



This work is protected by copyright and other intellectual property rights and duplication or sale of all or part is not permitted, except that material may be duplicated by you for research, private study, criticism/review or educational purposes. Electronic or print copies are for your own personal, non-commercial use and shall not be passed to any other individual. No quotation may be published without proper acknowledgement. For any other use, or to quote extensively from the work, permission must be obtained from the copyright holder/s.

HYPERSPACES OF UNIFORM SPACES

Thesis submitted for the Degree of Doctor of Philosophy

at the University of Keele

by

ANTHONY GAVIN HITCHCOCK, BA(Oxon).

Department of Mathematics,

University of Keele.

October, 1971.

DECLARATION

The work reported in this thesis is claimed as original except where explicitly stated otherwise. The thesis has not been submitted previously for a Higher Degree of this or any other University.

ACKNOWLEDGEMENTS

I wish to express my gratitude to the following:

Professor A. P. Robertson, my supervisor, for his inspiration and encouragement;

Dr. Wendy Robertson for some useful ideas;

Mrs. Sheila Pye and Miss Glenys Howell for the excellent typing of this thesis;

The University of Keele, and the Charelick Saloman Trust for financial assistance.

ABSTRACT

This thesis is concerned with the properties and uses of the so-called Hausdorff uniform structure on the set of subsets of a uniform space. Sometimes the results are for more specialised spaces - metric spaces, normed spaces or topological vector spaces, or for the more general proximity spaces. Historically the hyperspace has probably derived its importance mainly from the concept of hypercompleteness. Here a study is made of hyperspaces of Hausdorff completions and Hausdorff completions of hyperspaces, and, in particular, of a case where these spaces derive from two related uniform structures on a set. Certain Hausdorff completions are shown to be uniformly embedded in the 'hyperhyperspace', and some generalisations are proved of results of the Robertsons on sets of compact subsets. Making a completely different approach, J. L. Kelley's notion of a fundamental family of subsets is applied to uniform spaces and the Hausdorff completion of the hyperspace is constructed by means of fundamental families. A study is made of two conditions on a mapping between uniform spaces, the filter condition, introduced by Dr. Wendy Robertson, and the analogous fundamental family condition, which bears a relationship to hypercompleteness similar to that of the filter condition to completeness. Another subject which has attracted some interest recently is the comparability of the topologies induced on the set of subsets by various uniform structures on the original set. A survey of known results is given and the relationships between them are discussed. The question of when two uniform structures induce the same topology on sets of subsets of particular kinds is investigated, and finally a new condition is provided for a uniform structure to be unique in the sense that no other uniform structure

/Cont'd..

on the given set can induce the same topology on the set of subsets. The last two chapters deal with properties of mappings and sets of mappings, introducing two new concepts - the hypergraph of a mapping and a hypercompact set of mappings. The properties of the hypergraph are related to continuity and uniform continuity, giving rise to theorems similar to closed graph theorems. Hypercompactness is studied in relation to compactness and collective compactness of sets of mappings.

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

- 1.1 Prologue
- 1.2 Set-theoretical notions
- 1.3 Uniform spaces
- 1.4 Fundamental theory of uniform spaces
- 1.5 Metrics, pseudometrics, uniformities, proximities and topologies
- 1.6 Hyperspaces
- 1.7 Fundamental properties of the Hausdorff uniformity
- 1.8 Subspaces of the hyperspace
- 1.9 Induced mappings, and spaces of mappings
- 1.10 Topological vector spaces and normed spaces
- 1.11 Quotient spaces

CHAPTER 2 HYPERSPACES AND HAUSDORFF COMPLETIONS

- 2.1 Introduction
- 2.2 Preliminaries - extension of mappings
- 2.3 The construction of \hat{X} and its embeddings in $\mathcal{S}(S(X))$
- 2.4 The Hausdorff completions of $\mathcal{S}(X)$ and $\mathcal{E}(X)$
- 2.5 The Hausdorff completions of $\mathcal{C}(X)$, $\mathcal{P}(X)$, $\mathcal{R}(X)$ and $\mathcal{F}(X)$
- 2.6 Applications to topological vector spaces
- 2.7 Associated uniformities

CHAPTER 3 FUNDAMENTAL FAMILIES AND COMPLETIONS OF HYPERSPACES

- 3.1 Introduction
- 3.2 Fundamental families
- 3.3 The Hausdorff completion of the hyperspace
- 3.4 The filter condition and the fundamental family condition
- 3.5 The induced mapping t' and its extension
- 3.6 Hyperassociated uniformities

CHAPTER 4 TOPOLOGIES INDUCED ON HYPERSPACES

- 4.1 Introduction
- 4.2 The H-equivalence of uniformities
- 4.3 The H-equivalence of uniformities on subsets of $\mathcal{S}(X)$
- 4.4 The sufficiency of the set $\mathcal{S}(\mathcal{F}(X))$
- 4.5 The H-singularity of uniformities

CHAPTER 5 GRAPHS AND HYPERGRAPHS

- 5.1 Introduction
- 5.2 The closed hypergraph theorem
- 5.3 Weaker conditions on the hypergraph
- 5.4 The hypergraph of a relation
- 5.5 Examples

CHAPTER 6 COMPACT, COLLECTIVELY COMPACT AND HYPERCOMPACT SETS OF MAPPINGS

- 6.1 Introduction
- 6.2 Preliminary results about hyperspaces
- 6.3 Hypercompact and relatively hypercompact sets
- 6.4 Hyperprecompact and hyperbounded sets
- 6.5 Arzela-Ascoli Theorems and other results.

INTRODUCTION

1.1 Prologue. To appreciate the set of subsets of any mathematical space as a space in its own right involves more than a step-up of cardinality - it requires a conceptual leap which lends to a study of these 'hyperspaces' a character all its own. Apart from the intrinsic interest, such a study can bring fresh insight into familiar ideas and can sometimes draw out the common essence in seemingly diverse situations. It is hoped that this will emerge during the course of the present thesis, in which the set of subsets of a uniform space, endowed with a derived uniform structure, is investigated in relation to a number of different mathematical situations. The investigation sets out in a spirit which owes much to Ernest Michael, who really began it all, and J. L. Kelley, who observed the relevance of the hyperspace notion to the closed graph theorems of functional analysis.

In this first chapter the mathematical structures which underlie the work to follow are described. Since uniform spaces do not appear to have yet claimed their rightful place amongst the toys in the mathematician's nursery their description is given in some detail. A survey of fundamental results in the general theory of hyperspaces is included, both to place the ensuing work in its wider context and to allow greater freedom of movement in the future. Many of these results will be recalled frequently.

1.2 Set-theoretical notions. These are used naively. The terms *set*, *collection*, *family* and *class* will not be used in logically distinctive ways but simply to serve clarity. The set of non-empty subsets of a set X will be denoted by $\mathcal{S}(X)$, and in general capital letters, e.g. A , will be reserved for subsets of X and curly letters,

e.g. \mathcal{A} , for subsets of $\mathcal{S}(X)$. The set $\mathcal{S}(X)$ may also be regarded as the set of functions from X into the two-point set $\{0, 1\}$, sometimes written 2^X , ^{-∅}. When the set X is endowed with a mathematical structure and the set $\mathcal{S}(X)$ has a derived structure, it is called *the hyperspace* of the space X . The term hyperspace may also be used when speaking about subspaces of $\mathcal{S}(X)$.

Let U and V be relations on the set X - that is, subsets of the product $X \times X$. Then $U \circ V$ (or just UV) will denote the set $\{(x, y) : (x, z) \in U, (z, y) \in V, \text{ some } z \in X\}$. Also, $\overset{2}{U} = U \circ U$, $\overset{3}{U} = U \circ U \circ U$, etc., and $\overset{-1}{U} = \{(x, y) : (y, x) \in U\}$. The relation U is called symmetric if $U = \overset{-1}{U}$. For a relation U and a point x in X we write $U(x) = \{y \in X : (x, y) \in U\}$; for a subset A of X we write $U(A) = \{y \in X : (a, y) \in U \text{ for some } a \in A\}$. The set $\{(x, x) : x \in X\}$ is called the diagonal and denoted by Δ .

1.3 Uniform spaces. A uniform structure is the natural generalization of a metric structure, and is a special kind of topological structure in which the neighbourhoods of different points are comparable. The theory acquired its present form essentially at the hands of Weil in 1937. A *Uniform structure*, or *uniformity*, on a set X is a collection ξ of subsets of $X \times X$, each containing the diagonal Δ and together forming a filter, with the additional properties that, for each member U of ξ , the inverse $\overset{-1}{U}$ is also a member, and there is some member U' whose square $\overset{2}{U'}$ is contained in U . The members of ξ are called the *entourages* of the uniformity, following Bourbaki (4).

A *base* \mathcal{U} for the uniformity ξ is a collection of entourages such that a subset of $X \times X$ belongs to ξ if and only if it contains a member of the base. Conversely, a given collection of subsets of $X \times X$ will be a base for some uniformity on X if (1) each member contains the diagonal Δ , (2) the intersection of any two members contains a third, (3) each member

contains the inverse of some member, and (4) each member contains the square of some member. The entourages of the generated uniformity will then be just those subsets of $X \times X$ which contain a member of the base. An important fact is that every uniformity has a base consisting of symmetric entourages, closed with respect to the product topology (see below).

The set X , together with the uniformity ξ will be referred to as the uniform space (X, ξ) , or just the uniform space X , where there is no possibility of confusion with another uniformity. Sometimes, when the uniformity is specified by a base \mathcal{U} we will refer to the uniform space (X, \mathcal{U}) . The unique topology on X for which the neighbourhood filter of each point x is the collection $\mathcal{B}(x) = \{U(x) : U \in \xi\}$ is called the *uniform topology* and denoted $\tau(\xi)$. We may then refer to the topological space $(X, \tau(\xi))$. If this topology satisfied Hausdorff's separation axiom we call the uniformity *separated* or *Hausdorff*. This is the case if and only if the intersection of the entourages is the diagonal. The distinction between separated and non-separated uniformities will be important in this thesis.

1.4 Fundamental theory of uniform spaces. If A is a subset of the uniform space (X, ξ) then the *relative uniformity* on A is that whose entourages are the intersections of $A \times A$ with the entourages of ξ . The set A then becomes a uniform space itself.

Taking closures with respect to the uniform topology on X and the product topology on $X \times X$ we have the following useful identities:

$$\begin{aligned} \bar{A} &= \bigcap \{U(A) : U \in \xi\}, & A \text{ any subset of } X, \\ \bar{S} &= \bigcap \{U \circ S \circ U : U \in \xi\}, & S \text{ any subset of } X \times X. \end{aligned}$$

A mapping f from a uniform space (X, ξ) into a uniform space (Y, η) is called *uniformly continuous* if for each V in η there is some U in ξ such that $(f \times f)(U)$ is contained in V . It is then certainly continuous

with respect to the uniform topologies. A bijective bi-uniformly continuous mapping is called a *uniform isomorphism* and the two spaces are then called *uniformly isomorphic*. Since there will seldom be any possibility of confusion with other kinds of structural isomorphism we will usually abbreviate these to *isomorphism* and *isomorphic*. A uniform isomorphism is certainly a homeomorphism of the uniform topologies and so every topological invariant is also a uniform invariant.

There are certain concepts whose definition makes real use of the uniform structure, and which are thus uniform invariants but not topological invariants. Of particular importance are the concepts of *precompactness* and *completeness*. The uniform space (X, ξ) is called precompact if, for each U in ξ , there exists a finite subset F of X such that $X = U(F)$, or equivalently, if there exists a finite covering of X by U -small sets (a subset A is *U-small* if $A \times A \subseteq U$). This is the natural generalization of the concept of total boundedness in metric spaces. Completeness in uniform spaces may be formulated in terms of either filters or nets. Both formulations will be used freely. A filter is called a *Cauchy filter* if for each entourage U it contains a U -small set. The space is called complete if every Cauchy filter converges. A net (or directed set) $\{x_i : i \in I\}$ in X is called a *Cauchy net* if, for each entourage U there is i_0 in I such that (x_i, x_j) belongs to U whenever $i, j \geq i_0$. The space is complete if and only if every Cauchy net converges to some point of X .

1.5 Metrics, pseudometrics, uniformities, proximities and topologies.

If (X, ρ) is a metric (or pseudometric) space then the collection of subsets of $X \times X$ of the form $U(\rho, \epsilon) = \{(x, y) : \rho(x, y) < \epsilon\}$ forms a base for a uniformity on X , called the metric (or pseudometric) uniformity. A uniform space is called metrizable (pseudometrizable) if a metric (pseudometric) ρ can be defined on it so that the sets $U(\rho, \epsilon)$ form a

base for the uniformity. The space is pseudometrizable if and only if it has a countable base, and is metrizable if and only if it is separated as well. A topological space is called uniformizable if there exists a uniformity on it such that the uniform topology coincides with the given topology; this is the case if and only if the space is completely regular. On a compact separated topological space there is precisely one uniformity compatible with the topology. In this thesis we shall always use the term *compact* to mean what is sometimes called bcompact, or by Bourbaki, quasi-compact - that is, every open covering of the space has a finite subcovering. Note that although the closure of a compact subspace of an arbitrary topological space need not be compact, this is true in uniform spaces. Furthermore the closure of a relatively compact subspace is compact, even for non-separated spaces, and the closure of a precompact subspace is precompact. An important fact is that a uniform space is compact if and only if it is both precompact and complete.

There is a structural layer lying in between the topological and the uniform - that possessed by a *proximity space* (X, δ) , in which there is a relation δ in $\mathcal{S}(X)$ specifying which pairs of subsets of X are *in proximity* ($A \delta B$), and which are *remote* ($A \not\delta B$). The notion of proximity will occur in Chapters 3 and 4 but a detailed theory will not be required (see e.g. Thron (31)). Every uniform space (X, ξ) becomes a proximity space by putting $A \not\delta B$ whenever there exists U in ξ such that A and $U(B)$ do not meet, in which case A and B are called ξ -remote. If η is another uniformity on X we say that ξ is *proximity-finer* than η if every pair of η -remote subsets is also ξ -remote. The class of uniformities on X inducing the same proximity δ is denoted by $\pi(\delta)$ and has a unique coarsest member under which X is a precompact uniform space. Every proximity space (X, δ) becomes a topological space by taking, as *proximal neighbourhoods* of a point x , the subsets A for which $(X - A) \not\delta \{x\}$.

1.6 Hyperspaces. As with so many things, the study of hyperspaces had its genesis in the mind of Hausdorff, who defined a metric on the set of closed, nonempty subsets of a bounded metric space. When X is any topological space, we shall denote the set of closed nonempty subsets of X by $\mathcal{C}(X)$. When (X, ρ) is a pseudometrizable space we can follow Hausdorff and define the distance $d(A, B)$ between two subsets to be $\max \left\{ \sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A) \right\}$. This may take infinite values if the subsets are allowed to be unbounded, but then $\min\{d, 1\}$ defines a pseudometric on $\mathcal{S}(X)$ and a metric on $\mathcal{C}(X)$. We will refer to this metric as the *Hausdorff metric*.

Starting with a uniform space (X, ξ) , it is possible to define a number of structures for the set $\mathcal{S}(X)$, most of them derived from the topological structure on X . First of all, $\mathcal{S}(X)$ can be partially ordered by inclusion and the right and left order-topologies thereby defined. Ernest Michael, following Vietoris (33), studied the *finite* (or *Vietoris*) topology in (20); this topology is the supremum of two others, known as the upper-semi-finite and lower-semi-finite topologies, in the lattice of all topologies on $\mathcal{S}(X)$, and has been the one to attract most interest. Other interesting and useful topologies have been defined by Mrowka (21) and Fell (7).

It is natural to wonder what relationship any topological structure on $\mathcal{S}(X)$ has to the fairly well known notion of topological convergence of subsets (see Mrowka (21)). If $\{E_i\}$ is a net of subsets in a topological space X , let $\limsup E_i$ be the set of points of X for which E_i frequently intersects each neighbourhood, and $\liminf E_i$ be the set of points for which E_i eventually intersects each neighbourhood. Then, if $\limsup E_i = \liminf E_i = E$, the net $\{E_i\}$ is said to be topologically convergent to E . When X is a compact metric space, convergence in the Hausdorff metric on $\mathcal{C}(X)$ coincides with topological convergence of sets,

and when X is any separated compact space the finite topology on $\mathcal{E}(X)$ induces topological convergence of sets. However these results fail for locally compact spaces; when X is a separable, locally compact metric space Watson (37) defined a new metric on $\mathcal{E}(X)$ which induces topological convergence of sets, and when X is any locally compact space Mrowka defined what he called the lbc-topology on $\mathcal{E}(X)$ having the same property, and coinciding with the finite topology if X is compact. Effros (6) showed that for a separable complete metric space X the Borelian structure generated on $\mathcal{E}(X)$ by the topological convergence of sets is standard, and, if the space is locally compact, it is actually a topology, which turns out to be the same as that defined and studied by Fell (7), and only slightly different in construction from the finite topology.

When X is a regular space (and in particular when X is a uniform space) which is not locally compact then (21) there is no topology on $\mathcal{E}(X)$ which induces the topological convergence of sets.

The structure on $\mathcal{S}(X)$ with which this thesis is concerned is the natural generalization of the Hausdorff metric and was introduced by Bourbaki and further studied by Michael (20). It derives essentially from the uniform structure on X . Let \mathcal{U} be a base for the uniformity. For each U in \mathcal{U} let \tilde{U} be the collection of pairs (A, B) of subsets of X such that both $A \subset U(B)$ and $B \subset U(A)$. The sets \tilde{U} form a base for a uniformity $\tilde{\mathcal{U}}$ on $\mathcal{S}(X)$, called *the Hausdorff uniformity*, because it coincides for a metric space with the uniformity of the Hausdorff metric on $\mathcal{E}(X)$, and with the uniformity of the pseudometric on $\mathcal{S}(X)$. It is clearly independent of the choice of base for the uniformity ξ on X , and may be denoted by $\tilde{\xi}$.

For an arbitrary uniform space (X, ξ) it is true that convergence of sets with respect to the uniformity $\tilde{\xi}$ implies topological convergence, but in general the uniform topology $\tau(\tilde{\xi})$ does not induce topological

convergence of sets, as indicated by the remarks earlier. It is known (see Michael (20)) to coincide with the finite topology on the set of compact subsets, and so when X is a compact separated space these two topologies, Mrowka's lbc-topology, and Fell's topology, all coincide on $\mathcal{E}(X)$ and induce the topological convergence of sets. Interestingly, the topology $\tau(\tilde{\xi})$ does not characterize the uniformity ξ from which it is derived. This, and related topics, will be considered in Chapter 4.

1.7 Fundamental properties of the Hausdorff uniformity. The uniformity $\tilde{\xi}$ is not separated on $\mathcal{S}(X)$ but is easily seen to be separated when restricted to $\mathcal{E}(X)$. For the pair (A, B) of subsets belongs to the intersection of the entourages \tilde{U} if and only if $A \subset U(B)$ and $B \subset U(A)$ for each U , and since the intersection of the uniform neighbourhoods $U(B)$ of a subset B is just its closure, this will be true if and only if A and B have the same closure. For this reason most results concerning hyperspaces are formulated in $\mathcal{E}(X)$. For simplicity we shall also use the notation $\tilde{\xi}$, \tilde{U} , when talking about the Hausdorff uniformity restricted to $\mathcal{E}(X)$ or any other subspace of $\mathcal{S}(X)$, so long as there is no possibility of confusion. Most of the time from now on it will be understood that all hyperspaces are endowed with the Hausdorff uniformity.

The mapping $x \rightarrow \{x\}$ is a uniform isomorphism of X onto a subspace of $\mathcal{S}(X)$, and if X is separated, onto a closed subspace of $\mathcal{E}(X)$. When the Hausdorff metric is applicable it is actually an isometry. Thus X inherits many properties possessed by $\mathcal{E}(X)$; in particular X is metrizable, precompact or compact if and only if $\mathcal{E}(X)$ has each respective property (see Michael (20)). If $\mathcal{E}(X)$ is complete so is X . If X is a complete metrizable space then $\mathcal{E}(X)$ is complete, but this result is not true for arbitrary complete uniform spaces, a

fact which has interesting repercussions. The relation of the notion of completeness to hyperspaces will be the central theme in Chapters 2 and 3.

1.8 Subspaces of the hyperspace. When A is any subspace of X , the Hausdorff uniformity on $\mathcal{S}(A)$ derived from the relative uniformity on A is the same as that induced on $\mathcal{S}(A)$ as a subspace of $\mathcal{S}(X)$. If A is dense in X then $\mathcal{S}(A)$ is dense in $\mathcal{S}(X)$, and if A is closed in X then $\mathcal{C}(A)$ is closed in $\mathcal{C}(X)$. If A is compact and closed then the collection of sets of $\mathcal{C}(X)$ which intersect A is closed in $\mathcal{C}(X)$.

The union of a collection of closed subsets of X which forms a compact subset of $\mathcal{S}(X)$ is closed in X ; the union of a compact collection of compact subsets is compact. These facts were proved by Michael (20), and will be used frequently. They allow some interesting, simple proofs of results in many different contexts. For example, when A is a compact subset of a uniform space and U is a closed entourage then $U(A)$ is closed; the sum of a compact subset and a closed (resp. compact) subset of a topological vector space is closed (resp. compact); if G/H is the quotient of a topological group by a compact subset then the inverse image of any compact subspace of G/H under the canonical mapping $G \rightarrow G/H$ is compact. A number of Arzelà-Ascoli-type theorems can also be proved in this way (see Chapter 6).

The space $\mathcal{C}(X)$ is always dense in $\mathcal{S}(X)$. The set $\mathcal{F}(X)$ of non-empty finite subsets of X is dense in $\mathcal{S}(X)$ if and only if X is precompact; in fact the closure of $\mathcal{F}(X)$ in $\mathcal{S}(X)$ is just the set of all precompact subsets of X . It is interesting to contrast the situation when $\mathcal{S}(X)$ is regarded as a product 2^X of copies of the discrete space $\{0, 1\}$, with the product topology, $\prod_{x \in X} \{0, 1\}$. In this case $\mathcal{S}(X)$ is always compact, and $\mathcal{F}(X)$ is always a dense, precompact subspace. With the finite topology, too, $\mathcal{F}(X)$ is dense in $\mathcal{S}(X)$.

In this thesis we shall be particularly concerned with the subsets of $\mathcal{E}(X)$ consisting of the precompact sets and the compact sets, denoted respectively by $\mathcal{P}(X)$ and $\mathcal{C}(X)$; also with the set $\mathcal{R}(X)$ of relatively compact sets. The set $\mathcal{P}(X)$ is closed in $\mathcal{E}(X)$. If X is complete the set $\mathcal{C}(X)$ is closed in $\mathcal{E}(X)$. Any decreasing net of sets in $\mathcal{C}(X)$ which is Cauchy must converge to the intersection. Any decreasing net in $\mathcal{E}(X)$ which converges must converge to the intersection, and any increasing net whose union is precompact must converge to its closure. Note that if A_i converges to A in $\mathcal{S}(X)$ then it converges also to \bar{A} , and \bar{A}_i converges to \bar{A} in $\mathcal{E}(X)$.

1.9 Induced mappings, and spaces of mappings. Every mapping t from a uniform space X into a uniform space Y induces a mapping t° of $\mathcal{S}(X)$ into $\mathcal{S}(Y)$ and a mapping t' of $\mathcal{S}(X)$ into $\mathcal{E}(Y)$, defined by $t^\circ(A) = t(A)$ and $t'(A) = \overline{t(A)}$, for each subset A . This notation will be standard throughout the thesis, and the induced mappings t° and t' will be basic tools in the development. They will be involved particularly in Chapter 3, and the graph of t' (introduced as the *hypergraph* of t) will be the guest artist in Chapter 5.

Each of t° and t' is uniformly continuous if and only if t is uniformly continuous. Let T be a set of uniformly equicontinuous mappings (that is, for each entourage V of Y there is an entourage U of X such that $t^2(U) \subset V$ for each t in T); then the sets $T^\circ = \{t^\circ : t \in T\}$ and $T' = \{t' : t \in T\}$ are uniformly equicontinuous as well, and the mappings $A \mapsto T(A)$ and $A \mapsto \overline{T(A)}$ are both uniformly continuous on $\mathcal{S}(X)$, where $T(A) = \bigcup \{t(A) : t \in T\}$.

When X is any set and Y is a uniform space, let $F(X, Y)$ denote the set of all mappings from X into Y . For any collection \mathcal{A} of

subsets of X , a uniformity can be defined on this set, called *the uniformity of uniform convergence on the sets of \mathcal{A}* , and denoted by $\xi(\mathcal{A})$. It has base consisting of all finite intersections of sets of the form

$$W(A, V) = \{(f, g) : (f(a), g(a)) \in V \text{ for all } a \in A\},$$

where V is any entourage of Y and A any member of \mathcal{A} . When $\mathcal{A} = \{\{x\} : x \in B\}$, B some subset of X , then the uniformity $\xi(\mathcal{A})$ is called *the uniformity of pointwise convergence on B* . If $B = X$ it is known simply as the uniformity of pointwise convergence, and coincides with the product uniformity on $F(X, Y)$ as the product Y^X .

If $t_i \rightarrow t$ in $F(X, Y)$ with respect to the uniformity $\xi(\mathcal{A})$ then $t_i^\circ \rightarrow t^\circ$ in $F(S(X), S(Y))$ with respect to the uniformity of pointwise convergence on \mathcal{A} . The converse is not true. We can, however, define a new uniformity on $F(X, Y)$ so as to have the two kinds of convergence equivalent. Because of the way each mapping in $F(X, Y)$ induces, in a one-to-one manner, a mapping in $F(S(X), S(Y))$, the former set may be regarded as a subset of the latter. Then if \mathcal{A} is any subset of $S(X)$, the uniformity on $F(S(X), S(Y))$ of pointwise convergence on \mathcal{A} induces a uniformity on $F(X, Y)$ for which f_i converges to f if and only if $f_i(A)$ converges to $f(A)$ in $S(Y)$ for each A in \mathcal{A} . Other uniformities may be defined on $F(X, Y)$ in a similar manner.

Spaces of mappings and uniformly equicontinuous sets of mappings will come under consideration in Chapter 6.

1.10 Topological vector spaces; normed spaces. Some of the most interesting and useful uniform spaces are those which have topological and linear structures, compatible with each other in the sense that the vector addition and scalar multiplication

operators $(x, y) \rightarrow x + y$, $(x, \lambda) \rightarrow \lambda x$ are continuous. Such a space X is called a *topological vector space*. We will always assume the scalar field to be either real or complex. The topology on X is characterized by a base \mathcal{V} for the neighbourhoods of the origin 0 , for if x is any point in X the sets $x + V$, V in \mathcal{V} , form a base for the neighbourhoods of x . Consequently, any mapping of X into a uniform space is continuous if and only if it is continuous at 0 , and then it is actually uniformly continuous with respect to the natural uniform structure on X which has base consisting of the sets of the form $U(V) = \{(x, y) : y - x \in V\}$, for V in \mathcal{V} . The Hausdorff uniformity on $\mathcal{S}(X)$ is derived from this uniformity and then has base consisting of the sets.

$$\tilde{U}(V) = \{(A, B) : A \subset B + V \text{ and } B \subset A + V\},$$

for V in \mathcal{V} . For simplicity we shall denote this by \tilde{V} .

A subset A of X is called *convex* if $\mu a + \lambda b$ belongs to A whenever a, b belong to A , and μ, λ are non-negative real numbers summing to one. A subset A is called *balanced* if λa belongs to A whenever a belongs to A and λ has absolute value at most one; if A is both balanced and convex it is called *absolutely convex*. The balanced (convex) (absolutely convex) hull of A is the smallest set with each respective property containing A . The closure of any balanced (convex) (absolutely convex) subset has the same property. The space X is called *locally convex* if it has a base consisting of convex neighbourhoods of the origin. A subset A is said to absorb a subset B if there is some $\lambda > 0$ such that $B \subset \mu A$ for all μ with $|\mu| \geq \lambda$. A set which absorbs points is called *absorbent*. Each 0 -neighbourhood is absorbent. A set which is absorbed by each 0 -neighbourhood is called *bounded*; the set A is bounded if and only if for each 0 -neighbourhood V there exists $\lambda > 0$ such that $A \subset \lambda V$.

The set of nonempty, bounded subsets of X , and the set of

nonempty, balanced subsets of X are each closed in $\mathcal{S}(X)$. If X is locally convex the set of nonempty, convex closed subsets is closed in $\mathcal{E}(X)$, and hence the set of nonempty, absolutely convex, closed subsets is closed in $\mathcal{E}(X)$.

An absorbent, absolutely convex, closed set is called a barrel. Every locally convex space has a base consisting of barrels; a space is called *barrelled* if every barrel is a 0-neighbourhood.

A *pseudonorm* on a vector space X , is a real-valued function $x \mapsto \|x\|$ on X such that $\|x\| \geq 0$, $\|\lambda x\| = |\lambda| \|x\|$, and $\|x + y\| \leq \|x\| + \|y\|$. If $\|x\| = 0$ implies $x = 0$, the function is called a *norm*. A pseudonormed space is a topological vector space under the topology defined by the metric distance $\|x - y\|$. A topological vector space is pseudonormable if and only if it has a bounded, convex neighbourhood of the origin, and is normable if and only if it is separated in addition.

1.11 Quotient spaces. Since these can be regarded as subspaces of $\mathcal{S}(X)$ under certain circumstances, they provide a fruitful field for applications of the theory of hyperspaces. We shall investigate, particularly quotient spaces of topological vector spaces.

For an equivalence relation R on an arbitrary uniform space X , the quotient topology on X/R may differ from the topology induced by the Hausdorff uniformity, and from that induced by the finite topology on $\mathcal{S}(X)$ (see Bourbaki (4), Chapter 2, exercise 1.5). Applying a result of Michael ((20), Prop 5-11), we can deduce that for a compact, separated space X and a Hausdorff equivalence relation R , the quotient topology and Hausdorff uniform topology coincide on X/R , and the quotient space is a closed subspace of $\mathcal{E}(X)$. If X is a topological vector space and M a vector subspace, then the natural uniformity on the quotient space X/M

generated by the sets of the form $\{(a + M, b + M) : b - a + M \in Q(V)\}$, where Q is the quotient map $X \rightarrow X/M$ and V is any 0 -neighbourhood, is the same as that induced by the Hausdorff uniformity on $S(X)$, and the quotient topology is the uniform topology. If M is a closed vector subspace then the quotient space is a closed subspace of $\mathcal{E}(X)$.

CHAPTER 2

HYPERSPACES AND HAUSDORFF COMPLETIONS

2.1 Introduction. When X is a complete metric space the set $\mathcal{E}(X)$ of non-empty closed subsets of X with the Hausdorff metric is a complete space also (see e.g. Kuratowski (19), Price (25), Bourbaki (4)). However when X is any complete uniform space the set $\mathcal{E}(X)$ with the Hausdorff uniformity need not be complete. We shall call X *hypercomplete* if $\mathcal{E}(X)$ is complete. The term was introduced by J. L. Kelley to describe topological vector spaces X for which the collection of absolutely convex sets in $\mathcal{E}(X)$ is complete - we have widened the definition to make it applicable to uniform spaces. Each compact space is hypercomplete (see Bourbaki (4), Chapter 2, ex. 4.6), and each hypercomplete space is complete. For topological vector spaces hypercompleteness (even in Kelley's sense) implies full completeness (or B-completeness in the sense of Pták).

If a uniform space X is not complete then, in a sense, it has 'insufficient' points, and if it is not hypercomplete the collection of subsets is similarly deficient. This unsatisfactory state of affairs can be remedied in a manner typical of many others in mathematics, by first replacing X , if it is not already separated, by a separated uniform space X' closely associated with it, and then enlarging X' slightly to a complete separated space \hat{X} . This more pleasing space is called the Hausdorff completion of X . The hyperspaces $\mathcal{S}(X)$ and $\mathcal{E}(X)$ may be dealt with similarly, and the question arises at once what the relationship is between such spaces as $\mathcal{E}(X)$ and $\mathcal{E}(\hat{X})$. This chapter deals with questions like this, involving hyperspaces and the concept of Hausdorff completion.

The 'hyperhyperspace' $\mathcal{S}(\mathcal{S}(X))$ is neither separated nor complete in general, but it contains, uniformly embedded in it, a number of significant spaces including the Hausdorff completions \hat{X} and $\hat{\mathcal{C}}(X)$. More will be said about this in Chapter 3. It is shown in 2.3 that \hat{X} can be embedded in two ways in $\mathcal{S}(\mathcal{S}(X))$. Next, in 2.4, the relationships between the hyperspaces of X , X' and \hat{X} are investigated together with their Hausdorff completions, and this leads to a consideration of the problem of which spaces are such that the Hausdorff completion of the hyperspace can be realized as a hyperspace itself, and the related problem of which spaces have hyper-completions. In section 2.5 a study is made of various hyperspaces consisting of compact, precompact, relatively compact and finite sets, permitting some slight generalizations of Theorem 1 in the Robertsons' paper (28). It is shown that separatedness is unnecessary, and that for an arbitrary uniform space X the Hausdorff completion of the set of compact, closed subsets is isomorphic to the set of compact subsets of the Hausdorff completion of X . The results of 2.4 are applied in 2.6 to linear spaces, and in particular to the completions of quotient spaces. Finally in 2.7 an investigation is made of hyperspaces arising from two uniformities for a set X , the one associated in a certain way with the other, and as a consequence separatedness is removed from the hypotheses for Theorems 2 and 3 of (28).

2.2 Preliminaries - extension of mappings. In this section we state two well-known Theorems concerning the extension of mappings which will be useful throughout the Chapter, and incidentally probe a little deeper into the question of when

the extended mapping is injective. This question will arise again in the next Chapter. First the well-known results (see, for example Bourbaki (4), Chapter 2.):

THEOREM 2.2.1. Let A be a subspace of a uniform space X and let f be a uniformly continuous mapping from A into a complete Hausdorff uniform space Y . Then f can be extended by continuity to the closure of A in X , and the extended function f^* is uniformly continuous. The graph of f^* is the closure in $X \times Y$ of the graph of f .

THEOREM 2.2.2. Let A_1, A_2 be dense subsets of the complete Hausdorff uniform spaces X_1, X_2 respectively. Then every isomorphism f of A_1 onto A_2 extends by continuity to an isomorphism f^* of X_1 onto X_2 .

If, in this last result, f is given to be only a bijective uniformly continuous mapping then f^* is uniformly continuous but need be neither injective nor surjective. For (Bourbaki (4) Chapter 2, section 3, ex. 3) if \mathbb{R} denotes the real line with additive uniformity, $\overline{\mathbb{R}}$ the extended real line, and $\tilde{\mathbb{R}}$ its one-point compactification, and ξ is the uniformity induced on \mathbb{R} by the uniformity of $\overline{\mathbb{R}}$, then the identity mapping of \mathbb{R} is uniformly continuous in each of the cases $\mathbb{R} \rightarrow \overline{\mathbb{R}}$ and $(\mathbb{R}, \xi) \rightarrow \tilde{\mathbb{R}}$, and extends by continuity in the first case to an injective but not surjective mapping $\mathbb{R} \rightarrow \overline{\mathbb{R}}$, and in the second case to a surjective but not injective mapping $\overline{\mathbb{R}} \rightarrow \tilde{\mathbb{R}}$. If, rather, we allow X , in THEOREM 2.2.1, to be an arbitrary uniform space, and f is an isomorphism, then f^* still need not be injective or surjective, let alone an isomorphism. There will be an example of this in 2.4. However, we can clear up the

situation a little using the next Theorem, which is preceded by a Lemma isolating the main argument from any completeness condition.

LEMMA 2.2.3. Let X and Y be uniform spaces and let A be a dense subspace of X . If f is a uniformly continuous mapping of X into Y which induces an isomorphism between A and some subspace B of Y , then for x_1, x_2 in X , $f(x_1) = f(x_2)$ implies $(x_1, x_2) \in \bigcap \{U : U \in \mathcal{U}\}$ where \mathcal{U} is a base for the uniformity on X .

Proof. Let x_1, x_2 be points of X with $f(x_1) = f(x_2)$, and let $U \in \mathcal{U}$. Let \mathcal{V} be a base for the uniformity on Y . Choose a symmetric V in \mathcal{V} such that $(f(b_1), f(b_2))$ is in U whenever b_1 and b_2 are points of B with $(b_1, b_2) \in \overset{2}{V}$. Then choose U_1 in \mathcal{U} such that $(f(p), f(q))$ is in V whenever p and q are points of X with $(p, q) \in U_1$.

Now let U_2 be a symmetric member of \mathcal{U} with $U_2 \subset U \cap U_1$. There exist points a_1, a_2 in A such that a_1 is in $U_2(x_1)$ and a_2 is in $U_2(x_2)$, since A is dense in X . Then (a_1, x_1) and (a_2, x_2) are each in U_2 and so $(f(a_1), f(x_1))$ and $(f(a_2), f(x_2))$ are each in V . Hence $(f(a_1), f(a_2))$ is in $\overset{2}{V}$ since $f(x_1) = f(x_2)$, and therefore (a_1, a_2) is in U . Therefore (x_1, x_2) is in $U_2 \circ U \circ U_2 \subset \overset{3}{U}$, and the result follows.

THEOREM 2.2.4. Let A be a dense subspace of a uniform space X and let A be isomorphic to a subspace B of a complete Hausdorff uniform space Y . Then the isomorphism extends to a uniformly continuous mapping of X into Y which is injective on every Hausdorff subspace of X .

Proof. Let \mathcal{U} be a base for the uniformity on X . The isomorphism of A onto B can be extended to a uniformly continuous mapping $f : X \rightarrow Y$ by THEOREM 2.2.1. If x_1, x_2 are points of X with $f(x_1) = f(x_2)$, then by LEMMA 2.2.3 (x_1, x_2) is in $\bigcap \{U : U \in \mathcal{U}\}$. The result follows immediately.

2.3 The construction of \hat{X} and its embeddings in $\mathcal{S}(\mathcal{S}(X))$.

Let X be a uniform space and let \mathcal{U} be a base for the uniformity on X . Let \hat{X} denote the set of minimal Cauchy filters on X , with the uniformity, generated by the sets \hat{U} , where \hat{U} is the collection of pairs $(\mathcal{F}_1, \mathcal{F}_2)$ of minimal Cauchy filters such that \mathcal{F}_1 and \mathcal{F}_2 have a U -small set in common, and U runs through the symmetric entourages in \mathcal{U} . Let $i : X \rightarrow \hat{X}$ be the mapping which takes each x in X onto its neighbourhood filter. Then \hat{X} is a complete Hausdorff space, i is uniformly continuous, and $i(X)$ is dense in \hat{X} . \hat{X} is called the Hausdorff completion of X and the subspace $i(X)$ is called the Hausdorff space associated with X and denoted X' .

We note the following facts about these structures (see e.g. Bourbaki (4)).

(1) Given any uniformly continuous mapping f of X into a complete Hausdorff uniform space Y , there is a unique uniformly continuous mapping $g : \hat{X} \rightarrow Y$ such that $f = g \circ i$.

(2) If i is a uniformly continuous mapping of X into a complete Hausdorff uniform space X_1 , and the pair (i, X_1) has the property expressed for (i, \hat{X}) in (1), then there is a unique isomorphism $\phi : \hat{X} \rightarrow X_1$ such that $i_1 = \phi \circ i$.

(3) The graph of the equivalence relation $R : i(x) = i(x')$ is $\{U : U \in \mathcal{U}\}$ and X' and X/R are homeomorphic as topological

spaces. If X is a Hausdorff space then X is isomorphic to X' , and if X is a dense subspace of any complete Hausdorff space Y then the canonical injection $X \rightarrow Y$ extends to an isomorphism of \hat{X} onto Y .

(4) The uniform structure of X is the inverse image under i of that of \hat{X} (or X'). The entourages of X' are the images under $i \times i$ of the entourages of X , and the closures in $\hat{X} \times \hat{X}$ of the entourages of X' form a base of entourages for \hat{X} .

From (3) it is clear that if X is dense in a Hausdorff space Y then \hat{X} and \hat{Y} are the same (up to isomorphism). When Y is not Hausdorff we use (2). Let $i_1 : X \rightarrow Y$ be the composition of the two canonical mappings $k : X \rightarrow Y$, $j : Y \rightarrow \hat{Y}$. Then the pair (i_1, \hat{Y}) has the property expressed in (1) for (i, \hat{X}) , because if f is any uniformly continuous mapping of X into a complete Hausdorff uniform space Z , we can extend f by continuity to a uniformly continuous mapping $f' : Y \rightarrow Z$, and then by (1) there is a unique uniformly continuous mapping $g : \hat{Y} \rightarrow Z$ such that $f' = g \circ j$, and also $f = g \circ i$. Thus by (2) there is a unique isomorphism $\phi : \hat{X} \rightarrow \hat{Y}$ such that $i_1 = \phi \circ i$. We can now state:

(5) If X is dense in the uniform space Y then X and Y have the same Hausdorff completion (up to isomorphism).

Note that X and Y may have different associated Hausdorff spaces. For if X is a non-complete metric space and Y is the completion \hat{X} of X , then $\mathcal{E}(X)$ is not complete and $\mathcal{E}(\hat{X})$ is complete; but $\mathcal{S}(X)$ is dense in $\mathcal{S}(\hat{X})$ and $\mathcal{E}(X)$ and $\mathcal{E}(\hat{X})$ are their respective associated Hausdorff spaces, as will be shown in the next section.

We turn now to studying the relation between these ideas and the hyperspaces of the uniform space X . To begin with, there are two natural ways of embedding the Hausdorff completion \hat{X} in the space $\mathcal{S}(\mathcal{S}(X))$. The first is given as an exercise in Bourbaki (4), Chapter 2, section 3, ex. 7, and proceeds by extending the canonical mapping $x \mapsto \{\{x\}\}$ of X into $\mathcal{E}(\mathcal{E}(X))$ (we assume X to be Hausdorff for the moment) to an isomorphism of \hat{X} onto a closed uniform subspace of $\mathcal{E}(\mathcal{E}(X))$. As will be shown in the next section, for an arbitrary uniform space X , $\mathcal{E}(X)$ and $\mathcal{E}(X')$ are isomorphic, and this makes the embedding possible for any uniform space.

PROPOSITION 2.3.1. For any uniform space X there is an isomorphism of \hat{X} onto a closed uniform subspace of $\mathcal{E}(\mathcal{E}(X))$ which is an extension of the mapping $i(x) \mapsto \{\{\bar{x}\}\}$ of X' into $\mathcal{E}(\mathcal{E}(X))$.

The second embedding method proceeds directly from the definition given above of the Hausdorff completion.

PROPOSITION 2.3.2. For any uniform space X , the Hausdorff completion \hat{X} , considered as the set of minimal Cauchy filters on X , is a uniform subspace of $\mathcal{S}(\mathcal{S}(X))$.

Proof. Let \mathcal{U} be a base, consisting of symmetric entourages, for the uniformity on X . The uniformity on \hat{X} has a base $\tilde{\mathcal{U}}$ consisting of the sets \hat{U} of all pairs of minimal Cauchy filters which have a U -small set in common, as U runs through \mathcal{U} . As a subset of $\mathcal{S}(\mathcal{S}(X))$, \hat{X} can also be given the uniformity induced by the Hausdorff uniformity on $\mathcal{S}(\mathcal{S}(X))$, with base $\tilde{\tilde{\mathcal{U}}}$. We will show that these two uniformities on \hat{X} coincide.

Let $(\mathcal{F}_1, \mathcal{F}_2)$ belong to \hat{U} . Then let A be a U -small set in each of \mathcal{F}_1 and \mathcal{F}_2 , and choose F in \mathcal{F}_1 . We have $F \cup A \in \mathcal{F}_2$, and $F \cup A \subset U(F)$, since A is U -small and $A \cap F \neq \emptyset$. Clearly $F \subset U(F \cup A)$, so that $(F, F \cup A)$ is in \tilde{U} . Similarly for each F' in \mathcal{F}_2 , $F' \cup A$ is in \mathcal{F}_1 , and $(F', F' \cup A)$ is in \tilde{U} . Therefore $(\mathcal{F}_1, \mathcal{F}_2)$ is in \tilde{U} , and we have shown that $\hat{U} \subset \tilde{U}$, so that $\hat{\mathcal{U}}$ is finer than $\tilde{\mathcal{U}}$.

Now let \hat{U} be any member of $\hat{\mathcal{U}}$, and choose U_1 in \mathcal{U} with $\hat{U}_1 \subset U$. Let $(\mathcal{F}_1, \mathcal{F}_2)$ belong to \hat{U}_1 , and let F_1 be a U_1 -small set of \mathcal{F}_1 . Then there exists an F_2 in \mathcal{F}_2 such that $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \subset U_1 \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}$. Thus $U_1(F_1)$ belongs to both \mathcal{F}_1 and \mathcal{F}_2 , and is a U -small set, so that $(\mathcal{F}_1, \mathcal{F}_2)$ belongs to \hat{U} . This proves that $\tilde{\mathcal{U}}$ is finer than $\hat{\mathcal{U}}$, and so, finally, $\tilde{\mathcal{U}}$ and $\hat{\mathcal{U}}$ determine the same uniformity on \hat{X} .

2.4 The Hausdorff completions of $\mathcal{S}(X)$ and $\mathcal{E}(X)$. The main aim in this section is to show that for an arbitrary uniform space X , the hyperspaces $\mathcal{E}(X)$ and $\mathcal{E}(\hat{X})$ have the same Hausdorff completion, and this is achieved by first proving two Propositions to the effect that $\mathcal{S}(X)$ and $\mathcal{E}(X)$ have the same Hausdorff completion and that the hyperspaces $\mathcal{E}(X)$ and $\mathcal{E}(X')$ are isomorphic.

PROPOSITION 2.4.1. For any uniform space X , $\mathcal{E}(X)$ is isomorphic to the Hausdorff space $\mathcal{S}'(X)$ associated with $\mathcal{S}(X)$.

Proof. Let \mathcal{U} be a base for the uniformity on X . Let $j : \mathcal{S}(X) \rightarrow \mathcal{S}'(X)$ be the canonical mapping, and define a mapping $\phi : \mathcal{E}(X) \rightarrow \mathcal{S}'(X)$ by $\phi(M) = j(M)$ for each M in $\mathcal{E}(X)$. If M, N are in $\mathcal{E}(X)$ then $\phi(M) = \phi(N)$ if and only if $j(M) = j(N)$, if and only if $(M, N) \in \bigcap \{\tilde{U} : U \in \mathcal{U}\}$, if and only if $M \subset U(N)$ and $N \subset U(M)$ for each U in \mathcal{U} , if and only if $M = N$.

For each R in $\mathcal{S}(X)$, (R, \bar{R}) is in each \bar{U} , so that $\phi(\bar{R}) = j(R)$. Thus ϕ is a bijective mapping.

Since the uniformity on $\mathcal{S}(X)$ is the inverse image under j of that on $\mathcal{S}'(X)$, ϕ^{-1} is uniformly continuous, and since the entourages of $\mathcal{S}'(X)$ are the images under $j \times j$ of those of $\mathcal{S}(X)$ and hence of those of $\mathcal{E}(X)$, ϕ is uniformly continuous. Therefore ϕ is an isomorphism of $\mathcal{E}(X)$ onto $\mathcal{S}'(X)$.

COROLLARY 2.4.2. $\hat{\mathcal{E}}(X)$ is isomorphic to $\mathcal{S}(X)$.

This Corollary also follows directly from fact (5) in 2.3, because $\mathcal{E}(X)$ is dense in $\mathcal{S}(X)$.

PROPOSITION 2.4.3. For any uniform space X , $\mathcal{E}(X)$ is isomorphic to $\mathcal{E}(X')$.

Proof. Define a mapping $\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X')$ by $\alpha(M) = i(M)$ for each M in $\mathcal{E}(X)$, where i is the canonical mapping of X onto X' . Again, let \mathcal{U} be a base for the uniformity on X . Since the entourages of X' are the images under i^2 of those of X and since $i^2(U)(i(x)) \subset i^2(\bar{U}(x))$ for each x in X and each U in \mathcal{U} , i must be an open mapping.

Given M in $\mathcal{E}(X)$ and a point x in $X - M$, there is a U in \mathcal{U} such that $i(x)$ does not belong to $i(M)$, and hence $i(M) \cap i(X - M) = \emptyset$. Thus $i^{-1}(i(M)) = M$, and i , being open, must also be a closed mapping; α therefore maps $\mathcal{E}(X)$ onto $\mathcal{E}(X')$.

If M, N are in $\mathcal{E}(X)$ and $\alpha(M) = \alpha(N)$, then $i^{-1}(i(M)) = i^{-1}(i(N))$, so that $M = N$. If R is in $\mathcal{E}(X')$ then $i(i^{-1}(R)) = \alpha(i^{-1}(R))$, and $i^{-1}(R)$ belongs to $\mathcal{E}(X)$ since i is uniformly continuous. Thus α is bijective.

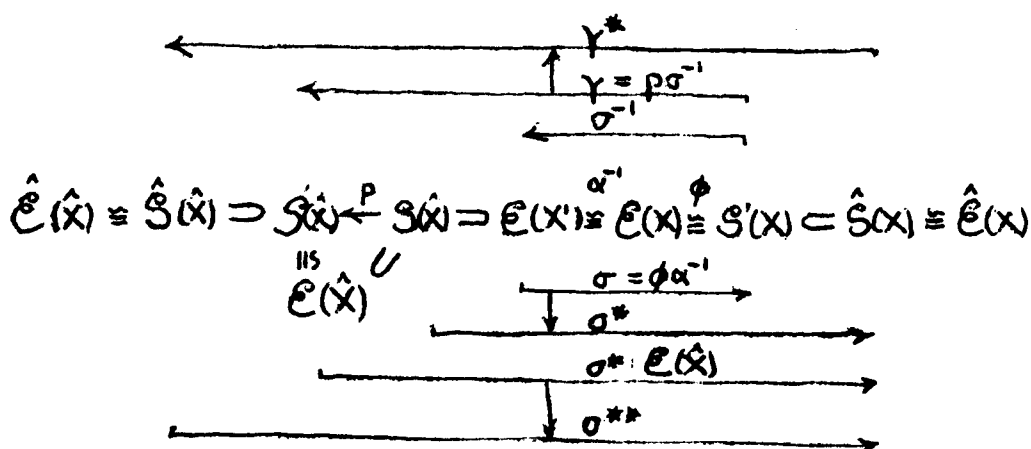
Since i is uniformly continuous, so is its induced mapping from $\mathcal{S}(X)$ to $\mathcal{S}(X')$, which coincides with α on $\mathcal{E}(X)$.

Now the sets $(i \times i)U$ form a base of entourages for X' , so the sets $V(U) = \{(P, Q) : P \subset i^2(U)(Q) \text{ and } Q \subset i^2(U)(P)\}$ form a base for the uniformity on $\mathcal{E}(X')$. Let P, Q belong to $\mathcal{E}(X')$ with (P, Q) in $V(U)$. Then $(i^{-1}(P), i^{-1}(Q))$ is in \tilde{U} , and therefore α^{-1} is a uniformly continuous mapping. We have now proved that α is an isomorphism of $\mathcal{E}(X)$ onto $\mathcal{E}(X')$.

THEOREM 2.4.4. For any uniform space X , the Hausdorff completion $\hat{\mathcal{E}}(X)$ of the hyperspace of non-empty closed subsets of X is isomorphic to the Hausdorff completion $\hat{\mathcal{E}}(\hat{X})$ of the hyperspace of non-empty closed subsets of the Hausdorff completion \hat{X} of X .

Proof. We have that X' , the Hausdorff space associated with X , is dense in \hat{X} , and so $\mathcal{S}(X')$ is dense in $\mathcal{S}(\hat{X})$. Using fact (5) of 2.3, $\hat{\mathcal{S}}(X')$ and $\hat{\mathcal{S}}(\hat{X})$ are isomorphic. But then, by COROLLARY 2.4.2., $\hat{\mathcal{E}}(X')$ and $\hat{\mathcal{E}}(\hat{X})$ must be isomorphic. Finally, since $\mathcal{E}(X)$ and $\mathcal{E}(X')$ are isomorphic by PROPOSITION 2.4.3., the result follows.

In order to actually exhibit the isomorphism between $\hat{\mathcal{E}}(X)$ and $\hat{\mathcal{E}}(\hat{X})$ and be able to use it in future, we resort to the rather terrifying diagram below, which attempts to demonstrate how the initial isomorphism between $\mathcal{E}(X')$ and $\mathcal{S}'(X)$, resulting from PROPOSITIONS 2.4.1. and 2.4.3., is successively extended or restricted in each direction.



We have here an example of the fact mentioned in 2.2, that an isomorphism between dense subspaces of two uniform spaces, one of which is a Hausdorff complete space, need not extend to an injective or a surjective mapping. For in the diagram

$$\mathcal{S}(\hat{X}) \supset \mathcal{E}(X') \cong \mathcal{S}'(X) \subset \hat{\mathcal{S}}(X)$$

the isomorphism certainly extends to the uniform continuous mapping σ^* of $\mathcal{S}(\hat{X})$ into $\hat{\mathcal{S}}(X)$, but this mapping does not distinguish between a subset of \hat{X} and its closure in \hat{X} , and does not map $\mathcal{S}(\hat{X})$ onto $\hat{\mathcal{S}}(X)$ unless $\mathcal{S}(\hat{X})$ is complete. THEOREM 2.2.4., however, insists that σ^* be injective on the separated subspace $\mathcal{E}(\hat{X})$, and in fact we know that $\mathcal{E}(\hat{X})$ is actually isomorphic to a subspace of $\hat{\mathcal{S}}(X)$.

If X is a metric space its Hausdorff completion \hat{X} is also a metric space, and $\mathcal{E}(X)$ is a complete metric space. Therefore, by THEOREM 2.4.4., $\hat{\mathcal{E}}(X)$ is isomorphic to $\mathcal{E}(\hat{X})$ so that the completion of the hyperspace is again a hyperspace. (It is easy to see that the isomorphism is actually an isometry.) The question arises - which spaces have this property? Equivalently, which spaces have the property that the Hausdorff completion of the hyperspace is the hyperspace of its Hausdorff completion? Clearly, by THEOREM 2.4.4., this class

of spaces is precisely the class for which the Hausdorff completion is hypercomplete.

There is another characterization of this class. We say that a uniform space X has a hypercompletion if X can be embedded in a hypercomplete space. It is well-known that there exist complete spaces which are not hypercomplete. That not every space has a hypercompletion then follows from the fact that a complete subspace of a hypercomplete space must be hypercomplete. This is not quite obvious and so we shall establish it precisely. If X is a complete uniform subspace of a hypercomplete uniform space Y then X' is complete and is a closed subspace of Y' . Therefore $\mathcal{E}(X')$ is a closed subspace of $\mathcal{E}(Y')$. Now $\mathcal{E}(Y)$ is complete, and so also, by PROPOSITION 2.4.3., is $\mathcal{E}(Y')$. Thus $\mathcal{E}(X')$ must be complete, and again by PROPOSITION 2.4.3, this implies that X is hypercomplete.

Which spaces, then, have a hypercompletion? Suppose that the uniform space X is embedded in a hypercomplete uniform space Y . Then \hat{X} is a closed subspace of \hat{Y} , which is hypercomplete, and so $\mathcal{E}(\hat{X})$ is a closed subspace of the complete space $\mathcal{E}(\hat{Y})$, and therefore is complete itself. Conversely, if $\mathcal{E}(\hat{X})$ is given to be complete and we let \ddot{X} denote the (non-Hausdorff) completion of X , so that \hat{X} is the Hausdorff space associated with \ddot{X} , then by PROPOSITION 2.4.3. $\mathcal{E}(\ddot{X})$ is complete, and \ddot{X} is a hypercompletion of X . We collect all these facts together in the following result.

THEOREM 2.4.5. The class of uniform spaces X which have a hypercompletion is precisely the class for which the Hausdorff completion of the hyperspace $\mathcal{E}(X)$ is the hyperspace $\mathcal{E}(\hat{X})$ of the Hausdorff completion, which is precisely the class for which the Hausdorff completion is hypercomplete.

It is easily seen that this class includes all metric spaces, and also all hypercomplete spaces, but it is very restricted nonetheless. If we narrow our demands to the compact, closed subsets in place of the closed subsets we might expect the corresponding class of spaces to expand enormously. Thus it turns out that for an arbitrary uniform space X , the Hausdorff completion of the space $\mathcal{C}(X)$ of compact, closed subsets of X is the same thing as the space $\mathcal{C}(\hat{X})$ of compact subsets of \hat{X} . This will appear in the next section.

2.5. The Hausdorff completions of $\mathcal{C}(X)$, $\mathcal{P}(X)$ and $\mathcal{R}(X)$.

It has been shown that for any uniform space X , the space of closed subsets of X is isomorphic to the space of closed subsets of the Hausdorff space X' associated with X . This isomorphism also takes those sets in $\mathcal{E}(X)$ which are compact onto those in $\mathcal{E}(X')$ which are compact.

PROPOSITION 2.5.1. For any uniform space X , the space $\mathcal{C}(X)$ of compact, closed subsets of X is isomorphic to the space $\mathcal{C}(X')$ of compact subsets of X' , the Hausdorff space associated with X .

Proof. The isomorphism $\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X')$ is defined by $\alpha(M) = i(M)$ for each M belonging to $\mathcal{E}(X)$, where $i : X \rightarrow X'$ is the canonical mapping. If C is a compact set in $\mathcal{E}(X)$, then $\alpha(C) = i(C)$ is compact, since i is uniformly continuous.

Now let D be a compact subset of X' , and let $E = \alpha^{-1}(D)$. Then $i(E) = D$. Let \mathcal{U} be an open covering of E . Let \mathcal{B} be an open covering of E such that for each B in \mathcal{B} there is a set A in \mathcal{U} with $\bar{B} \subset A$. This is possible by the regularity of X . Since i is an open mapping the collection $\{i(B) : B \in \mathcal{B}\}$ is an open covering of the subset D of X' , and since D is compact there is a finite subcovering $\{i(B_1), i(B_2), \dots, i(B_n)\}$. Let A_i be such that $\bar{B}_i \subset A_i$, for $i = 1, \dots, n$. If e is a point of E , then $i(e)$ is a point of $i(B_k)$, for some $1 \leq k \leq n$, and there is a point b of B_k such that (e, b) is in every entourage of the uniformity on X . This implies that e is in \bar{B}_k and hence in A_k . Therefore the collection $\{A_1, A_2, \dots, A_n\}$ covers E , and this proves that E is compact.

It now follows immediately that the isomorphism $\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X')$ takes $\mathcal{C}(X)$ onto $\mathcal{C}(X')$.

The Robertsons proved in (28) that if X is a complete, separated uniform space then the set of non-empty compact subsets of X forms a complete subspace of $\mathcal{E}(X)$. This was achieved by embedding the space X in a product of complete metric spaces, and thus making very real use of the separatedness of X . Using the foregoing PROPOSITION there is an immediate slight generalisation of their result - by elimination of the separatedness hypothesis.

THEOREM 2.5.2. For any complete uniform space (not necessarily separated), the set of non-empty, compact, closed subsets forms a complete space.

There is a further generalisation of this result, obtained by considering the Hausdorff completion of the space

of non-empty, compact closed subsets, as promised at the end of the previous section. The following Lemma will be needed in the proof.

LEMMA 2.5.3. Let Y be a uniform space and A a dense subspace of Y . Then the set of compact (resp. precompact) subsets of A forms a dense subspace of the space of compact (resp. precompact) subsets of Y .

Proof. Clearly any compact (resp. precompact) subset of A is a compact (resp. precompact) subset of Y . Let B be a compact (resp. precompact) subset of Y and let V be any entourage of the uniformity on Y . Choose a symmetric entourage V_1 with $\tilde{V}_1 \subset V$, and let b_1, b_2, \dots, b_n be points of B such that the sets $V_1(b_i)$, $i = 1, 2, \dots, n$, cover B .

Now for each i choose a point c_i in A such that c_i is in $V_1(b_i)$, and let $C = \{c_1, c_2, \dots, c_n\}$. Then C is a compact (resp. precompact) subset of A , and we have $C \subset V_1(B) \subset V(B)$, and also $B \subset \bigcup \{V_1(b_i) : i = 1, 2, \dots, n\} \subset \bigcup \{\tilde{V}_1(c_i) : i = 1, 2, \dots, n\} = \tilde{V}_1(C) \subset V(C)$. We have shown that C is in $\tilde{V}(B)$, and the result follows.

THEOREM 2.5.4. For any uniform space X , the Hausdorff completion of the space of non-empty, compact, closed subsets of X is isomorphic to the space of non-empty, compact subsets of its Hausdorff completion \hat{X} .

Proof. The space $\mathcal{C}(X)$, being Hausdorff, is dense in $\hat{\mathcal{C}}(X)$, and since X' is dense in \hat{X} it follows by LEMMA 2.5.3. that $\mathcal{C}(X')$ is dense in $\mathcal{C}(\hat{X})$. By PROPOSITION 2.5.1. there is an isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(X')$, and both $\hat{\mathcal{C}}(X)$ and $\mathcal{C}(\hat{X})$ are complete Hausdorff spaces, using THEOREM 2.5.2. for the latter. Therefore the isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(X')$ can be extended by THEOREM 2.2.2. to an isomorphism between $\hat{\mathcal{C}}(X)$ and $\mathcal{C}(\hat{X})$.

For the closing Theorem of this section we allow a few more of the common subspaces of $\mathcal{S}(X)$ to enter the picture.

A summary of notation follows.

$\mathcal{C}(X)$: the set of non-empty, compact, closed subsets of X ;

$\mathcal{C}_0(X)$: the set of non-empty, compact subsets of X ;

$\mathcal{P}(X)$: the set of non-empty, precompact, closed subsets of X ;

$\mathcal{P}_0(X)$: the set of non-empty, precompact subsets of X ;

$\mathcal{R}(X)$: the set of non-empty, relatively compact subsets of X ;

$\mathcal{F}(X)$: the set of non-empty, finite subsets of X .

First we prove a Proposition showing that the isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(X')$ not only takes the compact sets onto the compact sets, but also takes the precompact sets onto the precompact sets. Using this fact we can then give the relations between the Hausdorff completions and associated Hausdorff spaces for the various hyperspaces listed above.

PROPOSITION 2.5.5. For any uniform space X , the spaces $\mathcal{P}(X)$ and $\mathcal{P}(X')$ are each isomorphic to the subspace of $\mathcal{C}(\hat{X})$ consisting of those sets which are the closures of their intersections with X' .

Proof. Firstly, $\mathcal{P}(X)$ and $\mathcal{P}(X')$ are isomorphic. We already have the isomorphism $\alpha : \mathcal{C}(X) \rightarrow \mathcal{C}(X')$ defined by $\alpha(M) = i(M)$ for each M in $\mathcal{C}(X)$, where $i : X \rightarrow X'$ is the canonical mapping. If P belongs to $\mathcal{P}(X)$ then $\alpha(P) = i(P)$ is in $\mathcal{P}(X')$ since i is uniformly continuous. Now let Q belong to $\mathcal{P}(X')$ and let U be any entourage of X . Then $(i \times i)(U)$ is an entourage of X' and there is a finite covering $\{B_k\}$ for Q , consisting of $(i \times i)(U)$ -small sets. But then $\{i^{-1}(B_k)\}$ is a finite covering for $i^{-1}(Q)$, consisting of $\overset{3}{U}$ -small sets, and thus $i^{-1}(Q)$ is precompact.

Since $\alpha^{-1}(Q) = i^{-1}(Q)$, this proves that α takes $\mathcal{P}(X)$ onto $\mathcal{P}(X')$.

Next, $\mathcal{P}(X')$ is isomorphic to the subspace of $\mathcal{C}(\hat{X})$ described. To show this we use the diagram in 2.4 analysing the isomorphism between $\hat{\mathcal{C}}(X)$ and $\hat{\mathcal{C}}(\hat{X})$. It is clear, from the diagram, that in the construction of the isomorphism each set P belonging to $\mathcal{P}(X')$ is taken onto its closure in \hat{X} , which is the Hausdorff completion of P and so must be compact. Conversely, any compact subset C of \hat{X} which is the closure in \hat{X} of its intersection with X' must be the image, in the aforementioned sense, of this intersection, which is a precompact, closed subset of X' .

THEOREM 2.5.6. For any uniform space X the following isomorphisms are true:

$$\begin{aligned} (a) \quad \mathcal{R}'(X) &\cong \mathcal{C}(X), \quad \mathcal{C}'_0(X) \cong \mathcal{C}(X), \quad \mathcal{P}'_0(X) \cong \mathcal{P}(X), \\ (b) \quad \mathcal{F}(X) &\cong \mathcal{F}(\hat{X}) \cong \mathcal{C}(\hat{X}) \cong \hat{\mathcal{C}}(X) \cong \hat{\mathcal{P}}(X) \cong \hat{\mathcal{R}}(X) \cong \\ &\hat{\mathcal{C}}_0(X) \cong \hat{\mathcal{P}}_0(X) \cong \mathcal{P}'_0(\hat{X}) \cong \mathcal{R}'(\hat{X}). \end{aligned}$$

Proof. (a) This is immediate on observing that, when $\mathcal{G}'(X)$ and $\mathcal{E}(X)$ are identified using PROPOSITION 2.4.1., the canonical mapping $S(X) \rightarrow S'(X)$ takes each of $\mathcal{C}_0(X)$ and $\mathcal{R}(X)$ onto $\mathcal{C}(X)$, and takes $\mathcal{P}_0(X)$ onto $\mathcal{P}(X)$.

(b) Since the Hausdorff completion of any space is isomorphic to that of its associated Hausdorff space, part (a) yields the following isomorphisms:

$$\begin{aligned} \hat{\mathcal{C}}(X) &\cong \hat{\mathcal{R}}(X) \cong \hat{\mathcal{C}}_0(X), \quad \hat{\mathcal{P}}(X) \cong \hat{\mathcal{P}}_0(X), \\ \hat{\mathcal{C}}(\hat{X}) &\cong \hat{\mathcal{R}}(\hat{X}) \cong \hat{\mathcal{C}}_0(\hat{X}), \quad \hat{\mathcal{P}}(\hat{X}) \cong \hat{\mathcal{P}}_0(\hat{X}). \end{aligned}$$

Also by replacing X by \hat{X} in part (a), we have the isomorphisms:

$$\mathcal{C}(\hat{X}) \cong \mathcal{R}'(\hat{X}) \cong \mathcal{C}'_0(\hat{X}), \quad \mathcal{P}(\hat{X}) \cong \mathcal{P}'_0(\hat{X}).$$

Since $\mathcal{F}(X)$ is dense in $\mathcal{P}_0(X)$, they have the same Hausdorff completion, up to isomorphism, and similarly for $\mathcal{F}(\hat{X})$ and $\mathcal{P}_0(\hat{X})$. Since every precompact, closed subspace of a complete Hausdorff space must be compact, $\mathcal{P}(\hat{X})$ and $\mathcal{C}(\hat{X})$ are the same, and since every compact subset of a Hausdorff space must be closed, $\mathcal{C}(\hat{X})$ and $\mathcal{C}'_0(\hat{X})$ are the same. We also know that $\mathcal{C}(\hat{X})$ and $\hat{\mathcal{C}}(X)$ are isomorphic, by THEOREM 2.5.4. Consequently the situation simplifies, leaving two classes of isomorphic spaces:

$$\mathcal{F}(\hat{X}) \cong \mathcal{C}(\hat{X}) \cong \mathcal{R}(X) \cong \hat{\mathcal{C}}_0(X) \cong \mathcal{C}(\hat{X}) \cong \mathcal{R}'(\hat{X}) \cong \mathcal{P}'_0(\hat{X}),$$

$$\mathcal{F}(X) \cong \hat{\mathcal{P}}(X) \cong \hat{\mathcal{P}}_0(X).$$

To conclude the proof it is sufficient to show that $\hat{\mathcal{P}}(X)$ is isomorphic to $\mathcal{C}(\hat{X})$. Now, by LEMMA 2.5.3., $\mathcal{P}_0(X')$ is dense in $\mathcal{P}_0(\hat{X})$, and so these spaces have isomorphic Hausdorff completions; but then so also do their associated Hausdorff spaces which, by part (a), are $\mathcal{P}(X')$ and $\mathcal{P}(\hat{X})$ respectively. By PROPOSITION 2.5.5., $\mathcal{P}(X)$ is isomorphic to $\mathcal{P}(X')$, and so, finally, $\hat{\mathcal{P}}(X) \cong \hat{\mathcal{P}}(X') \cong \hat{\mathcal{P}}(\hat{X}) \cong \mathcal{C}(\hat{X})$, since $\mathcal{P}(\hat{X})$ and $\mathcal{C}(\hat{X})$ are the same Hausdorff complete space.

COROLLARY 2.5.7. For any complete uniform space the following hyperspaces are complete:

- (a) the space of non-empty compact (compact and closed) subsets;

- (b) the space of non-empty precompact subsets;
- (c) the space of non-empty relatively compact subsets.

Remark. An investigation relating to the results in this section was made by V. M. Ivanov in (14), who showed that the space of closed subsets of the Wallman compactification of a topological space X can be realized as the Wallman compactification of the space of closed subsets of X , using the finite topology. See also Flachsmeyer and Poppe (8).

2.6 Some applications to topological vector spaces.

A topological vector space X has a natural uniform structure as described in Section 1.10. Thus we may form the associated Hausdorff space X' , and the Hausdorff completion \hat{X} . These uniform spaces are, in fact, also topological vector spaces, for the linear structure on X is carried over directly to X' by the canonical mapping $i : X \rightarrow X'$, and the linear structure on \hat{X} is obtained by extending the operational mappings $(x, y) \rightarrow x + y$ and $(\lambda, x) \rightarrow \lambda x$ for X' by continuity, using THEOREM 2.2.1., to operational mappings for \hat{X} .

Using results of the previous two sections there is a uniform isomorphism between the hyperspaces $\mathcal{C}(X)$ and $\mathcal{C}(X')$, taking the compact sets onto the compact sets and the precompact sets onto the precompact sets. The next Proposition extends this "preservation" to all the common types of subsets deriving from the presence of the linear structure. The proof is straightforward and is omitted.

PROPOSITION 2.6.1. If X is a topological vector space and X' is its associated Hausdorff topological vector space, then the uniform isomorphism between $\mathcal{E}(X)$ and $\mathcal{E}(X')$ takes the convex (resp. balanced, bounded, absorbent) sets onto the convex (resp. balanced, bounded, absorbent) sets.

THEOREM 2.6.2. Let X be a locally convex topological vector space and let $\mathcal{K}(X)$ denote the collection of non-empty, absolutely convex closed subsets of X . Then $\mathcal{K}(X)$ is uniformly isomorphic to $\mathcal{K}(\hat{X})$, and so, if X has a hypercompletion in the sense of Kelley, to $\mathcal{K}(\hat{X})$ itself.

Proof. Clearly any absolutely convex subset of X' is also an absolutely convex subset of \hat{X} . Let $\mathcal{K}_0(X)$ denote the collection of all non-empty absolutely convex subsets of X . We show that $\mathcal{K}_0(X')$ is a dense subspace of $\mathcal{K}_0(\hat{X})$. For this purpose, let K be any set in the latter collection and let V be an absolutely convex 0-neighbourhood in \hat{X} . Then $K + V$ is absolutely convex and so also is the set $J = (K + V) \cap X'$. It is easy to see that (J, K) belongs to the entourage \bar{V} and so $\mathcal{K}_0(X')$ is dense in $\mathcal{K}_0(\hat{X})$, and they must have isomorphic Hausdorff completions.

Now, as in THEOREM 2.5.6, we can show that $\mathcal{K}(X')$ and $\mathcal{K}(\hat{X})$ are the respective Hausdorff spaces associated with $\mathcal{K}_0(X')$ and $\mathcal{K}_0(\hat{X})$, because the closure of any absolutely convex subset is absolutely convex. Also, by PROPOSITION 2.6.1, the spaces $\mathcal{K}(X)$ and $\mathcal{K}(X')$ are isomorphic. It follows that $\mathcal{K}(X)$ and $\mathcal{K}(\hat{X})$ have isomorphic Hausdorff completions.

For a normed space it is possible to prove a similar result for the closed, bounded subsets, using the method of the proof above and the fact that there is a base of bounded 0-neighbourhoods.

THEOREM 2.6.3. Let X be a normed vector space and let $\mathcal{B}(X)$ denote the collection of non-empty, closed, bounded subsets of X . Then the spaces $\hat{\mathcal{B}}(X)$ and $\mathcal{B}(\hat{X})$ are isometric.

We turn now to consideration of the Hausdorff completions of quotient spaces. Let M be a vector subspace of a topological vector space X , and form its associated Hausdorff space M' , and its Hausdorff completion \hat{M} . The space M' can be identified with the subspace $i(M)$ of X' , and the space \hat{M} can be identified with the closure of $i(M)$ in \hat{X} . Next, form the quotient spaces X/M , X/\bar{M} , X'/\bar{M}' and \hat{X}/\hat{M} , which are topological vector spaces, and uniform subspaces of $\mathcal{S}(X)$, $\mathcal{E}(X)$, $\mathcal{E}(X')$ and $\mathcal{E}(\hat{X})$ respectively (see section 1.11). Furthermore, the associated Hausdorff space $(X/M)'$ is embedded as a uniform subspace in $\mathcal{S}'(X)$, and the Hausdorff completions (X/M) and (\hat{X}/\hat{M}) are embedded as closed uniform subspaces in $\mathcal{E}(X)$ and $\mathcal{E}(\hat{X})$ respectively.

PROPOSITION 2.6.4. If M is a vector subspace of a topological vector space X , then the isomorphisms of section 2.4 between $\mathcal{S}'(X)$, $\mathcal{E}(X)$ and $\mathcal{E}(X')$ induce linear isomorphisms between $(X/M)'$, X/\bar{M} and X'/\bar{M}' .

Proof. If $(x + M)$ denotes a member of $(X/M)'$ with representative member $x + M$ of X/M , then, under the successive isomorphisms $\mathcal{S}'(X) \rightarrow \mathcal{E}(X) \rightarrow \mathcal{E}(X')$, we have

$$\begin{aligned} (x + M) &\rightarrow \overline{x + M} = x + \bar{M} \text{ (closures in } X) \\ x + \bar{M} &\rightarrow i(x + \bar{M}) = i(x) + i(\bar{M}) = i(x) + \overline{i(M)} \text{ (closure in } X') \\ &= i(x) + M'. \end{aligned}$$

Thus it is clear that $(X/M)'$ is taken onto X/\bar{M} , which, again, is taken onto X'/\bar{M}' . It is routine to verify that these mappings are linear.

THEOREM 2.6.5. If M is a vector subspace of a topological vector space X , then the isomorphism shown in section 2.4 between $\hat{\mathcal{S}}(X)$ and $\hat{\mathcal{E}}(\hat{X})$ takes (X/M) onto (\hat{X}/\hat{M}) and induces a linear isomorphism between them.

Proof. Define a mapping $f : X/\bar{M} \rightarrow \hat{X}/\hat{M}$ by $f(x + \bar{M}) = i(x) + \hat{M}$. This coincides with the restriction to X/\bar{M} of the mapping $p \circ \alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(\hat{X})$ described in the diagram in section 2.4., because

$$x + \bar{M} \xrightarrow{\alpha} i(x) + \overline{i(M)}^{X'} \xrightarrow{p} i(x) + \overline{i(M)}^{\hat{X}} = i(x) + \hat{M},$$

and therefore, by the discussion in section 2.4., f is a uniform isomorphism of X/\bar{M} onto the uniform subspace $L = \{i(x) + \hat{M} : x \in X\}$ of \hat{X}/\hat{M} . It is easy to see that, in fact, f is a linear isomorphism, and L is a vector subspace of \hat{X}/\hat{M} .

We next show that L is dense in \hat{X}/\hat{M} . Let \mathcal{U} be a 0-neighbourhood base for \hat{X} , and $Q : \hat{X} \rightarrow \hat{X}/\hat{M}$ the quotient map. Then (section 1.11) the collection of sets

$$\tilde{U} = \{(a + \hat{M}, b + \hat{M}) : b - a + \hat{M} \in Q(U)\},$$

where U runs through \mathcal{U} , forms a base for the uniformity on \hat{X}/\hat{M} . Let $a + \hat{M}$ be a point in \hat{X}/\hat{M} and let U belong to \mathcal{U} . Since X' is dense in \hat{X} , there is a point x of X such that $i(x)$ belongs to $a + U$, and then $i(x) - a + \hat{M}$ belongs to $Q(U)$. Therefore $(a + \hat{M}, i(x) + \hat{M})$ belongs to \tilde{U} , and since $i(x) + \hat{M}$ is a point of L , L must be dense in \hat{X}/\hat{M} .

We have, also, that \hat{X}/\hat{M} is dense in (\hat{X}/\hat{M}) , and so L is dense in (\hat{X}/\hat{M}) . By PROPOSITION 2.6.2., the space $(X/M)'$, which is dense in (\hat{X}/\hat{M}) , is linearly isomorphic to X/\bar{M} , and since we have shown that X/\bar{M} is linearly isomorphic to L , $(X/M)'$ must be linearly isomorphic to L . Finally, using THEOREM 2.2.2., this isomorphism can be extended to a uniform isomorphism of (\hat{X}/\hat{M}) onto (\hat{X}/\hat{M}) , and the extension is clearly also linear, and by its construction must coincide with the restriction to (\hat{X}/\hat{M}) of the isomorphism $\hat{\mathcal{S}}(X) \rightarrow \hat{\mathcal{E}}(\hat{X})$.

COROLLARY 2.6.6. The quotient space X/M can be embedded as a dense vector subspace in the quotient space \hat{X}/\hat{M} .

Remarks. The quotient space of a complete metrizable topological vector space by a closed vector subspace is known to be complete. Hence, if X is a metrizable topological vector space and M any vector subspace of X , the quotient space \hat{X}/\hat{M} must be complete, and so by THEOREM 2.6.3., (\hat{X}/\hat{M}) is isomorphic to \hat{X}/\hat{M} . That is, the Hausdorff completion of any quotient space of X is a quotient space of the Hausdorff completion of X . Which spaces, besides metrizable ones, have this property?

A topological vector space X will certainly be a successful candidate if all quotient spaces of its Hausdorff completion \hat{X} are complete, and in particular if \hat{X} is hypercomplete. For if $\hat{\mathcal{E}}(\hat{X})$ is complete and M is a vector subspace of X , then \hat{X}/\hat{M} is a closed uniform subspace of $\hat{\mathcal{E}}(\hat{X})$ (see Chapter 1), and must itself be complete. The class of spaces under consideration therefore

contains the class corresponding to THEOREM 2.4.5. - those with hypercompletions. Whether the two classes are the same, or how much larger the one is than the other, remain open questions.

An example of a complete locally convex topological vector space X possessing a closed vector subspace M such that X/M is not complete is given in Kelley and Namioka (18), 20D. It is interesting that while completeness is inherited by products but not by separated quotients, hypercompleteness is inherited by separated quotients but not by products.

2.7. Hyperspaces arising from two related uniformities for a set. For the remainder of this Chapter we shall be considering a set X endowed with two uniformities ξ and η which are related to each other in a particular way. Following the Robertsons in (28), we shall say that the uniformity η *is associated with the uniformity ξ* if ξ has a base consisting of subsets of $X \times X$ closed in the topology determined by η (briefly, η -closed). The motivation behind the study, here, of associated uniformities lies in an attempt to eliminate the hypothesis of separatedness in the second Theorem of (28), which says that if (X, ξ) is a complete separated uniform space and η is a coarser separated uniformity on X , associated with ξ , then the set of non-empty η -compact subsets of X is complete under the uniformity $\tilde{\xi}$.

The concept of associated uniformities probably arises most naturally in the search for a sufficient condition

for the completeness of a uniform space under the coarser of two uniformities to imply its completeness under the finer - all that is required is for the coarser uniformity to be associated with the finer. Bourbaki gives this result for topological vector spaces in his treatise on topological vector spaces, Chapter 1, section 1, Proposition 8. We shall explore this and related ideas in connection with the completeness and hypercompleteness of uniform spaces in Chapter 3.

When X is a topological vector space under each of two topologies ξ and η , we say that η is associated with ξ if the uniformity determined by η is associated with that determined by ξ , and this is true if and only if there is a base of η -closed ξ -neighbourhoods of the origin. The most obvious example is when η is taken to be the weak topology corresponding to a locally convex topology ξ . Dr. Wendy Robertson studied this association between topologies for a topological vector space in (29), under the name "closed neighbourhood condition". The following facts give some indication of the significance of the notion.

If η is associated with ξ then the filter condition holds for the identity mapping of (X, ξ) into (X, η) (see Chapter 3). If η is a convex topology, ξ is a weak topology, and η is associated with ξ , then ξ is coarser than η . If η is a convex topology then (X, η) is barrelled if and only if the only convex topologies with which η is associated are those coarser than η , and if η is any vector topology for X then (X, η) is ultra-barrelled if and only if the only topologies which are compatible with the

algebraic structure of X , and with which η is associated, are those coarser than η .

Let X be a set endowed with two uniformities ξ and η , and let η be associated with ξ . Thus let \mathcal{U} and \mathcal{V} be bases for ξ and η respectively, with the sets of \mathcal{U} \mathcal{V} -closed in $X \times X$. Form the Hausdorff spaces (X_1, ξ') and (X_2, η') associated with (X, ξ) and (X, η) respectively, and let i and j be the respective canonical mappings. We proceed to construct another uniformity for X_2 .

Consider the collection \mathcal{W} of sets of the form $(j \times j)(U)$, for U in \mathcal{U} . We show that this is a base for a uniformity on X_2 , by verifying the four requirements below.

(1) For any two members of \mathcal{W} there is a third member contained in their intersection; because if $U \subset U_1 \cap U_2$ then $j^2(U) \subset j^2(U_1) \cap j^2(U_2)$.

(2) Each member of \mathcal{W} contains the diagonal; this follows immediately from the fact that each U in \mathcal{U} contains the diagonal in $X \times X$.

(3) For each member of \mathcal{W} there is another member contained in its inverse; because if $U_2 \subset \overline{U_1}^1$ then $j^2(U_2) \subset j^2(\overline{U_1}^1) = (j^2(U_1))$.

(4) For each member of \mathcal{W} there is another member whose square is contained in the first. This is not so obvious as the previous three requirements, and, in fact, is the only one to make use of the association of η with ξ . That it is necessary to make use of it is why, in general, the image of a uniformity under a mapping from product space onto product space is not a uniformity. To prove the fourth requirement, let $W = j^2(U)$, and choose U_1 in

such that $U_1 \overset{3}{\subset} U$. Let $W_1 = j^2(U_1)$. Then if (p, q) belongs to $W_1 \overset{2}{\subset} W$ there is a point r in X_2 such that (p, r) and (r, q) are both in W_1 , so that there are four points a, b, c and d in X with $r = j(b) = j(c)$, $p = j(a)$, $q = j(d)$, and both of (a, b) and (c, d) belonging to U_1 . Since $j(b) = j(c)$, we have, for each V in \mathcal{V} , that (b, c) belongs to VU_1V , and hence also belongs to U_1 , because U_1 is \mathcal{V} -closed in $X \times X$. Thus (a, d) belongs to $U_1 \overset{3}{\subset} U$, and hence to U , and so (p, q) belongs to $j^2(U) = W$. This proves that $W_1 \overset{2}{\subset} W$.

The requirements (1)-(4) being satisfied, \mathcal{W} is the base of a uniformity for X_2 , and this uniformity is clearly independent of the particular choice of base \mathcal{U} for ξ ; we denote it by ω . We know that each U in \mathcal{U} is η -closed - it is also true that $j^2(U)$ is η' -closed. For if $(j(x_1), j(x_2))$ is a η' -limit point of $j^2(U)$, so that for each symmetric V in \mathcal{V} there is (u_1, u_2) belonging to U such that $(j(x_1), j(u_1))$ and $(j(x_2), j(u_2))$ both belong to $j^2(V)$, then (x_1, x_2) belongs to $V \overset{3}{U} V \overset{3}{U} V$, and since U is η -closed (x_1, x_2) belongs to U . Thus ω has a base of sets η' -closed in $X_2 \times X_2$, that is, η' is associated with ω .

PROPOSITION 2.7.1. The Hausdorff space (X'_2, ω') associated with (X_2, ω) is isomorphic to the Hausdorff space (X_1, ξ') associated with (X, ξ) .

Proof. Let $k : X_2 \rightarrow X'_2$ be the canonical mapping, and define a mapping f of X'_2 into X_1 as follows. Given z belonging to X'_2 , choose a point y belonging to X_2 such that $k(y) = z$, and then choose a point x belonging to X such that $j(x) = y$. Put $f(z) = i(x)$.

To show that this is well-defined, let x_1, x_2 be points in X with $kj(x_1) = kj(x_2)$, and let U be any member of \mathcal{U} . Since $kj(x_1) = kj(x_2)$, $(j(x_1), j(x_2))$ belongs to each entourage of ω , and in particular, belongs to $(j \times j)(U)$. Thus, for any V in \mathcal{V} , (x_1, x_2) belongs to $V \cup V$, and since U is \mathcal{V} -closed, (x_1, x_2) belongs to U . Hence $i(x_1) = i(x_2)$, and $fkj(x_1) = fkj(x_2)$.

To show that f is injective, let $f(z_1) = f(z_2)$, so that $i(x_1) = i(x_2)$. For each U in \mathcal{U} , (x_1, x_2) belongs to U , and so $(j(x_1), j(x_2))$ belongs to $(j \times j)(U)$. Hence $kj(x_1) = kj(x_2)$, which means $z_1 = z_2$.

To show that f is surjective, let p be any point belonging to X_1 . We can choose a point x in X such that $i(x) = p$, and then $kj(x)$ belongs to X'_2 and $f(kj(x)) = p$.

Now, since the entourages of ω' are the images under $k \times k$ of the entourages of ω , which are themselves the images under $j \times j$ of the entourages of ξ , and ξ is the inverse image under i of the uniformity ξ' , we conclude that ω' is the inverse image under f of ξ' , and hence f is an isomorphism of (X'_2, ω') onto (X_1, ξ') .

COROLLARY 2.7.2. If (X, ξ) is complete, then so also is (X_2, ω) .

As the notation is becoming rather complicated, the situation is represented in the following diagram:

$$\begin{array}{ccc} (X, \xi) & \xrightarrow{i} & (X_1, \xi') \\ \left\{ \begin{array}{l} (X, \eta) \end{array} \right. & \xrightarrow{j} & \left\{ \begin{array}{l} (X_2, \eta') \\ (X_2, \omega) \end{array} \right. \end{array} \quad \begin{array}{c} \nearrow \\ \xrightarrow{k} \end{array} \quad \begin{array}{c} (X'_2, \omega') \end{array}$$

PROPOSITION 2.7.3.

(a) The following four spaces are isomorphic:

$$(\mathcal{E}(X_2, \omega), \tilde{\omega}), (\mathcal{E}(X_2', \omega'), \tilde{\omega}'), (\mathcal{E}(X_1, \xi'), \tilde{\xi}'),$$

$$(\mathcal{E}(X, \xi), \tilde{\xi}).$$

(b) The following four spaces are isomorphic:

$$(\mathcal{C}(X_2, \omega), \tilde{\omega}), (\mathcal{C}(X_2', \omega'), \tilde{\omega}'), (\mathcal{C}(X_1, \xi'), \tilde{\xi}'),$$

$$(\mathcal{C}(X, \xi), \tilde{\xi}).$$

(c) The following four spaces are isomorphic:

$$(\mathcal{P}(X_2, \omega), \tilde{\omega}), (\mathcal{P}(X_2', \omega'), \tilde{\omega}'), (\mathcal{P}(X_1, \xi'), \tilde{\xi}'),$$

$$(\mathcal{P}(X, \xi), \tilde{\xi}).$$

Proof. The first and last pairs of spaces are isomorphic by PROPOSITION 2.4.3. for (a), PROPOSITION 2.5.1. for (b) and PROPOSITION 2.5.5. for (c). The middle pairs are isomorphic in (a), (b) and (c) by PROPOSITION 2.7.1.

COROLLARY 2.7.4. If (X, ξ) is complete then so also are the eight spaces in (b) and (c), by the results of section 2.5. If (X, ξ) is hypercomplete then so is (X_2, ω) .

PROPOSITION 2.7.5.

(a) $(\mathcal{E}(X, \eta), \tilde{\eta})$ is isomorphic to $(\mathcal{E}(X_2, \eta'), \tilde{\eta})$.

(b) $(\mathcal{C}(X, \eta), \tilde{\eta})$ is isomorphic to $(\mathcal{C}(X_2, \eta'), \tilde{\eta})$.

(c) $(\mathcal{P}(X, \eta), \tilde{\eta})$ is isomorphic to $(\mathcal{P}(X_2, \eta'), \tilde{\eta})$.

Proof. Define a mapping f of $\mathcal{E}(X, \eta)$ into $\mathcal{E}(X_2, \eta')$ by $f(M) = j(M)$ for each M in $\mathcal{E}(X, \eta)$. This is bijective by PROPOSITION 2.4.3., and takes $\mathcal{C}(X, \eta)$ onto $\mathcal{C}(X_2, \eta')$ and $\mathcal{P}(X, \eta)$ onto $\mathcal{P}(X_2, \eta')$ by PROPOSITIONS 2.5.1. and 2.5.5. respectively.

Now the entourages of ω are precisely the images under $j \times j$ of those of ξ , and it is also true that the

entourages of $\tilde{\omega}$ are the images under $f \times f$ of those of $\tilde{\xi}$. In fact, for each U in \mathcal{U} , $f^2(\tilde{U}) = j^2(\tilde{U})$. For if (A, B) is in \tilde{U} then $f(A) = j(A) \subset j(U(B)) = (j^2(U))(j(B))$, and similarly $f(B) \subset (j^2(U))(j(A))$, so that $(f(A), f(B))$ belongs to $j^2(\tilde{U})$. If, conversely, $(f(A), f(B))$ is in $j^2(\tilde{U})$, then $j(A) \subset (j^2(U))(j(B))$ so that for each a in A there is a b in B such that (a, b) belongs to $V \cup V$ for each V in \mathcal{V} . Thus, since U is \mathcal{V} -closed, (a, b) belongs to U , and hence $A \subset U(B)$. Similarly $B \subset U(A)$, so that (A, B) is in \tilde{U} . That f is an isomorphism of $(\mathcal{E}(X, \eta), \tilde{\xi})$ onto $(\mathcal{E}(X_2, \eta'), \tilde{\omega})$ follows immediately.

Thus far in this section we have required only that the uniformity η be associated with the uniformity ξ . At this point we demand more - from now on η will also be coarser than ξ . Since the entourages of the uniformities η' and ω are the images under $j \times j$ of the entourages of η and ξ , respectively, η' must be coarser than ω . Therefore ω is a separated uniformity, and by PROPOSITION 2.7.1., (X_2, ω) is the Hausdorff space associated with (X, ξ) . A summary of the situation is given in the next result.

PROPOSITION 2.7.6. If η, ξ are two uniformities for a set X , with η associated with ξ , and j is the canonical mapping of (X, η) into its associated Hausdorff space (X_2, η') , then the collection of subsets of $X_2 \times X_2$ of the form $(j \times j)(U)$, where U runs through a base for the uniformity ξ , is itself the base of a uniformity ω for X_2 , and η' is associated with ω . If, in addition, η is

coarser than ξ then η' is coarser than ω and (X_2, ω) is the Hausdorff space associated with (X, ξ) .

As mentioned at the beginning of this section Theorem 2 of the Robertsons' paper(28) states that if (X, ξ) is a complete, separated uniform space, and η is a coarser separated uniformity on X , associated with ξ , then the set of non-empty η -compact subsets of X is complete under the uniformity $\tilde{\xi}$. In proving this X is regarded as embedded in its η -Hausdorff completion \hat{X} , ξ and η are extended to \hat{X} , and use is made of Theorem 1 of the same paper, thus involving the separatedness hypotheses quite considerably. These hypotheses can now be eliminated.

THEOREM 2.7.7. Let (X, ξ) be a complete uniform space (not necessarily separated) and let η be a coarser uniformity on X , associated with ξ . Then the set of non-empty, η -compact, η -closed subsets of X is complete under the uniformity $\tilde{\xi}$.

Proof. Let (X_2, η') be the Hausdorff space associated with (X, η) , and construct the uniformity ω for X_2 as in PROPOSITION 2.7.6. By this PROPOSITION η' is coarser than ω and associated with ω , and clearly both uniformities are separated. Also by PROPOSITION 2.7.6. (X_2, ω) is the Hausdorff space associated with (X, ξ) and so is complete. Theorem 2 of (28) then implies that $\mathcal{C}(X_2, \eta')$ is complete under the uniformity $\tilde{\omega}$. But by PROPOSITION 2.7.5. (b) this space is isomorphic to the space $(\mathcal{C}(X, \eta), \tilde{\xi})$, which is therefore also complete. This was what we were trying to prove.

Remarks. It has already been observed in COROLLARY 2.5.7. that if (X, ξ) is complete then the set of non-empty ξ -relatively compact subsets of X is complete under $\tilde{\xi}$. Concerning the η -relatively compact subsets of X , the Robertsons give a Corollary to Theorem 2 of (28) which states that, under the same hypotheses as Theorem 2, the η -relatively compact, ξ -closed subsets form a closed subspace of $(\mathcal{K}(X, \xi), \tilde{\xi})$. Here again the separatedness is not necessary and the proof in (28) goes through almost unchanged, using THEOREM 2.7.7. instead of Theorem 2 of (28). That the set of η -relatively compact, ξ -closed subsets of X is not complete under $\tilde{\xi}$ is demonstrated by a counterexample in (28). The η -relatively compact, η -closed subsets are of course, just the η -compact, η -closed subsets.

There is a standard procedure for deducing results about the completeness of function spaces from results, like that mentioned in the previous paragraph, asserting closedness of a subspace of $\mathcal{C}(X)$. More will be said about this in section 6.5. Thus the Robertsons apply the Corollary of Theorem 2 in (28) to deduce their third theorem. Advancing further on the anti-separatedness crusade, this theorem can be released from separatedness hypotheses, together with its analogue for locally convex spaces. It then becomes the following.

Let S be a set and \mathcal{A} a family of subsets of S . Also let (X, ξ) be a complete uniform space and let η be a coarser uniformity on X associated with ξ . If F is the set of mappings from S into X which take the sets of \mathcal{A}

onto n -relatively compact subsets of X , then F is complete under the uniformity of ξ -uniform convergence on the sets of \mathcal{A} . For locally convex spaces this says that the space of linear mappings from a locally convex space into a complete locally convex space which take bounded sets onto weakly relatively compact subsets is complete under the topology of uniform convergence on the bounded subsets. Note that the corresponding analogue for THEOREM 2.7.7. says that the weakly compact and weakly closed subsets of a complete locally convex space form a complete space.

FUNDAMENTAL FAMILIES AND COMPLETIONS OF HYPERSPACES

3.1 Introduction. This chapter looks at completeness in hyperspaces from a rather different angle from that of the last chapter, but the results are intimately related. The notion of a fundamental family of subsets was introduced by J. L. Kelley (16) for families of nonempty, absolutely convex subsets of a locally convex space F . He showed that the collection of all such subsets of F is complete under the Hausdorff uniformity if and only if every fundamental family of them converges on a certain sense. These ideas are applied in section 2 to the collection $\mathcal{S}(X)$ of all nonempty subsets of a uniform space X , and it is shown that the space $\mathcal{S}(X)$ is complete if and only if every fundamental family on X converges. On an arbitrary uniform space every fundamental family of compact subsets converges.

The properties of fundamental families are investigated and shown to be closely related to those of Cauchy filter bases. This leads on to the construction by means of fundamental families of a uniform space $\Phi(X)$ which is uniformly isomorphic to $\mathcal{E}(X)$, the Hausdorff completion of the space of nonempty closed subsets of X . The spaces $\mathcal{E}(X)$, $\mathcal{E}(\hat{X})$, $\mathcal{C}(X)$, $\mathcal{C}(\hat{X})$ and $\hat{\mathcal{C}}(X)$ are each identified with natural subspaces of $\Phi(X)$, corresponding to particular kinds of fundamental families, and the latter two shown to be isomorphic by a method independent of that in chapter 2. When X is a locally convex topological vector space, then the space $\mathcal{K}(X)$ of nonempty, closed, absolutely convex subsets of X , its Hausdorff completion $\hat{\mathcal{K}}(X)$, and the space $\mathcal{L}(X)$ of closed vector subspaces of X are each likewise identified with subspaces of $\Phi(X)$, corresponding to particular kinds of fundamental families. As a bonus, the construction of $\Phi(X)$ allows an embedding of $\mathcal{E}(X)$ as a closed uniform subspace of

$\mathcal{E}(\mathcal{E}(X))$, bringing the range of spaces embeddable in the hyper-hyperspace to include the following (assuming X is separated) : $X, \hat{X}, X \times X, \mathcal{E}(X), \mathcal{E}(\hat{X})$ and $\hat{\mathcal{E}}(X)$.

In section 4 a study is made of two conditions on a mapping between uniform spaces - the filter condition, introduced by Dr. Wendy Robertson (29), and the analogous fundamental family condition. The latter is strictly stronger, and bears a relationship to hypercompleteness similar to that of the former to completeness. The fundamental family condition holds for a mapping if and only if the filter condition holds for the induced mapping between hyperspaces. Dr. Wendy Robertson showed that when $t : X \rightarrow Y$ is an injective, continuous linear mapping between topological vector spaces, the filter condition is necessary and sufficient for the extension $\hat{t} : \hat{X} \rightarrow \hat{Y}$ to be injective. For uniform spaces, sufficiency breaks down, and the only conditions I have been able to find which are sufficient achieve their ends in rather violent fashion. I do not know whether the fundamental family condition is sufficient.

Equivalent conditions for each of the filter condition and the fundamental family condition are given, in terms of the extension mappings $\hat{t} : \hat{X} \rightarrow \hat{Y}$ and $\hat{t}' : \hat{\mathcal{E}}(X) \rightarrow \hat{\mathcal{E}}(Y)$, respectively, when X and Y are uniform spaces. Applying these results to the case when $t : X \rightarrow Y$ is a continuous linear mapping between topological vector spaces, and M is a closed vector subspace of X , shows that, if the fundamental family condition holds and if t induces an injective mapping on the quotient space X/M , then its extension to the completion $\hat{X/M}$ is also injective. This motivates a general study in section 5 of induced mappings between hyperspaces, and their extensions.

Finally, in section 6 the concern is with two uniformities ξ, η on a set X such that the uniformity $\hat{\eta}$ on $\mathcal{S}(X)$ is *associated* with the uniformity $\tilde{\xi}$, in the sense of chapter 2. The uniformity η is then said to be *hyperassociated* with ξ . If η is associated (resp. hyperassociated) with ξ , the filter condition (resp. fundamental family condition) holds for the identity mapping $(X, \xi) \rightarrow (X, \eta)$, and if in addition η is coarser than ξ , then (X, η) complete (resp. hypercomplete) implies (X, ξ) complete (resp. hypercomplete). It is sufficient for hyperassociation that η be proximity-finer than ξ .

3.2 Fundamental families. A nonempty family \mathcal{F} of nonempty subsets of a uniform space (X, \mathcal{U}) will be called *fundamental* if

- (1) it is directed by \subset (that is, it is a filter base), and
- (2) for each U in \mathcal{U} there is a member A of \mathcal{F} such that $A \subset U(F)$ for all F in \mathcal{F} .

The family \mathcal{F} *converges* in X if, putting $C = \bigcap \{\bar{F} : F \in \mathcal{F}\}$, for each U in \mathcal{U} there is a member F of \mathcal{F} with $F \subset U(C)$; in other words, $U(C)$ eventually contains \mathcal{F} . Clearly every convergent family is fundamental and the closures of its members have non-empty intersection.

As a preliminary to dealing with the role of fundamental families in hypercompleteness, the following LEMMA records the facts relating them to Cauchy nets in $\mathcal{S}(X)$.

LEMMA 3.2.1 If \mathcal{F} is a fundamental family on X and I is an index set directing the members of \mathcal{F} by \subset , then $\{F_\alpha : \alpha \in I\}$ is a Cauchy net in $\mathcal{S}(X)$. If \mathcal{F} converges then $F_\alpha \rightarrow \bigcap \{\bar{F} : F \in \mathcal{F}\}$ in $\mathcal{S}(X)$. If, on the other hand, $F_\alpha \rightarrow A$ in $\mathcal{S}(X)$ then \mathcal{F} converges and $\bar{A} = \bigcap \{\bar{F} : F \in \mathcal{F}\}$.

If $\{A_\alpha : \alpha \in I\}$ is any Cauchy net in $\mathcal{S}(X)$ then, putting $F_\beta = \bigcup \{A_\alpha : \alpha \geq \beta\}$, $\mathcal{F} = \{F_\beta : \beta \in I\}$ is a fundamental family. If $A_\alpha \rightarrow A$ in $\mathcal{S}(X)$ then \mathcal{F} converges and $\bar{A} = \bigcap \{\bar{F}_\beta : \beta \in I\}$.

If, on the other hand, \mathcal{F} is known to converge, then

$$A_\alpha \rightarrow \bigcap \{\bar{F}_\beta : \beta \in I\}.$$

Proof Let \mathcal{F} be a fundamental family. Then for each U in \mathcal{U} there is α_0 in I such that $F_{\alpha_0} \subseteq U(F_\alpha)$ for all α in I , and thus, for $\alpha, \beta \geq \alpha_0$, $F_\beta \subseteq F_{\alpha_0} \subseteq U(F_\alpha)$, which proves $\{F_\alpha : \alpha \in I\}$ is a Cauchy net in $\mathcal{S}(X)$. Let \mathcal{F} converge. Then for each U in \mathcal{U} , if C denotes the set $\bigcap \{\bar{F} : F \in \mathcal{F}\}$, $F_\alpha \subseteq U(C)$ eventually. Since also $C \subseteq U(F_\alpha)$ for all α in I , it follows that $F_\alpha \rightarrow C$ in $\mathcal{S}(X)$. Now let it be given that $F_\alpha \rightarrow A$ in $\mathcal{S}(X)$. It is sufficient, for convergence of the family \mathcal{F} , that $\bar{A} = \bigcap \{\bar{F} : F \in \mathcal{F}\}$, for clearly $F_\alpha \subseteq U(\bar{A})$ eventually, for any given U in \mathcal{U} . But $\bigcap \{\bar{F} : F \in \mathcal{F}\} \subseteq \bar{F}_\alpha \subseteq \bar{U}(\bar{A})$, so that \bar{A} must contain the intersection. Also, for each U in \mathcal{U} , $U(F_\alpha)$ eventually, and hence always, contains A , so that \bar{A} is contained in the intersection, and equality is proved.

For the second part, let $\{A_\alpha : \alpha \in I\}$ be a Cauchy net in $\mathcal{S}(X)$ and put $F_\beta = \bigcup \{A_\alpha : \alpha \geq \beta\}$ for each β in I . Then $\mathcal{F} = \{F_\beta : \beta \in I\}$ is clearly a filter base. Given U in \mathcal{U} choose U_1 in \mathcal{U} with $U_1^2 \subseteq U$. There exists α_0 in I such that $A_\alpha \subseteq U_1(A_\beta)$ for all $\alpha, \beta \geq \alpha_0$, and hence $F_{\alpha_0} \subseteq U_1(A_{\alpha_0}) \subseteq U_1^2(A_\alpha) \subseteq U(A_\alpha)$ for all $\alpha \geq \alpha_0$. Therefore $F_{\alpha_0} \subseteq U(F_\beta)$ for all β in I , which proves that \mathcal{F} is a fundamental family.

Let $A_\alpha \rightarrow A$ in $\mathcal{S}(X)$. Then, for given U in \mathcal{U} , A is eventually contained in $U(A_\alpha)$, which implies that, eventually, $F_\beta \subseteq U(A)$. Therefore $\bigcap \{\bar{F}_\beta : \beta \in I\} \subseteq \bar{A}$, and equality is proved. Since it is shown that $F_\beta \subseteq U(\bar{A})$ eventually, the family \mathcal{F} converges.

Finally, let it be given that \mathcal{F} converges. Then, putting $C = \bigcap \{\bar{F}_\beta : \beta \in I\}$, and choosing U in \mathcal{U} , $F_\beta \subseteq U(C)$ eventually and hence $A_\alpha \subseteq U(C)$ eventually. Also, if α_0 is chosen such that

$A_\alpha \subseteq U(A_\beta)$ for all $\alpha, \beta \geq \alpha_0$, then, as shown above,
 $F_{\alpha_0} \subseteq U(A_\alpha)$ for all $\alpha \geq \alpha_0$. Hence $C \subseteq U(F_{\alpha_0}) \subseteq U(A_\alpha)$ for all
 $\alpha \geq \alpha_0$. It now follows that $A_\alpha \rightarrow C$ in $\mathcal{S}(X)$.

PROPOSITION 3.2.2. If X is a uniform space and \mathcal{M} is any complete uniform subspace of $\mathcal{S}(X)$, then every fundamental family on X consisting of elements of \mathcal{M} converges. If, conversely, every such family is known to converge and if \mathcal{M} is closed under unions of its members then every Cauchy net on \mathcal{M} converges in $\mathcal{S}(X)$ to some element of $\mathcal{S}(X)$.

Proof. The first part follows immediately from LEMMA 3.2.1.

For the second part, let $\{M_\alpha : \alpha \in I\}$ be a Cauchy net in \mathcal{M} , and put $F_\beta = \bigcup \{M_\alpha : \alpha \geq \beta\}$. Then $\mathcal{F} = \{F_\beta : \beta \in I\}$ is a fundamental family consisting of elements of \mathcal{M} , by LEMMA 3.2.1. By hypothesis \mathcal{F} converges, and again by the LEMMA,

$M_\alpha \rightarrow \bigcap \{F_\beta : \beta \in I\}$ in $\mathcal{S}(X)$.

COROLLARY 3.2.3. If \mathcal{M} is a dense subspace of $\mathcal{S}(X)$ closed under unions, and every fundamental family of elements of \mathcal{M} converges then $\mathcal{S}(X)$ is complete.

THEOREM 3.2.4. A uniform space X is hypercomplete if and only if every fundamental family (of closed sets, resp. of open sets) on X converges.

Proof. If $\mathcal{S}(X)$ is complete then by PROPOSITION 3.2.2 every fundamental family on X converges, and conversely. The proof is completed by observing firstly that the collection of open subsets of X satisfies the conditions in COROLLARY 3.2.3, and secondly that the closures of the members of any fundamental family form a fundamental family which converges if and only if the original family converges.

A little more than this can be said. The next result is a straightforward consequence of the fact that for any fundamental family \mathcal{F} on (X, \mathcal{U}) , the family $\{U(F) : F \in \mathcal{F}, U \in \mathcal{U}\}$ is fundamental and converges if and only if \mathcal{F} converges. The trace of this family on any dense subset of X will also be a fundamental family on X . A more elegant proof is by applying COROLLARY 3.2.3 to the collection of subsets of any dense subset of X .

PROPOSITION 3.2.5. If A is a dense subset of a uniform space X and if every fundamental family on X consisting of subsets of A converges, then X is hypercomplete.

Remarks Isbell, in (12), calls a filter *stable* if its members form a Cauchy net in the hyperspace, and *hyperconvergent* if this net converges. The terms are also meaningful for filter bases, and it is easy to see that a fundamental family is precisely a stable filter base, and a convergent family is precisely a hyperconvergent filter base. Thus THEOREM 3.2.4 corresponds to Isbell's result that a uniform space is hypercomplete if and only if every stable filter is hyperconvergent. We note here that Isbell uses the term *supercomplete* where we use *hypercomplete*; his usage does not seem to have general appeal.

The second part of LEMMA 3.2.1 remains true if, for a Cauchy net $\{A_\alpha : \alpha \in I\}$ of absolutely convex subsets of a locally convex topological vector space, we take F_β to be the convex hull of $\bigcup \{A_\alpha : \alpha \geq \beta\}$. With this alteration the LEMMA leads immediately to Kelley's result, mentioned in section 1, that the space of non-empty absolutely convex subsets of a locally convex space is complete if and only if every fundamental family of absolutely convex sets converges.

Knowing that the space of compact subsets of a complete uniform space is itself complete, we can expect any fundamental family consisting of compact sets on such a space to converge, by PROPOSITION 3.2.2. In fact there is the following stronger and rather surprising result.

PROPOSITION 3.2.6. On an arbitrary uniform space, every fundamental family of compact subsets converges.

Proof Let $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ be such a family on the uniform space X , directed by \subset . For each nonempty finite subset ϕ of I , $B_\phi = \bigcap \{\bar{F}_\alpha : \alpha \in \phi\}$ is a nonempty compact subset of X . Directing the finite subsets of I by \supset , $\{B_\phi\}$ is a decreasing net.

Given any entourage U of X there exists α_0 in I such that $F_{\alpha_0} \subset U(F_\alpha)$ for all α in I , and therefore, because each B_ϕ contains some F_α , $F_{\alpha_0} \subset U(B_\phi)$ for each finite subset ϕ . Let ψ_0 be any finite subset of I containing α_0 . Then, if $\phi_1, \phi_2 \supset \psi_0$, $B_{\phi_1} \subset U(B_{\phi_2})$ and $B_{\phi_2} \subset U(B_{\phi_1})$, so that $\{B_\phi\}$ is a Cauchy decreasing net of compact sets. It therefore converges in $\mathcal{S}(X)$ (see section 1.8) to the intersection, which is the set $C = \bigcap \{\bar{F}_\alpha : \alpha \in I\}$. Thus for each entourage U there exists ϕ such that $B_\phi \subset U(C)$. But there is some $F_\alpha \subset B_\phi$, and so $F_\alpha \subset U(C)$, and \mathcal{F} converges.

Remarks. The previous PROPOSITION and its proof suggest that there is a proof by this route of the Robertsons' result in (28) that the space of nonempty compact subsets of a complete separated uniform space is complete. But it does not appear to be easy to make the step from convergence of every fundamental family of compact subsets to completeness of the space of compact subsets of a complete space.

The notion of fundamental family is not greatly different from that of Cauchy filter base, and the similarity of their basic properties is illustrated by the remaining results of this section.

PROPOSITION 3.2.7. If t is a uniformly continuous mapping from a uniform space X into a uniform space Y , and \mathcal{F} is any fundamental family on X , then the family $t(\mathcal{F}) = \{t(F) : F \in \mathcal{F}\}$ is fundamental on Y . Furthermore if \mathcal{F} converges so does $t(\mathcal{F})$.

Proof. Clearly $t(\mathcal{F})$ is a filter base. For each entourage V of Y there is an entourage U of X with $t^2(U) \subset V$, by uniform continuity, and then since \mathcal{F} is fundamental there is some B in \mathcal{F} such that $B \subset U(F)$ for all F in \mathcal{F} . Then $t(B) \subset V(t(F))$ for all $t(F)$ in $t(\mathcal{F})$ and so $t(\mathcal{F})$ is fundamental.

Suppose \mathcal{F} converges. Let $C = \bigcap \{\bar{F} : F \in \mathcal{F}\}$, and $D = \bigcap \{t(F) : F \in \mathcal{F}\}$. When V and U are as above, there is some A in \mathcal{F} such that $A \subset U(C)$, and then $t(A) \subset V(t(C)) \subset V(D)$, using continuity of t . This means that $t(\mathcal{F})$ converges.

PROPOSITION 3.2.8. If t is a uniformly continuous mapping from a uniform space X into a uniform space Y , and maps entourages of X into entourages of Y , then for each fundamental family \mathcal{G} on Y , the family $\mathcal{F} = \{t^{-1}(G) : G \in \mathcal{G}\}$ is fundamental on X .

Proof. Clearly \mathcal{F} is a filter base. For each entourage U of X , $t^2(U)$ is an entourage of Y , so there is some A in \mathcal{G} such that $A \subset t^2(U)(G)$ for all G in \mathcal{G} . Then $t^{-1}(A) \subset U(t^{-1}(G))$ for all G in \mathcal{G} , and so \mathcal{F} is fundamental.

PROPOSITION 3.2.9. If \mathcal{F} is a Cauchy filter base on a uniform space X then \mathcal{F} is also a fundamental family on X , and \mathcal{F} converges as a fundamental family if and only if it converges as a Cauchy filter base to a point of X .

Proof. Let \mathcal{U} be a base for the uniformity on X . For each U in \mathcal{U} there is a U -small set B in \mathcal{F} , and then $B \subset U(C)$ for all F in \mathcal{F} , since $B \cap F \neq \emptyset$. Thus \mathcal{F} is a fundamental family. If \mathcal{F} converges as a fundamental family then $C = \bigcap \{\bar{F} : F \in \mathcal{F}\}$ is non-

empty, and so we can choose x in C . But x is a cluster point of the Cauchy filter base, and therefore a limit point of it. Conversely if $\mathcal{F} \rightarrow x$ as a Cauchy filter base, then x is a limit point of the filter base and so also a cluster point, so that x belongs to C . For each U in \mathcal{U} there is some F in \mathcal{F} such that $F \subset U(x)$, which implies that $C \subset \bigcup^2 U(x)$. The upshot of all this is that $\overline{\{x\}} = C$, and, for each U in \mathcal{U} , $U(C)$ eventually contains \mathcal{F} .

Thus there are more fundamental families on a uniform space than Cauchy filters - in fact the powers of the respective collections are related as the powers of hyperspace to space. This will become apparent in the next section. It is easy to see in this context how hypercompleteness implies completeness, for if every fundamental family converges so must every Cauchy filter.

There is a mild generalization of PROPOSITION 3.2.9. A filter (or filter base) \mathcal{F} on a uniform space X is called *semi-Cauchy* if for each entourage U there is some integer $n > 0$ such that \mathcal{F} contains a U^n -small set. Isbell uses the notion in (12) to show that the hyperspace of a compact space is compact, by first proving that each semi-Cauchy filter is a Cauchy net in the hyperspace, which converges when X is complete to a compact set. In our language, each semi-Cauchy filter base is also a fundamental family, convergent when X is complete. Notice that each filter base consisting of precompact sets is semi-Cauchy.

LEMMA 3.2.10. For each nonempty subset A of a uniform space X with uniformity base \mathcal{U} , the collection $\mathcal{B}(A) = \{U(A) : U \in \mathcal{U}\}$ is a convergent fundamental family on X .

Proof. If U_1, U_2 are in \mathcal{U} then there is some U in \mathcal{U} with $U \subset U_1 \cap U_2$ and then $U(A) \subset U_1(A) \cap U_2(A)$. Thus $\mathcal{B}(A)$ is a filter base. For each U in \mathcal{U} , $U(A) \subset U(\bar{A})$, and \bar{A} is the intersection

of the closures of the members of $\mathcal{B}(A)$, so $\mathcal{B}(A)$ converges, and is fundamental.

For comparison with this last result, if x is any point of X , the collection $\mathcal{B}(x) = \{U(x) : U \in \mathcal{U}\}$ is a Cauchy filter base convergent to x . But while $\mathcal{B}(x)$ is a neighbourhood base for x , $\mathcal{B}(A)$ is not in general a neighbourhood base for A .

3.3 The Hausdorff completion of the hyperspace. That fundamental families play a role in the hyperspace $\mathcal{S}(X)$ very like the role played by Cauchy filters in X is now quite obvious. Not only do they act analogously in determining hypercompleteness and completeness, but, as the first THEOREM of this section will show, the Hausdorff completion of the hyperspace can be constructed using fundamental families in much the same way that the Hausdorff completion of X can be constructed using Cauchy filters.

Let X be a uniform space with uniformity base \mathcal{U} . In the collection of fundamental families on X define a uniformity as follows. For each U in \mathcal{U} let \hat{U} be the collection of pairs $(\mathcal{F}, \mathcal{G})$ of fundamental families such that there exist F_0 in \mathcal{F} and G_0 in \mathcal{G} with $F_0 \subseteq U(G)$ for all G in \mathcal{G} and $G_0 \subseteq U(F)$ for all F in \mathcal{F} . To show that $\hat{\mathcal{U}} = \{\hat{U} : U \in \mathcal{U}\}$ is a base for a uniformity, four standard properties must be verified.

(1) By definition of a fundamental family $(\mathcal{F}, \mathcal{F})$ belongs to \hat{U} for each \hat{U} in $\hat{\mathcal{U}}$.

(2) If \hat{U}_1, \hat{U}_2 belong to $\hat{\mathcal{U}}$, there is some V in \mathcal{U} with $V \subseteq U_1 \cap U_2$, and then also $\hat{V} \subseteq \hat{U}_1 \cap \hat{U}_2$.

(3) Each \hat{U} is symmetric by definition.

(4) If \hat{U} belongs to $\hat{\mathcal{U}}$, there is some W in \mathcal{U} with $W^2 \subseteq U$.

Then suppose $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{H})$ both belong to \hat{W} . There exist F_0 in \mathcal{F} and G_0 in \mathcal{G} such that $F_0 \subseteq W(G)$ for all G in \mathcal{G} and

$G \subset W(F)$ for all F in \mathcal{F} , and there exist G_1 in \mathcal{G} and H_0 in \mathcal{H} such that $G_1 \subset W(H)$ for all H in \mathcal{H} and $H_0 \subset W(G)$ for all G in \mathcal{G} . The upshot of all this is that $F_0 \subset W(G_1) \subset \overset{2}{W}(H)$ for all H in \mathcal{H} , and $H_0 \subset W(G_0) \subset \overset{2}{W}(F)$ for all F in \mathcal{F} , so that $(\mathcal{F}, \mathcal{H})$ belongs to \hat{U} . We have shown that $\overset{2}{W} \subset \hat{U}$.

Next, form the associated Hausdorff space, denoted by $\phi(X)$. Let $[\mathcal{F}]$ denote the element of $\phi(X)$ determined by the fundamental family \mathcal{F} . It may be regarded as an equivalence class of fundamental families under the relation: $\mathcal{F} R \mathcal{G}$ if and only if the intersection of the filters generated by \mathcal{F} and \mathcal{G} is a fundamental family. We will continue to use the notation $\hat{\mathcal{U}}$ for the uniformity base on $\phi(X)$; thus a pair $([\mathcal{F}], [\mathcal{G}])$ belongs to a member \hat{U} of $\hat{\mathcal{U}}$ if there exist F_0 in \mathcal{F} and G_0 in \mathcal{G} such that $F_0 \subset U(G)$ for all G in \mathcal{G} and $G_0 \subset U(F)$ for all F in \mathcal{F} .

THEOREM 3.3.1. Let X be a uniform space with uniformity base \mathcal{U} , and construct the separated uniform space $(\phi(X), \hat{\mathcal{U}})$ as above.

(1) Let $h : \mathcal{S}(X) \rightarrow \phi(X)$ be the mapping which takes each subset A of X onto the element $[\mathcal{B}(A)]$ of $\phi(X)$, where $\mathcal{B}(A) = \{U(A) : U \in \mathcal{U}\}$. Then h is uniformly continuous and the uniformity on $\mathcal{S}(X)$ is the inverse image under h of that on $\phi(X)$.

(2) The space $\mathcal{C}(X)$ of nonempty closed subsets of X is uniformly isomorphic to the subspace $h(\mathcal{S}(X))$, which is dense in $\phi(X)$.

(3) The space $\phi(X)$ is complete.

Proof. From LEMMA 3.2.10, $\mathcal{B}(A)$ is a fundamental family on X for each nonempty subset A . Suppose \hat{V} is in $\hat{\mathcal{U}}$, with $(h(A), h(B))$ in \hat{V} . Then there exist U_1, U_2 in \mathcal{U} such that $U_1(A) \subset \overset{3}{V}(U_2(B))$ and $U_2(B) \subset \overset{3}{V}(U_1(A))$ for all U in \mathcal{U} . Thus $A \subset \overset{4}{V}(B)$ and $B \subset \overset{4}{V}(A)$, that is, (A, B) belongs to the entourage $\overset{4}{V}$ of $\mathcal{S}(X)$. Conversely,

if (A, B) belongs to \tilde{V} , then $A \subset V(B)$ and $B \subset V(A)$, so that $V(A) \subset \overset{2}{V}(U(B))$ and $V(B) \subset \overset{2}{V}(U(A))$ for all U in \mathcal{U} . Thus $(h(A), h(B))$ belongs to $\overset{2}{V}$. This has shown that the inverse images under $h \times h$ of the members of $\hat{\mathcal{U}}$ form a base for the uniformity $\tilde{\mathcal{U}}$ on $S(X)$, which is therefore the inverse image under h of the uniformity on $\phi(X)$. It is also the coarsest uniformity on $S(X)$ for which h is uniformly continuous. Thus (1) is proved.

To prove (2), let P, Q be nonempty subsets of X with $h(P) = h(Q)$. This means that for each V in \mathcal{U} there exist U_1, U_2 in \mathcal{U} such that $U_1(P) \subset V(U_2(Q))$ and $U_2(Q) \subset V(U_1(P))$ for all U in \mathcal{U} , so that $P \subset \overset{2}{V}(Q)$ and $Q \subset \overset{2}{V}(P)$. Therefore $\bar{P} = \bar{Q}$. Conversely if it is given that $\bar{P} = \bar{Q}$ then $h(P) = h(Q)$. Thus h is injective on $\mathcal{E}(X)$, and is an isomorphism, because of (1), between $\mathcal{E}(X)$ and $h(\mathcal{E}(X)) (= h(S(X)))$.

Next, we show that $h(S(X))$ is dense in $\phi(X)$. Let $[\mathcal{F}]$ be an arbitrary element of $\phi(X)$ and let \hat{V} belong to $\hat{\mathcal{U}}$. Choose V_0 in \mathcal{U} with $\overset{3}{V}_0 \subset V$. There exists F_0 in \mathcal{F} such that $F_0 \subset V_0(F)$ for all F in \mathcal{F} , since \mathcal{F} is fundamental. Put $A = V_0(F_0)$. Then $V_0(A) \subset \overset{3}{V}_0(F) \subset V(F)$ for all F in \mathcal{F} , and $F_0 \subset V(U(A))$ for all U in \mathcal{U} . Thus $([\mathcal{F}], [\mathcal{E}(A)])$ belongs to \hat{V} , and since $[\mathcal{E}(A)] = h(A)$, this proves that $h(S(X))$ is dense in $\phi(X)$.

It remains to prove (3). Since $h(S(X))$ is dense in $\phi(X)$ it will be sufficient to show that every Cauchy net in $h(S(X))$ converges to an element of $\phi(X)$. Let $\{[\mathcal{E}(A_\alpha)] : \alpha \in I\}$ be a Cauchy net. Then by (1), $\{A_\alpha : \alpha \in I\}$ is a Cauchy net in $S(X)$. Put $F_\beta = \bigcup \{A_\alpha : \alpha \geq \beta\}$ for each β in I . Then by LEMMA 3.2.1, $\mathcal{F} = \{F_\beta : \beta \in I\}$ is a fundamental family. We will show that $[\mathcal{E}(A_\alpha)] \rightarrow [\mathcal{F}]$ in $\phi(X)$.

Let V belong to \mathcal{U} , and choose V_0 in \mathcal{U} with $\overset{4}{V}_0 \subset V$.

Since $\{[\mathcal{B}(A_\alpha)] : \alpha \in I\}$ is Cauchy there exists α_0 in I such that $([\mathcal{B}(A_\alpha)], [\mathcal{B}(A_\beta)])$ belongs to \hat{V}_0 for $\alpha, \beta \geq \alpha_0$, and since \mathcal{F} is fundamental there exists β_0 in I such that $F_{\beta_0} \subseteq V_0(F_\beta)$ for all β in I . Take $\gamma \geq \alpha_0, \beta_0$. Then consider $\mathcal{B}(A_\gamma) = \{U(A_\gamma) : U \in \mathcal{U}\}$. We show that $([\mathcal{F}], [\mathcal{B}(A_\gamma)])$ belongs to \hat{V} . By choice of γ , $V_0(A_\gamma) \subseteq V_0(F_{\beta_0}) \subseteq \overset{2}{V}_0(F_{\beta_0}) \subseteq V(F_{\beta_0})$ for all β in I . If $\alpha \geq \alpha_0$ then there exists V_1 in \mathcal{U} such that $V_1(A_\alpha) \subseteq \overset{3}{V}_0(U(A_\gamma))$ for all U in \mathcal{U} , so that $A_\alpha \subseteq \overset{4}{V}_0(A_\gamma) \subseteq V(A_\gamma)$. Thus $F_\gamma \subseteq V(A_\gamma) \subseteq V(U(A_\gamma))$ for all U in \mathcal{U} . It has been proved that if $\gamma \geq \alpha_0, \beta_0$, then $([\mathcal{F}], [\mathcal{B}(A_\gamma)])$ belongs to \hat{V} , so that the net $\{[\mathcal{B}(A_\alpha)] : \alpha \in I\}$ converges to $[\mathcal{F}]$ in $\phi(X)$. Therefore $\phi(X)$ is complete.

COROLLARY 3.3.2. The Hausdorff completion $\hat{\mathcal{E}}(X)$ of the space of nonempty closed subsets of X is isomorphic to $\phi(X)$. This isomorphism takes $\hat{\mathcal{E}}(X)$ onto the subspace $\phi_0(X) = h(\hat{\mathcal{E}}(X))$ which consists of all elements of $\phi(X)$ determined by convergent fundamental families on X .

THEOREM 3.3.3. Let X be a locally convex topological vector space, and let $\mathcal{K}(X)$ denote the set of nonempty, absolutely convex, closed subsets of X . Then the Hausdorff completion $\hat{\mathcal{K}}(X)$ is uniformly isomorphic to the subspace $\phi_k(X)$ of $\phi(X)$ consisting of all elements of $\phi(X)$ determined by fundamental families of absolutely convex sets.

Proof. Let \mathcal{U} be a 0-neighbourhood base for X consisting of absolutely convex sets. The mapping $h : \mathcal{E}(X) \rightarrow \phi(X)$ of the previous THEOREM takes each closed, absolutely convex set A onto the element $[\mathcal{B}(A)]$, where $\mathcal{B}(A) = \{A + U : U \in \mathcal{U}\}$, and so, since $\mathcal{B}(A)$ is a fundamental family of absolutely convex sets, h takes $\mathcal{K}(X)$ into $\phi_k(X)$. Because of THEOREM 3.3.1 it will be sufficient to show that $h(\mathcal{K}(X))$ is dense in $\phi_k(X)$, and that $\phi_k(X)$ is closed in $\phi(X)$.

Let $[\mathcal{F}]$ be any element of $\Phi_k(X)$ with \mathcal{F} a fundamental family of absolutely convex sets, and let V belong to \mathcal{U} . Choose V_0 in \mathcal{U} such that $4V_0 \subset V$. Then there exists F_0 in \mathcal{F} such that $F_0 \subset F + V_0$ for all F in \mathcal{F} . Put $A = F_0 + V_0$. Then \bar{A} belongs to $\mathcal{K}(X)$, and $\bar{A} + V_0 \subset F + 4V_0 \subset F + V$ for all F in \mathcal{F} , and $F_0 \subset \bar{A} + U + V$ for all U in \mathcal{U} . Thus $([\mathcal{F}], [\mathcal{B}(\bar{A})])$ belongs to \hat{V} , so that $h(\mathcal{K}(X))$ is dense in $\Phi_k(X)$.

To show $\Phi_k(X)$ is closed, let $[\mathcal{G}]$ be an element in the closure. Thus, for each U in \mathcal{U} there is a fundamental family $\mathcal{F}(U)$, consisting of absolutely convex sets, such that there exists some $F_0(U)$ in $\mathcal{F}(U)$ with $F_0(U) \subset G + U$ for all G in \mathcal{G} , and there exists some $G_0(U)$ in \mathcal{G} with $G_0(U) \subset F(U) + U$ for all $F(U)$ in $\mathcal{F}(U)$.

Let \mathcal{H} be the collection of all sets of the form $F(U) + 2U$, as $F(U)$ runs through $\mathcal{F}(U)$ and U runs through \mathcal{U} . Then \mathcal{H} is a fundamental family of absolutely convex sets. For if $F(U_1) + 2U_1$, $F(U_2) + 2U_2$ are any two members, there exist $G_0(U_1)$ and $G_0(U_2)$ with $G_0(U_1) \subset F(U_1) + U_1$ and $G_0(U_2) \subset F(U_2) + U_2$. Let G_0 be a member of \mathcal{G} with $G_0 \subset G_0(U_1) \cap G_0(U_2)$, and let U_0 be a member of \mathcal{U} with $3U_0 \subset U_1 \cap U_2$. There is some $F(U_0)$ in $\mathcal{F}(U_0)$ with $F(U_0) \subset G_0 + U_0$, so that

$$\begin{aligned} F_0(U_0) + 2U_0 &\subset G_0 + 3U_0 \subset (G_0(U_1) + U_1) \cap (G_0(U_2) + U_2) \\ &\subset (F(U_1) + 2U_1) \cap (F(U_2) + 2U_2), \end{aligned}$$

and therefore \mathcal{H} is a filter base. Now if V is any member of \mathcal{U} , there is $F_0(V)$ in $\mathcal{F}(V)$ with $F_0(V) \subset G + V$ for all G in \mathcal{G} , and so also $F_0(V) + 2V \subset G + 3V$ for all G in \mathcal{G} . Hence

$F_0(V) + 2V \subset F(U) + U + 3V \subset F(U) + 2U + 3V$ for all U in \mathcal{U} , which shows that \mathcal{H} is a fundamental family.

Next, $[\mathcal{G}] = [\mathcal{H}]$. For if U_0 is any member of \mathcal{U} then there is some $H_0 = F(U) + 2U$ in \mathcal{H} with $H_0 \subset H + U_0$ for all H in \mathcal{H} .

But then there is $G_0(U)$ in \mathcal{G} with $G_0(U) \subset F(U) + U$, so that $G_0(U) \subset H + U_0$ for all H in \mathcal{H} . Also, if U_1 is a member of \mathcal{U} chosen so that $3U_1 \subset U_0$, there is $F_0(u_1)$ in $\mathcal{F}(U_1)$ with $F_0(U_1) \subset G + U_1$ for all G in \mathcal{G} . Hence

$$F_0(U_1) + 2U_1 \subset G + 3U_1 \subset G + U_0 \quad \text{for all } G \text{ in } \mathcal{G}.$$

Therefore, finally, $[\mathcal{G}]$ belongs to $\phi_k(X)$, which must then be closed in $\phi(X)$, and complete.

When X is a topological vector space over the real number field, a fundamental family \mathcal{F} on X which consists of absolutely convex sets and has the property that, for each F in \mathcal{F} and each positive real number r , rF is in \mathcal{F} , is called a scalar fundamental family, following Kelley in (17). The space X is called fully complete if every scalar fundamental family on X converges. For locally convex spaces this condition is equivalent to "full completeness" of H. Collins and "B-completeness" of V. Pták, and lies between completeness and hypercompleteness conditions. It is of some interest to look at the behaviour of the subspaces of $\phi(X)$ determined by the scalar fundamental families and convergent scalar fundamental families, under the identification with $\hat{\mathcal{E}}(X)$.

Denote the two subspaces by $\phi_s(X)$ and $\phi_{so}(X)$, respectively. Concerning the latter there is the following result.

PROPOSITION 3.3.4. The space $\mathcal{L}(X)$ of closed vector subspaces of the locally convex topological vector space X is uniformly isomorphic to the subspace $\phi_{so}(X)$ of $\phi(X)$, determined by the convergent scalar fundamental families.

Proof. Let \mathcal{U} be a 0-neighbourhood base for X , consisting of absolutely convex sets. The mapping $h : \mathcal{S}(X) \rightarrow \phi(X)$ of THEOREM 3.3.1 takes each M in $\mathcal{L}(X)$ onto the element $[\mathcal{B}(M)]$, where

$\mathcal{B}(M) = \{M + U : U \in \mathcal{U}\}$. Clearly the collection $\{M\}$ consisting of the one set M is a convergent scalar fundamental family on X , and $[\{M\}] = [\mathcal{B}(M)]$.

If \mathcal{F} is any convergent scalar fundamental family, then, putting $M = \bigcap \{\bar{F} : F \in \mathcal{F}\}$, the set M is easily seen to be a closed vector subspace of X . Also $[\mathcal{B}(M)] = [\mathcal{F}]$, for if U_0 is any member of \mathcal{U} then $M + U_0 \subset F + 2U_0$ for all F in \mathcal{F} , and there exists F_0 in \mathcal{F} with $F_0 \subset M + U \subset M + U + U_0$ for all U in \mathcal{U} .

It follows that h takes $\mathcal{L}(X)$ onto $\phi_{so}(X)$, and since h is a uniform isomorphism between $\mathcal{E}(X)$ and $h(\mathcal{E}(X))$ by THEOREM 3.3.1, the proof is complete.

PROPOSITION 3.3.5. If X is any uniform space, the space $\mathcal{C}(X)$ of nonempty, compact, closed subsets of X is isomorphic to the subspace $\phi_c(X)$ of $\phi(X)$, determined by the collection of fundamental families of compact subsets.

Proof. Let \mathcal{U} be a base for the uniformity on X . The isomorphism $h : \mathcal{E}(X) \rightarrow h(\mathcal{E}(X))$ of THEOREM 3.3.1 takes each nonempty, compact closed set C onto the element $[\mathcal{B}(C)]$, where $\mathcal{B}(C) = \{U(C) : U \in \mathcal{U}\}$. Clearly the collection $\{C\}$ consisting of the one set C is a fundamental family and $[\{C\}] = [\mathcal{B}(C)]$.

Now let \mathcal{F} be any fundamental family of compact subsets. Then by PROPOSITION 3.2.6 \mathcal{F} is convergent, and $C = \bigcap \{\bar{F} : F \in \mathcal{F}\}$ is a nonempty, compact, closed subset of X . Furthermore, as in the proof of the previous PROPOSITION, $\mathcal{B}(C)$ and \mathcal{F} determine the same element of $\phi(X)$.

Thus h takes $\mathcal{C}(X)$ onto $\phi_c(X)$, and the result is proved.

Remarks. If \mathcal{F} is a fundamental family on a uniform space X then it is easily seen that \mathcal{F} converges if and only if there is

a closed subset A of X such that \mathcal{F} and $\mathcal{B}(A)$ determine the same element of $\Phi(X)$, and then $A = \bigcap \{\bar{F} : F \in \mathcal{F}\}$. This observation leads immediately to the following two results of section 2, which can now be regarded as corollaries of THEOREMS 3.3.1 and 3.3.3.

The space $\mathcal{C}(X)$ of nonempty, closed subsets of a uniform space X is complete if and only if every fundamental family on X converges.

The space $\mathcal{K}(X)$ of nonempty absolutely convex, closed subsets of a locally convex topological vector space X is complete if and only if every fundamental family on X consisting of absolutely convex sets converges.

For a suitable space X the spaces $\hat{\mathcal{C}}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\hat{\mathcal{K}}(X)$ and $\mathcal{L}(X)$ have all been identified with natural subspaces of $\Phi(X)$ corresponding to various kinds of fundamental families on X . To do the same for the Hausdorff completions $\hat{\mathcal{L}}(X)$ and $\hat{\mathcal{C}}(X)$ is not so straightforward. They are, of course, the closures in $\Phi(X)$ of $\Phi_{so}(X)$ and $\Phi_c(X)$, respectively, by the previous two PROPOSITIONS. The space $\hat{\Phi}_s(X)$ corresponding to the scalar fundamental families is an appealing candidate for identification with $\hat{\mathcal{L}}(X)$, but Kelley has pointed out in (16) that although convergence of every scalar fundamental family implies completeness of $\mathcal{L}(X)$, the converse is not true. This does suggest, however, that the required set lies between $\Phi_{so}(X)$ and $\Phi_s(X)$.

When X is any uniform space, it has been shown that $\mathcal{C}(X)$ can be identified with $\Phi_c(X)$, and $\Phi_c(X)$ is easily seen to be dense in the space $\Phi_p(X)$ determined by the fundamental families of pre-compact sets. There is considerable attractiveness in the conjecture that $\hat{\mathcal{C}}(X)$ can be identified with $\Phi_p(X)$. If this could be done then the Robertsons' result, asserting completeness of $\mathcal{C}(X)$ if X is complete and separated, would be an easy consequence. The problem

is to show $\phi_p(X)$ is closed in $\phi(X)$.

It has only been possible to achieve the following PROPOSITION - the proof relying on the Robertsons' result. To state the PROPOSITION concisely, we introduce here a theoretical tool, which will be useful again later. If X and Y are uniform spaces and $t : X \rightarrow Y$ is a uniformly continuous mapping then we define the mapping $t^* : \phi(X) \rightarrow \phi(Y)$ by $t^*[\mathcal{F}] = [t(\mathcal{F})]$. It is easily seen to be well-defined and uniformly continuous.

PROPOSITION 3.3.6. Let X be a uniform space and i the canonical mapping of X into its Hausdorff completion \hat{X} . Then the closure in $\phi(X)$ of $\phi_c(X)$ is the inverse image of $\phi_c(\hat{X})$ under the mapping $i^* : \phi(X) \rightarrow \phi(\hat{X})$ induced by i .

Proof. Denote the inverse image by $\phi_q(X)$. First, it will be shown that $\phi_c(X)$ is dense in $\phi_q(X)$. Let $[\mathcal{G}]$ belong to $\phi_q(X)$, and let U_0 belong to the base \mathcal{U} for the uniformity on X . Now the closures in $\hat{X} \times \hat{X}$ of the sets $i^2(U)$, U in \mathcal{U} , form a base for the uniformity on \hat{X} , so, if bars denote closures in $\hat{X} \times \hat{X}$, there is some U_1 in \mathcal{U} and some entourage W of \hat{X} , such that $\bar{W} \subset \bar{i^2(U_1)} \subset \bar{i^2(U_0)}$. Since the fundamental family $i(\mathcal{G})$ is identified in $\phi(\hat{X})$ with a fundamental family of compact sets on \hat{X} , which must converge by PROPOSITION 3.2.6, it follows that $i(\mathcal{G})$ converges, and that the set $C = \bigcap \{\bar{i(G)} : G \in \mathcal{G}\}$ is a nonempty, compact closed subset of \hat{X} . Thus there is a finite subset D of C , with $C \subset W(D)$. Then, since $i(X)$ is dense in \hat{X} , there is a finite subset E of X such that $i(E) \subset W(C)$ and $C \subset \bar{W(i(E))}$.

The family $\{E\}$ consisting of the set E alone clearly determines an element of $\phi_c(X)$. This element will now be shown to belong to the U_0 -neighbourhood of $[\mathcal{G}]$. By the convergence of

$i(G)$, there exists G_0 in \mathcal{G} such that
 $i(G_0) \subset W(C) \subset \overset{3}{W}(i(E)) \subset \overset{3}{\overline{W(U_1)}}(i(E)) \subset \overline{W(U)}(i(E))$ so that, in
fact, $i(G_0) \subset \overset{2}{\overline{W(U)}}(i(E))$, and therefore $G_0 \subset \overset{3}{U}(E)$. Also
 $C \subset \overline{i(G)}$ for all G in \mathcal{G} and so $i(E) \subset W(C) \subset \overset{2}{W}(i(G))$ for all
 G in \mathcal{G} . As before this implies that $E \subset \overset{3}{U}(G)$ for all G in \mathcal{G} . This
proves that $([\mathcal{G}], [\{E\}])$ belongs to $\overset{3}{U}$, and hence $\phi_c(X)$ is
dense in $\phi_q(X)$.

Next to show that $\phi_q(X)$ is closed in $\phi(X)$ it is sufficient
to show that $\phi_c(\hat{X})$ is closed in $\phi(\hat{X})$, because the mapping i^*
is uniformly continuous. But $\phi_c(\hat{X})$ is isomorphic to $C(\hat{X})$ by
PROPOSITION 3.3.5, so complete by the Robertson's Theorem, and
hence closed in the separated space $\phi(\hat{X})$.

The mapping i^* can easily be shown to be an isomorphism
between $\phi(X)$ and $\phi(\hat{X})$, and corresponds to the isomorphism between
 $\mathcal{E}(X)$ and $\mathcal{E}(\hat{X})$ of Chapter 2. Denote by $\phi_r(X)$ the subspace of
 $\phi(X)$ determined by those fundamental families on X whose images
converge in \hat{X} . Then $\phi_r(X)$ can be identified with $\phi_o(\hat{X})$ and
hence with $\mathcal{E}(\hat{X})$. Under the same isomorphism, $\phi_q(X)$ is identified
with $\phi_c(\hat{X})$, which leads immediately, using the THEOREM above,
to the following result of Chapter 2.

COROLLARY 3.3.7. The space $\hat{\mathcal{C}}(X)$ is uniformly isomorphic to the
space $C(\hat{X})$.

Remark. Since the closures in \hat{X} of the images under i of
the members of any fundamental family of precompact sets on X
form a fundamental family of compact sets on \hat{X} , we have

$$\phi_c(X) \subset \phi_p(X) \subset \phi_q(X) \subset \phi_r(X).$$

In Chapter 2, section 3, it was shown that for any uniform
space X the Hausdorff completion \hat{X} can be embedded as a uniform

subspace in $S(S(X))$ and, in fact as a closed uniform subspace in $\mathcal{E}(\mathcal{E}(X))$. It now turns out that the Hausdorff completion $\hat{\mathcal{E}}(X)$ can also be embedded as a closed uniform subspace in $\mathcal{E}(\mathcal{E}(X))$. by means of the construction by fundamental families developed in this section.

PROPOSITION 3.3.8. Let X be a uniform space and \mathcal{U} a base for the uniformity. Construct the space $\Phi(X)$, and define a mapping $w : \Phi(X) \rightarrow S(S(X))$ by taking the image under w of an element $[F]$ to be the filter Ψ_F generated by the collection $\{U(F) : F \in \mathcal{F}, U \in \mathcal{U}\}$. Then w is an isomorphism of $\Phi(X)$ onto its image in $S(S(X))$.

Proof. It must first be shown that w is well-defined. Let \mathcal{F} and \mathcal{G} be fundamental families on X with $[\mathcal{F}] = [\mathcal{G}]$. If A belongs to the filter $\Psi_{\mathcal{F}}$, then there exist G in \mathcal{G} and U in \mathcal{U} with $U(G) \subseteq A$. Choose U_1 in \mathcal{U} such that $U_1^2 \subseteq U$, and then there is some F in \mathcal{F} such that $U_1(F) \subseteq U_1^2(G) \subseteq A$. Thus A belongs to $\Psi_{\mathcal{F}}$, and it follows that $\Psi_{\mathcal{F}} = \Psi_{\mathcal{G}}$.

Next, w is injective. For if $\Psi_{\mathcal{F}} = \Psi_{\mathcal{G}}$, and U is any member of \mathcal{U} , there exists F_0 in \mathcal{F} such that $F_0 \subseteq U(F)$ for all F in \mathcal{F} , and then $U(F_0)$ must belong to $\Psi_{\mathcal{G}}$, so that there exist G in \mathcal{G} and V in \mathcal{U} such that $G \subseteq V(G) \subseteq U(F_0) \subseteq U^2(F)$, for all F in \mathcal{F} . It follows that $[\mathcal{F}] = [\mathcal{G}]$.

Now suppose that $(\Psi_{\mathcal{F}}, \Psi_{\mathcal{G}})$ belongs to the entourage \hat{V} of $S(S(X))$. Choose F_0 in \mathcal{F} such that $F_0 \subseteq V(F)$ for all F in \mathcal{F} . Since $V(F_0)$ belongs to $\Psi_{\mathcal{F}}$, there is some A in $\Psi_{\mathcal{G}}$ such that $A \subseteq \hat{V}(F_0)$. Then there exist G in \mathcal{G} and U in \mathcal{U} such that $G \subseteq U(G) \subseteq A \subseteq \hat{V}(F_0) \subseteq \hat{V}^3(F)$ for all F in \mathcal{F} . Hence $([\mathcal{F}], [\mathcal{G}])$ belongs to the entourage $\hat{\hat{V}}$ of $\Phi(X)$.

Conversely, suppose $([\mathcal{F}], [\mathcal{G}])$ belongs to \hat{V} , and let A be any set in $\Psi_{\mathcal{G}}$. Then there exists F in \mathcal{F} such that $F \subseteq V(G)$ for all G in \mathcal{G} , so that $V(F) \subseteq \bar{V}(A)$. Put $B = A \cup V(F)$. Then clearly B is a member of $\Phi_{\mathcal{F}}$, and (A, B) belongs to \bar{V} . Hence $(\Psi_{\mathcal{F}}, \Psi_{\mathcal{G}})$ belongs to \bar{V} .

It has been proved that both w and its inverse are uniformly continuous, and the result follows.

COROLLARY 3.3.9. The Hausdorff completion $\hat{\mathcal{E}}(X)$ can be embedded as a closed uniform subspace in $\mathcal{E}(\mathcal{E}(X))$.

Proof. By COROLLARY 3.3.2, $\hat{\mathcal{E}}(X)$ is isomorphic to $\Phi(X)$. From the PROPOSITION, the mapping $[\mathcal{F}] \rightarrow \Psi_{\mathcal{F}}$ is an isomorphism of $\Phi(X)$ into its image in $\mathcal{S}(\mathcal{S}(X))$. But then, since $\Phi(X)$ is separated, and since the collection $\Psi_{\mathcal{F}}'$ of closures in X of the members of $\Psi_{\mathcal{F}}$ satisfies $(\Psi_{\mathcal{F}}', \Psi_{\mathcal{F}}) \in \bar{U}$ for each U in \mathcal{U} , the mapping $[\mathcal{F}] \rightarrow \Psi_{\mathcal{F}}'$ is an isomorphism of $\Phi(X)$ onto its image in $\mathcal{S}(\mathcal{E}(X))$. Furthermore if $\bar{\Psi}_{\mathcal{F}}'$ denotes the closure in $\mathcal{E}(X)$ of $\Psi_{\mathcal{F}}'$, then the mapping $[\mathcal{F}] \rightarrow \bar{\Psi}_{\mathcal{F}}'$ is an isomorphism of $\Phi(X)$ onto its image in $\mathcal{E}(\mathcal{E}(X))$, because $\mathcal{E}(\mathcal{E}(X))$ is the Hausdorff space associated with $\mathcal{S}(\mathcal{E}(X))$, by PROPOSITION 2.4.1.

The image in $\mathcal{E}(\mathcal{E}(X))$ must be closed, being a complete subspace of a Hausdorff space.

Remarks. Caulfield (5) has shown that the mapping $(x, y) \rightarrow \{\{x\}, \{x, y\}\}$ is an isomorphism of $X \times X$ into $\mathcal{S}(\mathcal{S}(X))$. The hyperhyperspace thus contains a surprising number of the significant spaces derivable from X . An application of Caulfield's embedding is given in Nachman's paper (22).

3.4 The filter condition and the fundamental family condition

Dr. Wendy Robertson discussed in (29) a condition on a mapping between

two topological vector spaces, called the filter condition. It is also meaningful for uniform spaces and will be said to hold for a mapping t from a uniform space X into a uniform space Y if the following is true.

FILTER CONDITION If ϕ is a Cauchy filter base on X and $t(\phi)$ is convergent to a point of $t(X)$ then ϕ is convergent to a point of X .

An analogous condition in terms of fundamental families is now introduced.

FUNDAMENTAL FAMILY CONDITION. If \mathcal{F} is a fundamental family on X and $t(\mathcal{F})$ is a convergent family on $t(X)$ then \mathcal{F} converges in X .

If A is a subset of X , the fundamental family condition will be said to hold on A if the above condition is satisfied with A in place of X . The fundamental family condition has, as might be expected, properties analogous to those of the filter condition, with hypercompleteness taking the place of completeness. The next result follows directly from the definitions and THEOREM 3.2.4.

PROPOSITION 3.4.1. If X is hypercomplete the fundamental family condition holds for every t . If t is uniformly continuous and the fundamental family condition holds, then if $t(X)$ is hypercomplete, X must be hypercomplete.

PROPOSITION 3.4.2. If the fundamental family holds on a subset A of X then it holds on any closed subset of A .

Proof. Let B be a subset of A closed in A , and let \mathcal{F} be a fundamental family consisting of subsets of B (just a fundamental family on B). Then clearly \mathcal{F} is also a fundamental family

on A . If t_A, t_B are the restrictions of t to A, B , respectively, then $t_A(\mathcal{F}) = t_B(\mathcal{F})$ and if $t_B(\mathcal{F})$ converges on $t_B(B)$ so does $t_A(\mathcal{F})$ on $t_A(A)$ because the intersection of the closures in $t_B(B)$ is contained in the intersection of the closures in $t_A(A)$. But then \mathcal{F} must converge in A , or, in other words, if C denotes the intersection of the closures in A of the members of \mathcal{F} and U is any entourage of X , $U(C)$ eventually contains \mathcal{F} . Since B is closed in A , C is also the intersection of the closures in B of the members of \mathcal{F} , and hence \mathcal{F} converges in B .

PROPOSITION 3.4.3. If the fundamental family condition holds for a mapping t between uniform spaces X and Y , then so does the filter condition.

Proof. Let \mathcal{F} be a Cauchy filter base on X with $t(\mathcal{F})$ converging to a point of $t(X)$. By PROPOSITION 3.2.9 \mathcal{F} is also a fundamental family on X and $t(\mathcal{F})$ is a convergent family on $t(X)$. If the fundamental family holds \mathcal{F} must converge in X as a fundamental family and so again by PROPOSITION 3.2.9 \mathcal{F} converges as a Cauchy filter base to a point of X . Thus the filter condition holds.

That the two conditions are not equivalent in general will now be demonstrated by a counterexample. Because of PROPOSITION 3.4.1 it will be sufficient to show the existence of a uniformly continuous surjective mapping from a complete but not hypercomplete uniform space X onto a hypercomplete uniform space Y . For then the filter condition obviously holds by the completeness of X , but if the fundamental family condition held X would be hypercomplete.

COUNTEREXAMPLE 3.4.4. Take for Y any infinite dimensional Banach space and

let X be the same vector space endowed with its finest locally convex topology $\tau(X, X^*)$. Then X is complete because it is a topological direct sum of copies of the scalar field, but if it were hypercomplete (or even fully complete) then a closed graph theorem of Kelley (16) could be applied to the identity mapping of Y onto X to imply that the norm topology and $\tau(X, X^*)$ were identical, which is not true (see for example (27)). However Y is hypercomplete being a complete metric space. Finally, the identity mapping of X onto Y is clearly uniformly continuous, and so the counterexample is supplied.

When X is a topological vector space over the real number field we can define a weakened sort of fundamental family condition on a mapping t from X into a uniform space Y .

SCALAR FUNDAMENTAL FAMILY CONDITION. If \mathcal{F} is a scalar fundamental family on X and $t(\mathcal{F})$ is a convergent family in $t(X)$ then \mathcal{F} converges in X .

If X is fully complete then, of course, the scalar fundamental family condition holds for every t . If, in addition, Y is a topological vector space over the real number field and t is a continuous linear mapping, then $t(\mathcal{F})$ is a scalar fundamental family for each scalar fundamental family \mathcal{F} on X . It follows that if, in this situation, the scalar fundamental family condition holds and $t(X)$ is fully complete, then X is fully complete.

The COUNTEREXAMPLE 3.4.4, provides a situation where the filter condition holds, but not the scalar fundamental family condition. That the latter condition is stronger for continuous mappings is the next result.

PROPOSITION 3.4.5. If X, Y are separated locally convex topological vector spaces and $t : X \rightarrow Y$ is a continuous linear mapping for which the scalar fundamental family condition holds, then the filter condition holds.

Proof. Let \mathcal{U} be a base of absolutely convex 0-neighbourhoods in X , and let $\hat{t} : \hat{X} \rightarrow \hat{Y}$ be the extension of t to the completions. Suppose that the filter condition does not hold. Then by Theorem 1 of Dr. Wendy Robertson's paper (29), there exists a point a in \hat{X} such that $\hat{t}(a) = 0$, but $t(a) \neq 0$. Clearly a does not belong to X . Let $A = \{\lambda a : \lambda \text{ any scalar}\}$. For each U in \mathcal{U} let $F(U) = (A + \hat{U}) \cap X$, where \hat{U} is the closure of U in $\hat{X} \times \hat{X}$. Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is a scalar fundamental family. For clearly each $F(U)$ is non-empty and absolutely convex, and $\lambda F(U) = F(\lambda U) \in \mathcal{F}$. Let U be any member of \mathcal{U} . If, for some scalar λ , y belongs to $(\lambda a + \hat{U}) \cap X$, then $y - \lambda a$ belongs to \hat{U} , and so for any U' in \mathcal{U} there exists z belonging to $U \cap (y - \lambda a + \hat{U}')$. Hence $y - z$ belongs to $(\lambda a + \hat{U}') \cap X$, so that y belongs to $(A + \hat{U}') \cap X + U$. Thus $F(U) \subseteq F(U') + U$, and \mathcal{F} is fundamental.

Next, since $\hat{t}(A) = \{0\}$, it is clear that $t(U) \subseteq t(F(U)) \subseteq \hat{t}(\hat{U}) \cap t(X)$, for each U in \mathcal{U} . By continuity of t and \hat{t} , therefore, $\bigcap \{\overline{t(F(U))} : U \in \mathcal{U}\} = \{0\}$, and $t(\mathcal{F})$ is a convergent scalar fundamental family on $t(X)$. By the hypothesis \mathcal{F} must converge on X . Now $\bigcap \{F(U) : U \in \mathcal{U}\} = \{0\}$. For if y is a point of X belonging to the intersection then y belongs to the closure of $A + \hat{U}$, for each U , and so belongs to $A + \hat{U} + \hat{U}$, and hence belongs to A itself, since A is one-dimensional and therefore closed in \hat{X} ; since a does not belong to X , and y does, we must have $y = 0$.

Choose U in \mathcal{U} such that $\hat{U} + \hat{U}$ does not contain a . Then because \mathcal{F} must converge to $\{0\}$, we can choose $U' \subset U$ with $F(U') \subset U$. But then there exists z in $(a + \hat{U}') \cap X \subset F(U') \subset U$, so that a belongs to $U + \hat{U}' \subset \hat{U} + \hat{U}$. This is a contradiction, and therefore the filter condition must hold.

We return to studying the behaviour of the fundamental family condition towards uniform spaces. The vague observation that fundamental families tend to behave towards the hyper-spaces as Cauchy filters towards the spaces themselves is given more substance by the next result.

THEOREM 3.4.6. Let t be a mapping from a uniform space X into a uniform space Y , and let t° be the induced mapping of $\mathcal{S}(X)$ into $\mathcal{S}(Y)$ defined by $t^\circ(A) = t(A)$ for each subset A . Then the fundamental family condition holds for t if and only if the filter condition holds for t° .

Proof. Suppose the fundamental family condition holds for t , and let $\{A_\alpha : \alpha \in I\}$ be a Cauchy net in $\mathcal{S}(X)$ with its image $\{t(A_\alpha) : \alpha \in I\}$ converging in $\mathcal{S}(Y)$ to $t(B)$, for some subset B of X . By LEMMA 3.2.1 the family $\mathcal{F} = \{F_\beta : \beta \in I\}$, where $F_\beta = \bigcup \{A_\alpha : \alpha \geq \beta\}$, is a fundamental family on X , and its image $t^\circ(\mathcal{F}) = t(\mathcal{F}) = \{\bigcup \{t(A_\alpha) : \alpha \geq \beta\} : \beta \in I\}$ is a convergent fundamental family on $t(X)$. By the fundamental family condition \mathcal{F} converges in X , and again by LEMMA 3.2.1 this implies that the net $\{A_\alpha : \alpha \in I\}$ converges to $\bigcap \{\overline{F_\beta} : \beta \in I\}$ in $\mathcal{S}(X)$. Therefore the filter condition holds for t° .

Conversely suppose the filter condition holds for t° and let \mathcal{F} be a fundamental family on X with $t(\mathcal{F})$ convergent in $t(X)$. Put $B = \bigcap \{\overline{t(F)} : F \in \mathcal{F}\}$, where the closures are in $t(X)$. By LEMMA 3.2.1, \mathcal{F} directed by \subset is a Cauchy net in $\mathcal{S}(X)$ and

$t(\mathcal{F})$ is a net converging to B in $\mathcal{S}(t(X))$. Since B belongs to $t^0(\mathcal{S}(X))$ the filter condition demands that the net \mathcal{F} converges to a point in $\mathcal{S}(X)$. But then by LEMMA 3.2.1 \mathcal{F} converges as a fundamental family on X , and therefore the fundamental family condition holds for t .

Dr. Wendy Robertson showed in (29) that when X and Y are separated topological vector spaces, $t : X \rightarrow Y$ is a continuous linear mapping and $\hat{t} : \hat{X} \rightarrow \hat{Y}$ is the extension to completions, then the filter condition holds if and only if $\hat{t}^{-1}(0) = t^{-1}(0)$. In particular, when t is injective the filter condition holds if and only if \hat{t} is injective. She gives the following example to demonstrate that, when t is a uniformly continuous mapping from a uniform space X into a uniform space Y , the filter condition is not sufficient to infer injectiveness of \hat{t} from injectiveness of t .

COUNTEREXAMPLE 3.4.7. Let X be the interval $-\pi < x < \pi$, and let $Y = \{e^{i\theta} : -\pi < \theta < \pi\}$, with topologies induced by real line and complex plane topologies, respectively. Define $t(x) = e^{ix}$. Then t is injective and uniformly continuous, and the filter condition holds, but $\hat{t}(-\pi) = \hat{t}(\pi)$.

The fundamental family condition does not hold for this example. For consider the family \mathcal{F} on X consisting of the sets

$$F_n = (-\pi, -\pi + \frac{1}{n}) \cup (\pi - x - \frac{1}{n}, \pi), n = 1, 2, 3, \dots$$

where x is any small positive number. Then clearly \mathcal{F} is fundamental. Furthermore,

$$t(F_n) = \{e^{i\theta} : -\pi < \theta < -\pi + \frac{1}{n} \text{ or } \pi - x - \frac{1}{n} < \theta < \pi\},$$

$$C = \bigcap \{\overline{F_n} : n = 1, 2, \dots\} = [\pi - x, \pi), \text{ and}$$

$$D = \bigcap \{\overline{t(F_n)} : n = 1, 2, \dots\} = \{e^{i\theta} : \pi - x \leq \theta < \pi\}.$$

Now any ϵ -neighbourhood of D eventually contains $t(F_n)$, so $t(\mathcal{F})$ converges in $t(X)$. But an ϵ -neighbourhood of C may not contain any F_n , because each F_n has component distant $2\pi - \frac{1}{n} - \frac{1}{n}$ from C . Thus \mathcal{F} does not converge in X .

If, in an attempt to find a situation where the fundamental family does hold and \hat{t} is still not injective, we take X to be the interval $(-\pi, \pi]$ and Y the circle $\{e^{i\theta} : -\pi < \theta \leq \pi\}$, with $t(x) = e^{ix}$, then the fundamental family \mathcal{F} described above suffices again to thwart the attempt.

I do not know if the fundamental family condition is sufficient for inferring injectiveness of \hat{t} from injectiveness of t . But there is a strengthening of the condition which does achieve this in a very heavy-handed way.

STRONG FUNDAMENTAL FAMILY CONDITION. If \mathcal{F} is a fundamental family on X and $t(\mathcal{F})$ is a convergent family in $\hat{t}(\hat{X})$, then \mathcal{F} converges in X .

This condition implies that every Cauchy filter on X which has image converging to a point of $\hat{t}(\hat{X})$ must converge to a point of X . Hence X must be complete.

There is also another kind of filter condition which achieves similar ends in more gentle fashion.

SECOND FILTER CONDITION. If ϕ is a filter base on X and $t(\phi)$ is a Cauchy filter base on $t(X)$, then ϕ is also Cauchy.

PROPOSITION 3.4.8. If t is a uniformly continuous mapping from X into Y , where X, Y are separated uniform spaces, and if the second filter condition holds, then t is open and injective and its extension $\hat{t} : \hat{X} \rightarrow \hat{Y}$ is also injective.

Proof. Let \mathcal{U}, \mathcal{V} be bases for the uniformities on $X, t(X)$,

respectively. Let x be a point of X . Then the collection $\{V(t(x)) : V \in \mathcal{V}\}$ is a Cauchy filter base on $t(X)$, so that $\{t^{-1}(V(t(x))) : V \in \mathcal{V}\}$ is also a Cauchy filter base on X , by the second filter condition. Thus for each U in \mathcal{U} there is some V in \mathcal{V} such that $V(t(x)) \subset t(U(x))$, and so t is open.

Now let \hat{y} belong to $\hat{t}(\hat{X})$, and put $C = \hat{t}^{-1}(\hat{y})$. Then C is a nonempty closed subset of \hat{X} , and there exists a fundamental family \mathcal{F} on X such that \mathcal{F} converges in \hat{X} and $\bigcap \{\bar{F}^{\hat{X}} : F \in \mathcal{F}\} = C$ (take the family $\{W(C) \cap X : W \text{ is an entourage of } \hat{X}\}$). By PROPOSITION 3.2.7, $\hat{t}(\mathcal{F}) = t(\mathcal{F})$ is a convergent family in $\hat{t}(\hat{X})$.

Let W_0 be any entourage of $\hat{t}(\hat{X})$. Choose an entourage W of \hat{X} such that $\hat{t}^2(W) \subset W_0$, using uniform continuity of \hat{t} . Then $\hat{t}(W(C)) \subset W_0(\hat{y})$. But since \mathcal{F} converges in \hat{X} , $W(C)$ eventually contains \mathcal{F} , and therefore $W_0(\hat{y})$ eventually contains $t(\mathcal{F})$. This implies that $t(\mathcal{F})$ is, in fact, a Cauchy filter base on $t(X)$, converging to \hat{y} in $\hat{t}(\hat{X})$.

By the second filter condition \mathcal{F} must be a Cauchy filter base on X , and thus for any entourage W of \hat{X} , there is some F in \mathcal{F} such that $\bar{F}^{\hat{X}}$ is W -small. Hence C must be a singleton, and \hat{t} is injective.

It has been observed that for topological vector spaces the filter condition is necessary and sufficient for a continuous, injective mapping $t : X \rightarrow Y$ to extend to an injective mapping $\hat{t} : \hat{X} \rightarrow \hat{Y}$, but that, for uniform spaces, sufficiency fails. However, there is another condition in terms of the behaviour of \hat{t} which is equivalent to the filter condition for uniform spaces.

THEOREM 3.4.9. Let X and Y be separated uniform spaces and let t be a uniformly continuous mapping from X into Y . Then

if $\hat{t} : \hat{X} \rightarrow \hat{Y}$ is the extension of t to the completions, the filter condition holds for t if and only if $\hat{t}^{-1}(t(X)) = X$.

Proof. Suppose the filter condition holds. Let \hat{x} belong to $\hat{t}^{-1}(t(X))$, so that $\hat{t}(\hat{x})$ belongs to $t(X)$. There is a Cauchy net $\{x_\alpha\}$ in X converging to the point \hat{x} in \hat{X} . Since \hat{t} is uniformly continuous $\hat{t}(x_\alpha) \rightarrow \hat{t}(\hat{x})$ in $\hat{t}(\hat{X})$, so that the net $\{t(x_\alpha)\}$ converges to a point of $t(X)$. Then, by the filter condition $x_\alpha \rightarrow x$ in X , and hence $\hat{x} = x$ and $\hat{t}^{-1}(t(X)) \subset X$. The reverse inclusion is trivial, and thus equality holds.

Conversely suppose that $\hat{t}^{-1}(t(X)) = X$. Let $\{x_\alpha\}$ be a Cauchy net in X with $\{t(x_\alpha)\}$ converging to a point $t(x)$ of $t(X)$. There is a point \hat{x} in \hat{X} such that $x_\alpha \rightarrow \hat{x}$ in \hat{X} , and then also $\hat{t}(x_\alpha) \rightarrow \hat{t}(\hat{x})$ in $\hat{t}(\hat{X})$. But then $t(x) = \hat{t}(\hat{x})$, and \hat{x} belongs to $\hat{t}^{-1}(t(X))$, and hence, by hypothesis, to X . Thus $\{x_\alpha\}$ converges to a point of X , and the filter condition holds.

There is an analogue of the foregoing result for the fundamental family condition. First, two lemmas are given concerning the induced mappings $t^0 : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ and $t' : \mathcal{S}(X) \rightarrow \mathcal{E}(Y)$. Recall that for each A in $\mathcal{S}(X)$, $t^0(A) = t(A)$ and $t'(A) = \overline{t(A)}$.

LEMMA 3.4.10. If t is uniformly continuous then, for the induced mapping t' , the filter condition holds on $\mathcal{S}(X)$ if and only if it holds on $\mathcal{E}(X)$.

Proof. Suppose it holds on $\mathcal{S}(X)$. Let $\{M_\alpha\}$ be a Cauchy net in $\mathcal{E}(X)$ with $t'(M_\alpha) \rightarrow t'(R)$ in $t'(\mathcal{E}(X))$. Then $\{M_\alpha\}$ is a Cauchy net in $\mathcal{S}(X)$ and $\{t(M_\alpha)\}$ converges to a point of $t'(\mathcal{S}(X))$, so by the filter condition on $\mathcal{S}(X)$, $M_\alpha \rightarrow M$ in $\mathcal{S}(X)$. But then $M \rightarrow \bar{M}$ in $\mathcal{E}(X)$, and therefore the filter condition holds on $\mathcal{E}(X)$.

Conversely, suppose the filter condition holds on $\mathcal{E}(X)$, and let $\{A_\alpha\}$ be a Cauchy net in $\mathcal{S}(X)$ with $t'(A_\alpha) \rightarrow t'(B)$ in $t'(\mathcal{S}(X))$. Then $\{\bar{A}_\alpha\}$ is a Cauchy net in $\mathcal{E}(X)$ and, by uniform continuity,

$\overline{t(A_\alpha)} = \overline{\tau(A_\alpha)}$ for each α , so that $t'(A_\alpha) \rightarrow t'(B) = t'(\overline{B})$ in $t(\mathcal{E}(X))$. Hence by the filter condition on $\mathcal{E}(X)$, $\{\overline{A_\alpha}\}$ converges to a point in $\mathcal{E}(X)$. But $\{A_\alpha\}$ must converge to the same point in $S(X)$, and therefore the filter condition holds on $S(X)$.

LEMMA 3.4.11. Let \mathcal{H} be a subset of $S(X)$. Then the filter condition holds for t' on \mathcal{H} if and only if it holds for t° on \mathcal{H} .

Proof. Suppose the former is true, and let $\{A_\alpha\}$ be a Cauchy net in \mathcal{H} with $t(A_\alpha) \rightarrow t(B)$, for some B in \mathcal{H} . Then $t'(A_\alpha) \rightarrow t'(B)$, so, by the supposition, $\{A_\alpha\}$ converges to a point of \mathcal{H} . Therefore the filter condition holds for t° on \mathcal{H} .

Conversely suppose the latter is true, and let $\{A_\alpha\}$ be a Cauchy net in \mathcal{H} with $t'(A_\alpha) \rightarrow t'(B)$, for some B in \mathcal{H} . Then $t(A_\alpha) \rightarrow t(B)$, and by supposition, $\{A_\alpha\}$ converges to a point of \mathcal{H} . Therefore the filter condition holds for t' on \mathcal{H} .

THEOREM 3.4.12. Let X, Y be uniform spaces and t be a uniformly continuous mapping from X into Y . Let $t' : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ be the induced mapping defined by $t'(M) = \overline{t(M)}$ for each M in $\mathcal{E}(X)$, and $\hat{t}' : \hat{\mathcal{E}}(X) \rightarrow \hat{\mathcal{E}}(Y)$ the extension to completions. Then the fundamental family condition holds for t if and only if $\hat{t}'^{-1}(t'(\mathcal{E}(X))) = \mathcal{E}(X)$.

Proof. The fundamental family condition holds for t if and only if the filter condition holds for the induced mapping t° on $S(X)$, by THEOREM 3.4.6, which is true if and only if the filter condition holds for t' on $\mathcal{E}(X)$, by LEMMA 3.4.10 and LEMMA 3.4.11, which in turn is true if and only if $\hat{t}'^{-1}(t'(\mathcal{E}(X))) = \mathcal{E}(X)$, by THEOREM 3.4.9.

Remarks. The foregoing result can be deduced quite easily from the development in section 3.3, and the method has some intrinsic interest. Thus if $t: X \rightarrow Y$ is a uniformly continuous

mapping between uniform spaces, define the mapping $t^* : \phi(X) \rightarrow \phi(Y)$ by putting $t^*[\mathcal{F}] = [t(\mathcal{F})]$, as in the preamble for PROPOSITION 3.3.6. The mapping t^* is easily seen to be well-defined and uniformly continuous. If h_1, h_2 are the canonical embeddings of $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ in $\phi(X)$ and $\phi(Y)$ respectively, the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{E}(X) & \xrightarrow{t'} & \mathcal{E}(Y) \\ h_1 \downarrow & & \downarrow h_2 \\ \phi(X) & \xrightarrow{t^*} & \phi(Y) \end{array}$$

Furthermore, on identifying $\hat{\mathcal{E}}(X)$ with $\phi(X)$ and $\hat{\mathcal{E}}(Y)$ with $\phi(Y)$, the mappings \hat{t}' and t^* coincide. Recall that $h_1(\mathcal{E}(X)) = \phi_0(X)$ and $h_2(\mathcal{E}(Y)) = \phi_0(Y)$, the respective subspaces of $\phi(X)$ and $\phi(Y)$ determined by the convergent families; by uniform continuity of t , t^* takes $\phi_0(X)$ into $\phi_0(Y)$.

Now the fundamental families which converge in $t(X)$ can easily be seen to determine the same elements of $\phi(Y)$ as the images $t(\mathcal{F})$ of families convergent in X , and so the fundamental family condition can be expressed in the form

$$t^{*-1}(t^*(\phi_0(X))) = \phi_0(X),$$

and the scalar fundamental family condition (for suitable X, Y), in the form

$$\phi_s(X) \cap t^{*-1}(t^*(\phi_0(X))) = \phi_{s0}(X).$$

But the first equation is just the condition in THEOREM 3.4.12 when the appropriate identifications are made.

It is noteworthy that in situations like COUNTEREXAMPLE 3.4.7, where $\mathcal{E}(\hat{X})$ and $\mathcal{E}(\hat{Y})$ are complete and can therefore be identified with $\hat{\mathcal{E}}(X)$ and $\hat{\mathcal{E}}(Y)$, the THEOREM provides a useful approach to the problem of ascertaining whether the fundamental family condition holds or not. It was the negative conclusion

arrived at by this approach that intensified the half-hearted search for the particular fundamental family exhibited in COUNTER-EXAMPLE 3.4.7.

When t is a continuous linear mapping from a topological vector space X into a topological vector space Y , the fundamental family condition has significant implications for the induced mappings between quotient spaces. Let M be a closed vector subspace of X . Then the quotient space X/M is a separated topological vector space, and a closed uniform subspace of $\mathcal{E}(X)$ (section 1.11). The mapping $t' : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ induces a continuous linear mapping $t'_1 : X/M \rightarrow Y/\overline{t(M)}$, and the extension $\hat{t}' : \hat{\mathcal{E}}(X) \rightarrow \hat{\mathcal{E}}(Y)$ induces the continuous linear mapping $\hat{t}'_1 : \hat{X/M} \rightarrow \hat{Y/\overline{t(M)}}$ which is the extension of t'_1 to the completions of the topological vector spaces X/M and $Y/\overline{t(M)}$. The mapping t'_1 is injective if and only if $t^{-1}(\overline{t(M)}) = M$. If this condition holds and if in addition t is injective on M , then t is injective on X .

Given that t'_1 is injective, we may ask under what circumstances the extension \hat{t}'_1 is also injective. An answer is given by the following result.

PROPOSITION 3.4.13. Let X and Y be topological vector spaces, $t: X \rightarrow Y$ a continuous linear mapping, and M a closed vector subspace of X . Then if the fundamental family condition holds and if the induced mapping t' is injective on X/M , its extension \hat{t}' must be injective on $\hat{X/M}$.

Proof. If the fundamental family condition holds for t then by THEOREM 3.4.6 the filter condition holds for $t^0 : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$, and by LEMMA 3.4.10 and LEMMA 3.4.11, the filter condition holds for $t' : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$. It follows easily that the filter condition

holds on every closed subspace of $\mathcal{E}(X)$, and in particular it must hold on the quotient space X/M . But the restriction $t'_1 : X/M \rightarrow Y/\overline{t(M)}$ is a continuous linear mapping between separated topological vector spaces, so if it is injective we can apply Dr. Wendy Robertson's Theorem 1 in (29) to deduce that the extension $\hat{t}'_1 : X/M \rightarrow Y/\overline{t(M)}$ is also injective.

3.5 The induced mapping t' and its extension. The mappings t' and \hat{t}' have figured quite considerably - notably in connection with the fundamental family condition. This section is devoted to further study of their properties.

Let X and Y be uniform spaces, let A be a uniform subspace of X and let $t : A \rightarrow Y$ be a uniformly continuous mapping with domain A . If Y is a complete separated space there exists a unique extension $\hat{t} : \bar{A} \rightarrow Y$ to the closure of A in X , whose graph is the closure in $X \times Y$ of the graph of t (see for example Kelley (15)). Similarly, if Y is also hypercomplete, the induced mapping $t' : \mathcal{S}(A) \rightarrow \mathcal{E}(Y)$ has a unique extension $\hat{t}' : \overline{\mathcal{S}(A)} \rightarrow \mathcal{E}(Y)$ to the closure of $\mathcal{S}(A)$ in $\mathcal{S}(X)$, whose graph is the closure in $\mathcal{S}(X) \times \mathcal{E}(Y)$ of the graph of t' (or the "hypergraph" of t , which is studied in Chapter 5). The next Proposition will show that this extension exists even if Y is not hypercomplete.

Now we can arrive at another mapping between $\mathcal{S}(X)$ and $\mathcal{E}(Y)$ by reversing the order of the inducing and extending operations. Thus the extended mapping $\hat{t} : \bar{A} \rightarrow Y$ induces the mapping $\hat{t}' : \mathcal{S}(\bar{A}) \rightarrow \mathcal{E}(Y)$. It turns out that these two operations are commutative, in the sense of the following result.

PROPOSITION 3.5.1. When t is a uniformly continuous mapping from a subspace A of a uniform space X into a hypercomplete separated uniform space Y , then the two derived mappings

$\hat{t}' : \overline{S(A)} \rightarrow \mathcal{E}(Y)$ and $\hat{t}' : S(\bar{A}) \rightarrow \mathcal{E}(Y)$ have the same domain and coincide thereon.

Proof. First we show that $\overline{S(A)} = S(\bar{A})$. Let B belong to $S(\bar{A})$. Then for each symmetric entourage U of X the set $C = U(B) \cap A$ belongs to $S(A)$, and $C \subset U(B)$. If b is a point of B there is some point a of A with $a \in U(b) \subset U(B)$. Thus a belongs to C , and since $b \in U(a) \subset U(C)$, we have $B \subset U(C)$. Therefore (B, C) belongs to \bar{U} and this shows that B is in $\overline{S(A)}$.

Conversely, if B belongs to $\overline{S(A)}$ and U is any entourage of X , there exists a subset C of A with $B \subset U(C) \subset U(A)$. Therefore $B \subset \bar{A}$, that is, B belongs to $S(\bar{A})$.

To show that \hat{t}' and \hat{t}' coincide on $\overline{S(A)}$, it is sufficient to show that they coincide on the dense subspace $S(A)$. If B is in $S(A)$ then

$$\begin{aligned} \hat{t}'(B) &= \overline{\hat{t}(B)}, \text{ where the closure is in } Y, \\ &= \overline{t(B)}, \text{ since } B \subset A, \\ &= t'(B) \\ &= \hat{t}'(B). \end{aligned}$$

COROLLARY 3.5.2. If Y is a separated, complete space (not necessarily hypercomplete) then the induced mapping t' of $S(A)$ into $\mathcal{E}(Y)$ still has a unique extension mapping $\overline{S(A)}$ into $\mathcal{E}(Y)$.

Proof. The mapping $t: A \rightarrow Y$ has a unique extension $\hat{t}: \bar{A} \rightarrow Y$, and the induced mapping $\hat{t}' : S(\bar{A}) \rightarrow \mathcal{E}(Y)$ is the required extension of t' , from the proof of the previous THEOREM.

Now suppose that X and Y are separated uniform spaces and that t is a uniformly continuous mapping from X into Y .

Then t induces the mapping t' of $\mathcal{E}(X)$ into $\mathcal{E}(Y)$, which extends to the mapping \hat{t}' of $\hat{\mathcal{E}}(X)$ into $\hat{\mathcal{E}}(Y)$. The mapping t itself extends to the mapping \hat{t} of \hat{X} into \hat{Y} , which induces the mapping \hat{t}' of $\mathcal{E}(\hat{X})$ into $\mathcal{E}(\hat{Y})$, and finally this also extends to the mapping $\hat{\hat{t}}$ of $\hat{\mathcal{E}}(\hat{X})$ into $\hat{\mathcal{E}}(\hat{Y})$. Out of this rather complex situation one consoling fact emerges. It was shown in Chapter 2 that the spaces $\hat{\mathcal{E}}(X)$ and $\hat{\mathcal{E}}(\hat{X})$ are isomorphic, and likewise the spaces $\hat{\mathcal{E}}(Y)$ and $\hat{\mathcal{E}}(\hat{Y})$ are isomorphic; this suggests that under these identifications the mappings \hat{t}' and $\hat{\hat{t}}$ may be the same.

PROPOSITION 3.5.3. When X and Y are separated uniform spaces and t is a uniformly continuous mapping from X into Y , the derived mappings $\hat{t}' : \hat{\mathcal{E}}(X) \rightarrow \hat{\mathcal{E}}(Y)$ and $\hat{\hat{t}}' : \hat{\mathcal{E}}(\hat{X}) \rightarrow \hat{\mathcal{E}}(\hat{Y})$ coincide on identification of $\hat{\mathcal{E}}(X)$ with $\hat{\mathcal{E}}(\hat{X})$ and $\hat{\mathcal{E}}(Y)$ with $\hat{\mathcal{E}}(\hat{Y})$.

Proof. This can be shown by the method of the previous PROPOSITION - identifying each closed subset of X with its closure in \hat{X} according to the isomorphism $\hat{\mathcal{E}}(X) \rightarrow \hat{\mathcal{E}}(\hat{X})$ exhibited in Chapter 2 section 4, and verifying that \hat{t}' and $\hat{\hat{t}}$ take the respective sets thus identified onto identified sets in $\hat{\mathcal{E}}(Y)$ and $\hat{\mathcal{E}}(\hat{Y})$, and so coincide on a dense subspace of $\hat{\mathcal{E}}(X)$.

It is more elegant, and simple, to deduce the result from the work in section 3. Recalling the remarks after THEOREM 3.4.12, the problem becomes that of showing that the mappings $t^* : \Phi(X) \rightarrow \Phi(Y)$ and $\hat{t}^* : \Phi(\hat{X}) \rightarrow \Phi(\hat{Y})$ coincide when the corresponding spaces are identified. By the remarks after THEOREM 3.3.6, an element $[\mathcal{F}]$ in $\Phi(X)$ and an element $[\mathcal{G}]$ in $\Phi(\hat{X})$ are identified if \mathcal{F} and \mathcal{G} determine the same element in $\Phi(\hat{X})$. But then $t(\mathcal{F})$ and $\hat{t}(\mathcal{G})$ must determine the same element in $\Phi(\hat{Y})$ by the very well-definedness of \hat{t}^* ! This completes the proof.

There is an immediate application of the foregoing result to quotient spaces of topological vector spaces.

PROPOSITION 3.5.4. If X and Y are separated topological vector spaces, M is a closed vector subspace of X , and t is a continuous linear mapping from X into Y for which the fundamental family condition holds, then provided that t' is injective on X/M , so is \hat{t}' on \hat{X}/\hat{M} and $\hat{\hat{t}}'$ on $\hat{\hat{X}}/\hat{\hat{M}}$.

Proof. By PROPOSITION 3.4.13, \hat{t}' is injective on X/M . But by PROPOSITION 3.5.3, \hat{t}' coincides with $\hat{\hat{t}}'$ under the identification of $\hat{\mathcal{C}}(X)$ and $\hat{\mathcal{C}}(\hat{X})$, and by THEOREM 2.6.5 this identification takes \hat{X}/\hat{M} onto $\hat{\hat{X}}/\hat{\hat{M}}$.

The last result of the section is a generalisation to uniform spaces of one of Dr. Wendy Robertson's results concerning the filter condition for topological vector spaces, allowing as a corollary an analogous result concerning the fundamental family condition.

PROPOSITION 3.5.5. Let X and Y be separated uniform spaces and let t be a surjective, uniformly continuous mapping from X to Y . If the filter condition holds then for any precompact subset A of X , $t(\overline{A}) = \overline{t(A)}$.

Proof. Since A is precompact \hat{A} , and hence also $\hat{\hat{t}}(\hat{A})$, are compact, and so complete. Here \hat{t} is the extension of t to the completions \hat{X} , \hat{Y} . But $t(A) \subset \hat{t}(\hat{A}) \subset \hat{\hat{t}}(\hat{\hat{A}})$, and therefore $\hat{t}(\hat{A}) = \hat{\hat{t}}(\hat{\hat{A}})$. Since t is surjective, $\overline{t(A)} = t(X) \bigcap \hat{t}(\hat{A}) = t(X) \bigcap \hat{\hat{t}}(\hat{\hat{A}})$.

Let y belong to $\overline{t(A)}$. Then $y = \hat{\hat{t}}(\hat{a})$ for some \hat{a} in $\hat{\hat{A}}$, and $\hat{t}(\hat{a})$ belongs to $t(X)$. By THEOREM 3.4.9 the filter condition is equivalent to $\hat{t}^{-1}(t(X)) = X$, and so a belongs to $X \bigcap \hat{A} = \overline{A}$. Thus y belongs to $t(\overline{A})$, and $\overline{t(A)} \subset t(\overline{A})$.

Since $t(\overline{A}) \subset \overline{t(A)}$ by continuity, the result follows.

COROLLARY 3.5.6. If the fundamental family condition holds for t , then for any precompact subset \mathcal{A} of $\mathcal{S}(X)$, $t'(\overline{\mathcal{A}}) = \overline{t'(\mathcal{A})}$.

Proof. By THEOREM 3.4.6 and LEMMAS 3.4.10 and 3.4.11, the filter condition holds for the mapping $t' : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$. If t is surjective so also is t' , and application of the PROPOSITION concludes the proof.

3.6 Hyperassociated uniformities. Let X be a set endowed with two uniform structures ξ and η . As in Chapter 2, section 7, we shall say that η is *associated with* ξ if ξ has a base consisting of sets closed in $X \times X$ relative to the topology derived from η .

Furthermore we shall say that η is *hyperassociated with* ξ on a subset \mathcal{A} of $\mathcal{S}(X)$ if the uniformity $\tilde{\eta}$ on \mathcal{A} is associated with the uniformity $\tilde{\xi}$ on \mathcal{A} .

Also, in future, we shall say that the filter condition (or the fundamental family condition) holds for ξ and η when it holds for the identity mapping $(X, \xi) \rightarrow (X, \eta)$.

PROPOSITION 3.6.1. If $\mathcal{D} = \{\{x\} : x \in X\} \subset \mathcal{A} \subset \mathcal{S}(X)$ and η is hyperassociated with ξ on \mathcal{A} , then η is associated with ξ .

Proof. The uniformity $\tilde{\xi}$ on \mathcal{A} has a base of $\tilde{\eta}$ -closed entourages, and induces a uniformity on \mathcal{D} which is isomorphic to ξ . The result follows easily.

PROPOSITION 3.6.2. If ξ and η are uniformities on a set X and η is associated with ξ , then the filter condition holds for ξ and η .

Proof. Let ϕ be a ξ -Cauchy filter and let $\phi \rightarrow x$ relative to η . Then for each V in η there exists F in ϕ such that $F \subset V(x)$, and for each U in ξ there exists G in ϕ which is U -small.

Then if U is chosen to be symmetric we have $G \subseteq U(V(x))$, since $\bigcap G \neq \emptyset$. Hence $G \subseteq (VUV)(x)$ for all V in η , and, provided U is a η -closed entourage, $G \subseteq U(x)$. But ξ has a base of such entourages, and therefore $\phi \rightarrow x$ relative to the uniformity ξ .

PROPOSITION 3.6.3. If ξ and η are uniformities on a set X with η hyperassociated with ξ on $\mathcal{S}(X)$, then the fundamental family condition holds for ξ and η .

Proof. Since $\tilde{\eta}$ is associated with $\tilde{\xi}$ on $\mathcal{S}(X)$ the filter condition must hold for the identity mapping $(\mathcal{S}(X), \tilde{\xi}) \rightarrow (\mathcal{S}(X), \tilde{\eta})$, by PROPOSITION 3.6.2. But this mapping is induced by the identity mapping $(X, \xi) \rightarrow (X, \eta)$, and so by THEOREM 3.4.6 the fundamental family condition holds for ξ and η .

COROLLARY 3.6.4. If η is hyperassociated with ξ and coarser than ξ then (X, η) hypercomplete implies (X, ξ) hypercomplete.

Proof. The result follows immediately from the PROPOSITION and PROPOSITION 3.4.1.

We set out now to find out when two uniformities are hyper-associated. To begin with, it is clear that if η induces a finer topology than ξ on a subset \mathcal{A} of $\mathcal{S}(X)$ then η will be hyperassociated with ξ on \mathcal{A} . For any base of $\tilde{\xi}$ -closed entourages for $\tilde{\xi}$ will be a base of $\tilde{\eta}$ -closed entourages for $\tilde{\xi}$. We can now make immediate use of results and notation of the next Chapter to obtain the following:

If η induces a finer topology than ξ on X then η is hyperassociated with ξ on the η -compact subsets. If ξ and η are two uniformities compatible with a given topology on X then they are each hyperassociated with the other on the compact subsets.

If η induces a finer proximity than ξ on X , and is uniformly finer over X on every ξ -discrete subset, then η is hyperassociated with ξ on $\mathcal{S}(X)$.

It is plainly desirable to find weaker conditions than these. The next PROPOSITION deals with hyperassociation on compact sets; this is followed by a LEMMA and THEOREM dealing with hyperassociation on $\mathcal{S}(X)$.

PROPOSITION 3.6.5. If ξ, η are two uniformities on a set X , and η is associated with ξ , then η is hyperassociated with ξ on the set of η -compact subsets of X .

Proof. Let U be an entourage of ξ , closed with respect to the topology derived from η . It is sufficient to show that the entourage \tilde{U} of the uniformity $\tilde{\xi}$ on the set of η -compact subsets is $\tilde{\eta}$ -closed. Let (A, B) be a pair of η -compact subsets belonging to the closure of \tilde{U} . Then for each V in η there exists (A', B') in \tilde{U} such that (A, A') belongs to \tilde{V} and (B, B') belongs to \tilde{V} . Hence we have

$$A \subset (V \circ U \circ V)(B) \quad \text{and} \quad B \subset (V \circ U \circ V)(A).$$

This being true for each V in η we have $A \subset \bigcap \{(V \circ U \circ V)(B) : V \in \eta\}$. But this intersection is just $U(B)$. For if x is any point in the intersection, there are points b_v in B such that (b_v, x) belongs to $V \circ U \circ V$. Since B is η -compact there is a limit point b of the set $\{b_v : V \in \eta\}$ in B , so that for each V there exists $V' \subset V$ with (b, b_v) in V' . Thus (b, x) belongs to $V' \circ U \circ V' \subset V \circ U \circ V$ for each V in η . Since U is η -closed, (b, x) belongs to U , and x belongs to $U(B)$.

Consequently $A \subset U(B)$, and similarly $B \subset U(A)$, so that (A, B) belongs to \tilde{U} and \tilde{U} is η -closed.

LEMMA 3.6.6. If ξ, η are two uniformities for a set X and η is proximity-finer than ξ on X , then for each subset A of X and each U belonging to ξ there exist V_1, V_2 belonging to η such that

$$(V_1 \cup V_2)(A) \subset \overset{3}{U}(A)$$

Proof. The sets A and $X - U(A)$ are ξ -remote, and must also be η -remote. Thus there is some V_2 in η such that $V_2(A) \subset U(A)$. Hence $(UV_2)(A) \subset \overset{2}{U}(A)$. Then, similarly, there is some V_1 in η such that $V_1((UV_2)(A)) \subset U((UV_2)(A))$. Hence $(V_1 \cup V_2)(A) \subset \overset{3}{U}(A)$.

THEOREM 3.6.7. If ξ, η are two uniformities for a set X and η is proximity-finer than ξ , then η is hyperassociated with ξ on $S(X)$.

Proof. For each U in ξ let $W(U)$ denote the $\tilde{\eta}$ -closure in $S(X) \times S(X)$ of the entourage \tilde{U} . Put $\mathcal{W} = \{W(U) : U \in \xi\}$. We will show that \mathcal{W} is a base for the uniformity $\tilde{\xi}$ on $S(X)$.

Let U belong to ξ and choose U_0 in ξ with $\overset{3}{U}_0 \subset U$. Let (A_1, A_2) belong to $W(U_0)$. Then for each V_1, V_2 in η there exists (B_1, B_2) in \tilde{U} such that (A_1, B_1) is in \tilde{V}_1 and (A_2, B_2) is in \tilde{V}_2 . It follows that $A_1 \subset (V_1 \cup U_0 V_2)(A_2)$ and $A_2 \subset (V_2 \cup U_0 V_1)(A_1)$. Now by the LEMMA 3.6.5, V_1 and V_2 can be chosen so that $(V_1 \cup U_0 V_2)(A_2) \subset \overset{3}{U}_0(A_2)$, which implies that $A_1 \subset U(A_2)$. Furthermore V_1 and V_2 can be chosen so that $(V_2 \cup U_0 V_1)(A_1) \subset \overset{3}{U}_0(A_1)$, which implies that $A_2 \subset U(A_1)$. Therefore (A_1, A_2) belongs to \tilde{U} . We have shown that $W(U_0) \subset \tilde{U}$. Trivially $\tilde{U} \subset W(U)$, and so \mathcal{W} is a base consisting $\tilde{\eta}$ -closed entourages for the uniformity $\tilde{\xi}$.

COROLLARY 3.6.8. If (X, δ) is a proximity space then the uniformities belonging to $\pi(\delta)$ (that is all uniformities inducing the proximity δ) are each hyperassociated with every other.

Remarks. It is not difficult to show that if η is proximity-finer than ξ then every η -convergent family of subsets of X is also ξ -convergent. I suspect there is partial converse but cannot find it. It is noteworthy that if η is associated with ξ and every η -convergent family of subsets is also ξ -convergent, then η induces a finer topology than ξ .

CHAPTER 4

TOPOLOGIES INDUCED ON HYPERSPACES

4.1. Introduction. Let ξ, η be two uniformities on a set X . In section 3.6. attention was paid to the problem of when $\tilde{\eta}$ is associated with $\tilde{\xi}$. We proceed now to consider conditions under which these respective Hausdorff uniformities and their induced topologies on various hyperspaces are comparable. This subject has had an interesting history over the last decade or so, and the relevant results are included and discussed in this Chapter. The central problem has been to determine when two uniformities on X induce the same topology on $\mathcal{S}(X)$. Such uniformities are called *H-equivalent*; if they induce the same topology on some $\mathcal{A} \subset \mathcal{S}(X)$ we shall call them *H-equivalent on \mathcal{A}* .

J. R. Isbell asserted (wrongly as it turned out) in (12), page 35, exercise 17, that distinct uniformities are not H-equivalent, with the implication that the topology on $\mathcal{S}(X)$ is "sufficient" to distinguish between uniformities on X . D. H. Smith in (30) showed that Isbell's proof was in error, but proved some partial results tending to support the conjecture. In particular, H-equivalent uniformities must lie in the same proximity class; this has two significant corollaries : that two distinct uniformities, at least one of which is precompact, cannot be H-equivalent, and that two distinct uniformities, each with countable base, cannot be H-equivalent. Two uniformities in the same proximity class must each be hyperassociated with the other (see section 3.6), but they need not be H-equivalent, as it will appear.

A. J. Ward (34), A. A. Ivanov (13) and Isbell himself (11) found counterexamples to the conjecture, and finally Ward gave, in (35), necessary and sufficient conditions for two uniformities to be H-equivalent. Strangely enough, F. Albrecht (1) had given slightly different equivalent conditions years before, but due (I suggest) to a misrepresentation in Maths. Reviews his result appears to have passed unnoticed. Ward's and Albrecht's results are included in section 2. In response to Ward's and Isbell's counterexamples, and in the following year 1967, J. L. Hirsch (10) brought his notion of "height" to the problem - this is a relation between uniformities which is dual in a sense to that of proximity. He showed that two uniformities equal in both proximity and height must satisfy Ward's conditions, and so are H-equivalent. That there exist distinct uniformities of this intimacy had previously been demonstrated by Hirsch.

In section 3 we ask when two uniformities are H-equivalent on various smaller sets than $\mathcal{S}(X)$. If ξ and η are simply given to induce the same topology on X (we shall describe them as *compatible* for brevity), then they are H-equivalent on the set of compact subsets. But they are not necessarily H-equivalent on the set of η -precompact subsets; for this to be true it is necessary that every η -precompact subset be ξ -precompact. Sufficient conditions involving the notion of height are given for ξ and η to be H-equivalent on the set of all precompact subsets. They may still not be H-equivalent on $\mathcal{S}(X)$, however. For this they must be proximity-equivalent and each must be uniformly finer than the other over X on each subset which is discrete with respect to the other (these are Ward's conditions - see section 2 for the definitions). Given all

this, ξ and η will certainly induce the same topology on the set of compact subsets of $S(X)$, but they may yet induce different topologies on $S(S(X))$.

In order to induce the same topology on $S(S(X))$, the Hausdorff uniformities $\tilde{\xi}$ and $\tilde{\eta}$ must be proximity-equivalent on $S(X)$, and V. Z. Poljakov (24) showed in 1968 that in that case ξ and η must coincide, so that of course all the induced uniformities and topologies coincide. Poljakov's proof is given in slightly different form in section 4, and it is apparent that even on the set $\mathcal{F}(X)$ of finite subsets of X the proximity structure distinguishes between uniformities on X .

Thus, where the topology on $S(X)$ is insufficient for distinguishing uniformities, the topology on $S(S(X))$, and even on $S(\mathcal{F}(X))$, is sufficient. While the topology on X is insufficient for determining the topology on $S(X)$, it is sufficient for determining the topology on $\mathcal{C}(X)$. In general, one topology on a set may arise from many uniform structures, unless it is separated and compact, when there is only one. Also one proximity structure may arise from many uniform structures, precisely one of which is precompact. Of the class of all induced uniformities on $S(X)$ or $\mathcal{F}(X)$, however, we can say that no two have the same proximity structure, and of the class of all induced uniformities on $S(S(X))$ or $S(\mathcal{F}(X))$, we can say that no two have the same topology.

Next we consider the problem of when H-equivalence of two uniformities implies identity. Ward has shown in (35) that when the uniform space (X, ξ) has a certain rather complicated homogeneity of structure, there can be no other uniformity on X H-equivalent to ξ . It is proved in section 5 that if (X, ξ) is the union of a compact collection of discrete sets then the same thing can be said. Ward's result follows from this.

4.2. The H-equivalence of uniformities. We call η *H-finer than ξ on a subset \mathcal{A} of $\mathcal{S}(X)$* if $\tilde{\eta}$ induces a finer topology than $\tilde{\xi}$ on \mathcal{A} . When $\mathcal{A} = \mathcal{S}(X)$ we merely say *H-finer*. The two uniformities ξ, η are H-equivalent if each is H-finer than the other. To begin with, it is obvious that if η is finer than ξ it must be H-finer. Before concentrating on topologies it seems logically desirable to state the case for comparability of uniformities; the following THEOREM is immediate.

THEOREM 4.2.1. If ξ, η are uniformities on a set X which are either compatible or both separated, then the following are equivalent:

- (1) η is finer than ξ on X ;
- (2) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on the set of ξ -compact subsets;
- (3) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on the set of η -compact subsets;
- (4) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on the set of ξ -precompact subsets;
- (5) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on the set of η -precompact subsets;
- (6) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on the set of ξ -closed subsets;
- (7) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on the set of η -closed subsets;
- (8) $\tilde{\eta}$ is finer than $\tilde{\xi}$ on $\mathcal{S}(X)$.

For η to be H-finer than ξ on $\mathcal{S}(X)$ it is clearly necessary that η induce a finer topology on X . However it is not necessary that the various conditions of THEOREM 4.2.1. should hold, as was demonstrated by the counterexamples of Ward, Ivanov and Isbell. It was Albrecht (1) who first found necessary and sufficient conditions, in the following form:

Given that ξ, η induce the same topology on X , η is H-finer than ξ if and only if, for each M in $\mathcal{C}(X)$ and each U in ξ , there exists V in η such that

- (i) $V(M) \subset U(M)$, and
- (ii) for each point x in M there is some point x' in M with $V(x') \subset U(x)$.

Ward (35) gave necessary and sufficient conditions in somewhat more elegant form and we include his result, together with a slightly altered version of Albrecht's, in the next THEOREM and (lest there be any further confusion on the matter!) prove them equivalent.

First, some definitions are required, for use throughout the Chapter.

Let A be any subset of X . Then η is said to be *uniformly finer than ξ on A over X* if, given any U in ξ , there exists V in η such that $V \bigcap (A \times X) \subset U$ (or, equivalently, $V(a) \subset U(a)$ for each point a in A).

A subset B of X is called *U -discrete*, for U in ξ , if, for each point b in B , $U(b) \bigcap B = \{b\}$. The subset B is called *ξ -discrete* if it is U -discrete for some U in ξ .

We recall that η is *proximity-finer* than ξ if and only if for each $A \subset X$ and each U in ξ there exists a V in η such that $V(A) \subset U(A)$. We describe this by saying that η induces finer uniform neighbourhoods of A than ξ .

THEOREM 4.2.2. Let ξ, η be two uniformities on a set X . Then the following are equivalent:

- (1) η is H -finer than ξ ;
- (2) (WARD'S CONDITIONS) η is proximity-finer than ξ and uniformly finer over X on every ξ -discrete set;
- (3) (ALBRECHT'S CONDITIONS) for each $A \subset X$ and U in ξ there exists V in η such that

- (i) $V(A) \subset U(A)$, and
- (ii) for each x in A there is some x' in A with $V(x') \subset U(x)$.

Proof. The equivalence of (1) and (2) was proved by Ward (35).

We shall simply show that (2) and (3) are equivalent.

Suppose that (2) is true, and let $A \subset X$ and U in ξ be given. Let U_1 be a symmetric entourage in ξ with $U_1^2 \subset U$, and let B be a maximal U_1 -discrete subset of A . By (2) there exists V_1, V_2 in η such that $V_1(A) \subset U(A)$ and $V_2 \bigcap (B \times X) \subset U_1$. Put $V = V_1 \bigcap V_2$. Then V belongs to η and $V(A) \subset U(A)$. Furthermore, if x is any point in A , there is a point x' in B such that (x, x') belongs to U_1 , by the maximality of B ; also $V_2(x') \subset U_1(x')$, so that $V(x') \subset V_2(x') \subset U_1(x') \subset U_1^2(x) \subset U(x)$. Therefore (3) is true.

Conversely, suppose that (3) is true. Then, quite trivially, η is proximity-finer than ξ . Let A be a U -discrete subset of X , for some U in ξ . Then there exists V in η such that condition (ii) of (3) holds, where in fact $x' = x$ since A is U -discrete. Thus $V(x) \subset U(x)$ for each x in A , which means that η is uniformly finer than ξ on A over X .

Remark. It is easy to see that if ξ and η are compatible uniformities on X then η is H -finer than ξ on $S(X)$ if and only if it is H -finer on $\mathcal{C}(X)$, and this is true if and only if η induces a finer proximity between the closed subsets of X and is uniformly finer over X on every ξ -discrete subset.

4.3 The H -equivalence of uniformities on subsets of $S(X)$.

When two uniformities are not H -equivalent on $S(X)$ then looking for subsets of $S(X)$ on which they are H -equivalent can yield a measure of their failure in this respect. The next result fixes

the zero - any two compatible uniformities must at least be H-equivalent on the set of compact subsets.

THEOREM 4.3.1. Let ξ, η be two uniformities on a set X .

Then the following are equivalent:

- (1) η induces a finer topology than ξ on X ;
- (2) η induces finer uniform neighbourhoods of η -compact subsets;
- (3) η is uniformly finer over X on every finite ξ -discrete subset;
- (4) η is uniformly finer over X on every finite subset;
- (5) η is H-finer on the set of η -compact subsets.

Proof. Trivially, (2) implies (1). To show that (1) implies (2) let C be any η -compact subset and let U in ξ be given. Then $A = X - U(C)$ is a ξ -closed set and so is also η -closed. Since C is η -compact and disjoint from A , there exists V in η such that $V(C) \cap A = \emptyset$ (see e.g. Bourbaki (4)). Thus $V(C) \subset U(C)$, and (2) follows. Next, it is obvious that (4) implies (3) and clear that (3) implies (1). We prove that (1) implies (4). Let B be a finite subset and let U in ξ be given. For each b_i in B there is a V_i in η such that $V_i(b_i) \subset U(b_i)$. Then $V = \bigcap_i V_i$ belongs to η , and if b is any point in B , $V(b) \subset U(b)$ so that $V \bigcap (B \times X) \subset U$. We now have (1), (2), (3) and (4) equivalent.

That (5) implies (1) is obvious, so it remains to prove that the truth of the conditions (1) - (4) implies the truth of (5). Let C_0 be an η -compact subset and let U in ξ be given. By (2) there is some V_0 in η such that $V_0(C_0) \subset U(C_0)$. Choose a symmetric U_1 in ξ with $U_1^2 \subset U$. Then, since C_0 must also be ξ -compact by (1), there is a finite ξ -discrete set $B \subset C_0$ such that $C_0 \subset U_1(B)$. By (4) we can choose a symmetric V in η such

that both $V \subset V_0$ and $V \cap (B \times X) \subset U_1$. Now let C be any η -compact set such that (C, C_0) belongs to \tilde{V} . For any point b in B there exists a point c in C such that (b, c) belongs to V , because $B \subset C_0 \subset V(C)$. Then (b, c) must also belong to U_1 , because $V \cap (B \times X) \subset U_1$. Since U_1 is symmetric we have $B \subset U_1(C)$, and hence $C_0 \subset \overset{2}{U}_1(C) \subset U(C)$. Also, $C \subset V(C_0) \subset V_0(C_0) \subset U(C_0)$, so that (C, C_0) belongs to \tilde{U} , and the truth of (5) is established.

It is natural to ask whether this result can be extended - whether compatible uniformities must be H -equivalent on the set of ξ -precompact subsets or the set of η -precompact subsets. An examination of the foregoing proof yields the following:

COROLLARY 4.3.2. If η induces finer uniform neighbourhoods of ξ -precompact subsets than does ξ , then η is H -finer on the set of ξ -precompact sets. If η induces finer uniform neighbourhoods of η -precompact subsets and if every η -precompact subset is also ξ -precompact, then η is H -finer on the set of η -precompact subsets.

However, in general, compatible uniformities are not H -equivalent on the set of all precompact subsets. This is a consequence of the following two LEMMAS, the first of which is proved using ideas of Ward's paper (35).

LEMMA 4.3.3. If η is H -finer than ξ on the set of η -precompact subsets, then η is uniformly finer over X on every ξ -discrete η -precompact subset.

Proof. Let A be an η -precompact and U_0 -discrete subset of X , where U_0 belongs to ξ , and let U in ξ be given. Let U_1 be a symmetric entourage of ξ contained in $U_0 \cap U$. Then there is some symmetric V in η such that any η -precompact set B belonging to $\tilde{V}(A)$ must belong also to $\tilde{U}_1(A)$. In particular, the set $B = \{y_0\} \cup (A - \{x_0\})$, where x_0 belongs to A and (x_0, y_0)

belongs to V , must belong to $\tilde{U}_1(A)$, and therefore $A \subset U_1(B)$.

Thus there exists y' in B such that (x_0, y') belongs to $U_1 \subset U_0$. Since A is U_0 -discrete, $y' = y_0$. Thus for any point x in A and any point y in $V(x)$ we must have y belonging to $U_1(x) \subset U(x)$, and so $V \bigcap (A \times X) \subset U$.

LEMMA 4.3.4. If η is uniformly finer than ξ over X on every ξ -discrete η -precompact subset, then every η -precompact subset is ξ -precompact.

Proof. Let A be η -precompact, and let U_0 in ξ be given. Choose a symmetric U_1 in ξ with $\tilde{U}_1^2 \subset U_0$, and choose a maximal U_1 -discrete subset of A , say B . Then $A \subset U_1(B)$ and there exists by hypothesis some V in η such that $V \bigcap (B \times X) \subset U_1$. Since B is η -^{pre}compact, there exists a finite set $F \subset B$ such that $B \subset V(F)$. But then also $B \subset U_1(F)$, and so $A \subset \tilde{U}_1^2(F) \subset U_0(F)$. Therefore A is ξ -precompact.

By the previous two LEMMAS, in order that ξ and η be H -equivalent on the set of η -precompact subsets, every η -precompact subset must also be ξ -precompact, and in order that ξ and η be H -equivalent on the set of subsets which are either ξ -precompact or η -precompact, we must have the respective collections of precompact subsets coinciding. If, on the other hand, these collections are known to coincide, and if the respective systems of uniform neighbourhoods of each precompact set induced by ξ and η form equivalent filter bases, then by COROLLARY 4.3.2, ξ and η are H -equivalent on the set of all precompact subsets.

At this point we recall Hursch's notion of height. The uniformity ξ is said to be *less than or equal in height* (\leq) to η if for each U in ξ there exists a finite covering $\{A_i\}$ of X

and a V in η such that $V \bigcap (A_i \times A_i) \subset U$ for each i . Hursch showed in (10) that if $\xi \leq \eta$ and $\eta \leq \xi$, and ξ, η are proximity-equivalent, then they are H-equivalent. By means of COROLLARY 4.3.2 and the following LEMMA, we can state a sort of half-way result.

LEMMA 4.3.5. If $\xi \leq \eta$ then every η -precompact subset is also ξ -precompact.

Proof. Let A be η -precompact and let U be any symmetric entourage in ξ . Then there exists a finite covering $\{A_i\}$ of X and a V in η such that $V \bigcap (A_i \times A_i) \subset U$ for each i . Let B be a maximal U -discrete subset of A , and choose V_1 in η with $V_1^2 \subset V$. Since A is η -precompact, there exists a finite set $F \subset A$ such that $A \subset V_1(F)$.

Now, for each f in F , $V_1(f)$ contains at most a finite number of points of B . For if b_1, b_2 are two points of B in $V_1(f)$ which belong to the same member A_i of the finite covering, then (b_1, b_2) belongs to $V \bigcap (A_i \times A_i) \subset U$, and since B is U -discrete, $b_1 = b_2$. Hence $V(F)$ contains at most a finite number of points of B ; but $B \subset V(F)$, so B must be a finite set. By maximality of B , $A \subset U(B)$, and therefore A is ξ -precompact.

PROPOSITION 4.3.6. If ξ is less than or equal in height to η , and if η induces finer uniform neighbourhoods of η -precompact subsets, then η is H-finer than ξ on the set of η -precompact subsets.

Remarks. It is quite possible that the part of the hypothesis concerning uniform neighbourhoods could be considerably weakened. Perhaps compatible uniformities which are equal in height must be H-equivalent on the set of precompact subsets? It is also tempting to conjecture that compatible uniformities must be H-equivalent on the set of subsets which are precompact with respect to each uniformity.

4.4 The sufficiency of the set $\mathcal{S}(\mathcal{F}(X))$. After Ward, Ivanov and Isbell had succeeded in showing the "insufficiency of the hyperspace" (Isbell's phrase), Poljakov proved in (24) that if ξ, η are distinct compatible uniformities on X then the uniformities $\tilde{\xi}$ and $\tilde{\eta}$ on $\mathcal{S}(X)$ are not proximity-equivalent, and hence by Ward's conditions in THEOREM 4.2.2. the uniformities $\tilde{\xi}$ and $\tilde{\eta}$ induce distinct topologies on $\mathcal{S}(\mathcal{S}(X))$. Actually, his proof shows that even the set of subsets of $\mathcal{F}(X)$ (the set of finite subsets of X) is sufficient for distinguishing uniformities on X .

We present here a slight adaptation of Poljakov's original proof.

THEOREM 4.4.1. Let ξ, η be two uniformities on a set X . Then $\tilde{\eta}$ is proximity-finer than $\tilde{\xi}$ on the set $\mathcal{F}(X)$ if and only if η is finer than ξ .

Proof. If η is finer than ξ then it is clear that $\tilde{\eta}$ is proximity-finer than $\tilde{\xi}$ on $\mathcal{F}(X)$. Conversely, suppose the latter is true and let U in ξ be given. Choose a symmetric U_1 in ξ such that $\overset{2}{U}_1 \subset U$.

Consider the following two subsets of $\mathcal{F}(X)$:

$$\mathcal{A} = \{\{x\} : x \in X\},$$

$$\mathcal{B} = \{\{x, y\} : (x, y) \notin U\}.$$

We show that \mathcal{A} and \mathcal{B} are $\tilde{\xi}$ -remote. Let $\{x, y\}$ belong to $\tilde{U}_1(\mathcal{A})$, so that there exists A in \mathcal{A} with $\{x, y\}$ belonging to $\tilde{U}_1(A)$; this means that there is some point z in X such that $\{x, y\} \subset U_1(z)$. Then (x, y) belongs to $\overset{2}{U}_1 \subset U$, so that $\{x, y\}$ does not belong to \mathcal{B} . Thus $\tilde{U}_1(\mathcal{A}) \cap \mathcal{B} = \phi$.

Since $\tilde{\eta}$ is proximity-finer than $\tilde{\xi}$ on $\mathcal{F}(X)$, \mathcal{A} and \mathcal{B} must also be $\tilde{\eta}$ -remote. Thus there exists a symmetric V in η such that $\tilde{V}(\mathcal{A}) \cap \mathcal{B} = \phi$. That is, for each x, y in X , (x, y)

belongs to U whenever $\{x, y\}$ belongs to $\tilde{V}(\{z\})$ for some z in X .
 Let (x, y) belong to V . Then $\{x, y\}$ belongs to $\tilde{V}(\{x\})$, and so
 (x, y) must belong to U . Therefore $V \subset U$, and η is finer than ξ .

COROLLARY 4.4.2. If η induces a finer topology than ξ on $S(\mathcal{F}(X))$, then η is a finer uniformity than ξ .

4.5 The H-singularity of uniformities. When can we say of a uniform space (X, ξ) that there is no other uniformity on X H-equivalent to ξ ? Such a uniformity is called *H-singular*, and Ward (35) has given sufficient conditions for H-singularity, in the following form:

Let (X, ξ) be a uniform space such that there exists a compact uniform space K and a family of mappings $f_i : K \rightarrow X$ ($i \in I$), satisfying:

- (1) $\bigcup f_i(K) = X$,
 - (2) the set $E_x = \{f_i(x) : i \in I\}$ is ξ -discrete for every x in K ,
 - (3) the mappings f_i , $i \in I$, are uniformly equicontinuous.
- Then ξ is H-singular.

These conditions can be made intrinsic for X at the expense of their strength by supposing K to be a subspace of X , as pointed out by Ward. In any case they are rather complex. A more general result is possible, which can be stated much more simply. The following PROPOSITION will be the basic tool, and Ward's original proof will beam the guiding light.

PROPOSITION 4.5.1. If ξ, η are two uniformities on a set X , and if X is the union of a $\tilde{\xi}$ -compact collection of subsets on each of which ξ is uniformly finer than η over X , then ξ is finer than η on X .

Proof. Let $\mathcal{D} = \{D_i : i \in I\}$ be the collection of subsets, and let V in η be given. Choose a symmetric V_0 in η such that $V_0^2 \subset V$. Then there exists, for each i in I , a U_i in ξ such that $U_i \cap (D_i \times X) \subset V_0$. The collection $\{\tilde{U}_i(D_i) : i \in I\}$ is an open covering of \mathcal{D} , and since \mathcal{D} is $\tilde{\xi}$ -compact there is a finite subcover $\tilde{U}_{i_1}(D_{i_1}), \dots, \tilde{U}_{i_n}(D_{i_n})$. Let U be a symmetric entourage in ξ such that $U \subset \bigcap \{U_{i_k} : 1 \leq k \leq n\}$.

Now suppose (x, y) belongs to U . For some i in I , x belongs to D_i , and for some $1 \leq k \leq n$, D_i belongs to $\tilde{U}_{i_k}(D_{i_k})$. Thus y belongs to $U(D_i) \subset U(U_{i_k}(D_{i_k})) \subset \tilde{U}_{i_k}^2(D_{i_k})$, and so there is some d in D_{i_k} such that (d, y) belongs to $\tilde{U}_{i_k}^2$. Since $U \subset U_{i_k}$ and (y, x) belongs to U , we also have that (d, x) belongs to $\tilde{U}_{i_k}^3$. But then both (d, x) and (d, y) must belong to V_0 , so that (x, y) belongs to V .

It has been proved that $U \subset V$, and therefore ξ is finer than η .

THEOREM 4.5.2. If (X, ξ) is any uniform space which is the union of a compact collection of discrete subsets, then ξ is H -singular

Proof. Suppose that η is a uniformity on X H -equivalent to ξ , and let \mathcal{D} be the $\tilde{\xi}$ -compact collection of ξ -discrete subsets. By Ward's Theorem 1 in (35), each ξ -discrete subset is η -discrete, and ξ is uniformly finer over X than η on each η -discrete set. Hence by PROPOSITION 4.5.1, ξ must be finer than η on X .

But since ξ and η are H -equivalent, \mathcal{D} is also an $\tilde{\eta}$ -compact set, and again by Ward's theorem η is uniformly finer than ξ over X on each ξ -discrete set. Therefore PROPOSITION 4.5.1 can again be applied, with ξ and η in reversed positions, to show that η is finer than ξ on X .

That Ward's result on H -singularity follows from THEOREM 4.5.2 can be seen by taking $\mathcal{D} = \{E_x : x \in X\}$, and observing that

the mapping $x \rightarrow E_x$ of K onto \mathcal{D} is continuous by the uniform equicontinuity of the mappings f_i , $i \in I$.

In view of another result of Ward - Theorem 3 in (36) - I make the following conjecture:

CONJECTURE 4.5.3. Let ξ be a uniformity on X with countable base, and let (X, ξ) be the union of a precompact collection of discrete sets. Then ξ is H -singular, provided that the cardinal of X is non-measurable.

GRAPHS AND HYPERGRAPHS

5.1 Introduction. Let X and Y be sets with topological structures and let t be a mapping of X into Y . The graph $G(t)$ of t is the subset $\{(x, t(x)) : x \in X\}$ of $X \times Y$, and we say that t has a closed graph if $G(t)$ is closed in $X \times Y$ with respect to the product topology. We define the *hypergraph* $\mathcal{H}(t)$ of t to be the subset $\{(A, \overline{t(A)}) : A \in \mathcal{S}(X)\}$ of $\mathcal{S}(X) \times \mathcal{E}(Y)$, that is, the graph of the induced mapping $t' : \mathcal{S}(X) \rightarrow \mathcal{E}(Y)$.

Provided that Y is separated, every continuous mapping has a closed graph, but the converse is not true in general and the many Closed-graph Theorems of functional analysis have been concerned with situations in which continuity and closed-graph conditions are equivalent. It will be shown in section 2 that when X and Y are arbitrary uniform spaces and $\mathcal{S}(X)$ and $\mathcal{E}(Y)$ have their Hausdorff uniformities, then any uniformly continuous mapping has hypergraph closed in $\mathcal{S}(X) \times \mathcal{E}(Y)$, and any mapping with a closed hypergraph is continuous. Thus, in particular, when X and Y are topological vector spaces the closed-hypergraph condition is equivalent to continuity for a linear mapping.

If \mathcal{A} is any collection of subsets of X we will say that $\mathcal{H}(t)$ is *closed on the sets of* \mathcal{A} if $\mathcal{H}(t) \cap (\mathcal{A} \times \mathcal{E}(Y))$ is closed in $\mathcal{A} \times \mathcal{E}(Y)$. This is a weaker condition than the closed-hypergraph condition and for a suitable choice of \mathcal{A} lies in between closed-graph and closed-hypergraph conditions when Y is separated. In section 3 some situations are found in which such a condition implies continuity. For example, when X and Y are locally convex topological vector spaces and X is barrelled then a linear mapping is continuous if and only if its

hypergraph is closed on the absolutely convex absorbent subsets of X .

In section 4 the hypergraph of a relation R on a uniform space X is defined and it is shown that the separatedness of the quotient space X/R of an equivalence relation is in a somewhat similar position, with regard to closed graph and closed hypergraph conditions, to continuity in the case of a mapping. Some examples are given in section 5.

5.2 The closed hypergraph theorem. Let (X, ξ) and (Y, η) be uniform spaces and let $\tilde{\xi}$ and $\tilde{\eta}$ be the Hausdorff uniformities on $\mathcal{S}(X)$ and $\mathcal{E}(Y)$ respectively. There are simple characterizations of mappings with closed graph, and mappings with closed hypergraph.

LEMMA 5.2.1. The graph $G(t)$ of t is closed if and only if

$\bigcap \{V(t(U(x))) : U \in \xi, V \in \eta\} = \{t(x)\}$ for each x in X , and the hypergraph $\mathcal{H}(t)$ of t is closed in $\mathcal{S}(X) \times \mathcal{E}(Y)$ if and only if $\bigcap \{\tilde{V}(t'(\tilde{U}(A))) : U \in \xi, V \in \eta\} = \{\overline{t(A)}\}$ for each A in $\mathcal{S}(X)$.

Proof. The two results are parallel statements about the graphs of t and t' ; it is enough to prove the second. Let $\mathcal{H}(t)$ be closed. If B is a closed subset of Y belonging to the given intersection then, for each $U, V, t'(\tilde{U}(A)) \cap \tilde{V}(B) \neq \emptyset$ since V is symmetric, and so $(\tilde{U}(A) \times \tilde{V}(B)) \cap \mathcal{H}(t) \neq \emptyset$. Thus (A, B) belongs to the closure of the hypergraph, and by hypothesis must actually belong to hypergraph itself, so that $B = \overline{t(A)}$.

Conversely if we assume the intersection is $\{\overline{t(A)}\}$ and if (P, Q) belongs to the closure of the hypergraph, then for each $U, V, (\tilde{U}(P) \times \tilde{V}(Q)) \cap \mathcal{H}(t) \neq \emptyset$, so that $t'(\tilde{U}(P)) \cap \tilde{V}(Q) \neq \emptyset$, and hence Q belongs to $\tilde{V}(t'(\tilde{U}(P)))$. Thus $Q = \overline{t(P)}$ and (P, Q) belongs to the hypergraph, which must therefore be closed.

Remarks. There are analogous characterizations for

relations, which will be given later. When X and Y are topological vector spaces and \mathcal{U} and \mathcal{V} are their respective 0-neighbourhood bases, the above result about the graph of t takes the well-known form:

a linear mapping t has closed graph if and only if

$$\bigcap \{t(U) + V : U \in \mathcal{U}, V \in \mathcal{V}\} = \{0\}.$$

If, back in the uniform space situation, \mathcal{A} is any collection of subsets of X then a slight modification of the foregoing proof shows that the hypergraph of a mapping t is closed on the sets of \mathcal{A} if and only if $\bigcap \{\tilde{V}(t'(\tilde{U}(A))) : U \in \xi, V \in \eta\} = \{\overline{t(A)}\}$ for each A in \mathcal{A} , where \tilde{U} denotes the entourage induced on \mathcal{A} by the entourage \tilde{U} on $S(X)$.

THEOREM 5.2.2. Let X and Y be uniform spaces and t be any mapping of X into Y . If t is uniformly continuous then its hypergraph is closed. If the hypergraph is closed, on the other hand, then t is continuous.

Proof. If t is uniformly continuous then the induced mapping t' of $S(X)$ into $\mathcal{E}(Y)$ is uniformly continuous, and since $\mathcal{E}(Y)$ is separated the graph of t' , which is the hypergraph of t , must be closed.

Now let the hypergraph be closed, and let A be any subset of X . By LEMMA 5.2.1 the intersection of the sets $\tilde{V}(t'(\tilde{U}(A)))$, where U and V are any entourages of X and Y respectively, is $\{\overline{t(A)}\}$. But consider the set $\overline{t(A)}$. Clearly \bar{A} belongs to $\tilde{U}(A)$ for each U , and so $\overline{t(\bar{A})} = t'(\bar{A})$ belongs to $t'(\tilde{U}(A))$; thus $\overline{t(\bar{A})}$ belongs to $\tilde{V}(t'(\tilde{U}(A)))$ for each U, V and $\overline{t(\bar{A})} = \overline{t(A)}$ by the LEMMA. This being so for each subset A of X , t must be continuous.

COROLLARY 5.2.3. The closed hypergraph condition is equivalent to continuity for a mapping t of X into Y , in the following situations:

- (1) X, Y topological vector space, t a linear mapping,
- (2) X, Y uniform spaces, and the uniformity on X the finest inducing the given topology;
- (3) X, Y uniform spaces, and X compact.

Proof. In each case it is well-known that continuity and uniform continuity are equivalent.

Remarks. I have not been able to find any continuous mapping with non-closed hypergraph, nor any mapping with closed hypergraph which is not uniformly continuous, but I conjecture that the converses of the THEOREM are false. Note that the use of the separated space $\mathcal{E}(Y)$ in the LEMMA, instead of $\mathcal{S}(Y)$, and the definition of the hypergraph as the graph of t' (as opposed to t^0), are essential for the theory.

5.3. Weaker conditions on the hypergraph. If \mathcal{A} is any subset of $\mathcal{S}(X)$, it is clear that the hypergraph of a mapping will be closed on the sets of \mathcal{A} if it is closed in $\mathcal{S}(X) \times \mathcal{E}(Y)$. If Y is separated and $\{ \{x\} = x \in X \} \subset \mathcal{A}$, then the condition that the hypergraph be closed on the sets of \mathcal{A} is intermediate to the closed graph condition and uniform continuity. This follows from the identity $G(t) = \mathcal{H}(t) \cap (\mathcal{A} \times \mathcal{E}(Y)) \cap (X \times Y)$, where X and Y are regarded as isomorphically embedded in \mathcal{A} and $\mathcal{E}(Y)$ respectively. Similarly the condition that $\mathcal{H}(t) \cap (\mathcal{A} \times \mathcal{E}(Y))$ be a Borel set in $\mathcal{A} \times \mathcal{E}(Y)$ is intermediate to the condition that $G(t)$ be a Borel set in $X \times Y$ and the condition that t be uniformly continuous. It is natural to ask what structures on X and Y will enable continuity or uniform

continuity to be inferred from any of these intermediate conditions.

When X and Y are general uniform spaces it is sufficient for continuity of t that its hypergraph be closed on $\mathcal{E}(X)$.

Consider the following diagram:

$$\begin{array}{ccc} \mathcal{S}(X) & \xrightarrow{t^0} & \mathcal{S}(Y) \\ j_1 \downarrow & \searrow t' & \downarrow j_2 \\ \mathcal{E}(X) & \xrightarrow{t''} & \mathcal{E}(Y) \end{array}$$

Here we regard $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ as the Hausdorff spaces associated with $\mathcal{S}(X)$ and $\mathcal{S}(Y)$ respectively, and j_1, j_2 are the respective canonical mappings, taking each set onto its closure (see section 2.4). The mappings t^0 and t' are the usual induced mappings, and t'' is the restriction of t' to $\mathcal{E}(X)$. Now if A belongs to $\mathcal{S}(X)$ we have $(j_2 \circ t^0)(A) = \overline{t(A)} = t'(A)$, and $(t'' \circ j_1)(A) = \overline{t(A)}$. Hence t is continuous if and only if the diagram is commutative, which is true if and only if saying $(t(A), t(B))$ belongs to the intersection of the entourages of $\mathcal{S}(Y)$ is equivalent to saying $(t(\overline{A}), t(\overline{B}))$ belongs to the intersection of the entourages of $\mathcal{S}(Y)$.

Also, because j_1 is continuous, the mapping t' has closed graph if t'' has closed graph, and because j_1 is open, t'' has closed graph if t' has closed graph. Thus $\mathcal{G}(t)$ is closed in $\mathcal{S}(X) \times \mathcal{E}(Y)$ if and only if it is closed on the sets of $\mathcal{E}(X)$.

PROPOSITION 5.3.1. If X is a metrizable space and Y any uniform space then a mapping t of X into Y is continuous if its hypergraph is closed on the bounded subsets of X .

Proof. By the remarks after LEMMA 5.2.1, the condition implies that the intersection of the sets $\tilde{V}(t'(\tilde{U}(B)))$, where U is any entourage of the Hausdorff uniformity on the set \mathcal{B} of bounded

subsets of X , and \tilde{V} is any entourage of the Hausdorff uniformity on $\mathcal{E}(Y)$, must be $\{\overline{t(B)}\}$, for each B in \mathcal{B} . Since the closure of any bounded set is bounded, this implies that $\overline{t(B)} = \overline{t(B)}$ for each B in \mathcal{B} . Now if B_0 is a fixed member of \mathcal{B} and B is any subset of B_0 , then B belongs to \mathcal{B} and $t(B^B_0) \subset t(B^X) \subset \overline{t(B)}$. Hence t is continuous on B_0 . But each point of X has a neighbourhood belonging to \mathcal{B} , and so t must be continuous on X .

Remarks. It is clear from the proof of the PROPOSITION that if, for any $\epsilon > 0$, we let \mathcal{B} be the collection of subsets of X with diameter less than or equal to ϵ , then the condition that t has hypergraph closed on the sets of \mathcal{B} is sufficient for continuity of t . This approximates, intuitively at least, remarkably closely to the closed graph condition.

We can extract from the PROPOSITION some closure conditions on the collection \mathcal{B} and local conditions on X which enable the following more general result to be proved by exactly the same method.

THEOREM 5.3.2. Let X and Y be uniform spaces and let \mathcal{A} be a collection of subsets of X such that for each A in \mathcal{A} the closure of A belongs to \mathcal{A} and any subset of A belongs to \mathcal{A} . Suppose also that each point of X has a neighbourhood base consisting of sets of \mathcal{A} . Then a mapping t of X into Y is continuous if its hypergraph is closed on the sets of \mathcal{A} .

COROLLARY 5.3.3. If X is a locally compact (resp. locally precompact uniform space and Y is any uniform space then a mapping t of X into Y is continuous if its hypergraph is closed on the relatively compact (resp. precompact) subsets of X .

Note that we could replace the condition "X is a locally compact space" with "X is a k-space" - that is, one in which every set intersecting each compact, closed set in a closed set must be closed. For then continuity on X can be inferred from continuity on each compact subset. Clearly the topology of a k-space is determined by the collection of compact closed sets. One of the most important properties of a k-space is that the set of all continuous mappings from a k-space to a complete uniform space is complete with respect to the uniformity of uniform convergence on the compact subsets. If X is separated, and is either locally compact or first-countable, then it is a k-space; in particular a pseudometrizable space qualifies (see e.g. Kelley (15)).

There is a further, slightly different, result along the lines of the previous THEOREM.

THEOREM 5.3.4. If X and Y are locally convex topological vector spaces and X is barrelled, then a linear mapping t of X into Y is continuous if and only if its hypergraph is closed on the absolutely convex, absorbent subsets of X.

Proof. The condition implies, as in the proof of PROPOSITION 5.3.1, that $\overline{t(A)} = \overline{t(A)}$ for each absolutely convex, absorbent subset A, because the closure of such a subset is also absolutely convex and absorbent. Let V be any neighbourhood of the origin in Y. Any locally convex space has a neighbourhood base consisting of barrels, so we can choose $V_1 \subset V$ where V_1 is a member of such a base. Then $t^{-1}(V_1)$ is absolutely convex and absorbent by linearity of t , and hence $\overline{t(t^{-1}(V_1))} \subset \overline{t(t^{-1}(V_1))} = V_1 \subset V$. But $\overline{t^{-1}(V_1)}$ is a barrel and consequently a neighbourhood of the origin in X, because X is barrelled. Therefore t is continuous.

The converse follows immediately from THEOREM 5.2.2.

5.4 The hypergraph of a relation. Let X be a set and R a relation on X . We denote by xRy the statement " x is in the relation to y ", or just " x is related to y ". Let $R(x)$ denote the set of points to which x is related, and for any subset A of X let $R(A)$ denote the set of points to which some point of A is related. The graph $G(R)$ of the relation is the subset $\{(x, y) = xRy\}$ of $X \times X$; the distinction between the concepts of a relation and its graph is essentially one of convenience.

The relation R induces a natural relation R' on $\mathcal{S}(X)$ as follows. For A, B in $\mathcal{S}(X)$ we put $AR'B$ if for each point a in A there exists b in B such that aRb and for each point b' in B there exists a' in A such that $a'Rb'$. We then define the hypergraph $\mathcal{H}(R)$ of the relation R to be the subset $\{(A, B) : A R'B\}$ of $\mathcal{S}(X) \times \mathcal{S}(X)$, that is, the graph of R' .

If R is reflective, symmetric or transitive then R' inherits the corresponding property. Thus if R is an equivalence relation, so is R' . In this case, for A in $\mathcal{S}(X)$, the set $R(A)$ is the same thing as what is usually called the saturation of A with respect to R , and

$$\mathcal{H}(R) = \{(A, B) : R(A) = R(B)\}.$$

A set A is called saturated with respect to R if $A = R(A)$. Note that this will be so if and only if each member of $R'(A)$ is contained in A .

Now let X be a uniform space with uniformity ξ . In this section $\tilde{\xi}$ will denote the Hausdorff uniformity on $\mathcal{S}(X)$. There are simple characterizations, analogous to those for mappings, of those relations with graph closed in $X \times X$ and those with hypergraph closed in $\mathcal{S}(X) \times \mathcal{S}(X)$.

LEMMA 5.4.1. The graph $G(R)$ of R is closed in $X \times X$ if and only if $\bigcap \{V(R(U(X))) : U, V \in \xi\} = R(x)$ for each x in X , and the hypergraph $\mathcal{H}(R)$ of R is closed in $\mathcal{S}(X) \times \mathcal{S}(X)$ if and only if $\bigcap \{\tilde{V}(R'(\tilde{U}(A))) : U, V \in \xi\} = R'(A)$, for each A in $\mathcal{S}(X)$. If R is symmetric, the latter follows if $\bigcap \{V(R(U(A))) : U, V \in \xi\} = R(A)$.

Proof. As with LEMMA 5.2.1, it is enough to prove the result for $\mathcal{H}(R)$. Let $\mathcal{H}(R)$ be closed, and let A and B be subsets such that for each U, V in ξ , B belongs to $\tilde{V}(R'(\tilde{U}(A)))$. Then, for each U, V in ξ , $(R'(\tilde{U}(A))) \cap \tilde{V}(B) \neq \phi$, since V is symmetric, so that $(\tilde{U}(A) \times \tilde{V}(B)) \cap \mathcal{H}(R) \neq \phi$. Thus (A, B) belongs to $\overline{\mathcal{H}(R)}$ and hence to $\mathcal{H}(R)$, so that $AR'B$ and B belongs to $R'(A)$. Since $R'(A)$ is easily seen to be contained in the intersection of the sets $\tilde{V}(R'(\tilde{U}(A)))$, this concludes one part of the proof.

Conversely, assume that the intersection is $R'(A)$, and let (P, Q) belong to $\overline{\mathcal{H}(R)}$. Then for each U, V in ξ , $(\tilde{U}(P) \times \tilde{V}(Q)) \cap \mathcal{H}(R) \neq \phi$, that is $(R'(\tilde{U}(P))) \cap \tilde{V}(Q) \neq \phi$, and so Q belongs to $\tilde{V}(R'(\tilde{U}(P)))$, \tilde{V} being symmetric. By assumption, then, Q belongs to $R'(P)$, so that (P, Q) belongs to $\mathcal{H}(R)$, which must therefore be closed.

Now suppose that R is a symmetric relation, and that $\bigcap \{V(R(U(A))) : U, V \in \xi\} = R(A)$ for each A in $\mathcal{S}(X)$. Let (P, Q) belong to $\overline{\mathcal{H}(R)}$. Then for each U, V , there exists (A, B) in $\mathcal{H}(R)$ such that (A, P) belongs to \tilde{U} and (B, Q) belongs to \tilde{V} . We have $AR'B$ and since R is symmetric this implies both $A \subseteq R(B)$ and $B \subseteq R(A)$. We also have $A \subseteq U(P)$, $P \subseteq U(A)$, $B \subseteq V(Q)$ and $Q \subseteq V(B)$, and hence $P \subseteq U(R(V(Q)))$ and $Q \subseteq V(R(U(P)))$. By the supposition, $P \subseteq R(Q)$ and $Q \subseteq R(P)$, and so (P, Q) belongs to $\mathcal{H}(R)$, which is therefore closed.

We shall now show that, if R is an equivalence relation, the relation of closed graph and closed hypergraph conditions to

the separatedness of the quotient space X/R closely parallels the relation of these conditions to continuity in the case of a mapping.

If X is a uniform space and $Q : X \rightarrow X/R$ is the quotient mapping, then the quotient topology on X/R is the finest for which Q is continuous. The equivalence relation R is called open (resp. closed) if the mapping Q is open (resp. closed) with respect to the quotient topology. If X/R is known to be separated then $G(R)$ is closed for it is the inverse image of the diagonal under the continuous mapping $Q \times Q : X \times X \rightarrow X/R \times X/R$. The converse is not true in general, but is true if R is open (see e.g. Bourbaki (4), Chapter 1, section 8, no. 3). However if we assume that $\mathcal{V}(R)$ is closed, then the quotient space becomes as separated as a space can be.

THEOREM 5.4.2. If R is an equivalence relation on a uniform space X and if R has a closed hypergraph, then the quotient topology on X/R is the discrete topology, thus making X/R separated, every saturated subset of X both open and closed, and R both an open and a closed relation.

Proof. By LEMMA 5.4.1 if R has closed hypergraph then the intersection of the sets $\tilde{V}(R'(\tilde{U}(A)))$ is $R'(A)$, for each subset A . For each U in ξ , $R(A)$ belongs to $R'(\tilde{U}(A))$ and so for each U , V in ξ , $\overline{R(A)}$ belongs to $\tilde{V}(R'(\tilde{U}(A)))$. Thus $\overline{R(A)}$ belongs to $R'(A)$, which implies that $R(\overline{R(A)}) = R(A)$ and therefore $\overline{R(A)} = R(A)$. That is, the saturation of each subset of X is closed in X . All the statements of the THEOREM follow immediately from this.

5.5 Examples. (1) Some interesting things can be said about relations on a topological vector space X . A relation R on X is called *linear* if $G(R)$ is a vector subspace of $X \times X$.

A symmetric and reflexive linear relation must also be transitive. It is easily seen that an equivalence relation R is linear if and only if it is a relation modulo some vector subspace M of X ; then $M = \{m : m = x - y \text{ and } xRy\} = \{m : mRo\}$. The graph $G(R)$ of R is closed if and only if M is closed, if and only if the quotient space X/M is separated. The latter equivalence is very well known and the former is a consequence of the quotient mapping being open (see the remarks preceding THEOREM 5.4.2.). It follows also from LEMMA 5.4.1, since $\bigcap \{x + U + M + V : U, V \text{ } 0\text{-neighbourhoods}\} = x + M$ if and only if $x + M$ is closed, if and only if M is closed.

The hypergraph $\mathcal{H}(R)$ of the equivalence relation R is the set $\{(A, B) : A + M = B + M\}$. By the proof of the last part of LEMMA 5.4.1, if $A + M$ is closed (so that $\bigcap \{A + M + U : U \text{ } 0\text{-neighbourhood}\} = A + M$) for each A in a collection \mathcal{A} of subsets of X , then $\mathcal{H}(R) \cap (\mathcal{A} \times \mathcal{A})$ is closed in $\mathcal{A} \times \mathcal{A}$. In particular this will be true when $\mathcal{A} = \mathcal{C}(X)$, the collection of compact, closed subsets of X . If X is complete, $\mathcal{C}(X)$ is closed in $\mathcal{E}(X)$, and so also $\mathcal{H}(R) \cap (\mathcal{E}(X) \times \mathcal{E}(X))$ is closed in $\mathcal{E}(X) \times \mathcal{E}(X)$.

When R is any linear relation on X with closed graph, we can no longer talk about the quotient mapping, but we may still ask whether $R(A)$ is open for each open subset A . This is not true in general, but (see Kelley and Namioka (18), section 11E) is true when X is a complete metrizable space and $R(X)$ is of the second category in X . The latter condition will certainly be satisfied if R is reflexive. It might be expected from experience with mappings, that the answer to our question is "yes" for arbitrary X provided that the linear relation R has closed hypergraph; this is certainly true for equivalence relations, but I do not know if it is true for arbitrary linear relations.

(2) Let (X, ξ) be a uniform space and X' its associated Hausdorff space. Define an equivalence relation R by xRy if $i(x) = i(y)$, where i is the canonical mapping of X into X' . Then X' is homeomorphic to X/R , and i is the quotient mapping. Thus R is open and closed, and X/R is separated. The graph $G(R)$ of R is the set $\bigcap \{U : U \in \xi\}$, and is closed.

Consider the induced relation R' on $S(X)$. We have $AR'B$ if and only if $i(A) = i(B)$, if and only if $\alpha(A) = \alpha(B)$, where $\alpha : S(X) \rightarrow S(X')$ is the mapping whose restriction to $\mathcal{E}(X)$ has been proved in section 2.4 to be an isomorphism onto $\mathcal{E}(X')$. Thus R' restricted to $\mathcal{E}(X)$ becomes the equality relation. We know also that $\alpha : S(X) \rightarrow S(X')$ is an open mapping, and therefore R' is an open relation and $S(X)/R'$ is homeomorphic with $S(X')$ (see Bourbaki (4), Chapter 1, section 5.2).

If the hypergraph $\mathcal{H}(R)$ is closed then, by THEOREM 5.4.2, X' must be a discrete space.

(3) Let (X, ξ) be a pre-ordered uniform space, so that the pre-order \leq is a reflexive, transitive relation. It is not symmetric unless it reduces to the equality relation. For each subset A let $d(A)$ denote the smallest decreasing set containing A , and let $i(A)$ denote the smallest increasing set containing A . These coincide with $R^{-1}(A)$ and $R(A)$, where R^{-1} is the inverse relation.

Define a pre-order on $S(X)$, by putting $A \leq B$ if $A \subseteq d(B)$ and $B \subseteq i(A)$. This coincides with the induced relation R' . If $G(R)$ is closed then for each x in X the sets $d(x)$ and $i(x)$ are closed; if $\mathcal{H}(R)$ is closed then for each $A \subseteq X$ the sets $d(A)$ and $i(A)$ are closed, and also $d(\bar{A}) = d(A)$ and $i(\bar{A}) = i(A)$.

We could define another pre-order on $S(X)$ by putting $A \leq B$

if $A \subseteq D(B)$ and $B \subseteq I(A)$, where $D(B)$ denotes the smallest closed decreasing set containing B , and $I(A)$ the smallest closed increasing set containing A . If $\mathcal{K}(R)$ is closed, this relation coincides with R' .

There is yet another obvious pre-order relation on $\mathcal{S}(X)$ - that of inclusion. This is anti-symmetric, sometimes referred to as an order. The graph of this relation is not closed in $\mathcal{S}(X) \times \mathcal{S}(X)$, for any topological space equipped with an order with closed graph must be separated. However its intersection with $\mathcal{E}(X) \times \mathcal{E}(X)$ is closed in this space.

COMPACT, COLLECTIVELY COMPACT AND HYPERCOMPACT
SETS OF MAPPINGS

6.1. Introduction. This chapter is a study of sets of mappings characterized by looking at the collections of images of various subsets of the domain space as subsets of the hyperspace of the range space. Many of the concepts involved, such as compact mappings and collectively compact sets of mappings, usually appear in the context of normed vector spaces or topological vector spaces, but they are treated here in a very general way. The unification achieved seems worth the sacrifice of notational simplicity.

Let X be a set and Y a uniform space, and let $F(X, Y)$ denote the set of all mappings from X into Y . For any (non-empty) collection \mathcal{A} of subsets of X we make the following definitions.

A mapping t in $F(X, Y)$ is called \mathcal{A} -compact if it maps the sets of \mathcal{A} onto relatively compact subsets of Y .

A set of mappings $T \subset F(X, Y)$ is called collectively \mathcal{A} -compact if, for each A in \mathcal{A} , $T(A) = \{t(a) : a \in A, t \in T\}$ is a relatively compact subset of Y .

A set of mappings T is called \mathcal{A} -hypercompact if, for each A in \mathcal{A} , $\{t(A) : t \in T\}$ is a compact subset of $\mathcal{S}(Y)$, and relatively \mathcal{A} -hypercompact if, for each A in \mathcal{A} , $\{t(A) : t \in T\}$ is a relatively compact subset of $\mathcal{S}(Y)$.

The set $F(X, Y)$ can be endowed with the uniformity $\xi(\mathcal{A})$ of uniform convergence on the sets of \mathcal{A} (see section 1.9), which has a subbase consisting of all sets of the form

$$W(A, V) = \{(f, g) : (f(a), g(a)) \in V \text{ for all } a \in A\}$$

where A is a member of \mathcal{A} and V is an entourage of Y .

Where there is no possibility of ambiguity about the collection \mathcal{A} , it will be understood that $F(X, Y)$ has the uniformity $\xi(\mathcal{A})$, and the prefixes \mathcal{A} will be omitted in the terms just defined.

In section 6.3, hypercompact and relatively hypercompact sets are investigated, and in particular it is shown that for compact mappings hypercompactness is an intermediate condition to compactness and collective compactness; if Y is complete then for compact mappings relative hypercompactness is equivalent to collective compactness. In section 6.4 the concepts of hyperprecompact and hyperbounded sets of mappings are introduced and studied. The theory parallels that of section 6.3 much of the way, and proofs are omitted. A set of mappings is collectively precompact if and only if it is a hyperprecompact set of precompact mappings, and collectively bounded (or uniformly bounded on the sets of \mathcal{A}) if and only if it is a hyperbounded set of bounded mappings.

6.2. Preliminary results about hyperspaces. Throughout this section Y will be a uniform space. The proof of the first lemma is omitted; the remaining results supplement the two well-known facts that the union of a compact collection of closed subsets is closed and the union of a compact collection of compact subsets is compact.

LEMMA 6.2.1. A subset \mathcal{M} of $\mathcal{S}(Y)$ is relatively compact in $\mathcal{S}(Y)$ if and only if $\{\bar{M} : M \in \mathcal{M}\}$ is relatively compact in

$\mathcal{B}(Y)$ if and only if the latter set is relatively compact in $\mathcal{E}(Y)$. The same is true if relatively compact is replaced by compact.

PROPOSITION 6.2.2. The union of a compact collection of relatively compact subsets is relatively compact.

Proof. Let \mathcal{R} be such a collection. Then each R in \mathcal{R} has compact closure in the uniform space Y , and by LEMMA 6.2.1. the set $\{\bar{R} : R \in \mathcal{R}\}$ is compact. But then $\bigcup\{R : R \in \mathcal{R}\} \subset \bigcup\{\bar{R} : R \in \mathcal{R}\}$ which is the union of a compact collection of compact sets, and so is compact.

PROPOSITION 6.2.3. The union of a precompact collection of precompact subsets is precompact.

Proof. Let \mathcal{P} be such a collection, and let U be any entourage of Y . Choose a symmetric entourage V such that $\tilde{V} \subset U$. Then there exists a finite subcollection $\{P_i\}$ of \mathcal{P} such that the union of the sets $\tilde{V}(P_i)$ covers \mathcal{P} , and for each i there exists a finite set $F_i \subset P_i$ such that $P_i \subset V(F_i)$. Let F be the union of the sets F_i . Then for each P in \mathcal{P} there is some i such that $P \subset V(P_i) \subset \tilde{V}(F_i) \subset U(F)$. Since F is finite the result follows.

A subset A of Y is called bounded if for each entourage U there exists a finite set $F \subset A$ and an integer $n > 0$ such that $A \subset \bigcap_{i=1}^n U(F)$. Every precompact subset is bounded, and the closure of every bounded subset is bounded. The image of a bounded set under a uniformly continuous mapping is bounded. In a product of non-empty uniform spaces, a subset is bounded if and only if each projection is bounded.

For topological vector spaces this definition of boundedness (with respect to the natural uniformity) coincides with the usual definition.

PROPOSITION 6.2.4. The union of a bounded collection of bounded subsets is bounded.

Proof. Let \mathcal{B} be such a collection, and let U be any entourage of Y . Then there exists a finite subcollection $\{B_i\}$ of \mathcal{B} and an integer $n > 0$ such that the union of the sets $\bigcup_n U(B_i)$ covers \mathcal{B} , and for each i there exists a finite set $F_i \subset B_i$ and an integer $m_i > 0$ such that $B_i \subset \bigcup_{m_i} U(F_i)$. Let F be the union of the sets F_i and let $N = n + \max\{m_i\}$. Then for any B in \mathcal{B} , and for some i , $B \subset \bigcup_n U(B_i) \subset \bigcup_{n+m_i} U(F_i) \subset \bigcup_N U(F)$. Since F is finite the result follows.

6.3. Hypercompact and relatively hypercompact sets

LEMMA 6.3.1. The mapping θ_B of $F(X, Y)$ into $\mathcal{S}(Y)$ defined by $\theta_B(t) = t(B)$ is uniformly continuous relative to the uniformity $\xi(\mathcal{A})$ on $F(X, Y)$, for each subset B of X which is covered by a finite subcollection of \mathcal{A} .

Proof. Let V be an entourage of Y and let $\{A_i\}$ be a finite subcollection of \mathcal{A} covering B . Let (f, g) belong to the intersection of the sets $W(A_i, V)$, so that $(f(x), g(x))$ belongs to V for each x in the union of the sets A_i , and so for each x in B . Then $(f(B), g(B))$ belongs to \tilde{V} , and θ_B is uniformly continuous.

THEOREM 6.3.2. If T is a set of mappings from the set X into the uniform space Y , and if T is (relatively) compact

with respect to the uniformity of uniform convergence on the sets of $\mathcal{A} \subset \mathcal{S}(X)$, then T is (relatively) \mathcal{A} -hypercompact.

Proof. The mapping θ_A of LEMMA 6.3.1. is uniformly continuous, for each A in \mathcal{A} , and since $\theta_A(T) = \{t(A) : t \in T\}$, the result follows.

THEOREM 6.3.3. If T is an \mathcal{A} -hypercompact set of \mathcal{A} -compact mappings from the set X into the uniform space Y then T is collectively \mathcal{A} -compact. If Y is complete and T is only a relatively \mathcal{A} -hypercompact set of \mathcal{A} -compact mappings then T is again collectively \mathcal{A} -compact.

Proof. For the first part, observe that, for each A in \mathcal{A} , $T(A) = \bigcup \{t(A) : t \in T\}$ is the union of a compact collection of relatively compact sets, and so is relatively compact by PROPOSITION 6.2.2.

For the second part, when Y is complete, it is sufficient to show that $T(A)$ is precompact. But $T(A)$ is clearly the union of a precompact collection of precompact sets, so PROPOSITION 6.2.3. brings us home.

THEOREM 6.3.4. If T is a collectively \mathcal{A} -compact set of mappings from the set X into the uniform space Y then T is a relatively \mathcal{A} -hypercompact set of \mathcal{A} -compact mappings.

Proof. Since $T(A)$ is relatively compact, there is a compact subset K such that $T(A) \subset K \subset Y$. Then for each t in T $t(A) \subset K$, so t is \mathcal{A} -compact. Also $\{t(A) : t \in T\} \subset \mathcal{S}(T(A)) \subset \mathcal{S}(K)$ which is compact (see section 1.7), and therefore T is relatively \mathcal{A} -hypercompact.

From the previous two THEOREMS the next is immediate:

THEOREM 6.3.5. If T is a set of mappings from the set X into the complete uniform space Y then T is collectively \mathcal{A} -compact if and only if T is a relatively \mathcal{A} -hypercompact set of \mathcal{A} -compact mappings.

The completeness hypothesis here is not superfluous - that is, the converse of THEOREM 6.3.4. is not true; this will be demonstrated by the examples to follow. They will also demonstrate that the converses of THEOREM 6.3.2. and the first part of THEOREM 6.3.3. are not true. First we consider the situation when X and Y have additional structure.

If X and Y are both topological vector spaces and \mathcal{A} is the collection of bounded subsets of X then every linear mapping t from X into Y which is compact in the usual sense is also \mathcal{A} -compact. For this means that there is some \mathcal{A} -neighbourhood U of X and some compact subset K of Y such that $t(U) \subset K$. If A is any bounded subset of X there is a real number $\lambda > 0$ such that $A \subset \lambda U$, and then $t(A) \subset t(\lambda U) \subset \lambda t(U) \subset \lambda K$ which is compact. If X and Y are actually normed vector spaces then the uniformity $\xi(\mathcal{A})$ of uniform convergence on the bounded subsets of X is normable on the set $C(X, Y)$ of continuous linear mappings, with norm $\|t\| = \sup \{\|t(x)\| : \|x\| \leq 1\}$. In this case the \mathcal{A} -compact and usual compact (sometimes called completely continuous) mappings are the same - they are just the ones for which the image of the unit ball $B = \{x \in X : \|x\| \leq 1\}$ has compact closure. Similarly a collectively \mathcal{A} -compact set of mappings T is just a set for which $T(B)$ has compact closure; this is the definition used by Anselone and Palmer (3).

Thus THEOREMS 6.3.2. and 6.3.3. together yield the result for normed spaces in (3) that every compact set of compact linear mappings is collectively compact.

When talking about normed spaces it will always be understood that \mathcal{A} is the collection of bounded sets, unless otherwise stated. Observe that by taking $\mathcal{A} = \{B\}$ the uniformity $\xi(\mathcal{A})$ is unchanged, and THEOREM 6.3.3. implies that a set T of compact mappings for which the set of images of the unit ball is compact must be collectively compact. A set which is thus "hypercompact on the unit ball" is not necessarily hypercompact, even if it consists of compact mappings; this will be demonstrated in EXAMPLE 6.3.7. By taking $\mathcal{A} = \{B\}$ in this EXAMPLE, a counterexample is constructed for the converse of THEOREM 6.3.2. Placing various conditions on the collection \mathcal{A} , such as covering X or being closed under scalar multiplication, effectively invalidates the counterexample and leaves it open whether the converse of the THEOREM then holds. EXAMPLES 6.3.6. and 6.3.7. do show, however, that in general hypercompactness lies strictly in between compactness and collective compactness, for compact mappings.

EXAMPLE 6.3.6. One of the most fruitful fields for applications of the theory of compact mappings, and collectively compact sets of mappings, is the theory of integral equations of Fredholm or Volterra type. We proceed now to construct a sequence of mappings, each representing an integral equation of Volterra type. Let $X = Y = C[a, b]$, the set of continuous, real-valued functions on the closed interval $[a, b]$; this is a normed

vector space with norm $\|x(t)\| = \sup\{|x(t)| : t \in [a, b]\}$

For each $m = 1, 2, \dots$, and each x in X , define

$$y_m(u) = \int_a^b k(u, v)x(v)dv, \text{ where } k(u, v) = \begin{cases} 1, & a \leq v \leq a + \frac{u-a}{m} \\ 0, & a + \frac{u-a}{m} < v \leq b \end{cases}$$

$$= \int_a^{a + (u-a)/m} x(v)dv.$$

Then, for each m , y_m belongs to X , and the mapping t_m defined by $t_m(x) = y_m$ is a compact (continuous) mapping of X into itself. Let $T = \{t_m : m \text{ a positive integer}\}$.

Then T is collectively compact. To show this, let $\{x_n\}$ be any bounded sequence in X , and $\{m(n)\}$ any sequence of positive integers. It must be proved that the sequence $y_{m(n),n} = t_{m(n)}(x_n)$ has a subsequence convergent in X . By Ascoli's Theorem it is sufficient to prove that the sequence is bounded and that its members form an equicontinuous family. Now

$$\|y_{m(n),n}\| \leq \|t_{m(n)}\| \|x_n\|$$

$$\leq R \sup \left\{ \left| \int_a^{a+(u-a)/m(n)} x(v)dv \right| : u \in [a, b], \|x\| \leq 1 \right\}$$

where $R = \sup \{\|x_n\| : n > 0\}$,

$$\leq \frac{R(b-a)}{m(n)}$$

$$\leq R(b-a).$$

Therefore the sequence is bounded. If u_1, u_2 are points of $[a, b]$ with $|u_2 - u_1| < \epsilon/R$, ϵ any positive number, then

$$\left| y_{m(n),n}(u_2) - y_{m(n),n}(u_1) \right| = \left| \int_a^{a+(u_2-a)/m(n)} x_n(v)dv - \int_a^{a+(u_1-a)/m(n)} x_n(v)dv \right|$$

$$= \left| \int_{a+(u_1-a)/m(n)}^{a+(u_2-a)/m(n)} x_n(v)dv \right| < \epsilon, \text{ for all } n > 0.$$

Therefore the family $\{y_{m(n)}, n\}$ is equicontinuous. This completes the proof that T is collectively compact.

It follows that T is relatively hypercompact, but it is not hypercompact - in fact it is not even hypercompact on the unit ball. For

$$\|t_m(x)\| = \sup\{y_m(u) : u \in [a, b]\} \leq \frac{(b-a)}{m} \|x\|,$$

and hence for any bounded subset A of X , $t_m(A)$ is eventually contained in any ϵ -neighbourhood of the origin. Thus $t_m(A)$ converges in $\mathcal{S}(X)$ to $\{0\}$. Since $t_m(A) \neq \{0\}$ for all $m > 0$, for suitable A (the unit ball, for instance), the set $\{t_m(A) : m > 0\}$ is not compact, and so T is not hypercompact.

Although T is not a compact set it is relatively compact, for if T^* is the set obtained by appending the zero mapping t_∞ to T , then because $\|t_m - t_\infty\| \rightarrow 0$, T^* is compact (and so also hypercompact).

EXAMPLE 6.3.7. Let $X = Y = \ell^p$, $1 \leq p \leq \infty$, the set of infinite sequences $x = \{x_n\}$ of real numbers satisfying $\sum_{n=1}^{\infty} |x_n|^p < \infty$; this is a normed vector space with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}, \text{ and basis } (\phi_1, \phi_2, \dots), \text{ where } \phi_i$$

is the sequence with unit i th component and zeros elsewhere. Define a sequence of mappings of X into itself by putting $t_n(x) = x_n \phi_1$, $n > 0$. Then for each n t_n is a compact linear mapping.

The set $T = \{t_n : n > 0\}$ is hypercompact on the unit ball. For $t_n(B) = \{\alpha \phi_1 : |\alpha| \leq 1 \text{ for each } n\}$, so that $\{t_n(B) : n > 0\}$ is a singleton in $\mathcal{S}(X)$.

But T is not hypercompact. For if a is the element of X with $a_n = \frac{1}{n}$ then $\{t_n(a) : n > 0\} = \{\{\frac{1}{n} \phi_1\} : n > 0\}$ which is not closed in $\mathcal{E}(X)$ and so not compact.

It can be inferred now by THEOREM 6.3.2. that T is not a compact set. In fact it is not even precompact, for

$$\|t_n - t_m\| = \sup\{\|(x_n - x_m)\phi_1\| : \|x\| \leq 1\} = 2^{\frac{1}{p}}, m \neq n.$$

In spite of not being relatively compact, T is relatively hypercompact, because it is collectively compact. This follows from the fact that $T(B)$ is bounded and one-dimensional, or from T being a set of compact mappings, hypercompact on the unit ball.

EXAMPLE 6.3.8. Let X and Y be normed vector spaces, and let $\{t_n\}$ be a sequence of compact linear mappings in $C(X, Y)$ converging (in norm) to the non-compact continuous linear mapping t . This situation is known to exist when Y is not complete. Then for each bounded subset A of X , $t_n(A) \rightarrow t(A)$ in $S(Y)$ by LEMMA 6.3.1., and so the collection $\{t_n(A)\} \cup \{t(A)\}$ is compact.

Putting $T = \{t_n\}$, the set $T \cup \{t\}$ is hypercompact and the set T is relatively hypercompact. But T is not collectively compact; for if so t would be a compact mapping since $t(B) \subset \overline{T(B)}$. This falsifies the converse of THEOREM 6.3.4.

This section concludes with some properties of hypercompact and relatively hypercompact sets. In many respects they behave like compact and collectively compact sets. Any finite union and any intersection of such sets is a set of the same type. If Y is a topological vector space and $F(X, Y)$ has its natural linear structure, then any scalar multiple of a hypercompact or relatively hypercompact set is again a set of the same type. Any

subset of a relatively hypercompact set must be relatively hypercompact, but a subset of a hypercompact set need not be hypercompact whether closed or not. Of particular interest is the fact that the union of a compact set and a collectively compact set need inherit neither property, but must be relatively hypercompact.

PROPOSITION 6.3.9. If T is (relatively) \mathcal{R} -hypercompact then so is the $\xi(\mathcal{R})$ -closure \bar{T} . When X and Y are normed vector spaces, the balanced hull of any set in $C(X, Y)$ which is relatively hypercompact, hypercompact, or hypercompact on the unit ball, is a set of the same type.

Proof. The mapping $t \rightarrow \overline{t(A)}$ is uniformly continuous from $F(X, Y)$ into $\mathcal{E}(Y)$ with respect to the uniformity $\xi(\mathcal{R})$, for each A in \mathcal{R} . This follows from LEMMA 6.3.1. and the uniform continuity of the closure operation from $\mathcal{S}(Y)$ into $\mathcal{E}(Y)$. Hence the set $\{\overline{t(A)} : t \in \bar{T}\}$ is contained in the closure in $\mathcal{E}(Y)$ of the set $\{\overline{t(A)} : t \in T\}$. If the latter set is compact then it must coincide with the former set, and if relatively compact its closure is compact and contains the former set. With the aid of LEMMA 6.2.1. this proves the first part.

For the second part, when X and Y are normed vector spaces the mapping $(\lambda, B) \rightarrow \lambda B$ is continuous from $S_0 \times \mathcal{B}$ into \mathcal{B} where S_0 is the set of scalars with absolute value ≤ 1 and \mathcal{B} is the set of bounded subsets of Y , and hence if $\{t(A) : t \in T\}$ is (relatively) compact so is $\{(\lambda t)(A) : |\lambda| \leq 1, t \in T\}$.

Remark. I do not know if, for suitable spaces, the convex hull or the pointwise closure of a (relatively) hypercompact

set must be of the same type, although these hold for a collectively compact set.

Now let Y and Z be uniform spaces, while X is any set. Again let $F(X, Y)$ denote the set of all mappings from X into Y , and let $C(Y, Z)$ denote the set of all uniformly continuous mappings from Y into Z . As always, \mathcal{A} is a non-empty collection of subsets of X , and \mathcal{B} will now be any non-empty collection of subsets of Y . It is easily shown that if T is a $\xi(\mathcal{A})$ -compact subset of $F(X, Y)$ and S is a $\xi(\mathcal{B})$ -compact subset of $C(Y, Z)$ and if, in addition for each A in \mathcal{A} there exists B in \mathcal{B} such that $T(A) \subset B$, then ST is a $\xi(\mathcal{A})$ -compact subset of $F(X, Z)$. Also, if T is collectively \mathcal{A} -compact and \bar{S} is $\xi(\mathcal{B})$ -compact then ST is collectively \mathcal{A} -compact, provided that for each A in \mathcal{A} there exists B in \mathcal{B} such that $\bar{T}(A) \subset B$. This last provision is automatically made when considering only continuous linear mappings between normed vector spaces, and the two foregoing results may then be stated briefly : the composite of two compact sets is compact, and the composite of a collectively compact set and a set with compact closure is collectively compact. This last result for normed spaces was proved by Anselone and Palmer in (3). A further result along these lines follows.

PROPOSITION 6.3.10. Let X be a set, Y and Z uniform spaces, \mathcal{A} a collection of subsets of X and \mathcal{B} a collection of subsets of Y . If $T \subset F(X, Y)$ is (relatively) \mathcal{A} -hypercompact and $S \subset C(Y, Z)$ is a $\xi(\mathcal{B})$ -compact set of uniformly continuous mappings, then ST is (relatively) \mathcal{A} -hypercompact, provided that for each A in \mathcal{A} there exists B in \mathcal{B} such that $T(A) \subset B$.

Proof. For each A in \mathcal{A} the set $\{t(A) : t \in T\}$ is (relatively) compact and so, for each s in S , is the set $\{st(A) : t \in T\}$, since the latter set is the image of the former under the uniformly continuous mapping induced by s between $\mathcal{S}(Y)$ and $\mathcal{S}(Z)$. For each A in \mathcal{A} define a mapping α of $C(Y, Z)$ into $\mathcal{S}(\mathcal{S}(Z))$ by putting $\alpha(f) = \{ft(A) : t \in T\}$. We will show that α is uniformly continuous, with respect to $\xi(\mathcal{B})$.

Let B be a member of \mathcal{B} containing $T(A)$. For any entourage V of Z choose the entourage $W(B, V)$ of $\xi(\mathcal{B})$. If (f, g) belongs to $W(B, V)$ we must have $(ft(a), gt(a))$ belonging to V for each a in A and each t in T . Hence $(ft(A), gt(A))$ belongs to the entourage \hat{V} of $\mathcal{S}(Z)$ for each t in T , and so, finally, $(\alpha(f), \alpha(g))$ belongs to the entourage \hat{V} of $\mathcal{S}(\mathcal{S}(Z))$.

Since α is uniformly continuous, $\alpha(S)$ is compact, and hence $\{st(A) : s \in S, t \in T\} = \bigcup \{\alpha(s) : s \in S\}$ is the union of a compact collection of (relatively) compact sets, and so is (relatively) compact.

COROLLARY 6.3.11. Let X, Y, Z be topological vector spaces and let \mathcal{A}, \mathcal{B} be the collections of bounded subsets. If T is a (relatively) hypercompact set of continuous mappings and S is a compact set of continuous mappings then ST is (relatively) hypercompact.

Proof. It is clearly sufficient to show that $T(A)$ is bounded for each bounded subset A of X . But each t in T is continuous, so $t(A)$ is bounded, and $\{t(A) : t \in T\}$ is bounded by hypothesis; hence $T(A)$ is the union of a

bounded collection of bounded sets and must be bounded by PROPOSITION 6.2.4.

It is easy to see that the statement of the COROLLARY will still be true when \mathcal{A} and \mathcal{B} are the collections of precompact subsets, and the hypercompactness part will still be true when \mathcal{A} is the collection of compact (or even relatively compact) subsets of X and \mathcal{B} is the collection of compact subsets of Y .

6.4. Hyperprecompact and hyperbounded sets. In this section some further definitions are made and the general theory extended to include what are sometimes called totally bounded mappings and totally bounded sets of mappings (see e.g. Palmer (23)), and also bounded mappings and bounded sets of mappings. Since the proofs follow much the same pattern as the proofs in the previous section for the theory of relatively hypercompact sets, they are omitted. Suffice it to say that they rely heavily on PROPOSITIONS 6.2.3. and 6.2.4. and LEMMA 6.3.1.

Let X be a set and Y a uniform space, and let \mathcal{A} be any (non-empty) collection of subsets of X . We make the following definitions.

A mapping t in $F(X, Y)$ is called \mathcal{A} -precompact (\mathcal{A} -bounded) if it maps the sets of \mathcal{A} onto precompact (bounded) subsets of Y .

A set $T \subset F(X, Y)$ is called collectively \mathcal{A} -precompact (collectively \mathcal{A} -bounded) if $T(A)$ is precompact (bounded) for each A in \mathcal{A} .

A set $T \subset F(X, Y)$ is called \mathcal{A} -hyperprecompact

(\mathcal{A} -hyperbounded) if $\{t(A) : t \in T\}$ is precompact (bounded) for each A in \mathcal{A} .

It is clear that the conditions \mathcal{A} -hypercompact, relatively \mathcal{A} -hypercompact, \mathcal{A} -hyperprecompact and \mathcal{A} -hyperbounded become progressively weaker. When Y is hypercomplete the middle two are equivalent. Also (collectively) \mathcal{A} -compact implies (collectively) \mathcal{A} -precompact, which implies (collectively) \mathcal{A} -bounded, and when Y is complete the first two coincide.

Now let Y be a topological vector space. The set of \mathcal{A} -bounded mappings is a closed subspace of $F(X, Y)$ with respect to the uniformity $\xi(\mathcal{A})$, which actually induces a vector topology on it. The set of \mathcal{A} -precompact mappings is a closed subspace as well. Both subspaces inherit the property of completeness from Y (see e.g. Kelley and Namioka (18), section 8).

Next, let X also be a topological vector space, and let us consider only linear mappings. Let \mathcal{A} be the collection of bounded subsets of X . Then the \mathcal{A} -precompact mappings are just the usual precompact (or totally bounded) mappings, and the \mathcal{A} -bounded mappings are the usual bounded mappings. A set of mappings is collectively \mathcal{A} -bounded (sometimes called uniformly bounded on the sets of \mathcal{A}) if and only if bounded with respect to the topology of uniform convergence on the sets of \mathcal{A} . In particular this is true for every equicontinuous set. Each continuous mapping is bounded, and if X is pseudo-normable the converse holds. If X and Y are normed the \mathcal{A} -precompact

and collectively \mathcal{A} -precompact sets of mappings coincide with Palmer's, Anselone's and Vala's precompact mappings and collectively precompact sets (see (23), (2), (32)) - just those for which the (collective) image of the unit ball is precompact.

For the following theorems we return to the general situation in which the definitions were framed.

THEOREM 6.4.1. If T is $\xi(\mathcal{A})$ -precompact (-bounded) then it is \mathcal{A} -hyperprecompact (-hyperbounded). If Y is λ -complete^{hyper} and T is $\xi(\mathcal{A})$ -precompact then \bar{T} is \mathcal{A} -hypercompact and T is relatively \mathcal{A} -hypercompact.

THEOREM 6.4.2. The set T is collectively \mathcal{A} -precompact if and only if it is an \mathcal{A} -hyperprecompact set of \mathcal{A} -precompact mappings. If Y is complete these conditions are equivalent to collective \mathcal{A} -compactness.

The last two results together show that a precompact set of precompact mappings is collectively precompact - proved for normed spaces by Palmer (23).

THEOREM 6.4.3. The set T is collectively \mathcal{A} -bounded if and only if it is an \mathcal{A} -hyperbounded set of \mathcal{A} -bounded mappings. If Y is a topological vector space and the uniformity $\xi(\mathcal{A})$ induces a vector topology on T , then T is collectively \mathcal{A} -bounded if and only if it is \mathcal{A} -hyperbounded if and only if it is bounded with respect to $\xi(\mathcal{A})$.

Thus THEOREM 6.4.3. includes a partial converse for the "boundedness" part of THEOREM 6.4.1. The EXAMPLE 6.3.7. provides a case of a relatively hypercompact (and so, of

course, hyperprecompact) set which is not a precompact set, even though Y is complete and all the mappings are compact.

The permanence properties of hyperprecompactness and hyperboundedness are similar to those of relative compactness; any finite union or any intersection of such sets is of the same type, and any subset inherits the type. If Y is a topological vector space, any scalar multiple of such a set is of the same type. Although the union of a precompact set and a collectively precompact set need have neither property, it is hyperprecompact, and likewise the union of a bounded set and a collectively bounded set is hyperbounded.

PROPOSITION 6.4.4. If T is \mathcal{A} -hyperprecompact (\mathcal{A} -hyperbounded) then so is the $\xi(\mathcal{A})$ -closure \bar{T} . When X and Y are normed vector spaces the balanced hull of a hyperprecompact (hyperbounded) subset of $C(X, Y)$ is of the same type.

PROPOSITION 6.4.5. Let Z be another uniform space, and \mathcal{B} any collection of subsets of Y . Let $T \subset F(X, Y)$ and $S \subset C(Y, Z)$ and suppose that for each A in \mathcal{A} there exists B in \mathcal{B} such that $T(A) \subset B$. Then

T is \mathcal{A} -hyperprecompact and S is $\xi(\mathcal{B})$ -precompact implies ST is \mathcal{A} -hyperprecompact;

T is \mathcal{A} -hyperbounded and S is $\xi(\mathcal{B})$ -bounded implies ST is \mathcal{A} -hyperbounded.

COROLLARY 6.4.6. If X, Y and Z are topological vector spaces and \mathcal{A}, \mathcal{B} are the collections of bounded subsets of X and Y respectively, then the two implications of the PROPOSITION hold provided that S and T are sets of continuous mappings.

6.5. Arzela-Ascoli Theorems and other results. Let X and Y be uniform spaces. Some results of the Arzela-Ascoli type can be proved very simply by means of hyperspace techniques.

PROPOSITION 6.5.1. If $T \subseteq F(X, Y)$ is uniformly equicontinuous on each set in \mathcal{K} , the collection of compact subsets of X , and if $T(x)$ is relatively compact in Y for each $x \in X$, then T is collectively \mathcal{K} -compact.

Proof. For each A in \mathcal{K} and each a in A the mapping $f : A \rightarrow \mathcal{S}(Y)$ defined by $f(a) = T(a)$ is uniformly continuous, and $T(A)$ is the union of the collection $\{T(a) : a \in A\}$. The result follows by PROPOSITION 6.2.2.

COROLLARY 6.5.2. The PROPOSITION holds also if "compact" is replaced throughout by "precompact" or "bounded".

PROPOSITION 6.5.3. Let Y be complete and let $\mathcal{K} \subseteq \mathcal{S}(X)$. If t is an \mathcal{K} -compact mapping in $F(X, Y)$ and is uniformly continuous, then it is also $\overline{\mathcal{K}}$ -compact. If $T \subseteq F(X, Y)$ is uniformly equicontinuous and collectively \mathcal{K} -compact, then it is also collectively $\overline{\mathcal{K}}$ -compact.

Proof. The mappings $A \mapsto \overline{t(A)}$ and $A \mapsto \overline{T(A)}$ are uniformly continuous, and the set $\mathcal{C}(Y)$ is closed in $\mathcal{E}(Y)$ (see sections 1.8 and 1.9).

PROPOSITION 6.5.4. Let $\mathcal{K} \subseteq \mathcal{S}(X)$ and let \mathcal{B} be any closed subspace of $\mathcal{E}(Y)$. Then the set of mappings in $F(X, Y)$ which take the sets of \mathcal{K} onto sets with closures in \mathcal{B} is closed in $F(X, Y)$ with respect to the uniformity $\xi(\mathcal{K})$.

Proof. If $t_\alpha \rightarrow t$ in $F(X, Y)$ then $\overline{t_\alpha(A)} \rightarrow \overline{t(A)}$ for each A in \mathcal{K} , by LEMMA 6.3.1.

COROLLARY 6.5.5. If Y is complete then the set of \mathcal{Q} -compact mappings is complete; the sets of \mathcal{Q} -precompact and \mathcal{Q} -bounded mappings are closed in $F(X, Y)$ whether Y is complete or not.

Remark. The Robertsons proved in (28) that if η is a coarser uniformity on Y than ξ , associated with ξ , then the η -relatively compact subsets in $\mathcal{E}(Y, \xi)$ form a closed subspace of $\mathcal{E}(Y, \xi)$. Their Theorem 3, discussed in section 2.7., is then a direct application of the principle of PROPOSITION 6.5.4.

REFERENCES

- (1) F. Albrecht "Au sujet de l'espace des ensembles fermés,"
Com. Acad. R. P. Roumaine 2 (1952), 209-212.
- (2) P. M. Anselone "Collectively compact and totally bounded sets
of linear operators," J. Math. Mech. 17 (1968),
613-622.
- (3) P. M. Anselone and T. W. Palmer "Collectively compact sets of linear operators,"
Pacific J. Math. 25 (1968), 417-422.
- (4) N. Bourbaki "General Topology," Part 1, Hermann, Addison-
Wesley (translation).
- (5) P. J. Caulfield "Multi-valued functions and uniform spaces,"
Thesis, The Ohio State University, Columbus,
Ohio (1967).
- (6) E. G. Effros "Convergence of closed subsets in a topological
space," Proc. Amer. Math. Soc. 16 (1965), 929-931.
- (7) J.M.G. Fell "A Hausdorff topology for the closed subsets of
a locally compact non-Hausdorff space, Proc.
Amer. Math. Soc. 13 (1962), 472-476.
- (8) J. Flachsmeyer and H. Poppe "On extensions of hyperspaces," Contributions
to the Extension Theory of Topological Structures
(Proc. Sympos. Berlin 1967) VEB Deutscher Verlag
Der Wissenschaften, Berlin (1969), pp 73-76.
- (9) F. Hausdorff "Mengenlehre", 2nd ed., Berlin, Gruyter (1927).
- (10) J. L. Hursch "A Theorem about Hyperspaces", Proc. Camb. Phil.
Soc. 63 (1967), 597-598.
- (11) J. R. Isbell "Insufficiency of the hyperspace", Proc. Camb.
Phil. Soc. 62 (1966), 685-686.
- (12) J. R. Isbell "Uniform Spaces", (Math Surveys 12), Amer. Math.
Soc., Providence, Rhode Island, 1964.
- (13) A. A. Ivanov "On spaces of subsets defined by uniform structures"
Vestnik Leningrad Univ. 21 (1966) no. 19, 151-153
(Russian, English Summary).
- (14) V. M. Ivanov "Spaces of closed subsets of compact extensions",
Mat. Sb (N.S.) 50(92) (1960), 91-100. (Russian).
- (15) J. L. Kelley "General Topology", Van Nostrand (1955), Toronto-
New York-London.
- (16) J. L. Kelley "Hypercomplete linear topological spaces",
Michigan Math. J. 5 (1958), 235-246.
- (17) J. L. Kelley "Hyperspaces of a continuum", Trans. Amer. Math.
Soc. 52 (1942), 23-26.
- (18) J. L. Kelley and I. Namioka and
co-authors. "Linear Topological Spaces", Van Nostrand (1963),
Toronto-New York-London.

- (19) K. Kuratowski "Topology, Vol. I", Academic Press (1966), New York-London, PWN Polish Scientific Publishers, Warszawa.
- (20) E. Michael "Topologies on spaces of subsets", Trans. Amer. Math. Soc. 71 (1951), 152-182.
- (21) S. Mrowka "On the convergence of nets of sets", Fund. Math. 45 (1957-58).
- (22) L. J. Nachman "Hyperspaces of proximity spaces", Math. Scand. 23 (1968), 201-213.
- (23) T. W. Palmer "Totally bounded sets of precompact linear operators", Proc. Am. Math. Soc. 20 (1969), 101-106.
- (24) V. Z. Poljakov "Uniformities on X and $\mathcal{F}(\mathcal{F}(X))$ ", Soviet Math. Dokl. 9 (1968), 318-319.
- (25) G. Baley Price "On the completeness of a certain metric space", Bull. Amer. Math. Soc. 46 (1940), 278-280.
- (26) A. P. Robertson "Hypercomplete spaces", Extrait du deuxième colloque sur l'analyse fonctionnelle, tenu à Liège, Mai 1964 (Librairie Universitaire, 10 Rue de las Monnaie, Louvain-Belgique).
- (27) A. P. Robertson & W. Robertson "Topological Vector Spaces".
- (28) A. P. Robertson & W. Robertson "On the space of subsets of a uniform space", Proc. Amer. Math. Soc. 12 (1961) 321-326.
- (29) W. Robertson "Completions of topological vector spaces", Proc. London. Math. Soc. 8 (1958), 242-257.
- (30) D. Hammond Smith "Hyperspaces of a uniformizable space", Proc. Camb. Philos. Soc. 62 (1966), 25-28.
- (31) W. J. Thron "Topological Structures", Holt, Rinehart and Winston (1966), New York-Toronto-London.
- (32) K. Vala "On compact sets of compact operators", Ann. Acad. Sci. Fenn., Ser A1, 351 (1964).
- (33) L. Vietoris "Bereiche Zweiter Ordnung, Monatshefte für Mathematik und Physik, Vol. 33 (1923), 49-62.
- (34) A. J. Ward "A counterexample in uniformity theory", Proc. Camb. Phil. Soc. 62 (1966), 207-208.
- (35) A. J. Ward "On H-equivalence of uniformities (the Isbell-Smith problem)", Pacific J. Math. 22 (1967), 189-196.
- (36) A. J. Ward "On H-equivalence of uniformities (II)", Pacific J. Math. 28 (1969), 207-216.
- (37) P. D. Watson "On the limits of sequences of sets", Quart. J. Math. (2), 4(1953), 1-3.