space. Since $\tau(E',E)$ is coarser than $\tau\{E',(E,u^{oo})^\wedge\}$, it follows from Proposition 6.4.1 that for every convex (semiconvex, linear) topology w on E' which is coarser than $\tau(E',E)$, (E',w) is a $D(\mathcal{C};(\mathcal{A},T)_3)$-$(D(\mathcal{C}_h;(\mathcal{A},T)_2)$-$D(\mathcal{C}_u;(\mathcal{A},T)_1)$-) space.

<u>Proposition 6.4.2</u>.

Let $(\mathcal{A},T)$ be admissible. If $\mathcal{A}_1$ is a subclass of $\mathcal{A}$ such that every quotient by a closed subspace of each member of $\mathcal{A}_1$ is also in $\mathcal{A}_1$ then, $D(\mathcal{A}_1;\mathcal{A},T) = D(\mathcal{A}_1^*;\mathcal{A},T)$.

<u>Proof</u>: It is sufficient to prove that $D(\mathcal{A}_1;\mathcal{A},T) \subseteq D(\mathcal{A}_1^*;\mathcal{A},T)$. Let f be a closed linear map from a $D(\mathcal{A}_1;\mathcal{A},T)$-space E onto some E_1 in $\mathcal{A}_1^*$. There is no loss of generality in assuming that f is (1 - 1) since every quotient of E by a closed linear subspace is also a $D(\mathcal{A}_1;\mathcal{A},T)$-space. We therefore make this assumption. Since by the hypothesis every quotient by a closed linear subspace of a member of $\mathcal{A}_1$ is also in $\mathcal{A}_1$, we may assume that E_1 is the $(\mathcal{A},T)$-inductive limit of some $(F_\gamma; i_\gamma : \gamma\epsilon\Gamma)$, where each $F_\gamma \epsilon \mathcal{A}_1$ and each i_γ is a (1 - 1) linear map of F_γ into E_1.

The graph of the linear map $i_\gamma^{-1} of$ of E onto F_γ is closed for each γ in Γ, since it is the inverse image of the graph of f by the continuous map $(x,y) \to (x,i_\gamma(y))$ of $E \times F_\gamma$ into $E \times E_1$. Therefore $i_\gamma^{-1} of$ is open and thus for every neighbourhood V in E, $i_\gamma^{-1} of(V)$ $(=(f^{-1} o i_\gamma)^{-1}(V))$ is a neighbourhood in F_γ. This implies that $f^{-1} o i_\gamma$ is a continuous map of F_γ into E for each γ in Γ. Therefore f^{-1} is continuous and thus f is open and E is a $D(\mathcal{A}_1^*;\mathcal{A},T)$-space.

<u>Corollary</u>:

(a) $D(\mathcal{B}_1;(\mathcal{A},T)_1) = D(\mathcal{B}_1^*;(\mathcal{A},T)_1)$.

(b) $D(\mathfrak{I}_1;(\mathcal{A},T)_1) = D(\mathfrak{I}_1^*;(\mathcal{A},T)_1)$.

(c) $D(\mathcal{B};(\mathcal{A},T)_3) = D(\mathcal{B}^*;(\mathcal{A},T)_3)$.

(d) $D(\mathcal{B}_n(\mathcal{A},T)_3) = D(\mathfrak{I};(\mathcal{A},T)_3) = D(\mathcal{N};(\mathcal{A},T)_3)$

$= D(\mathcal{B}_n^*;(\mathcal{A},T)_3)$.

(e) $D(\mathcal{L};(\mathcal{A},T)_1) = D(\mathcal{N}_1;(\mathcal{A},T)_1) = D(\mathcal{L}^*;(\mathcal{A},T)_1)$.

The proof of the following result is easy and is therefore omitted.

Proposition 6.4.3.

Let $(\mathcal{A}, T)$ be admissible and let $\mathcal{A}_1$ be a subclass of $\mathcal{A}$. If every separated continuous (continuous (1 - 1)) linear nearly open image of each member of $\mathcal{A}_1$ is also in $\mathcal{A}_1$ then, every $D(\mathcal{A}_1; \mathcal{A}, T)$-($D_r(\mathcal{A}_1; \mathcal{A}, T)$-) space which is in $\mathcal{A}_1$ is B-complete (B_r-complete).

In ((18), page 195, problem D(a)), $G = E_1 \times E_2$, where E_1 is a B-complete barrelled space, being a countable direct sum of reflexive Banach spaces and E_2 is a Fréchet space. Since G is barrelled (hyper-barrelled, ultrabarrelled) but not B-complete, it follows by Proposition 6.4.3 that a product of two $D(\mathcal{C}; (\mathcal{A}, T)_3)$ -($D(\mathcal{C}_h; (\mathcal{A}, T)_2)$-, $D(\mathcal{C}_u; (\mathcal{A}, T)_1)$-) spaces does not necessarily belong to the class.

We observe that in the example referred to in the last paragraph, E_1, E_2 are strictly hypercomplete. Thus a product of two hypercomplete spaces need not be hypercomplete. This answers (negatively) a question of Kelley's in ((17), page 236).

By Proposition 6.4.3, an incomplete quotient of an L.F. space is not in $D_r(\mathcal{C}; (\mathcal{A}, T)_3)$, or $D_r(\mathcal{C}_u; (\mathcal{A}, T)_1)$. But by Theorem 6.3.1, such a space is necessarily in $D(\mathcal{B}; (\mathcal{A}, T)_3)$, and $D(\mathcal{B}_1; (\mathcal{A}, T)_1)$. It is easy to see that every locally convex space is in $D(\mathcal{F}_\delta; (\mathcal{A}, T)_1)$ and $D(\mathcal{F}_\delta; (\mathcal{A}, T)_3)$. But an infinite dimensional Banach space under its finest convex topology is not in $D_r(\mathcal{F}_1; (\mathcal{A}, T)_1)$ or $D_r(\mathcal{B}_n; (\mathcal{A}, T)_3)$. In Proposition 6.4.4, (i), (ii), (v) and (vi) follow from these observations and the corollary of Propositon 6.4.2. Parts (iii) and (iv) of the same result are easily established.

Proposition 6.4.4.

(i) $D_r(\mathcal{C}_u;(\mathcal{A},T)_1) \subsetneq D_r(\mathcal{B}_1;(\mathcal{A},T)_1) \subseteq D_r(\mathcal{F}_1;(\mathcal{A},T)_1) \subsetneq D_r(\mathcal{F}_\delta;(\mathcal{A},T)_1)$.

(ii) $D(\mathcal{C}_u;(\mathcal{A},T)_1) \subsetneq D(\mathcal{B}_1;(\mathcal{A},T)_1) = D(\mathcal{B}_1{}^*;(\mathcal{A},T)_1) \subseteq D(\mathcal{F}_1;(\mathcal{A},T)_1) = D(\mathcal{F}_1{}^*;(\mathcal{A},T)_1) \subsetneq D(\mathcal{F}_\delta;(\mathcal{A},T)_1)$.

(iii) $D(\mathcal{B};(\mathcal{A},T)_3) \subsetneq D(\mathcal{B};(\mathcal{A},T)_1)$.

(iv) $D(\mathcal{B}_1;(\mathcal{A},T)_1) \subseteq D(\mathcal{B};(\mathcal{A},T)_1)$.

(v) $D_r(\mathcal{C};(\mathcal{A},T)_3) \subsetneq D_r(\mathcal{B}^*;(\mathcal{A},T)_3) \subseteq D_r(\mathcal{B}_n{}^*;(\mathcal{A},T)_3) \subseteq D_r(\mathcal{N};(\mathcal{A},T)_3) \subseteq D_r(\mathcal{F},(\mathcal{A},T)_3) \subseteq D_r(\mathcal{B}_n;(\mathcal{A},T)_3) \subsetneq D_r(\mathcal{F}_\delta;(\mathcal{A},T)_3)$.

(vi) $D(\mathcal{C};(\mathcal{A},T)_3) \subsetneq D(\mathcal{B}^*;(\mathcal{A},T)_3) = D(\mathcal{B};(\mathcal{A},T)_3) \subseteq D(\mathcal{B}_n{}^*;(\mathcal{A},T)_3) = D(\mathcal{B}_n;(\mathcal{A},T)_3) = D(\mathcal{N};(\mathcal{A},T)_3) = D(\mathcal{F};(\mathcal{A},T)_3) \subsetneq D(\mathcal{F}_\delta;(\mathcal{A},T)_3)$.

6.5 The case when $\mathcal{A}_1$ is an ω cum inductive class

If $(\mathcal{A},T)$ is admissible then, on any linear space E there is a finest topology for which the space is in $\mathcal{A}$. This topology shall be denoted by ω. The topology ω is s, sc or $\tau(E,E^*)$ respectively according as we are considering the admissible case (1), (2) or (3).

Let $(\mathcal{A},T)$ be admissible. We call a subclass $\mathcal{A}_1$ of $\mathcal{A}$ <u>an ω cum inductive class in $\mathcal{A}$</u> if $(E,\omega) \in \mathcal{A}_1$ for every linear space E and for every $(E_n : n=1,2, \ldots N)$ in $\mathcal{A}_1$, any $(\mathcal{A},T)$-inductive limit of $(E_n : n=1,2, \ldots N)$ is in $\mathcal{A}_1$.

For the admissible case (1), $\mathcal{C}_u$, $\mathcal{B}_n^*$, $\mathcal{F}_1^*$, $\mathcal{F}^*$, $\mathcal{L}^*$ $\mathcal{B}^*$, $\mathcal{B}_1^*$ are ωcum inductive classes. For the admissible case (2), similar examples are $\mathcal{C}_h$, $\mathcal{B}_n^*$, $\mathcal{F}_{11}^*$, $\mathcal{F}^*$, $\mathcal{L}^*$, $\mathcal{B}^*$, $\mathcal{B}_{11}^*$ and for case (3), $\mathcal{C}$, $\mathcal{B}_n^*$, $\mathcal{B}^*$, $\mathcal{F}^*$.

Now, any inductive limit of a sequence of Banach spaces is a quotient of their direct sum. Since an inductive limit of a sequence of Banach spaces need not be sequentially complete ((23), page 437), we see that $\mathcal{N}$, $\mathcal{N}_1$, $\mathcal{N}_{11}$ are not ωcum inductive classes for the admissable cases (3), (1) and (2) respectively.

<u>Theorem 6.5.1.</u>

Let $(\mathcal{A}, T)$ be admissible and let $\mathcal{A}_1$ be an ω cum inductive class in $\mathcal{A}$. Then, (E,τ) in $\mathcal{A}$ is a $D_r(\mathcal{A}_1;\mathcal{A},T)$-space if and only if every closed linear map from any F in $\mathcal{A}_1$ into (E,τ) is continuous.

<u>Proof</u>: Suppose that (E,τ) is a $D_r(\mathcal{A}_1;\mathcal{A},T)$-space. Let f be a closed linear map from some F in $\mathcal{A}_1$ into (E,τ). Since $F/f^{-1}(0)$ is also in $\mathcal{A}_1$, we may also suppose that f is a (1 - 1) map. As f is closed and linear, it follows by Lemma 2.6.5 that f is continuous from F into (E,v_1) where v_1 is coarser than τ and (E,v_1) is in $\mathcal{A}$. Since f is (1 - 1), we may identify F with the linear subspace $E_1 = f(F)$ of E. Let (E_1,p) be this space with the topology of F. The space (E_1,p) is in $\mathcal{A}_1$ and p is clearly finer than the v_1-induced topology on E_1. Let E_2 be an algebraic supplement of E_1 in E. As E is algebraically isomorphic to $E_1 \times E_2$, we may identify $E_1 \times E_2$ with E. With this identification, let (E,q) be the $(\mathcal{A},T)$-direct sum of (E_1,p) and (E_2,ω).

Clearly q is finer than v_1, (E,q) is in $\mathcal{A}_1$ and f is continuous from F into(E,q). Now the identity map i say, from (E,τ) onto (E,v_1) is closed, being continuous and thereforethe graph of i is closed in $(E,\tau) \times (E,q)$. Since (E,τ) is a $D_r(\mathcal{A}_1;\mathcal{A},T)$-space and $(E,q) \varepsilon \mathcal{A}_1$ it follows that τ is coarser than q. Therefore f is continuous from F into (E,τ).

The converse is easy.

Cf. (33), Theorem 4.9.

By using Theorem 6.5.1, one can prove the following result.

<u>Theorem 6.5.2</u>.

Let $(\mathcal{A},T)$ be admissible and let $\mathcal{A}_1$ be an ω cum inductive class in $\mathcal{A}$. Then, E in $\mathcal{A}$ is a $D(\mathcal{A}_1;\mathcal{A},T)$-space if and only if every closed linear map from any F in $\mathcal{A}_1$ into each quotient of E by a closed linear subspace is continuous.

It is not difficult to deduce from Theorem 6.5.1 that every closed linear subspace of a $D_r(\mathcal{A}_1;\mathcal{A},T)$-space is also a $D_r(\mathcal{A}_1;\mathcal{A},T)$-space whenever $\mathcal{A}_1$ is an ω cum inductive class in $\mathcal{A}$. By using Theorem 6.5.2 and Lemma 2.6.4 one can show that in a similar situation, every closed linear subspace of a $D(\mathcal{A}_1;\mathcal{A},T)$-space is also a $D(\mathcal{A}_1;\mathcal{A},T)$-space. From Theorem 6.5.2 and Proposition 6.4.2, the following is immediate.

<u>Corollary.</u> Let $(\mathcal{A},T)$ be admissible and let $\mathcal{A}_1$ be an ω cum inductive class in $\mathcal{A}$. Suppose that $\mathcal{A}_2$ is a subclass of $\mathcal{A}_1$ such that every quotient by a closed linear subspace of each member of $\mathcal{A}_2$ is also in $\mathcal{A}_2$. If every member of $\mathcal{A}_1$ is the $(\mathcal{A},T)$-inductive limit of some $(E_\gamma;\ u_\gamma\ :\gamma\varepsilon\Phi)$

where each $E_\gamma \in \mathcal{A}_2$ then, E in $\mathcal{A}$ is a D($\mathcal{A}_2$; $\mathcal{A}$,T)-space if and only if every closed linear map from any member of $\mathcal{A}_2$ into each quotient of E by a closed linear subspace is continuous.

For the admissible case (1), the hypothesis of the above Corollary is satisfied if $\mathcal{A}_1$ is $\mathcal{B}_n^*$, $\mathcal{F}^*$, $\mathcal{F}_1^*$, $\mathcal{F}_{11}^*$, $\mathcal{B}^*$, $\mathcal{B}_1^*$, $\mathcal{B}_{11}^*$ or $\mathcal{L}^*$, and $\mathcal{A}_2$ is respectively chosen to be $\mathcal{B}_n$, $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_{11}$, $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{B}_{11}$ or $\mathcal{L}$.

Similarly for the admissible case (2) when $\mathcal{A}_1$ is $\mathcal{B}_n^*$, $\mathcal{F}^*$, $\mathcal{F}_{11}^*$, $\mathcal{B}^*$, $\mathcal{B}_{11}^*$ or $\mathcal{L}^*$, and $\mathcal{A}_2$ is $\mathcal{B}_n$, $\mathcal{F}$, $\mathcal{F}_{11}$, $\mathcal{B}$, $\mathcal{B}_{11}$ or $\mathcal{L}$.

And for the admissible case (3) when $\mathcal{A}_1$ is $\mathcal{B}_n^*$, $\mathcal{F}^*$, or $\mathcal{B}^*$, and $\mathcal{A}_2$ is $\mathcal{B}_n$, $\mathcal{F}$, or $\mathcal{B}$.

In the next two theorems we shall take $\mathcal{N}'$ to be $\mathcal{N}_1$, $\mathcal{N}_{11}$ or $\mathcal{N}$ according as we are considering the admissible case (1), (2) or (3). For case (1), every member of $\mathcal{N}_1$ is the ($\mathcal{A}$,T)-inductive limit of some $(E_\gamma;\ u_\gamma:\ \gamma\in\Phi)$, where each $E_\gamma \in \mathcal{L}$. It is then easy to see that for F in $\mathcal{A}$, every closed linear map from each member of $\mathcal{N}_1$ into F is continuous if and only if every closed linear map from each member of $\mathcal{L}$ into F is continuous. There is a similar remark for case (2), when " $\mathcal{N}_1$" is replaced by " $\mathcal{N}_{11}$" and for case (3), when " $\mathcal{N}_1$" and "$\mathcal{L}$" are respectively replaced by "$\mathcal{N}$" and "$\mathcal{B}_n$". With these observations, one can prove the following result by a method similar to that used in Theorem 6.5.1.

Theorem 6.5.3.

Let $(\mathcal{A}, T)$ be admissible. Then, E in $\mathcal{A}$ is a $D_r(\mathcal{N}'; \mathcal{A}, T)$-space if and only if every closed linear map from any member of $\mathcal{N}'$ into E is continuous.

Corollary 1. (i) If $(\mathcal{A}, T)$ is admissible then,

$$D_r(\mathcal{N}'; \mathcal{A}, T) = D_r(\mathcal{N}'^*; \mathcal{A}, T) .$$

(ii) $D_r(\mathcal{N}_1; (\mathcal{A}, T)_1) = D_r(\mathcal{L}^*; (\mathcal{A}, T)_1) = D_r(\mathcal{N}_1^*; (\mathcal{A}, T)_1)$.

(iii) $D_r(\mathcal{N}_{11}; (\mathcal{A}, T)_2) = D_r(\mathcal{L}^*; (\mathcal{A}, T)_2) = D_r(\mathcal{N}_{11}^*; (\mathcal{A}, T)_2)$.

(iv) $D_r(\mathcal{N}; (\mathcal{A}, T)_3) = D_r(\mathcal{B}_n^*; (\mathcal{A}, T)_3) = D_r(\mathcal{N}^*; (\mathcal{A}, T)_3)$.

(see Proposition 6.4.4).

Corollary 2.

(a) $D_r(\mathcal{L}; (\mathcal{A}, T)_1) = D_r(\mathcal{L}^*; (\mathcal{A}, T)_1) = D_r(\mathcal{N}_1; (\mathcal{A}, T)_1)$ if and only if for each $D_r(\mathcal{L}; (\mathcal{A}, T)_1)$-space F, every closed linear map from any E in $\mathcal{L}$ into F is continuous.

(b) $D_r(\mathcal{L}; (\mathcal{A}, T)_2) = D_r(\mathcal{L}^*; (\mathcal{A}, T)_2) = D_r(\mathcal{N}_{11}; (\mathcal{A}, T)_2)$ if and only if for each $D_r(\mathcal{L}; (\mathcal{A}, T)_2)$-space F, every closed linear map from any E in $\mathcal{L}$ into F is continuous.

(c) $D_r(\mathcal{B}_n^*; (\mathcal{A}, T)_3) = D_r(\mathcal{N}; (\mathcal{A}, T)_3) = D_r(\mathcal{J}; (\mathcal{A}, T)_3) = D_r(\mathcal{B}_n; (\mathcal{A}, T)_3)$ if and only if for each $D_r(\mathcal{B}_n; (\mathcal{A}, T)_3)$-space F, every closed linear map from any E in $\mathcal{B}_n$ into F is continuous.

Using Theorem 6.5.3, one can prove the following result.

Theorem 6.5.4.

Let $(\mathcal{A}, T)$ be admissible. Then E in $\mathcal{A}$ is a $D(\mathcal{N}'; \mathcal{A}, T)$-space if and only if every closed linear map from any member of $\mathcal{N}'$ into each quotient of E by a closed linear subspace is continuous.

6.6 The case when $\mathcal{A}_1$ is a second category class

If $(\mathcal{A},T)$ is admissible, we shall throughout this section assume that $\mathcal{A}_1$ is the class of all second category linear topological spaces in $\mathcal{A}$. We say that E in $\mathcal{A}$ is a $D_1(\mathcal{A}_1;\mathcal{A},T)$-space if there exists a continuous linear map from some F onto E, where F is either an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space or is the generalized strict *-inductive limit of a sequence of extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-spaces.

If (E,u) is a B-complete l.t.s. or an L.F. space then, for any linear (semiconvex, convex) topology v on E coarser than u, (E,v) is a $D_1(\mathcal{A}_1;(\mathcal{A},T)_1)$- ($D_1(\mathcal{A}_1;(\mathcal{A},T)_2)$- , $D_1(\mathcal{A}_1;(\mathcal{A},T)_3)$-) space. Also, if (E,u) is the *-direct sum of a sequence (E_i) of linear topological spaces, where for each i, $E_i = l^p$ or H^p for some p in the open interval (0, 1) then, for any linear (semiconvex, convex) topology v on E coarser than u^{oo}, (E,v) is a $D_1(\mathcal{A}_1;(\mathcal{A},T)_1)$- ($D_1(\mathcal{A}_1;(\mathcal{A},T)_2)$- , $D_1(\mathcal{A}_1;(\mathcal{A},T)_3)$-) space.

There may not exist a continuous linear map from a B-complete locally convex space onto a $D_1(\mathcal{A}_1;(\mathcal{A},T)_3)$-space. For, let (E,u) be the sequence space $l^{\frac{1}{2}}$. Then, the incomplete barrelled normed linear space (E,u^{oo}) is a $D_1(\mathcal{A}_1;(\mathcal{A},T)_3)$-space. If there were to exist a continuous linear map f say, from a B-complete locally convex space onto (E,u^{oo}), then f would be open and this would imply that (E,u^{oo}) is complete.

From the corollary of Theorem 6.5.2, we derive the following result (which we denote by (R)).

($\underline{R}$): Let $(\mathcal{A},T)$ be admissible. Then, E in $\mathcal{A}$ is a $D(\mathcal{A}_1;\mathcal{A},T)$-space if and only if every closed linear map from any F in $\mathcal{A}_1$ into each quotient of E by a closed linear subspace is continuous.

Let F be an l.t.s. and (F_n) a sequence of $D_1(\mathcal{A}_1;\mathcal{A},T)$-spaces. Suppose that for each n, u_n is a continuous linear map from F_n into F and that F is the union of $(u_n(F_n))$. For each n, there exists a continuous linear map g_n say, from some G_n onto F_n, where either (a) G_n is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space or (b) G_n is the generalized strict *-inductive limit of some $(G_{n_i};\ w_{n_i} : i = 1,2,\ldots)$, where each G_{n_i} is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space.

For each n where G_n is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space, the induced map v_n of the continuous linear map $\zeta_n = u_n o g_n$ from G_n into F is continuous, and $G_n/\zeta_n^{-1}(o)$ is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space. Let J_1 be the union over n of $v_n(G_n/\zeta_n^{-1}(o))$.

For each n where G_n is the generalized strict *-inductive limit of some $(G_{n_i};\ w_{n_i} : i = 1, 2, \ldots)$ (each G_{n_i} is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space), the induced map v'_{n_i} of the map $\zeta'_{n_i} = u_n o g_n o w_{n_i}$ of G_{n_i} into F is continuous. Also, $G_{n_i}/\zeta'^{-1}_{n_i}(o)$ is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space. Let J_2 be the union over n and i of $v'_{n_i}(G_{n_i}/\zeta'^{-1}_{n_i}(o))$.

Clearly $J_1 \cup J_2 = F$. We may thus suppose that each u_n is a continuous linear (1 - 1) map and that each F_n is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space. With this observation, on using the method of proof of Theorem 6.3.1, but this time applying ($\underline{R}$) instead of Theorem 6.2.2, one can prove the following result.

Theorem 6.6.1.

Let $(\mathcal{A},T)$ be admissible and let E be the $(\mathcal{A},T)$-inductive limit of some $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$, where each E_γ is in $\mathcal{A}_1$. Suppose that $F \varepsilon \mathcal{A}$ and that for each positive integer n, u_n is a continuous linear map from a $D_1(\mathcal{A}_1;\mathcal{A},T)$-space F_n into F. If F is the union of $(u_n(F_n))$, then any closed linear map from E into F is continuous and any closed linear map of F onto E is open.

Corollary. Let $(\mathcal{A},T)$ be admissible. If E is the $(\mathcal{A},T)$-inductive limit of $(E_\gamma; u_\gamma : \gamma\varepsilon\Phi)$ and F, the generalized strict $(\mathcal{A},T)$-inductive limit of $(F_n; v_n : n = 1, 2, \ldots)$, where each $E_\gamma \varepsilon \mathcal{A}_1$ and each F_n is a $D_1(\mathcal{A}_1;\mathcal{A},T)$-space then, any closed linear map from E into F is continuous and any closed linear map from F onto E is open.

Cf. Theorems 2 and 3(ii) of (35).

Theorem 6.6.2.

Let $(\mathcal{A},T)$ be admissible and suppose that E, F $\varepsilon\ \mathcal{A}$. For each positive integer n, let u_n be a continuous linear map from a $D_1(\mathcal{A}_1;\mathcal{A},T)$-space E_n into E, and suppose that E is the union of $(u_n(E_n))$. If t is a closed linear map from E into F such that t(E) is in $\mathcal{A}_1$, then t(E) is closed in F.

Proof: By an argument similar to that preceding Theorem 6.6.1, one can show that we may assume that each E_n is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space and that E is the union of subspaces (E_n) such that the topology of E_n is finer than that induced from E.

Since $t(E) = \bigcup_{n \geq 1} t(E_n)$ is of the second category in itself, there exists a positive integer N such that $t(E_N)$ is of the second category in $t(E)$ and $t(E_N)$ is dense in $t(E)$. ($t(E)$, $t(E_N)$ are assumed to have the topologies induced from F). The space $t(E_N)$ is clearly in $\mathcal{A}_1$ and the graph of the map t of E_N onto $t(E_N)$ is closed in $E_N \times t(E_N)$.

As E_N is a $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space and $t(E_N)$ is in $\mathcal{A}_1$, t is an open map from E_N onto $t(E_N)$.

Since $E_N/t^{-1}(o)$ is an extracomplete $D(\mathcal{A}_1;(\mathcal{A},T)_1)$-space, we may assume that t is (1 - 1) and thus consider E_N as the same space $t(E_N)$ under a coarser topology v. Moreover, $(t(E_N),v)$ is complete and the identity map i say, from $(t(E_N),v)$ into F is closed.

Let $(y_\alpha : \alpha\varepsilon\Psi)$ be a net in $t(E_N)$ converging to y_o in F. Because i is an open map from $(t(E_N),v)$ onto $t(E_N)$, $(y_\alpha : \alpha\varepsilon\Psi)$ is v-Cauchy and must therefore converge to some point y_o' in $(t(E_N),v)$, since this space is complete. As the graph of i is closed in $(t(E_N),v) \times F$, $y_o = y_o'$ and thus $t(E_N)$ is closed in F. The result now follows from this, since $t(E_N)$ is dense in $t(E)$.

Corollary. Let $(\mathcal{A},T)$ be admissible and suppose that F is in $\mathcal{A}$. Let E be generalized strict $(\mathcal{A},T)$-inductive limit of $(E_n; u_n : n = 1, 2, \ldots)$, where each E_n is a $D_1(\mathcal{A}_1;\mathcal{A},T)$-space. If t is a closed linear map of E into F then, either $t(E)$ is of first category in F or $t(E) = F$.

Proof: If $t(E)$ is of the second category in F then, $t(E) \varepsilon \mathcal{A}_1$ and $t(E)$ is dense in F. By the theorem, $t(E)$ is closed in F and this gives the result.

Cf. (35), Theorem 3, Corollary.

REFERENCES

1 A. Alexiewicz, On sequences of operations II, Studia Math. 11 (1950) 200-236.

2 A. Alexiewicz, On the two-norm convergence, Studia Math 14 (1954) 49-56.

3 S. Banach, Théorie des opérations lineares, Warsaw (1932).

4 N. Bourbaki, Eléments de Mathématique Livre V Espaces vectoriels topologiques, Chapitres III-V, Paris (1955).

5 H.S. Collins, Completeness and compactness in linear topological spaces, Trans. Amer. Math. Soc. 79 (1955) 256-280.

6 M.M. Day, The spaces L^p, $o<p<1$, Bull. Amer. Soc. 49 (1940) 816-823.

7 J. Dieudonné and L. Schwartz, La dualité des espaces (F) et (LF), Ann. Inst. Fourier Grenoble 1 (1949) 61-101.

8 A. Grothendieck, Sur les espaces (F) et (DF), Summa Brasiliensis Math. 3 (1954) 57-123.

9 T. Husain, Locally convex spaces with the B(t)-property, Math. Ann. 146 (1962) 413-422.

10 T. Husain, B(t)-spaces and the closed graph theorem, Math. Ann. 153 (1964) 293-298.

11 T. Husain, B(t)-spaces and the closed graph theorem II, Math. Ann. 156 (1964) 334-336.

12 T. Husain, On completion and completeness of B(σ)-spaces, Math. Ann. 154 (1964) 73-76.

13 T. Husain, S-spaces and the open mapping theorem, Pacific J. Math. 12 (1962) 253-271.

14 T. Husain, Two new classes of locally convex spaces, Math. Ann. 166 (1966) 289-299.

15 T. Husain and M. Mahowald, Barrelled spaces and the open mapping theorem, Proc. Amer. Math. Soc. 13 (1962) 423-424.

16 J.L. Kelley, General topology, New York (1955).

17 J.L. Kelley, Hypercomplete linear topological spaces, Michigan Math. J. 5 (1958) 235-246.

18 J.L. Kelley and I. Namioka, Linear topological spaces, New York (1963).

19 V.L. Klee, Convex sets in linear spaces III, Duke Math. J. 20 (1953) 105-110.

20 V.L. Klee, Shrinkable neighbourhoods in Hausdorff linear spaces, Math. Ann. 141 (1960) 281-285.

21 A. Kolmogoroff, Zur Normier barkeit eines allgemeinen topologischen Linearen Räumes, Studia Math, 5 (1934) 29-33.

22 Y. Komura, On linear topological spaces, Kumamoto J. Sci. Series A 5(3) (1962) 148-157.

23 G. Köthe, Topologische Lineare Räume 1, Berlin (1960).

24 M.G. Krein and V.L. Smulian, On regularly convex sets in the space conjugate to a Banach space, Ann. Math. 41 (1940) 556-583.

25 G.W. Mackey, On convex topological linear spaces, Proc. Nat. Acad. Sci. U.S.A. 29 (1943) 315-319.

26 M. Mahowald, Barrelled spaces and the closed graph theorem, J. Lond. Math. Soc. 36 (1961) 108-110.

27 M. Mahowald and G. Gould, Quasi-barrelled locally convex spaces, Proc. Amer. Math. Soc. 2 (1960) 811-816.

28 J. Von Neumann, On complete topological spaces, Trans. Amer. Math. Soc. 37 (1935) 1-20.

29 N.T. Peck, Non-locally convex spaces, Math. Ann. 161 (1965) 102-115.

30 A. Persson, A generalization of two-norm spaces, Ark. Mat. 5 (1962) 27-36.

31 B.J. Pettis, On continuity and openness of homomorphisms in topological groups, Ann. of Math. 52 (1950) 293-308.

32 V. Pták, On complete topological linear spaces, Čehoslovak Math. Z. 3 (78) (1953) 301-364.

33 V. Pták, Completeness and the open mapping theorem, Bull. Soc. Math. Fr. 86 (1958) 41-74.

34 D.A. Raikov, Closed graph theorem and completeness in linear topological spaces, Proc. 4th All Union Math. Congress, Leningrad (1961).

35 A.P. Robertson and W. Robertson, On the closed graph theorem, Proc. Glasgow Math. Assoc. 3, (1956) 9-12.

36 A.P. Robertson and W. Robertson, Topological vector spaces, Cambridge Tract (1963).

37 W. Robertson, Completions of topological vector spaces, Proc. Lond. Math. Soc. 8 (1958) 242-257.

38 I. Singer, Completeness and aw*-topologies, Arch. Math. 15 (1964) 42-45.

39 N.Th. Varopoulos, Studies in harmonic analysis, Proc. Camb. Phil. Soc. 60 (1964) 465-516.

40 S.S. Walters, The space H^p with $o<p<1$, Proc. Amer. Math. Soc. 1 (1950) 800-805.

41 J.D. Weston, The principle of equicontinuity for topological vector spaces, Proc. Univ. Durham Philos. Soc. Series A 13 (1957) 1-5.

42 A. Wiweger, Linear spaces with mixed topology, Studia Math 20 (1951) 47-68.

43 V.L. Klee, Exotic topologies in Hausdorff linear spaces, Proc. Symp. on General topology and its relation to modern analysis and algebra, Prague (1961) 238-249.

44 S. Simons, Boundedness in linear topological spaces, Trans. Amer. Math. Soc. 113 (1964) 169-180.