SET-VALUED FUNCTIONS AND SELECTORS

by

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ABSTRACT

Let S and X be any two sets; then a mapping Γ which assigns to each point t in S a set $\Gamma(t)$ of points in X is called a multifunction (or set-valued function) from S into X. A selector for Γ is a function f from S into X such that $f(t)$ belongs to the set $\Gamma(t)$ for each t. This thesis contains a systematic study of multifunctions, especially measurable multifunctions, and a number of instances are given where a multifunction Γ has a selector which inherits the good properties of Γ, or at least is not much worse. The problem of proving that selectors exist can be approached from more than one direction; in particular the class of multifunctions of Souslin type is introduced. This class is comprehensive, containing the kinds of measurable multifunction most commonly studied previously, it is closed under the usual operations of analysis and set-theory, and yet it is well-supplied with measurable selectors.

PREFACE

This thesis describes the results of research work carried out by the author at the University of Keele (1971 - 73) and at the University of Western Australia (1973 - 74). I am grateful to the Science Research Council for providing financial support during the whole of this period, and to my supervisor, Professor A.P. Robertson, for his constant interest and encouragement. Thanks are also due to Miss Glenys Dalziell whose careful typing is evident in the pages that follow.

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INTRODUCTION

Set-valued functions, or multifunctions, have in recent years been turning up in several branches of applicable mathematics; for instance, R.J. Aumann (1) and G. Debreu (14) have used them in mathematical economics, and a number of papers have been written about measurable multifunctions by authors who have an interest in control theory. Among these are C. Castaing (6-10), M.Q. Jacobs (24, 25), R.T. Rockafellar (42) and M. Valadier (50, 51). Drawing on the work of A.F. Filippov (15), H. Hermes and J.P. LaSalle (18) have produced an exposition of time optimal control which makes use of set-valued functions.

What I have attempted here is a systematic study of the properties of measurable multifunctions; I have looked at continuous and semicontinuous multifunctions as well, especially where the methods overlap. Clearly the most general setting for the study of measurable multifunctions is an abstract measurable space, and this is the setting I have used; however it is occasionally useful to place restrictions on the kind of measurable space we use, and I have included one or two results which are valid for multifunctions defined on a locally compact Hausdorff space with a Radon measure.

The first chapter of this thesis is concerned mainly with the stability properties of various classes of measurable multifunctions; that is, whether they are closed under such operations as taking intersections or products.

The selection problem for multifunctions is interesting for its own sake as well as for its importance in applications; Chapter II gives an introduction to the problem and follows this with a number of selection theorems for closed-valued or compact-valued multifunctions.

In Chapter III I have taken a different approach to measurable selectors and introduced a new class of multifunctions called multifunctions of Souslin type. Every multifunction with non-empty values which belongs to this class has at least a countable dense collection of measurable selectors (Theorem 11.1), and so in order to prove that a multifunction has a measurable selector it is enough to show that it is of Souslin type. It turns out that the class of multifunctions of Souslin type contains within it most of the kinds of set-valued function which have been studied previously, including some which arise naturally in implicit function problems. It has good stability properties as well, admitting all the commonly used operations of set-theory and analysis.

Chapter IV is about multifunctions with values which are compact convex subsets of a topological vector space. It is shown in particular that in some cases measurable multifunctions with non-empty compact convex values have extreme-valued measurable selectors. Results of this kind lead to the bang-bang principle of control theory (cf. Hermes and LaSalle (18), and Himmelberg and Van Vleck (22)).

The four chapters are divided into sections which are numbered throughout. The first section of Chapter I consists of a summary of facts and definitions from elsewhere which are used repeatedly, and the second section contains the main definitions for the present work. Theorems, lemmas and propositions are numbered consecutively in each section; thus Lemma 16.5 is the fifth result displayed in section 16, for example. All other numbered references are to the bibliography at the end of the thesis.

I. MULTIFUNCTIONS

0. Preliminaries. We mention here various notations and facts of topology which will be used in the following sections.

(i) A Polish space is a separable metrisable space which is complete with respect to some compatible metric. A Souslin space is a metrisable space which is the continuous image of a Polish space.

(ii) The product of countably many Polish spaces is a Polish space, and the product of countably many Souslin spaces is a Souslin space. Similarly the topological sum $\Sigma_i X_i$ of a countable collection (X_i) of Polish spaces is a Polish space. We mean by the topological sum that $\Sigma_i X_i$ is the disjoint union of the spaces X_i, and that a set G in $\Sigma_i X_i$ is open if and only if $G \cap X_i$ is open in X_i for each i. The topological sum of a countable collection of Souslin spaces is also a Souslin space.

(iii) Let $\mathbf{N}$ denote the space of positive integers $\{1,2, \ldots\}$. Let I denote the interval $[0,1]$, and $I^{\mathbf{N}}$ the topological product of I with itself countably many times. Then any separable metrisable space can be embedded in $I^{\mathbf{N}}$ (sometimes called the Hilbert cube), and hence has a compatible totally bounded metric. An open subset of a Polish space is a Polish subspace.

(iv) Let S be any set, and $\mathcal{A}$ a class of subsets. Then $\mathcal{A}$ is said to be a ring if for any A,B in $\mathcal{A}$, $A \cup B$ and $A \setminus B$ are also in $\mathcal{A}$. If also $\mathcal{A}$ is closed under the operation of forming countable unions, then $\mathcal{A}$ is called a σ-ring. If a ring $\mathcal{A}$ in S contains S as an element, then $\mathcal{A}$ is called an algebra. An algebra which is closed under the formation of countable unions is called a σ-algebra.

(v) If X is a topological space, then the σ-algebra $\mathcal{B}_X$ generated by the closed sets of X is called the σ-algebra of Borel sets. If X is a Polish space, every Borel set in X is a Souslin subspace. The

Souslin subspaces of a Polish space are frequently referred to as the <u>analytic</u> sets, or Souslin sets. We, however, shall be defining the term Souslin set somewhat differently, in a later section.

(vi) If S is any set, and $\mathcal{A}$ any class of subsets, then $\mathcal{A}_\sigma$ denotes the class of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_\delta$ denotes the class of countable intersections of sets in $\mathcal{A}$. Similarly, $\mathcal{A}_{\sigma\delta}$ is the class of countable intersections of sets in $\mathcal{A}_\sigma$. We define in the same way $\mathcal{A}_{\delta\sigma}$, $\mathcal{A}_{\delta\sigma\delta}$, and so on.

(vii) If X is a topological space, $\mathcal{F}(X)$, $\mathcal{G}(X)$, $\mathcal{K}(X)$ shall denote respectively the closed, open and compact sets of X. We shall refer to these classes more briefly as $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$.

(viii) If X and Y are topological spaces, and $f : X \to Y$ is a mapping, then f is said to be of the <u>first class</u> of Baire if $f^{-1}(G)$ is an $\mathcal{F}_\sigma$ set for every open set G in Y. It is of the <u>second class</u> if for every open set G in Y, $f^{-1}(G)$ is a $\mathcal{G}_{\delta\sigma}$ set.

(ix) Any Polish space is homeomorphic to a $\mathcal{G}_\delta$ set in the Hilbert cube I^N.

(x) A topological space X will be said to be <u>perfectly normal</u> if it is normal and if every open set in X is an $\mathcal{F}_\sigma$. In particular, every metrisable space is perfectly normal. If F is closed in X, then there exists a sequence (G_i) of open sets such that $F = \bigcap_i G_i$ and

$$G_1 \supset \bar{G}_2 \supset G_2 \supset \bar{G}_3 \supset \dots \ .$$

(xi) Let X be any separable metrisable space, and d a compatible metric. Then there exists a countable collection (U_i) of closed sets, each of diameter $\leq \frac{1}{2}$, which cover X. Similarly, for each i, U_i has a covering $\{U_{ij} : j = 1, 2, \dots\}$ by closed sets of diameter $\leq \frac{1}{4}$. Carrying on in this way, we obtain a collection $(U_{\sigma_1 \dots \sigma_n})$ of closed sets, indexed by the set of finite sequences of positive integers. For convenience, we shall denote the sequence $\sigma_1 \dots \sigma_n$ by $\sigma|n$, a notation which is used in (43), pp. 44-49. We may clearly choose the sets $\{U_{ij} : j = 1,2, \ldots\}$ to be subsets of U_i. Then, if $\sigma = (\sigma_1, \sigma_2, \ldots)$ is a sequence of positive integers, the sets $U_{\sigma|1}, U_{\sigma|2}, \ldots$ form a descending sequence. We call the collection $(U_{\sigma|n})$ a <u>sifting</u> of X. Clearly if X is a Polish space, it has a sifting $(U_{\sigma|n})$ such that for each infinite sequence σ, the sets $U_{\sigma|1}, U_{\sigma|2}, \ldots$ intersect in a single point, which we shall call x_σ.

(xii) A topological space will be said to be <u>second countable</u> or to be a 2C-<u>space</u> if it has a countable base of open sets.

(xiii) Let X be any topological space and let $\mathcal{F}^*$ and $\mathcal{K}^*$ denote the spaces of non-empty, closed and non-empty compact subsets of X. Then the <u>Vietoris topology</u> (or <u>exponential topology</u>) on $\mathcal{F}^*$ is defined to be the topology generated by the sets

$$\{F \in \mathcal{F}^* : F \subset G\} \quad \text{and} \quad \{F \in \mathcal{F}^* : F \cap G \neq \phi\}$$

where G ranges over all the open sets in X. Similarly we may define the Vietoris topology on the space $\mathcal{K}^*$. Clearly the Vietoris topology on $\mathcal{F}^*$ (and similarly on $\mathcal{K}^*$) is the upper bound of the topologies τ^+ and τ^- where τ^+ is generated by the sets $\{F \in \mathcal{F}^* : F \subset G\}$ and τ^- is generated by the sets $\{F \in \mathcal{F}^* : F \cap G \neq \phi\}$.

(xiv) Let (X,d) be a metric space. Then we define a metric δ, called the <u>Hausdorff metric</u>, on the space $\mathcal{F}^*$ by

$$\delta(A,B) = \sup \{\sup_{x \in A} d(x,B) \, , \, \sup_{y \in B} d(y,A)\} \, ,$$

where for each $x \in X$, $C \subseteq X$, $d(x,C) = \inf_{c \in C} d(x,c)$. δ can take the value $+\infty$, and it determines a topology on $\mathcal{F}^*$, which in general depends on the metric d. The subspace topology induced by the Hausdorff topology on the subspace $\mathcal{K}^*$ of $\mathcal{F}^*$ is identical to the Vietoris topology on $\mathcal{K}^*$ (Lemma 4.2 of (8)), and hence is the same for any equivalent metric on X.

For a non-compact metric space, the Hausdorff topology on $\mathcal{F}^*$ is in general different from the Vietoris topology, as is shown by the following examples in $\mathbf{R}^2$ with the Euclidean norm:

<u>Example 1.</u> Let $A = \{(x,y) : y \geq \frac{1}{x}, x > 0\}$.

For each positive integer n let

$$A_n = \{(x,y) : y \geq \frac{1}{x} - \frac{1}{n}, x > 0\}.$$

Then if δ is the Hausdorff metric determined by the Euclidean metric on $\mathbf{R}^2$, we see that $\delta(A_n, A) \leq \frac{1}{n}$, whence $A_n \to A$ in the Hausdorff topology. However $A_n \not\to A$ in the Vietoris topology, since $A \subseteq G$, where $G = \{(x,y) : x > 0, y > 0\}$, but $A_n \not\subseteq G$ for all n.

<u>Example 2.</u> Let $A = \mathbf{R}^2$ and let $A_n = \{(x,y) : x \leq n\}$, for each n. Then it is clear that $A_n \to A$, as $n \to \infty$, in the Vietoris topology. But $A_n \not\to A$ in the Hausdorff topology as $\delta(A_n, A)$ is infinite for each n.

(xv) If X is a topological space, then a real-valued function f on X is said to be <u>upper semicontinuous</u> at $x_0 \in X$ if for all $\epsilon > 0$ the set $\{x \in X : f(x) < f(x_0) + \epsilon\}$ is a neighbourhood of x_0. It is said to be <u>lower semicontinuous</u> at x_0 if for all $\epsilon > 0$ the set $\{x \in X : f(x) > f(x_0) - \epsilon\}$ is a neighbourhood of x_0. f is continuous at x_0 if and only if it is both upper and lower semicontinuous there. An upper semicontinuous function attains a maximum on any compact subset of X ((2), p. 76).

(xvi) A <u>measurable space</u> S is a set with a σ-algebra $\mathcal{M}$ of subsets, called the <u>measurable sets</u>. If $(S, \mathcal{A})$ and $(T, \mathcal{B})$ are measurable spaces, $\mathcal{A} \otimes \mathcal{B}$ shall denote the σ-algebra in $S \times T$ generated by the class $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$. If S is a measurable space and X a topological space, a function $f : S \to X$ will be said to be <u>measurable</u> if $f^{-1}(F)$ is measurable for every closed set F in X. A function f is then measurable if and only if $f^{-1}(G)$ is measurable for every <u>open</u> set G in X.

Proofs of the results mentioned in paragraphs (ii), (iii), (v) and (ix) can be found in references (3) and (28).

1. Basic Properties of Multifunctions. Let S, X be any two sets. Then a mapping Γ which assigns to each $t \in S$ a subset $\Gamma(t)$ of X is called a multifunction (or set-valued mapping or correspondence) from S into X, and we express this by writing $\Gamma : S \to X$. To avoid confusion with the notation used for mappings we shall denote multifunctions by upper-case Greek letters and mappings by lower case Greek or Roman letters. If (Γ_n) is a sequence of multifunctions from S into X, then $\cup_n \Gamma_n$ shall denote the multifunction

$$t \to \bigcup_{n=1}^{\infty} \Gamma_n(t),$$

and $\cap_n \Gamma_n$ shall denote the multifunction

$$t \to \bigcap_{n=1}^{\infty} \Gamma_n(t).$$

If, for each positive integer n, Γ_n is a multifunction from S into a space X_n, then $\Pi_n \Gamma_n$ shall denote the multifunction

$$t \to \prod_{n=1}^{\infty} \Gamma_n(t)$$

from S into $\Pi_n X_n$.

If $\varphi : X \to Z$ is a mapping, then $\varphi \circ \Gamma$ denotes the multifunction

$$t \to \varphi(\Gamma(t))$$

from S into Z.

If $\Phi : X \to Z$ is a multifunction, then $\Phi \circ \Gamma$ denotes the multifunction

$$t \to \cup \{\Phi(x) : x \in \Gamma(t)\}$$

from S into Z.

If $\Gamma : S \to X$ is a multifunction and $B \subseteq X$, then we define the <u>upper inverse image</u> of B to be the set

$$\Gamma^{+}(B) = \{t \in S : \Gamma(t) \subseteq B\} ,$$

and the <u>lower inverse image</u> of B to be the set

$$\Gamma^{-}(B) = \{t \in S : \Gamma(t) \cap B \neq \phi\} .$$

$\Gamma^{+}(B)$ is the complement in S of the set $\Gamma^{-}(X \setminus B)$.

Now let X be a topological space. Then a multifunction $\Gamma : S \to X$ will be said to be <u>closed-valued</u> if, for each t, $\Gamma(t)$ is a closed set. It will be said to be <u>compact-valued</u> if, for each t, $\Gamma(t)$ is a <u>compact set</u>. Similarly, if X is a linear space, we may speak of <u>convex</u>-valued multifunctions. If Γ is a multifunction, we define $\bar{\Gamma}$ to be the multifunction whose value at a point $t \in S$ is the closure of $\Gamma(t)$.

PROPOSITION 1.1. *If $\Gamma : S \to X$ is a multifunction, where X is a topological space, then $\bar{\Gamma}^{-}(G) = \Gamma^{-}(G)$ for every open set G in X.*

<u>Proof</u>. This proposition follows at once from the fact that, if $\bar{\Gamma}(t)$ meets G, then $\Gamma(t)$ meets G, by the definition of the closure of a set.

PROPOSITION 1.2. *If (Γ_n) is a sequence of multifunctions from S into the space X, then for any set B in X,*

$$(\cup_n \Gamma_n)^{-}(B) = \cup_n \Gamma_n^{-}(B) .$$

<u>Proof</u>. This follows from the definition of lower inverse image.

PROPOSITION 1.3. *If $\Gamma : S \to X$, $\Phi : X \to Y$ are multifunctions, then for any set B in Y*

$$(\Phi \circ \Gamma)^{-}(B) = \Gamma^{-}(\Phi^{-}(B)) .$$

Proof. $(\Phi \circ \Gamma)^{-}(B) = \{t : \Phi(x) \cap B \neq \phi \text{ for some } x \in \Gamma(t)\}$

$$= \{t : \Gamma(t) \cap \Phi^{-}(B) \neq \phi\}$$

$$= \Gamma^{-}(\Phi^{-}(B)) .$$

PROPOSITION 1.4. *If* $\Gamma : S \to X$ *is a multifunction and* (B_n) *a sequence of sets in* X ,

$$\Gamma^{-}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} \Gamma^{-}(B_n) .$$

Proof. $\Gamma(t)$ meets $\bigcup_{n=1}^{\infty} B_n$ if and only if $t \in \Gamma^{-}(B_n)$ for some n .

This proposition clearly holds for arbitrary unions of sets in X .

PROPOSITION 1.5. *Let* (Γ_n) *be a descending sequence of closed-valued multifunctions from the set* S *into the topological space* X *such that, for each* t , $\Gamma_n(t)$ *is compact for some* n . *Then for any closed set* F *in* X ,

$$(\cap_n \Gamma_n)^{-}(F) = \bigcap_{n=1}^{\infty} \Gamma_n^{-}(F) .$$

Proof. If $\Gamma_n(t)$ meets F for each n , then the sets $(F \cap \Gamma_n(t))$ form a descending sequence of closed sets. Since one of these is compact, this sequence has a non-empty intersection. Hence $(\cap_n \Gamma_n)(t)$ meets F , as required.

It is clear that if $\Gamma_n(t)$ is non-empty for each n , then $(\cap_n \Gamma_n)(t)$ is also non-empty. This proposition also holds for a descending transfinite sequence of closed-valued multifunctions.

The following result is a version of Théorème 1.1 of (8).

THEOREM 1.6. <u>Let</u> S <u>be any set,</u> X <u>a perfectly normal space,</u> <u>and</u> $\Gamma : S \to X$ <u>a compact-valued multifunction.</u> <u>Then if</u>

$$\mathcal{A} = \{\Gamma^-(G) : G \text{ open in } X\},$$

$\Gamma^-(F) \in \mathcal{A}_\delta$ <u>for every closed set</u> F <u>in</u> X .

<u>Proof.</u> Let F be a closed set in X . Then from §0, (x), there exists a sequence (G_i) of open sets such that

$$G_1 \supset \overline{G}_2 \supset G_2 \supset \overline{G}_3 \supset \dots$$

and
$$F = \bigcap_{n=1}^{\infty} G_n .$$

Then
$$\Gamma^-(F) = \bigcap_{n=1}^{\infty} \Gamma^-(G_n) . \qquad \dots(i)$$

It is clear that if $\Gamma(t) \cap F \neq \phi$, then $\Gamma(t) \cap G_n \neq \phi$ for all n . Conversely, if t belongs to $\Gamma^-(G_n)$, all n , then $\Gamma(t) \cap \overline{G}_n \neq \phi$ all n . Since $\Gamma(t)$ is compact, $\cap_n(\Gamma(t) \cap \overline{G}_n) \neq \phi$. Thus $\Gamma(t)$ meets $\cap_n \overline{G}_n = F$, which completes the proof of statement (i). Hence $\Gamma^-(F) \in \mathcal{A}_\delta$, as required.

One technique which we shall use for manipulating set-valued functions is that of forming "refinements" of multifunctions. This idea has been applied in (41). Let S and X be sets, and $\Gamma : S \to X$ a multifunction. If B is a set in X , we define the <u>refinement of</u> Γ <u>by</u> B to be the multifunction Γ_B where

$$\Gamma_B(t) = \Gamma(t) \cap B \qquad \text{for } t \in \Gamma^-(B)$$

and
$$\Gamma_B(t) = \Gamma(t) \qquad \text{otherwise.}$$

For any set C in X we have the formula:

$$\Gamma_B^-(C) = \Gamma^-(B \cap C) \cup (\Gamma^-(C) \setminus \Gamma^-(B)) .$$

We conclude from this:

PROPOSITION 1.7. Let S be a space on which is defined an algebra $\mathcal{A}$ of subsets, X a topological space, and $\Gamma : S \to X$ a multifunction such that $\Gamma^-(F) \in \mathcal{A}$ for every closed set F in X. Then if B is a closed set in X, Γ_B also has this property.

In the same way we have:

PROPOSITION 1.8. Let S be a space on which is defined a ring $\mathcal{C}$ of subsets, X a topological space, and $\Gamma : S \to X$ a multifunction such that $\Gamma^-(K) \in \mathcal{C}$ for every compact set K in X. Then if H is a closed compact set in X, Γ_H also has this property.

We now introduce the notions of continuity and measurability for multifunctions. Let T and X be topological spaces. Then, following (2) and (28), we define a multifunction $\Gamma : T \to X$ to be upper semicontinuous (u.s.c.) if $\Gamma^+(G)$ is open in T for every open set G in X; Γ is said to be lower semicontinuous (l.s.c.) if $\Gamma^-(G)$ is open for every open set G in X. We shall use the abbreviations u.s.c. and l.s.c. to avoid confusion with the different, but not unrelated, notion of semicontinuity for real-valued functions. If Γ is both u.s.c. and l.s.c., we shall say that it is continuous. Clearly, if Γ is closed-valued, it is continuous in this sense if and only if it is continuous as a point-valued function into $\mathcal{H}(X)$ with the Vietoris topology.

If $t_0 \in T$, Γ is said to be upper semicontinuous at t_0 if for any open set G, such that $\Gamma(t_0) \subset G$, $\Gamma^+(G)$ is a neighbourhood of t_0. Similarly, Γ is said to be lower semicontinuous at t_0 if, for any open set G, such that $\Gamma(t_0) \cap G \neq \phi$, $\Gamma^-(G)$ is a neighbourhood of t_0.

If S is a measurable space and X a topological space, then following (8), (18), (19), (41) and (42) we define a multifunction $\Gamma : S \to X$ to be measurable if $\Gamma^-(F)$ is measurable for every closed set F in X.

Γ will be said to be $\mathcal{G}$-<u>measurable</u> if $\Gamma^{-}(G)$ is measurable for every open set G in X.

We shall call Γ $\mathcal{K}$-<u>measurable</u> if $\Gamma^{-}(K)$ is measurable for every compact set K in X. In this case, we usually take the measurable sets of S to form a σ-ring rather than a σ-algebra.

It follows from Proposition 1.1 that if Γ is $\mathcal{G}$-measurable, then so is $\overline{\Gamma}$.

If (Γ_n) is a sequence of measurable multifunctions, it follows from Proposition 1.2 that $\bigcup_n \Gamma_n$ is measurable. The same thing is true for $\mathcal{G}$- and $\mathcal{K}$-measurable multifunctions.

If S is a measurable space, X, Y topological spaces, $\Gamma : S \to X$ a measurable multifunction and $\Phi : X \to Y$ is u.s.c., then $\Phi \circ \Gamma$ is measurable (Proposition 1.3). Similarly, if Γ is $\mathcal{G}$-measurable and Φ is l.s.c., then $\Phi \circ \Gamma$ is $\mathcal{G}$-measurable.

If X is a topological space (for instance, a metrisable space) in which every open set is an F_σ, then we see from Proposition 1.4 that every measurable multifunction into X is also $\mathcal{G}$-measurable.

Theorem 1.6 shows that every compact-valued $\mathcal{G}$-measurable multifunction into a perfectly normal space is also measurable.

From Propositions 1.7 and 1.8 we see that the refinement of a measurable multifunction by a closed set is itself measurable, and that the refinement of a $\mathcal{K}$-measurable multifunction by a closed compact set is $\mathcal{K}$-measurable.

2. <u>Measurability and the Souslin Operation</u>. Let $(A_{\sigma_1 \ldots \sigma_n})$ be a countable collection of sets in a given space, indexed by the set of all finite sequences $\sigma_1, \ldots, \sigma_n$ of positive integers. Then the set

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} A_{\sigma_1 \ldots \sigma_n},$$

the union being taken over the collection of all infinite sequences σ of positive integers, is said to be obtained from the family $(A_{\sigma_1 \ldots \sigma_n})$ by the Souslin operation. If the sets $(A_{\sigma_1 \ldots \sigma_n})$ belong to a given class $\mathcal{N}$ of sets, then A will be said to belong to the class Souslin-$\mathcal{N}$. This operation appears in a paper of M. Souslin, (49). We shall use the convenient notation adopted by C.A. Rogers in (43), pp. 44-49, and write

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} A_{\sigma|n} ,$$

where $\sigma|n$ denotes the finite sequence $\sigma_1, \ldots, \sigma_n$.

The measurable space S will be said to admit the Souslin operation if every subset formed in this way from measurable sets is measurable. Every measurable space derived from an outer measure admits the Souslin operation; this is proved in pp. 44-49 of (43). Moreover, it follows from the remarks on p. 95 of (28) that, if S is a locally compact Hausdorff space with a Radon measure μ, then the class of μ-measurable sets admits the Souslin operation.

Suppose that the class $\mathcal{N}$ is closed under the formation of finite intersections; then every set A in Souslin-$\mathcal{N}$ can be represented in the form

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} A_{\sigma|n} ,$$

where, for each sequence σ, the sets $A_{\sigma|1}, A_{\sigma|2}, \ldots$ form a descending sequence. If the sets $(A_{\sigma|n})$ do not have this property we replace them by the sets $(B_{\sigma|n})$ where, for each finite sequence $\sigma|n$,

$$B_{\sigma|n} = A_{\sigma|1} \cap \ldots \cap A_{\sigma|n} .$$

We shall refer to the Souslin-$\mathcal{F}$ sets of a topological space as the "Souslin sets". We shall make use of the following important result, which is proved in (17), §19:

PROPOSITION 2.1. *If $\mathcal{A}$ is any class of subsets of a given set, and if $\mathcal{B}$ = Souslin-$\mathcal{A}$, then Souslin $\mathcal{B}$ = Souslin $\mathcal{A}$.*

That is, we obtain no new sets by the iteration of the Souslin operation.

PROPOSITION 2.2. *If X is a topological space in which every open set is a Souslin set, then every Borel set in X is a Souslin set.*

Proof. Let $\mathcal{C}$ be the class of all Souslin sets whose complements are also Souslin sets. Since, by Proposition 2.1, the class of Souslin sets is closed under the formation of countable unions and intersections, $\mathcal{C}$ is closed under the formation of countable unions. As $\mathcal{C}$ is also closed under the operation of complementation, it is a σ-algebra. It contains the closed sets of X and hence it contains the Borel sets.

The conclusion of Proposition 2.2 holds in particular when X is a metrisable space. Clearly, in the statement we can replace "open set" by "closed set" and "Souslin" by "Souslin-$\mathcal{G}$". The proof is unaltered.

PROPOSITION 2.3. *If S is a measurable space which admits the Souslin operation, X a topological space, and $\Gamma : S \to X$ a measurable compact-valued multifunction, then $\Gamma^{-}(A)$ is measurable for every Souslin set A in X.*

Proof. Let $A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} A_{\sigma|n}$,

where the sets $(A_{\sigma|n})$ are closed subsets of X, and the sequence $(A_{\sigma|1}, A_{\sigma|2}, \ldots)$ is descending for every sequence σ of positive integers. Then

$$\Gamma^{-}(A) = \bigcup_{\sigma} \Gamma^{-}\Big(\bigcap_{n=1}^{\infty} A_{\sigma|n}\Big) \qquad \text{(Proposition 1.4)},$$

$$= \bigcup_{\sigma} \bigcap_{n=1}^{\infty} \Gamma^{-}(A_{\sigma|n}) \qquad \text{(cf. Proposition 1.5)},$$

which, being the result of applying the Souslin operation to measurable sets, is measurable.

Similarly we have:

PROPOSITION 2.4. <u>If</u> S <u>is a measurable space which admits the Souslin operation,</u> X <u>a Hausdorff space, and</u> $\Gamma : S \to X$ <u>a</u> $\mathcal{K}$<u>-measurable closed-valued multifunction, then</u> $\Gamma^{-}(A)$ <u>is measurable for every Souslin-</u>$\mathcal{K}$ <u>set</u> A <u>in</u> X .

It is a trivial consequence of Proposition 2.3 that if X is a space in which every open set is a Souslin set, then a compact-valued measurable multifunction from S to X , where S admits the Souslin operation, is $\mathcal{G}$-measurable. We can obtain a partial "converse" of this as follows:

PROPOSITION 2.5. <u>Let</u> S <u>be a measurable space which admits the Souslin operation,</u> X <u>a topological space in which every closed set is a Souslin-</u>$\mathcal{G}$ <u>set, and</u> $\Gamma : S \to X$ <u>a</u> $\mathcal{G}$<u>-measurable multifunction such that, for each</u> t , $\Gamma(t)$ <u>is finite (though the cardinal number of</u> $\Gamma(t)$ <u>may vary)</u>. <u>Then</u> Γ <u>is measurable</u>.

<u>Proof</u>. Let F be a closed set in X , with the Souslin representation

$$F = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} G_{\sigma|n} ,$$

where $G_{\sigma|n}$ is open for each $\sigma|n$, and the sequences $\{G_{\sigma|1}, G_{\sigma|2}, \ldots\}$ are descending.

Suppose that, for some sequence σ , $\Gamma(t)$ meets $G_{\sigma|n}$ for each n , in the point x_n say. Since $\Gamma(t)$ is finite, there is a point y in $\Gamma(t)$ which occurs infinitely many times in the sequence (x_n) . This $y \in \bigcap_n G_{\sigma|n}$, and so

$$t \in \Gamma^{-}(\bigcap_n G_{\sigma|n}) .$$

Therefore

$$\Gamma^{-}(\cap_n G_{\sigma|n}) = \cap_n \Gamma^{-}(G_{\sigma|n}) ,$$

and so

$$\Gamma^{-}(F) = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} \Gamma^{-}(G_{\sigma|n}) ,$$

which is a measurable set.

We conclude this section with a result on the composition of measurable multifunctions:

PROPOSITION 2.6. *Let* S *be a measurable space which admits the Souslin operation,* X *a topological space in which every open set is a Souslin set, and* Y *any other topological space. Let* $\Gamma : S \to X$ *be a measurable compact-valued multifunction and* $\Phi : X \to Y$ *a multifunction which is measurable with respect to the Borel sets of* X *. Then* $\Phi \circ \Gamma$ *is measurable.*

Proof. This result follows at once from Propositions 1.3, 2.2, and 2.3.

If in Proposition 2.6 Φ is $\mathcal{G}$-measurable, then we conclude that $\Phi \circ \Gamma$ is $\mathcal{G}$-measurable.

3. *The Graph of a Measurable Multifunction.* If S and X are any two sets and $\Gamma : S \to X$ a multifunction, then the *graph* of Γ is defined to be the set

$$G(\Gamma) = \{(x,y) \in X \times Y : y \in \Gamma(x)\} .$$

If M is any subset of $S \times X$, then M is the graph of the multifunction

$$\Omega : t \to \{x : (t,x) \in M\} ,$$

and this multifunction, obtained by taking cross-sections of M , is uniquely defined. We shall denote the natural projection from $S \times X$ onto S by π_1. Then if B is any subset of X,

$$\Gamma^{-}(B) = \pi_1(G(\Gamma) \cap (S \times B)) .$$

In what follows, S will be a measurable space and X a topological space. $\mathcal{R}$ will denote the class of all sets $A \times B$ where A is a measurable set in S and B is a closed set in X. We shall need the following lemma, which is a generalization of result (3.4) of (14), and of the main result of (34):

LEMMA 3.1. Let X_1 be a topological space and $\mathcal{K}_1$ the class of sets which are closed and compact in X_1. Let Y be a Souslin-$\mathcal{K}_1$ subspace of X_1. Then if S is a measurable space which admits the Souslin operation, $\pi_1(A)$ is measurable for every Souslin-$\mathcal{R}$ set A in $S \times Y$.

Proof. Let $A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (A_{\sigma|n} \times B_{\sigma|n})$,

where each $A_{\sigma|n}$ is measurable and each $B_{\sigma|n}$ is closed relative to Y. Now for each $\sigma|n$, $B_{\sigma|n} = C_{\sigma|n} \cap Y$, say, where $C_{\sigma|n}$ is closed in X_1. Hence each $B_{\sigma|n}$ is Souslin-$\mathcal{K}_1$ in X_1, and as the iteration of the Souslin operation produces no new sets (Proposition 2.1), A is Souslin-$\mathcal{R}$ in $S \times X_1$. Thus, with new notation, we may write

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (D_{\sigma|n} \times E_{\sigma|n}) ,$$

where each $D_{\sigma|n}$ is measurable, and each $E_{\sigma|n}$ is a closed compact set in X_1. For each finite sequence $\sigma|n$, we define

$$E^*_{\sigma|n} = E_{\sigma_1} \cap \dots \cap E_{\sigma_1 \dots \sigma_n} ,$$

and we put $D^*_{\sigma|n} = \phi$ if $E^*_{\sigma|n} = \phi$, $D^*_{\sigma|n} = D_{\sigma|n}$ otherwise.

Then

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (D^*_{\sigma|n} \times E^*_{\sigma|n}) ,$$

$$= \bigcup_{\sigma} (\bigcap_{n=1}^{\infty} D^*_{\sigma|n}) \times (\bigcap_{n=1}^{\infty} E^*_{\sigma|n}) .$$

Thus
$$\pi_1(A) = \bigcup_{\sigma} \pi_1(\bigcap_{n=1}^{\infty} D^*_{\sigma|n} \times \bigcap_{n=1}^{\infty} E^*_{\sigma|n}) ,$$

$$= \bigcup_{\sigma} \bigcap_{n=1}^{\infty} D^*_{\sigma|n} ,$$

since, if $\bigcap_{n=1}^{\infty} E^*_{\sigma|n}$ is empty, $E^*_{\sigma|n} = \phi$, for some n, by compactness, in which case $D^*_{\sigma|n} = \phi$ by our definition.

We may extend this result a little:

LEMMA 3.2. Let X_1, S and Y be as in Lemma 3.1. Then if $Z = \varphi(Y)$, where φ is a continuous mapping, and A is a Souslin-$\mathcal{R}$ set in $S \times Z$, $\pi_1(A)$ is measurable.

Proof. Let $A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (A_{\sigma|n} \times B_{\sigma|n})$,

where each $A_{\sigma|n}$ is measurable and each $B_{\sigma|n}$ is closed. We define

$$A_1 = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (A_{\sigma|n} \times \varphi^{-1}(B_{\sigma|n})) , \quad \text{in } S \times Y .$$

It is easily shown that $\pi_1(A) = \pi_1(A_1)$; the result follows at once from Lemma 3.1.

Now let X be a Polish space. We know that X is homeomorphic to a Borel subset of the compact metrisable space I^N (§0, (ix)). Hence, by Proposition 2.2, any Polish space is homeomorphic to a Souslin-$\mathcal{K}$ subset of I^N and so satisfies the hypotheses of Lemma 3.1. We deduce from Lemma 3.2 the following:

LEMMA 3.3. If S admits the Souslin operation and X is the *continuous image of a Polish space, then* $\pi_1(A)$ *is measurable for every Souslin-*$\mathcal{R}$ *set* A *in* $S \times X$.

COROLLARY. *If* S *admits the Souslin operation,* X *is the continuous image of a Polish space and* $\Gamma : S \to X$ *has Souslin-*$\mathcal{R}$ *graph, then* Γ *is measurable.*

Proof. If F is a closed set in X,

$$\Gamma^{-}(F) = \pi_1(G(\Gamma) \cap (S \times F)),$$

which is the projection of a Souslin-$\mathcal{R}$ set.

LEMMA 3.4. *Let* $(S,\mathcal{M})$ *be a measurable space and* X *a topological space in which every open set is a Souslin set. Then every set in the* σ*-algebra* $\mathcal{M} \otimes \mathcal{B}_X$ *is Souslin-*$\mathcal{R}$.

Proof. Let $\mathcal{A}$ be the class of sets A in $S \times X$ such that both A and its complement A' are Souslin-$\mathcal{R}$. This class is a σ-algebra, since by Proposition 2.1 any Souslin class is invariant under the formation of countable unions and intersections. $\mathcal{A}$ also contains the sets of $\mathcal{R}$, for if M is measurable and F is closed, the complement of $M \times F$ is

$$(M \times F)' = (M' \times X) \cup (S \times F'),$$

which is clearly Souslin-$\mathcal{R}$.

The sets $M \times F$ generate the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$, since the σ-algebra generated by the sets $\{M \times F : F \text{ closed in } X\}$ for a fixed set M in S contains all the sets $M \times B$ where B is a Borel set in X.

Hence $\mathcal{A}$ contains $\mathcal{M} \otimes \mathcal{B}_X$, and so every set in $\mathcal{M} \otimes \mathcal{B}_X$ is Souslin-$\mathcal{R}$.

In particular, the conclusion holds if X is metrisable.

PROPOSITION 3.5. *If* $(S,\mathcal{M})$ *is a measurable space which admits the Souslin operation and* X *is a topological space which is the continuous*

<u>image of a Polish space, then the projection into</u> S <u>of any set in</u> $\mathcal{M} \otimes \mathcal{B}_X$ <u>is measurable</u>.

<u>Proof</u>. Let $X = \varphi(P)$, where P is a Polish space. Consider the mapping ψ from $S \times P$ into $S \times X$ defined by

$$\psi(t,p) = (t, \varphi(p)) .$$

Then if $A \in \mathcal{M}$ and $B \in \mathcal{B}_X$,

$$\psi^{-1}(A \times B) = A \times \varphi^{-1}(B) ,$$

which belongs to $\mathcal{M} \otimes \mathcal{B}_P$, φ being continuous. Thus if $M \in \mathcal{M} \otimes \mathcal{B}_X$, $\psi^{-1}(M) \in \mathcal{M} \otimes \mathcal{B}_P$. Therefore

$$\begin{aligned} \pi_1(M) &= \{t \in S : (t,x) \in M \text{ some } x \in X\} , \\ &= \{t \in S : (t, \varphi(p)) \in M \text{ some } p \in P\} , \\ &= \{t \in S : (t,p) \in \psi^{-1}(M) \text{ some } p \in P\} , \\ &= \pi_1(\psi^{-1}(M)) , \end{aligned}$$

and this is measurable, by Lemma 3.4 and Lemma 3.3.

We now consider the relationship between the measurability of a multifunction and the nature of its graph. The following theorem comes from statement 4.3 of (14) and from Theorem 2 of (42).

THEOREM 3.6. <u>If</u> $(S,\mathcal{M})$ <u>is any measurable space</u>, X <u>a</u> 2C-<u>space and</u> Γ <u>any</u> $\mathcal{G}$-<u>measurable closed-valued multifunction from</u> S <u>into</u> X, $G(\Gamma)$ <u>belongs to the</u> σ-<u>algebra</u> $\mathcal{M} \otimes \mathcal{B}_X$.

<u>Proof</u>. Let (U_n) be a countable collection of open sets forming a base for the topology of X.

Then $(t,x) \notin G(\Gamma)$ if and only if $x \in U_n$ for some n, where $U_n \cap \Gamma(t) = \phi$.

That is, $(t,x) \notin G(\Gamma)$ if and only if $(t,x) \in \Gamma^+(U_n') \times U_n$ for some n. Thus the complement of the graph is

$$G(\Gamma)' = \bigcup_{n=1}^{\infty} (\Gamma^+(U_n') \times U_n) ,$$

and so $G(\Gamma)$ is measurable.

Similarly we have:

THEOREM 3.7. <u>If</u> $(S,\mathcal{M})$ <u>is any measurable space,</u> X <u>any regular space which is the continuous image of a</u> 2C-<u>space, and</u> Γ <u>a closed-valued measurable multifunction from</u> S <u>into</u> X , <u>then</u> $G(\Gamma)$ <u>belongs to the</u> σ-<u>algebra</u> $\mathcal{M} \otimes \mathcal{B}_X$.

<u>Proof</u>. Let (U_n) be a countable base of open sets for the 2C-space Y , where $X = \varphi(Y)$, φ being a continuous mapping. For each n , let $V_n = \overline{\varphi(U_n)}$. Then, since φ is continuous, if x is any point of X and U a neighbourhood of x , there exists an integer n such that $x \in V_n \subseteq U$. Therefore, if $x \notin \Gamma(t)$, $V_n \cap \Gamma(t) = \phi$ for some n and so

$$G(\Gamma)' = \bigcup_{n=1}^{\infty} (\Gamma^+(V_n') \times V_n) ,$$

which clearly belongs to $\mathcal{M} \otimes \mathcal{B}_X$.

THEOREM 3.8. <u>Let</u> $(S,\mathcal{M})$ <u>be a measurable space which admits the Souslin operation,</u> X <u>a Souslin space and</u> $\Gamma : S \to X$ <u>a closed-valued multifunction. Then the statements</u> (i) <u>to</u> (iv) <u>are equivalent</u>:

(i) Γ <u>is measurable</u>;

(ii) Γ <u>is</u> $\mathcal{G}$-<u>measurable</u>;

(iii) $G(\Gamma) \in \mathcal{M} \otimes \mathcal{B}_X$;

(iv) $\Gamma^-(B)$ <u>is measurable for every Souslin set</u> B <u>in</u> X .

<u>Proof</u>. From Proposition 1.4, we deduce that (i) implies (ii). The fact that (ii) implies (iii) has already been shown (Theorem 3.6). Suppose that (iii) holds. Then $G(\Gamma)$ is a Souslin-$\mathcal{R}$ set, by Lemma 3.4, and so if B is a Souslin set in Y ,

$$\Gamma^-(B) = \pi_1(G(\Gamma) \cap (S \times B)) ,$$

which is the projection of a Souslin-$\mathcal{R}$ set, and so is measurable, by Lemma 3.3.

The chain of implications is closed by the trivial fact that (iv) implies (i).

The equivalence of (i) and (ii) has also been shown by A.P. Robertson, in Theorem 3 of (41).

THEOREM 3.9. Let $(S,\mathcal{M})$ be a measurable space which admits the Souslin operation, X the regular continuous image of a Polish space and $\Gamma : S \to X$ a closed-valued multifunction. Then the following three statements are equivalent:

(i) Γ is measurable;

(ii) $G(\Gamma) \in \mathcal{M} \otimes \mathcal{B}_X$;

(iii) $\Gamma^-(B)$ is measurable for every Souslin set B in Y .

Proof. It follows from Theorem 3.7 that (i) implies (ii). If (ii) holds, then (iii) follows from Lemma 3.3 and Lemma 3.4, because

$$\Gamma^-(B) = \pi_1(G(\Gamma) \cap (S \times B)) .$$

It is trivial that (iii) implies (i).

In this case, Γ is also $\mathcal{G}$-measurable, a conclusion which holds under more general hypotheses, as is shown in Theorem 3.11 below. First we need :

LEMMA 3.10. If X is the regular continuous image of a 2C-space, then every open set in X is an $\mathcal{F}_\sigma$ set.

Proof. Let $X = \varphi(Y)$, where Y is a 2C-space and φ is a continuous mapping. Let (U_n) be a countable base of open sets for Y . Now if G is open in X , and $x \in G$, there is a closed neighbourhood V of x contained in G . Now there is a point y in Y such that $x = \varphi(y)$. Since φ is continuous, $\varphi(U_n) \subset V$ for some neighbourhood U_n of y . Hence $\overline{\varphi(U_n)} \subset G$. It follows that every open set G is a union of the sets $\overline{\varphi(U_n)}$, and so is an $\mathcal{F}_\sigma$.

THEOREM 3.11. _If_ S _is any measurable space, and_ X _is the regular continuous image of a 2C-space, then every measurable multifunction from_ S _into_ X _is_ $\mathcal{G}$_-measurable_.

Proof. This follows immediately from Proposition 1.4 and Lemma 3.10.

We end this section with two results which are related to Theorems 3.8 and 3.9 but are not covered by them.

THEOREM 3.12. _Let_ $(S,\mathcal{M})$ _be any measurable space,_ X _any topological space and_ $\Gamma : S \to X$ _a multifunction such that_ $G(\Gamma) \in \mathcal{M} \otimes \mathcal{B}_X$. _Then_ (i) _for each_ $t \in S$, $\Gamma(t)$ _is a Borel measurable set,_ _and_ (ii) _for each_ $x \in X$, $\Gamma^-(\{x\}) = \{t \in S : x \in \Gamma(t)\}$ _is measurable_.

Proof. This theorem is a restatement in the language of multifunctions of (16), §34, Theorem A.

THEOREM 3.13. _Let_ $(S,\mathcal{M})$ _be a measurable space which admits the Souslin operation,_ X _the continuous image of a Polish space, and_ Γ _a multifunction with graph in_ $\mathcal{M} \otimes \mathcal{B}_X$. _Then_ $\Gamma^-(B)$ _is measurable for every Borel set_ B _in_ X.

Proof. If B is a Borel set in X,

$$\Gamma^-(B) = \pi_1(G(\Gamma) \cap (S \times B)) ,$$

and this is the projection onto S of a set in $\mathcal{M} \otimes \mathcal{B}_X$. This set $\Gamma^-(B)$ is measurable, by Proposition 3.5.

4. _Intersections and Products of Measurable Multifunctions_. If S is a measurable space, X a topological space and (Γ_n) a sequence of measurable multifunctions from S into X, then it follows from Proposition 1.2 that $\cup_n \Gamma_n$ is also a measurable multifunction. In general, in order to prove similar theorems for intersections and products, we have

to place restrictions on the kind of spaces we use.

LEMMA 4.1. Let S be any measurable space, X a separable metrisable space and Γ_1, Γ_2 two measurable compact-valued multifunctions from S into X. Then the set $\{t : \Gamma_1(t) \cap \Gamma_2(t) = \phi\}$ is measurable.

Proof. Let $A = \{t : \Gamma_1(t) \cap \Gamma_2(t) = \phi\}$

Let (U_n) be a countable family of open sets forming a base for the topology of X. Then $t \in A$ if and only if there exists a finite sub-collection $U_{i_1}, \ldots, U_{i_k}$ of these sets such that

$$\Gamma_1(t) \subseteq U_{i_1} \cup \ldots \cup U_{i_k} \quad \text{and} \quad \Gamma_2(t) \cap (U_{i_1} \cup \ldots \cup U_{i_k}) = \phi .$$

Hence

$$A = \bigcup_{i_1,\ldots,i_n} (\Gamma_1^+(U_{i_1} \cup \ldots \cup U_{i_n}) \setminus \Gamma_2^-(U_{i_1} \cup \ldots \cup U_{i_n})) ,$$

the union being taken over the countable set of finite sets of positive integers. A is a countable union of measurable sets, and so is measurable.

PROPOSITION 4.2. Let S be any measurable space, X a separable metrisable space and Γ_1, Γ_2 two measurable compact-valued multifunctions from S into X. Then the multifunction

$$\Gamma : t \to \Gamma_1(t) \cap \Gamma_2(t)$$

is also measurable.

Proof. If F is any closed set in X,

$$\Gamma^-(F) = \{t : \Gamma_1(t) \cap \Gamma_2(t) \cap F \neq \phi\} ,$$

$$= \{t : \Gamma_1(t) \cap \Gamma_{2,F}(t) \neq \phi\} \cap \Gamma_2^-(F) ,$$

where $\Gamma_{2,F}$ is the refinement (§1) of Γ_2 by the closed set F. This is a measurable compact-valued multifunction, by Proposition 1.7, and hence $\Gamma^-(F)$ is a measurable set, by Lemma 4.1.

THEOREM 4.3. Let S be any measurable space, X a separable metrisable space and (Γ_n) a sequence of measurable compact-valued *multifunctions from* S *into* X . *Then the multifunction* $\cap_n \Gamma_n$ *is also measurable.*

Proof. It follows from Proposition 4.2, by induction on n, that the multifunction

$$\Omega_n : t \to \Gamma_1(t) \cap \dots \cap \Gamma_n(t)$$

is measurable for each n. Hence (Ω_n) is a descending sequence of compact-valued measurable multifunctions. Now $\cap_n \Gamma_n = \cap_n \Omega_n$, and the measurability of $\cap_n \Omega_n$ follows immediately from Proposition 1.5.

This last result generalizes Théorème 4.10 of (8), where it is stated for the case where S is a locally compact Hausdorff space with a Radon measure. In fact, our hypotheses can be relaxed; Lemma 4.1 still holds if X is a 2C-space and Γ_2 any $\mathcal{G}$-measurable closed-valued multifunction. The same applies to Proposition 4.2. We state Theorem 4.3 with relaxed conditions as follows; the proof is unchanged.

THEOREM 4.4. *Let* S *be any measurable space,* X *a* 2C-*space, and* (Γ_n) *a sequence of multifunctions from* S *into* X . *Suppose that* Γ_1 *is measurable and compact-valued, and that all the others are* $\mathcal{G}$-*measurable and closed-valued. Then the multifunction* $\cap_n \Gamma_n$ *is measurable.*

We now turn to the question of closed-valued multifunctions; we place more restrictions on S and X .

THEOREM 4.5. *Let* S *be a measurable space which admits the Souslin operation,* X *the regular continuous image of a Polish space, and* (Γ_n) *a sequence of closed-valued measurable multifunctions from* S *into* X . *Then the multifunction* $\cap_n \Gamma_n$ *is also measurable.*

Proof. $G(\cap_n \Gamma_n) = \bigcap_{n=1}^{\infty} G(\Gamma_n)$.

Each Γ_n has measurable graph, by Theorem 3.9, and hence so has $\cap_n \Gamma_n$.

It follows from the same theorem that $\cap_n \Gamma_n$ is measurable.

If (Γ_n) is any sequence of multifunctions which satisfy the hypotheses of Theorem 4.5, then we can show in the same way that the result of applying any finite sequence of countable operations to (Γ_n) is a measurable multifunction. For instance, the multifunction $t \to \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \Gamma_n(t)$ is measurable in this case. We next consider the question of products of set-valued functions.

PROPOSITION 4.6. <u>Let</u> S <u>be any measurable space,</u> X <u>any topological space and</u> Y <u>any</u> 2C-<u>space. Then if</u> $\Gamma_1 : S \to X$ <u>and</u> $\Gamma_2 : S \to Y$ <u>are</u> $\mathcal{G}$-<u>measurable multifunctions, the multifunction</u>

$$\Gamma : t \to \Gamma_1(t) \times \Gamma_2(t)$$

<u>from</u> S <u>to</u> $X \times Y$ <u>is also</u> $\mathcal{G}$-<u>measurable</u>.

<u>Proof</u>. Let G be any open set in $X \times Y$. Then

$$G = \bigcup_{i \in I} (A_i \times B_i) , \quad \text{say,}$$

where A_i is open in X, all i, and B_i is open in Y. The index set I need not be countable. Let (U_n) be a countable family of open sets forming a base for the topology of Y. Then, if we define for each integer n

$$O_n = \bigcup \{A_i : U_n \subset B_i\} ,$$

we have

$$G = \bigcup_{n=1}^{\infty} (O_n \times U_n) .$$

Therefore
$$\Gamma^-(G) = \bigcup_{n=1}^{\infty} (\Gamma_1^-(O_n) \cap \Gamma_2^-(U_n)) ,$$

which is clearly measurable.

COROLLARY. <u>If</u> S <u>is any measurable space,</u> X <u>any</u>

topological space, Y any 2C-space, and $f : S \to X$, $g : S \to Y$ are measurable functions, then the function

$$t \to (f(t), g(t)),$$

is also measurable.

PROPOSITION 4.7. Let S be any measurable space, (X_i) a sequence of 2C-spaces, and (Γ_i) a sequence of $\mathcal{G}$-measurable multifunctions from S into X_i respectively. Then $\Pi_i\Gamma_i$ is $\mathcal{G}$-measurable with respect to the product topology on $\Pi_i X_i$.

Proof. Let $G = \Pi_i G_i$ be any basic open set in $\Pi_i X_i$ (that is, $G_i = X_i$ for all but finitely many i; each G_i is open). Then it is easily seen that

$$(\Pi_i\Gamma_i)^-(G) = \bigcap_{j=1}^{\infty} \Gamma_j^-(G_j),$$

which is a measurable set. Since any open set H in $\Pi_i X_i$ is a countable union of basic ones $H_i (i = 1,2, \ldots)$,

$$(\Pi_i\Gamma_i)^-(H) = \bigcup_{j=1}^{\infty} (\Pi_i\Gamma_i)^-(H_j),$$

which is measurable.

COROLLARY. Let S be any measurable space, (X_i) a sequence of 2C-spaces, and (f_i) a sequence of measurable functions from S into X_i, respectively. Then the function

$$t \to (f_1(t), f_2(t), \ldots)$$

from S into $\Pi_i X_i$ is measurable with respect to the product topology on $\Pi_i X_i$.

THEOREM 4.8. Let S be any measurable space, (X_i) a sequence of separable metrisable spaces and (Γ_i) a sequence of measurable compact-valued multifunctions from S into X_i, respectively. Then the multifunction $\Pi_i \Gamma_i$ is also measurable, with respect to the product topology on $\Pi_i X_i$.

Proof. Γ is $\mathcal{G}$-measurable, by Proposition 4.7. Γ is also compact-valued, and so is measurable by Theorem 1.6, the space $\Pi_i X_i$ being metrisable.

We now consider the graph of a product of measurable multifunctions; this will enable us to prove stronger results than those obtained so far.

THEOREM 4.9. Let $(S, \mathcal{M})$ be any measurable space and (X_i) any sequence of topological spaces. Let $X = \Pi_i X_i$, with the product topology. Then if, for each i, Γ_i is a multifunction from S into X_i with $G(\Gamma_i) \in \mathcal{M} \otimes \mathcal{B}_{X_i}$, we have $G(\Pi_i \Gamma_i) \in \mathcal{M} \otimes \mathcal{B}_X$.

Proof. Consider the mapping $\varphi : S \times X \to \Pi_i (S \times X_i)$ defined by

$$\varphi(s, (x_i)) = ((s, x_i)),$$

where $s \in S$ and $(x_i) \in \Pi_i X_i$.

Let π_j be the natural projection of $\Pi_i (S \times X_i)$ onto $S \times X_j$; let $\mathcal{A}$ be the smallest σ-algebra on $\Pi_i (S \times X_i)$ such that, for all j, $A \in \mathcal{M} \otimes \mathcal{B}_{X_j}$ implies $\pi_j^{-1}(A) \in \mathcal{A}$. Suppose that j is fixed; then the set $\{B \subseteq S \times X_j : \pi_j^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra. It contains $\mathcal{M} \otimes \mathcal{B}_{X_j}$ and hence all the sets $C \times D_j$ where $C \in \mathcal{M}$ and D_j is closed in X_j. Now these sets $\{C \times D_j\}$ generate $\mathcal{M} \otimes \mathcal{B}_{X_j}$. Thus $\mathcal{A}$ is generated by the sets $\pi_j^{-1}(C \times D_j)$, as j runs through the positive integers.

Now $\varphi^{-1}(\pi_j^{-1}(C \times D_j)) = C \times (X_1 \times \ldots \times D_j \times X_{j+1} \times \ldots)$, which clearly belongs to $\mathcal{M} \otimes \mathcal{B}_X$.

Therefore, if $E \in \mathcal{A}$, $\varphi^{-1}(E) \in \mathcal{M} \otimes \mathcal{B}_X$.

Now

$$G(\Pi_i \Gamma_i) = \varphi^{-1}(\Pi_i G(\Gamma_i)) ,$$

which therefore belongs to $\mathcal{M} \otimes \mathcal{B}_X$.

THEOREM 4.10. Let $(S,\mathcal{M})$ be a measurable space which admits the Souslin operation and (X_i) a sequence of regular topological spaces, each of which is the continuous image of some Polish space. Then, if $\Gamma_i : S \to X_i$ is for each i a measurable closed-valued multifunction, $\Pi_i \Gamma_i$ is also measurable with respect to the product topology on $\Pi_i X_i$.

Proof. Suppose that for each i $X_i = \psi_i(P_i)$, where P_i is a Polish space and ψ_i is a continuous mapping. Then $\Pi_i X_i$ is the image of the Polish space $\Pi_i P_i$ (§0, (ii)) under the continuous mapping

$$\psi : (p_i) \to (\psi_i(p_i)) .$$

Now, for each i, $G(\Gamma_i) \in \mathcal{M} \otimes \mathcal{B}_{X_i}$, by Theorem 3.9. Thus if $X = \Pi_i X_i$,

$$G(\Gamma) \in \mathcal{M} \otimes \mathcal{B}_X ,$$

by Theorem 4.9, and so by Theorem 3.9 Γ is measurable.

Clearly all the results from Proposition 4.7 to Theorem 4.10 hold for finite products of multifunctions as well as for countably infinite products.

Theorem 4.9 is also true for uncountably infinite products, as nowhere in the proof do we use the fact that (X_i) is a countable family.

5. The Limit of a Sequence of Measurable Multifunctions. Let X by any topological space and (A_n) any sequence of subsets of X . Then, following K. Kuratowski ((28), §29) and C. Berge ((2) pp. 118-119), we make the following definitions:

A point x_o in X is a limit point of (A_n) if to each neighbourhood U of x_o there corresponds an integer N such that $k \geq N$ implies $A_k \cap U \neq \phi$.

A point x_o in X is a <u>cluster point</u> of (A_n) if, for each neighbourhood U of x_o , A_k meets U for infinitely many k .

The set of limit points of (A_n) is called the <u>lower limit</u> of (A_n) and is denoted by $Li(A_n)$; the set of cluster points of (A_n) is called the <u>upper limit</u> of (A_n) and is denoted by $Ls(A_n)$.

It follows from these definitions that $Li(A_n) \subseteq Ls(A_n)$ for every sequence (A_n) , and that the two limits are always closed sets. It is possible that one or both of them may be empty.

If $Li(A_n) = Ls(A_n) = A_o$, we shall say that the sequence (A_n) <u>converges</u> to A_o , or that A_o is the <u>limit</u> of the sequence (A_n) . We write $A_n \to A_o$ or $A_o = \text{Lim}\ (A_n)$ to express this.

The following proposition, due to Hausdorff, is quoted in (28), p. 337.

PROPOSITION 5.1. <u>If</u> (A_n) <u>is any sequence of sets in a topological space, then</u>

$$Ls(A_n) = \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{n=k}^{\infty} A_n} \right) .$$

We can get a similar formula for $Li(A_n)$ if we admit uncountable operations (cf. (28), p. 337, footnote).

PROPOSITION 5.2. <u>If</u> (A_n) <u>is a sequence of sets in a topological space, then</u>

$$Li(A_n) = \bigcap_{\sigma} \left(\overline{\bigcup_{k=1}^{\infty} A_{\sigma_k}} \right) ,$$

<u>the intersection being taken over all strictly increasing sequences</u> σ <u>of positive integers</u>.

<u>Proof</u>. If x is a limit point, then it follows from the definition that x belongs to the right-hand side of the above formula.

Now suppose that x is not a limit point of (A_n). Then there is a neighbourhood U of x_o, and an increasing sequence (σ_k) of positive integers such that, for each k, $A_{\sigma_k} \cap U = \phi$. Thus U does not meet $\cup_k A_{\sigma_k}$ and so x does not belong to the right-hand side of the formula, which is all we need to prove the proposition.

PROPOSITION 5.3. <u>Let</u> S <u>be any measurable space, and</u> (Γ_n) <u>a sequence of measurable or</u> $\mathcal{G}$<u>-measurable multifunctions from</u> S <u>into a perfectly normal space</u> X, <u>such that for each</u> t <u>the set</u> $\cup_i \Gamma_i(t)$ <u>is relatively compact. Then the multifunction</u> $t \to Ls(\Gamma_n(t))$ <u>is measurable</u>.

<u>Proof</u>. Γ_n is $\mathcal{G}$-measurable for each n, and so, by Propositions 1.1 and 1.2 the multifunction

$$t \to \overline{\bigcup_{n=k}^{\infty} \Gamma_n(t)}$$

is $\mathcal{G}$-measurable for each k. Since it is compact-valued, it is measurable, by Theorem 1.6. The fact that $t \to Ls(\Gamma_n(t))$ is measurable then follows from Proposition 1.5.

PROPOSITION 5.4. <u>Let</u> S <u>be a measurable space which admits the Souslin operation, and</u> (Γ_n) <u>a sequence of measurable or</u> $\mathcal{G}$<u>-measurable multifunctions from</u> S <u>into the Souslin space</u> X. <u>Then the multifunction</u> $t \to Ls(\Gamma_n(t))$ <u>is measurable</u>.

<u>Proof</u>. For each n, Γ_n is $\mathcal{G}$-measurable. Therefore for each k the multifunction

$$t \to \overline{\bigcup_{n=k}^{\infty} \Gamma_n(t)}$$

is $\mathcal{G}$-measurable, and hence measurable, by Theorem 3.8. Thus $t \to Ls(\Gamma_n(t))$ is a measurable multifunction, by Theorem 4.5.

PROPOSITION 5.5. *Let* S *be a measurable space which admits the Souslin operation,* X *a Polish space, and* (Γ_n) *a sequence of measurable multifunctions from* S *into* X. *Then the multifunction* $\Gamma : t \to Li(\Gamma_n(t))$ *is also measurable.*

Proof. Define $\Gamma_{\sigma|n}(t) = \Gamma_{\sigma_1 + \ldots + \sigma_n}(t)$, for each $t \in S$. Then, with this re-indexing, we have

$$Li(\Gamma_n(t)) = \bigcap_{\sigma} \overline{\bigcup_{n=1}^{\infty} \Gamma_{\sigma|n}(t)} ,$$

from Proposition 5.2, the intersection being taken over all sequences of positive integers.

Let F be a closed subset of X. Since X is a separable metrisable space, F has a countable covering (U_i) by open sets of diameter less than $\frac{1}{2}$. We suppose that $U_i \cap F \neq \phi$ for each i. Similarly, for each i, $F \cap U_i$ has a countable covering $\{U_{ij} : j = 1,2, \ldots\}$ by open sets of diameter less than $\frac{1}{4}$, where again $U_{ij} \cap F \neq \phi$ for each j.

Continuing in this way we obtain a family $(U_{\sigma|n})$ of open sets, where $U_{\sigma|n} \cap F \neq \phi$ for each n, and $U_{\sigma|n}$ has diameter less than 2^{-n}. For each infinite sequence σ of positive integers $\bigcap_n \overline{U}_{\sigma|n}$ contains a single point of F, x_σ say, as X is a complete space. Conversely, if $x \in F$, there exists at least one sequence σ such that $x \in U_{\sigma|n}$ for all n.

Now $\Gamma(t) \cap F \neq \phi$ if and only if for every σ there exists a sequence τ of positive integers such that

$$x_\sigma \notin \overline{\bigcup_{n=1}^{\infty} \Gamma_{\tau|n}(t)} .$$

This is true if and only if there exists an integer k such that $U_{\sigma|k} \cap \Gamma_{\tau|n}(t) = \phi$ for all n. Therefore

$$\Gamma^+(F') = \bigcap_{\sigma} \bigcup_{k=1}^{\infty} [\bigcup_{\tau} \bigcap_{n=1}^{\infty} \Gamma^+_{\tau|n}(\overline{U}'_{\sigma|k})] ,$$

where $\bar{U}'_{\sigma|n}$ denotes the complement of $\bar{U}_{\sigma|n}$. The term in square brackets is measurable; thus $\Gamma^+(F')$ is the complement of a set obtained by applying the Souslin operation to measurable sets. It is therefore measurable, and so Γ is a measurable multifunction.

Let (A_n) be a sequence of closed sets in a topological space X, and suppose that $A_n \to A$ in the Vietoris topology on $\mathcal{H}(X)$. It is easy to show that $A \subset \mathrm{Li}(A_n)$. Moreover, if X is a regular space, $\mathrm{Ls}(A_n) \subset A$ and so $A = \mathrm{Lim}\,(A_n)$.

Similarly, if (X,d) is a metric space and (A_n) is a sequence of closed sets in X such that $A_n \to A$ in the Hausdorff topology on $\mathcal{H}(X)$, then again $A = \mathrm{Lim}\,(A_n)$. However, $\mathrm{Lim}\,(A_n)$ may exist even though the sequence does not converge in the Hausdorff topology.

<u>Example</u>. Take $X = \mathbb{R}$, and let $A_n = \{\frac{1}{n}, n\}$. Then $\mathrm{Lim}\,(A_n) = \{0\}$, but $\delta(A_n, \{0\}) = n$, and so there is no convergence in the Hausdorff, or the Vietoris, topology.

Hence convergence in the Vietoris or Hausdorff topology is a stronger condition than the existence of $\mathrm{Lim}\,(A_n)$, and so we expect to be able to prove stronger results for convergent sequences of multifunctions than we have obtained so far.

THEOREM 5.6. <u>Let</u> S <u>be a measurable space</u>, X <u>a perfectly normal space and</u> (Γ_n) <u>a sequence of measurable (or</u> $\mathcal{G}$<u>-measurable) closed-valued multifunctions from</u> S <u>to</u> X <u>such that for each</u> t <u>the sequence</u> $(\Gamma_n(t))$ <u>converges in the Vietoris topology to the closed set</u> $\Gamma(t)$. <u>Then the multifunction</u> Γ <u>is</u> $\mathcal{G}$<u>-measurable</u>.

<u>Proof</u>. If F is a closed set in X, there exists a sequence (G_n) of open sets such that $F = \bigcap_n G_n$ and

$$G_1 \supset \bar{G}_2 \supset G_2 \supset \bar{G}_3 \supset \dots$$

Then it follows that

$$\Gamma^+(F) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Gamma_n^+(\overline{G}_k) , \qquad \ldots(i)$$

from which we deduce that Γ is $\mathcal{G}$-measurable. It remains to prove that (i) holds.

If $t \in \Gamma^+(F)$, then $\Gamma(t) \subseteq F$ and so $\Gamma(t) \subseteq \overline{G}_k$ all k. Moreover, for each k, there exists an integer $m(k)$ such that $n \geqslant m(k)$ implies that $\Gamma_n(t) \subset G_k$, and hence $t \in \Gamma_n^+(\overline{G}_k)$. Thus t belongs to the right-hand side of (i).

Conversely if $\Gamma(t) \not\subseteq F$, then $\Gamma(t) \not\subseteq \overline{G}_k$, for some k.

Thus $\Gamma(t) \cap \overline{G}_k' \neq \phi$.

Hence there exists an integer $m(k)$ such that $n \geqslant m(k)$ implies that $\Gamma_n(t) \not\subseteq \overline{G}_k$ for $n \geqslant m(k)$. In particular there is for each m an integer $m_1 > m$ such that $\Gamma_{m_1}(t) \not\subseteq \overline{G}_k$.

Therefore, if t does not belong to the left-hand side of (i), it does not belong to the right-hand side.

COROLLARY. <u>If</u> S <u>is a measurable space,</u> X <u>a perfectly normal space and</u> (f_n) <u>a sequence of measurable functions from</u> S <u>into</u> X <u>such that for each</u> t <u>the sequence</u> $(f_n(t))$ <u>converges to the point</u> $f(t)$, <u>then the function</u> f <u>is measurable</u>.

PROPOSITION 5.7. <u>If</u> T <u>is a topological space,</u> X <u>a perfectly normal space and</u> (f_n) <u>a sequence of continuous functions from</u> T <u>into</u> X <u>such that for each</u> t <u>the sequence</u> $(f_n(t))$ <u>converges to the point</u> $f(t)$, <u>then the function</u> f <u>is a Baire class</u> 2 ; <u>that is, for every open set</u> G <u>in</u> X, $f^{-1}(G)$ <u>is a</u> $\mathcal{G}_{\delta\sigma}$ <u>set</u>.

<u>Proof</u>. If F is a closed set in X, and $F = \bigcap_n G_n$ where

(G_n) is a sequence of open sets such that

$$G_1 \supset \bar{G}_2 \supset G_2 \supset \bar{G}_3 \supset G_3 \supset \dots$$

then statement (i) of Theorem 5.6 holds in the form:

$$f^{-1}(F) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}(\bar{G}_k) ,$$

which is an $\mathcal{F}_{\sigma\delta}$ set. Hence $f^{-1}(F')$ is a $\mathcal{G}_{\delta\sigma}$, which is the required result.

It is well-known that a function which is the pointwise limit of a sequence of continuous functions need not be continuous. This last proposition sets a bound to the possible bad behaviour of f.

We now examine the case where a sequence of multifunctions converges in the Hausdorff topology.

THEOREM 5.8. <u>Let</u> S <u>be a measurable space,</u> (X,d) <u>a metric space and</u> (Γ_n) <u>a sequence of measurable or</u> $\mathcal{G}$<u>-measurable closed-valued multifunctions. If for each</u> t $(\Gamma_n(t))$ <u>converges, in the Hausdorff topology induced by</u> d, <u>to the closed set</u> $\Gamma(t)$, <u>then</u> Γ <u>is a</u> $\mathcal{G}$<u>-measurable multifunction</u>.

<u>Proof</u>. Let F be a closed set in X. Then let

$$F_n = \{x \in X : d(x,F) \leq \frac{1}{n}\}$$

for each positive integer n. We then have

$$\Gamma^+(F) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \Gamma_k^+(F_n) . \qquad \dots(i)$$

If $\Gamma(t) \subset F$, then $\Gamma(t) \subset F_n$ for all n. Let δ be the Hausdorff metric induced by d on $\mathcal{H}(X)$. Then there exists a positive integer n_1 such that $\delta(\Gamma_k(t), \Gamma(t)) \leq \frac{1}{n}$ for all $k \geq n_1$. This implies that $\Gamma_k(t) \subset F_n$ for $k \geq n_1$, and so t belongs to the right-hand side of (i).

Conversely, suppose that $\Gamma(t) \not\subseteq F$. Then there exists an integer n such that $\Gamma(t) \not\subseteq F_n$. Suppose that $x \in \Gamma(t) \setminus F_n$. Then there exists a real number $\epsilon > 0$ such that the set

$$B_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$$

does not meet F_n . However, there exists an integer m_1 such that for $m \geq m_1$, $\delta(\Gamma_m(t) , \Gamma(t)) < \epsilon$. Thus for $m \geq m_1$, $\Gamma_m(t) \cap B_\epsilon(x) \neq \phi$, from the definition of the Hausdorff metric. In particular, for every integer m , there exists an integer $k \geq m$ such that $\Gamma_k(t) \not\subseteq F_n$.

Thus if $\Gamma(t) \not\subseteq F$, t does not belong to the right-hand side of (i).

This proves statement (i), from which it follows at once that Γ is $\mathcal{G}$-measurable.

II. THEOREMS ON SELECTORS

6. The selection problem. Let S, X be two sets and $\Gamma : S \to X$ a multifunction with non-empty values. Then a selector for Γ is a function $\gamma : S \to X$ such that, for all $t \in S$, $\gamma(t) \in \Gamma(t)$. We shall be concerned mainly with the problem of the existence of measurable selectors for measurable multifunctions; though, following the example of K. Kuratowski and C. Ryll-Nardzewski (in (29)), we shall state the results in a more general form. This measurable selection problem has been investigated by several authors (for example in (6) - (10), (13), (20), (21), (29), (31), (41), (42), (45), (50) and (51), some of whom have also considered applications of the theory. We shall also touch on the question of the existence of continuous selectors; much work has been done on this by E. Michael ((35) - (38)). M.M. Čoban ((11), (12)) has shown that under quite general conditions continuous or semicontinuous multifunctions have Borel measurable selectors.

The difference between the selection problem and the uniformization problem, which has been studied by several authors (see for instance (44), (47) and (52)), is mainly of approach; there it is not the continuity or measurability of Γ and its selector γ which are studied so much as the topological properties of $G(\Gamma)$ and $G(\gamma)$.

A selection of results from the works cited above is presented by T. Parthasarathy in his lecture notes ((40)).

7. Compact-valued multifunctions.

THEOREM 7.1. Let S be a space on which is defined an algebra $\mathcal{L}$ of subsets, X a 2C-space, and $\Gamma : S \to X$ a multifunction with non-empty compact values such that, for any closed set F in X, $\Gamma^{-}(F) \in \mathcal{L}$. Then Γ has a selector γ such that, for every open set G in X, $\gamma^{-1}(G) \in \mathcal{L}_\sigma$.

<u>Proof</u>. Let (U_n) be a sequence of open sets forming a base for the topology of X. We write $\Gamma_o = \Gamma_1$ and then define a sequence (Γ_n) of multifunctions such that, for each $n \geq 1$, Γ_n is the refinement (§1) of Γ_{n-1} by the closed set $(X \setminus U_n)$. Thus (Γ_n) is a descending sequence of multifunctions such that, for each t, $\Gamma_n(t)$ is non-empty and closed relative to $\Gamma_o(t)$. Hence the multifunction $\Delta = \bigcap_n \Gamma_n$ has non-empty values. A selector γ of Δ exists, by the Axiom of Choice. We now examine the properties of γ. Let F be any closed subset of X. Then

$$F = \bigcap_{i \in I} (X \setminus U_i) ,$$

where I is some subset of the positive integers, and so

$$\gamma^{-1}(F) = \bigcap_{i \in I} \gamma^{-1}(X \setminus U_i) .$$

To complete the proof of the theorem, it is sufficient to show that for each i, $\gamma^{-1}(X \setminus U_i) \in \mathcal{L}$, for then $\gamma^{-1}(F) \in \mathcal{L}_\delta$ and hence, for any open set G, $\gamma^{-1}(G) \in \mathcal{L}_\sigma$. We observe that, for any n and any closed set B, $\Gamma_n^-(B) \in \mathcal{L}$; this follows from Proposition 1.7, by induction on n.

We now show that $\gamma^{-1}(X \setminus U_i) \in \mathcal{L}$ for each i : if $\gamma(t) \in X \setminus U_i$, then $\Delta(t)$ meets $X \setminus U_i$, and hence so does $\Gamma_{i-1}(t)$, *a fortiori*. Conversely, if $\Gamma_{i-1}(t)$ meets $(X \setminus U_i)$, then $\Gamma_i(t) \subseteq (X \setminus U_i)$ by definition, and so $\Delta(t) \subseteq (X \setminus U_i)$. We have proved therefore that

$$\gamma^{-1}(X \setminus U_i) = \Gamma_{i-1}^-(X \setminus U_i) ,$$

which belongs to $\mathcal{L}$.

Any selector γ for Δ would have given the same result.

THEOREM 7.2. <u>Let</u> S <u>be a space on which is defined an algebra</u> $\mathcal{L}$ <u>of subsets</u>, X <u>the regular continuous image of a</u> 2C-<u>space, and</u>

$\Gamma : S \to X$ <u>a multifunction with non-empty compact values such that, for any closed set</u> F <u>in</u> X , $\Gamma^{-}(F) \in \mathcal{L}$. <u>Then</u> Γ <u>has a selector</u> γ <u>such that for any open set</u> G <u>in</u> X , $\gamma^{-1}(G) \in \mathcal{L}_\sigma$.

<u>Proof</u>. Let $X = \varphi(Y)$, where Y is a 2C-space and φ a continuous mapping. Let (U_n) be a countable base for the topology of X . We define $V_n = \overline{\varphi(U_n)}$ for each n . Writing $\Gamma_o = \Gamma$, we define a sequence (Γ_n) of multifunctions such that, for each $n \geq 1$, Γ_n is the refinement of Γ_{n-1} by the closed set V_n . Thus (Γ_n) is a descending sequence of multifunctions such that for each t $\Gamma_n(t)$ is non-empty and closed relative to $\Gamma_o(t)$, and so the multifunction $\Delta = \cap_n \Gamma_n$ has non-empty values. Let γ be any selector for Δ . If G is an open set in X , then

$$G = \bigcup_{i \in I} V_i ,$$

for some set I of positive integers (cf. Lemma 3.10). We now show that $\gamma^{-1}(V_i) \in \mathcal{L}$ for each i : if $\gamma(t) \in V_i$, then $\Delta(t)$ meets V_i , and hence so does $\Gamma_{i-1}(t)$. Conversely, if $\Gamma_{i-1}(t)$ meets V_i , $\Gamma_i(t) \subseteq V_i$ by definition and so $\Delta(t) \subseteq V_i$. We have proved therefore that

$$\gamma^{-1}(V_i) = \Gamma^{-}_{i-1}(V_i) ,$$

which belongs to $\mathcal{L}$, by Proposition 1.7.

In fact, rather more is true in this case:

THEOREM 7.3. <u>If</u> S,X <u>and</u> Γ <u>are as in the statement of Theorem</u> 7.2, <u>then</u> Γ <u>has a countable family</u> (γ_n) <u>of selectors such that for each</u> t <u>the set</u> $\{\gamma_n(t) : n = 1,2, \ldots\}$ <u>is dense in</u> $\Gamma(t)$ <u>and such that for any</u> n <u>and any open set</u> G <u>in</u> X , $\gamma_n^{-1}(G) \in \mathcal{L}_\sigma$.

<u>Proof</u>. As before, let $X = \varphi(Y)$, where Y is a space with a countable base (U_n) of open sets, and φ is a continuous mapping. We

define $V_n = \varphi(U_n)$, for each n, and take Γ_n to be the refinement of Γ by V_n. Γ_n has a selector γ_n such that, for any open set G in X, $\gamma_n^{-1}(G) \in \mathcal{L}_\sigma$, by Theorem 7.2.

Let $t \in S$ be fixed, and let $x \in \Gamma(t)$. If N is any neighbourhood of x, then, since φ is continuous and since X is regular, $x \in V_i \subseteq N$ for some integer i. Therefore N contains $\Gamma_i(t)$ and hence the point $\gamma_i(t)$. It follows that the set $\{\gamma_i(t) : i = 1, 2, \ldots\}$ is dense in $\Gamma(t)$.

Let $\mathcal{K}^*(X)$ denote the space of non-empty compact subsets of a topological space X with the Vietoris topology. A <u>choice function</u> is a function f from $\mathcal{K}^*(X)$ into X such that, for every $K \in \mathcal{K}^*(X)$, $f(K) \in K$.

COROLLARY. <u>If</u> X <u>is a regular</u> 2C-<u>space, then there is a choice function</u> $f : \mathcal{K}^*(X) \to X$ <u>of the first class</u>.

<u>Proof</u>. In Theorem 7.1 put $S = \mathcal{K}^*(X)$, $\mathcal{L}$ the algebra of subsets of $\mathcal{K}^*(X)$ which are both $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$, and define $\Gamma(K) = K$ for every $K \in \mathcal{K}^*(K)$. For any closed set F in X,

$$\Gamma^-(F) = \{K : K \cap F \neq \phi\},$$

which is a closed set in the Vietoris topology. Now let (U_i) be a countable base of open sets in X, and G an open set. Since X is a regular space, a compact set K is contained in G if and only if there exists a finite subset I of the positive integers such that

$$K \subseteq \bigcup_{i \in I} \overline{U}_i \subseteq G.$$

Therefore $\Gamma^+(G) = \bigcup_I \Gamma^+(\bigcup_{i \in I} \overline{U}_i)$,

the union being taken over the countable set of all such finite sequences of positive integers. Thus $\Gamma^+(G)$ is an $\mathcal{F}_\sigma$, and it follows that, for any closed set F in X, $\Gamma^{-}(F)$ is a $\mathcal{G}_\delta$ as well as being closed. By Theorem 7.1, Γ has a selector f such that for any open set G in X, $f^{-1}(G) \in \mathcal{L}_\sigma$. Since every set in $\mathcal{L}$ is an $\mathcal{F}_\sigma$, $f^{-1}(G)$ is an $\mathcal{F}_\sigma$, and hence f is of the first class of Baire.

Let X be any space. Then we shall say that a family $\{B_\lambda : \lambda \in \Lambda\}$ of subsets of X <u>separates</u> the points of X if, for any two distinct points x and y, there is a λ in Λ such that B_λ contains one or other of x,y but not both. We shall say that a topological space X satisfies <u>condition</u> (S) if there exists a countable family of closed sets which separates the points of X.

PROPOSITION 7.4. <u>Any Hausdorff continuous image of a</u> 2C-<u>space satisfies condition</u> (S).

<u>Proof</u>. Let X be a Hausdorff space, and suppose that $X = \varphi(Y)$, where φ is a continuous mapping and Y is a 2C-space. Let (U_n) be a countable base of open sets for Y and let $V_i = \overline{\varphi(U_i)}$ for each i. If $x \neq y$ in X, there exists a closed neighbourhood V of x which does not contain y. Now $x = \varphi(z)$, for some $z \in Y$, and since φ is continuous there is a neighbourhood U_i of z such that $\varphi(U_i) \subseteq V$. Hence $x \in V_i \subseteq V$. Therefore the points of X are separated by the countable family (V_i).

PROPOSITION 7.5. <u>A topological space satisfies condion</u> (S) <u>if and only if there is a countable family of upper semicontinuous real-valued functions which separates points</u>.

<u>Proof</u>. If (B_n) is a countable family of closed sets which separates the points of the topological space X, then their characteristic functions also separate the points of X, and are upper semicontinuous.

Conversely, if (f_n) is a sequence of upper semicontinuous real-valued functions which separates the points of X, the closed sets

$$\{x \in X : f_n(x) \geq r\} ,$$

where r is a rational number, form a countable family of sets which separate points.

THEOREM 7.6. <u>Let</u> S <u>be a space on which is defined an algebra</u> $\mathcal{L}$ <u>of subsets and</u> X <u>a topological space which satisfies condition</u> (S). <u>Then if</u> $\Gamma : S \to X$ <u>is a multifunction with non-empty compact values such that, for any closed set</u> F, $\Gamma^-(F) \in \mathcal{L}$, <u>there is a selector</u> γ <u>for</u> Γ <u>such that</u> $\gamma^{-1}(G) \in \mathcal{L}_\sigma$ <u>for any open set</u> G <u>in</u> X.

<u>Proof</u>. Let (B_n) be a countable family of closed sets which separate the points of X. We then write $\Gamma_o = \Gamma$ and define a sequence (Γ_n) of multifunctions such that, for each $n \geq 1$, Γ_n is the refinement of Γ_{n-1} by the closed set B_n. As in Theorem 7.1 or 7.2 we define $\Delta = \cap_n \Gamma_n$, and $\Delta(t)$ is non-empty for every t. If F is a closed set in X, it follows from Proposition 1.5 that

$$\Delta^-(F) = \bigcap_{n=1}^{\infty} \Gamma_n^-(F) ,$$

and so the result will follow when we have shown that, for every t, $\Delta(t)$ consists of a single point, $\gamma(t)$, say.

Suppose that $x,y \in \Delta(t)$ and that $x \neq y$. Then there exists a closed set B_i which contains x, say, but not y. Now B_i meets $\Delta(t)$ and hence $\Gamma_{i-1}(t)$. Therefore $\Delta(t) \subseteq B_i$, which contradicts our assumption that $y \in \Delta(t)$.

γ is then the required selector.

COROLLARY. <u>Let</u> X <u>be a</u> 2C-<u>space, or the regular continuous image of a</u> 2C-<u>space, or any space which satisfies condition</u> (S). <u>Then if</u> T <u>is a topological space in which every open set is an</u> $\mathcal{F}_\sigma$, <u>every upper</u>

semicontinuous multifunction from T into X with non-empty compact values has a selector of the first class.

Proof. This result follows from Theorems 7.1, 7.2, and 7.6 on taking in each case $\mathcal{L}$ to be the algebra of subsets of T which are both $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$.

The method of Theorem 7.6 has been used previously by A.P. Robertson to prove a measurable selection theorem ((41), Theorem 1) in the case where X is the Hausdorff continuous image of a separable metrisable space. From Proposition 7.5 we see that the conclusion of Theorem 7.6 holds if X is a locally convex Hausdorff topological vector space with separable dual.

So far we have used only countable processes in proving the existence of selectors. We now consider transfinite methods. We shall say that X satisfies condition (B) if it has a family (B_α) of closed sets, whose power is at most that of the first uncountable cardinal, which generates the Borel σ-algebra on X . The following theorem is in effect a generalization of a theorem of M. Sion ((46), Theorem 4.1). Condition (B) is satisfied by all spaces which satisfy Sion's condition (I). We do not require the space to be either regular or Hausdorff.

THEOREM 7.7. Let S be any measurable space, and X any topological space which satisfies condition (B) . Let $\Gamma : S \to X$ be a measurable multifunction with non-empty compact values. Then Γ has a measurable selector.

Proof. Let $\{B_\alpha : 1 \leq \alpha < \alpha_1\}$ be a family of closed sets which generates the Borel σ-algebra, indexed by the ordinals less than α_1 , where α_1 is not greater than the first uncountable ordinal. We write $\Gamma_0 = \Gamma$, and we define a sequence $(\Gamma_\alpha : 0 \leq \alpha < \alpha_1)$ of measurable multifunctions recursively as follows. We assume that Γ_β is defined and is a measurable multifunction for $\beta < \alpha$, where $\alpha < \alpha_1$.

Then the multifunction

$$\Omega_\alpha : t \to \bigcap_{\beta<\alpha} \Gamma_\beta(t)$$

is measurable, by Proposition 1.5, since the set $\{\beta : \beta < \alpha\}$ is at most countable. We then define Γ_α to be the refinement of Ω_α by the closed set B_α. Clearly Γ_α is a measurable multifunction.

Define $\Delta(t) = \bigcap_{\alpha<\alpha_1} \Gamma_\alpha(t)$, all t.

For each t, $\Delta(t)$ is non-empty since it is the intersection of a descending (transfinité) sequence of sets which are closed relative to the compact set $\Gamma_o(t)$.

Let δ be any selector for the multifunction Δ. Then δ is a measurable selector. To prove this, we first show that, for any α,

$$\delta^{-1}(B_\alpha) = \Delta^-(B_\alpha) = \Omega_\alpha^-(B_\alpha),$$

and so is a measurable set. This last step follows from the definition of refinement. It is clear that $\delta^{-1}(B_\alpha) \subseteq \Omega_\alpha^-(B_\alpha)$. Conversely, if $t \in \Omega_\alpha^-(B_\alpha)$, then $\Gamma_\alpha(t) \subseteq B_\alpha$, whence $\delta(t) \in \Gamma_\alpha(t) \subseteq B_\alpha$, as required.

Therefore $\delta^{-1}(B_\alpha)$ is measurable for all α; since the sets (B_α) generate $\mathcal{B}_X$, $\delta^{-1}(B)$ is measurable for every Borel set B in X, and so δ is measurable.

COROLLARY 1. _Let_ S _be any measurable space and_ X _any topological space in which the family of all closed sets has power less than or equal to the first uncountable cardinal. Then if_ $\Gamma : S \to X$ _is a measurable multifunction with non-empty compact values,_ Γ _has a measurable selector_.

In order to test whether the spaces which arise in analysis satisfy conditions such as our condition (B), it is useful to assume the continuum hypothesis, as is done by McShane and Warfield ((33), Theorem 4). As an example of the use of this device we have:

COROLLARY 2. <u>If</u> S <u>is a measurable space, and</u> X <u>is the continuous image of a 2C-space, then every measurable multifunction from</u> S <u>into</u> X <u>with non-empty compact values has a measurable selector</u>.

<u>Proof</u>. The class of closed subsets of X has at most the same cardinal number as the space of real numbers and, under the continuum hypothesis, this is the first uncountable cardinal. The existence of a measurable selector then follows from Corollary 1.

We conclude this section with an example which shows that a topological space can satisfy condition (B) without being the continuous image of a 2C-space. We take X to be the set of all ordinals less than the first uncountable ordinal, with the topology in which all non-empty open sets are of the form $\{\beta \in X : \beta \geq \alpha\}$ for some ordinal number α. This space is not separable and so cannot be the continuous image of a 2C-space; however it satisfies the hypotheses of Corollary 1.

Consider now the space $X \times [0,1]$, where the interval $[0,1]$ has the usual topology. This has a base of open sets of the form

$$\{\beta \in X : \beta \geq \alpha\} \times (r,s) ,$$

where α runs through all the ordinals up to the first uncountable ordinal, and r,s are rational. Any open set in $X \times [0,1]$ is the union of a countable subfamily of these basic sets, and so this space satisfies condition (B). This provides an example of a space which satisfies condition (B) and which is not a 2C-space or a continuous image of one. Moreover, the class of closed sets in this space has power at least equal to the power of the continuum.

8. Closed-valued multifunctions. The selection theorems which we have proved for compact-valued multifunctions enable us to deduce corresponding theorems for closed-valued multifunctions with values in certain σ-compact topological spaces. Any topological space for which the conclusion of Theorem 7.1 holds will be said to satisfy condition (A). Condition (A) is satisfied by any 2C-space, or the regular continuous image of one (Theorem 7.2), or by any space which satisfies condition (S) (Theorem 7.6).

THEOREM 8.1. Let S be a space on which is defined an algebra $\mathcal{L}$ of subsets, X a topological space which is the union of countably many closed compact subspaces each satisfying condition (A), and $\Gamma : S \to X$ a multifunction with non-empty closed values such that $\Gamma^-(F) \in \mathcal{L}$ for every closed set F in X. Then Γ has a selector γ such that for every open set G in X, $\gamma^{-1}(G) \in \mathcal{L}_\sigma$.

Proof. Let $X = K_1 \cup K_2 \cup \dots$ where the sets K_i are closed, compact, and satisfy condition (A). For each positive integer i we define

$$E_i = \Gamma^-(K_i) \setminus (\bigcup_{j<i} \Gamma^-(K_j)) ;$$

that is, E_i is the set of t in S such that K_i is the first of the sequence $K_1, K_2, \dots$ to intersect $\Gamma(t)$. Clearly $E_i \in \mathcal{L}$ for each i. We define multifunctions $\Gamma_i : E_i \to K_i$ by writing $\Gamma_i(t) = \Gamma(t) \cap K_i$ for each $t \in E_i$. Then, for each i, Γ_i has a selector δ_i such that, for any set G_i open relative to K_i, $\gamma_i^{-1}(G_i) \in \mathcal{L}_\sigma$. Hence, since the sets E_i form a partition of S, Γ has a selector γ defined by

$$\gamma(t) = \delta_i(t) \quad \text{for} \quad t \in E_i .$$

If G is any open set in X,

$$\gamma^{-1}(G) = \bigcup_{i=1}^{\infty} \delta_i^{-1}(G \cap K_i) ,$$

which clearly belongs to $\mathcal{L}_\sigma$.

If in the above theorem S is a topological space in which every open set is an $\mathcal{F}_\sigma$ and Γ is upper-semicontinuous, then a selector γ can be found of the first Baire class.

We now consider other cases in which the selection problem for a closed-valued multifunction can be reduced to a selection problem for compact-valued multifunctions. We shall need the following set-theoretic lemma:

LEMMA 8.2. <u>Let $\mathcal{L}$ be a ring of sets in a space S, and (A_n) any sequence of sets in $\mathcal{L}_\sigma$. Then there exists a partition of $\bigcup_n A_n$ by a sequence (B_n) of sets in $\mathcal{L}_\sigma$ such that $B_n \subseteq A_n$ for each</u> n.

<u>Proof</u>. For each n, $A_n = \bigcup_{r=1}^{\infty} A_{nr}$, say, where $A_{nr} \in \mathcal{L}$ for each n,r. We rearrange the sets A_{nr} as a single sequence (C_k); from this we obtain a disjoint sequence (D_k) where, for each integer k,

$$D_k = C_k \setminus \bigcup_{h<k} C_h .$$

Then $D_k \in \mathcal{L}$, for all k. Now, for each k, $D_k \subseteq A_n$ for some n, and so there is a first such n. We define, for each n,

$$B_n = \bigcup\{D_k : D_k \subseteq A_n \text{ but } D_k \nsubseteq A_m \text{ for } m < n\} .$$

Clearly $B_n \subseteq A_n$ for each n, the sets $\{B_n\}$ are disjoint, and

$$\bigcup_n B_n = \bigcup_k D_k = \bigcup_k C_k = \bigcup_{n,r} A_{nr} = \bigcup_n A_n .$$

THEOREM 8.3. <u>Let S be a space on which is defined an algebra $\mathcal{L}$ of subsets, and Y a topological space. Let (K_{ij}) be a double</u>

sequence of closed compact subsets of Y, and let X be the subspace $\bigcap_i \bigcup_j K_{ij}$ of Y. Then if $\Gamma : S \to X$ is a multifunction with non-empty closed values such that $\Gamma^-(F) \in \mathcal{L}_\sigma$ for every closed set F in X, there exists a multifunction $\Delta : S \to X$ with non-empty compact values such that $\Delta(t) \subseteq \Gamma(t)$ for all $t \in S$ and $\Delta^-(F) \in \mathcal{L}_{\sigma\delta}$ for every closed set F in X.

Proof. For each finite sequence $\sigma|n$ of positive integers we define $H_{\sigma|n} = K_{1,\sigma_1} \cap \dots \cap K_{n,\sigma_n}$.

Thus

$$X = \bigcup_\sigma \bigcap_{n=1}^{\infty} H_{\sigma|n} ,$$

the union being taken over all sequences σ of positive integers. Now the sets $\Gamma^-(X \cap H_i)$ form a covering of S by sets in $\mathcal{L}_\sigma$. Therefore, by Lemma 8.2, there exists a partition (A_i) of S by sets in $\mathcal{L}_\sigma$ such that $A_i \subseteq \Gamma^-(X \cap H_i)$ for each i.

If we fix i, then the sets $X \cap H_{ij}$ $(j = 1,2, \dots)$ cover $X \cap H_i$. Therefore A_i is covered by the sets $\Gamma^-(X \cap H_{ij})$, and so, applying Lemma 8.2 again, this time to the sets $A_i \cap \Gamma^-(H_{ij} \cap X)$, we obtain a sequence $\{A_{ij} : j = 1,2, \dots\}$ of sets in $\mathcal{L}_\sigma$ which form a partition of A_i. Continuing in this way we obtain a family $(A_{\sigma|n})$ of sets in $\mathcal{L}_\sigma$, such that $A_{\sigma|n} \subseteq \Gamma^-(X \cap H_{\sigma|n})$ for each $\sigma|n$, and such that for each fixed sequence $\sigma|n$,

$$A_{\sigma|n} = \bigcup_{i=1}^{\infty} A_{\sigma_1 \dots \sigma_n i} ,$$

this being a disjoint union. Thus, if $t \in S$, there is a unique sequence σ of positive integers such that $t \in A_{\sigma|n}$ for all n, and we define

$$\Delta(t) = \Gamma(t) \cap \bigcap_{n=1}^{\infty} H_{\sigma|n} . \qquad \dots(i)$$

Since, for this value of t, $\Gamma(t)$ meets $H_{\sigma|n}$ for all n, $\Delta(t)$ is compact and non-empty. Moreover, if F is any closed set in X,

$$\Delta^{-}(F) = \bigcap_{n=1}^{\infty} \left[\bigcup_{\sigma|n} (A_{\sigma|n} \cap \Gamma^{-}(F \cap H_{\sigma|n})) \right],$$

where the union inside the square brackets is taken over all finite sequences $\sigma|n$ of fixed length n. It remains to prove this statement, for then it is clear that $\Delta^{-}(F) \in \mathcal{L}_{\sigma\delta}$.

Suppose that $t \in \Delta^{-}(F)$. Then there is a unique sequence σ such that $t \in A_{\sigma|n}$, all n, and so, from statement (i), $\Gamma(t)$ meets $F \cap H_{\sigma|n}$ for all n. Hence t belongs to the right-hand side of the above formula.

Conversely, suppose that t belongs to the right-hand side. There is a unique sequence σ such that $t \in A_{\sigma|n}$ for all n. Since for each fixed m the sets $\{A_{\tau|m}\}$ are disjoint, $t \in \Gamma^{-}(F \cap H_{\sigma|n})$ for each n. Therefore $\Gamma(t) \cap F$ meets $H_{\sigma|n}$ for each n, and since the sets $\{H_{\sigma|n}\}$ are compact

$$\Gamma(t) \cap F \cap \bigcap_{n=1}^{\infty} H_{\sigma|n} \neq \phi .$$

Therefore $\Delta(t)$ meets F, as required.

We may use the main idea of the previous proof to give a new proof of the theorem of K. Kuratowski and C. Ryll-Nardzewski ((29)):

THEOREM 8.4. (Kuratowski and Ryll-Nardzewski). *Let S be a space on which is defined an algebra $\mathcal{L}$ of subsets, let X be a Polish space, and let $\Gamma : S \to X$ be a multifunction with non-empty closed values such that, for every open set G in X, $\Gamma^{-}(G) \in \mathcal{L}_{\sigma}$. Then Γ has a selector γ such that $\gamma^{-1}(G) \in \mathcal{L}_{\sigma}$ for every open set G in X.*

Proof. Let d be a metric such that (X,d) is a complete space. Let (U_i) be a covering of X by open sets of diameter $\leq \frac{1}{2}$.

Similarly for each i, we take (U_{ij}) to be a covering of U_i by open sets of diameter $\leq \frac{1}{4}$. Continuing in this way, we obtain a family of sets $(U_{\sigma|n})$, where diam $(U_{\sigma|n}) \leq 2^{-n}$.

Now the sets $\Gamma^{-}(U_i)$ $(i = 1,2, \ldots)$ are all in $\mathcal{L}_\sigma$ and their union is S. Therefore, by Lemma 8.2, there is a partition (A_i) of S by sets in $\mathcal{L}_\sigma$ such that $A_i \subseteq \Gamma^{-}(U_i)$ for each i.

The sets $\Gamma^{-}(U_{ij})$ $(j = 1,2, \ldots)$ for a fixed i are also in $\mathcal{L}_\sigma$, and they cover A_i. Hence, applying Lemma 8.2 to the sets $\{A_i \cap \Gamma^{-}(U_{ij})\}$, we obtain a partition (A_{ij}) of A_i by sets in $\mathcal{L}_\sigma$. Continuing, we obtain a family $(A_{\sigma|n})$ of sets in $\mathcal{L}_\sigma$ such that, for each $\sigma|n$, $A_{\sigma|n} \subseteq \Gamma^{-}(U_{\sigma|n})$, and such that for every $\sigma|n$ the sets $\{A_{\sigma_1 \ldots \sigma_n i} : i = 1,2, \ldots\}$ form a partition of $A_{\sigma|n}$.

If $t \in S$, there is a unique sequence σ such that $t \in A_{\sigma|n}$ for all n. If $t \in A_{\sigma|n}$ for all n, we define

$$\gamma(t) = x_\sigma ,$$

where x_σ is the unique point of X which is contained in $\overline{U}_{\sigma|n}$ for all n. The sets $(U_{\sigma|n})$ clearly form a base of open sets for X. Therefore if G is any open set in X,

$$\gamma^{-1}(G) = \cup \{A_{\sigma|n} : \overline{U}_{\sigma|n} \subseteq G\} ,$$

which clearly belongs to the class $\mathcal{L}_\sigma$. It is clear that if $\gamma(t) \in G$, where $t \in \cap_n A_{\sigma|n}$, then $\overline{U}_{\sigma|n} \subseteq G$ for some n, and so t belongs to the right-hand side of the above formula. Conversely, if t belongs to $A_{\sigma|n}$, where $\overline{U}_{\sigma|n} \subseteq G$, then $\sigma|n$ consists of the first n terms of a uniquely defined sequence σ such that $t \in A_{\sigma|m}$, all m. Hence $\gamma(t) = x_\sigma$, which belongs to G.

It remains to prove that γ is a selector for Γ. This is so because if $t \in \cap_n A_{\sigma|n}$, then $\Gamma(t)$ meets $\overline{U}_{\sigma|n}$ for each n; as (X,d)

is a complete metric space and $\operatorname{diam}(\Gamma(t) \cap \overline{U}_{\sigma|n}) \to 0$, $\Gamma(t)$ contains the point x_σ , which is equal to $\gamma(t)$.

The proof of the above theorem relates the existence of the selector γ to the "sifting" property of Polish spaces. The conclusion also holds under the following hypotheses:

(i) X is a topological space with a family $(U_{\sigma|n})$ of subsets such that $X = \cup_i U_i$ and, for each $\sigma|n$, $U_{\sigma|n} = \cup_i U_{\sigma_1 \ldots \sigma_n i}$;

(ii) For each sequence σ , there exists an element x_σ in $\cap_n \overline{U}_{\sigma|n}$ such that, if $x_n \in U_{\sigma|n}$ for each n , then x_σ is the limit of the sequence (x_n) ;

(iii) S is a space with an algebra $\mathcal{L}$, and $\Gamma : S \to X$ is a multifunction with non-empty closed values such that $\Gamma^-(U_{\sigma|n}) \in \mathcal{L}_\sigma$ for each $\sigma|n$.

LEMMA 8.5. Let S be a measurable space which admits the Souslin operation, and let Y be a Souslin-$\mathcal{K}_1$ subset of a topological space, where $\mathcal{K}_1$ is the class of closed, compact subsets. Let also X be a regular space, the image of Y by a continuous mapping φ , and let $\Gamma : S \to X$ be a closed-valued measurable multifunction. Then $\Gamma^-(\varphi(F))$ is measurable for every closed set F in Y .

Proof. Let

$$Y = \bigcup_\sigma \bigcap_{n=1}^{\infty} K_{\sigma|n} ,$$

where $(K_{\sigma|n})$ is a family of closed compact subsets of a topological space. For each $\sigma|n$, let $B_{\sigma|n} = \overline{\varphi(F \cap K_{\sigma|n})}$, where F is a closed subset of Y . Then we show that

$$\Gamma^-(\varphi(F)) = \bigcup_\sigma \bigcap_{n=1}^{\infty} \Gamma^-(B_{\sigma|n}) , \qquad \ldots(\mathrm{i})$$

which is a measurable set. Now

$$\varphi(F) = \varphi(\bigcup_{\sigma} \bigcap_{n=1}^{\infty} (F \cap K_{\sigma|n})) = \bigcup_{\sigma} \varphi(\bigcap_{n=1}^{\infty} (F \cap K_{\sigma|n})) \subseteq \bigcup_{\sigma} \bigcap_{n=1}^{\infty} B_{\sigma|n} .$$

Therefore, if $\Gamma(t)$ meets $\varphi(F)$, it must meet $\bigcap_n B_{\sigma|n}$ for some σ, and hence $t \in \Gamma^-(B_{\sigma|n})$, all n.

Conversely, suppose that $\Gamma(t)$ does not meet $\varphi(F)$, but that t belongs to the right-hand side of the formula (i). Then $\Gamma(t)$ meets $B_{\sigma|n}$ for all n, for some sequence σ, but $\Gamma(t)$ does not meet $\varphi(\bigcap_n (F \cap K_{\sigma|n}))$. This set is compact, and as X is a regular space there exists an open set G such that

$$\Gamma(t) \subset G \text{ and } \bar{G} \cap \varphi(\bigcap_n (F \cap K_{\sigma|n})) = \phi . \qquad \text{...(ii)}$$

Since $\Gamma(t)$ meets $B_{\sigma|n}$ for each n, G meets $\varphi(F \cap K_{\sigma|n})$ for each n, in $\varphi(y_n)$ say, where $y_n \in K_{\sigma|n} \cap F$. The sequence (y_n) has a cluster point, y_0 say, which must be contained in $\bigcap_n (F \cap K_{\sigma|n})$. Now $\varphi(y_n) \in G$ for all n, and so $\varphi(y_0) \in \bar{G}$, φ being a continuous mapping. This contradicts statement (ii), and so our assumption that $\Gamma(t) \cap \varphi(F) = \phi$ is false.

This conclusion holds in particular if X is the regular continuous image of a Polish space, as any Polish space is homeomorphic to a $\mathcal{K}_{\sigma\delta}$ subset of the Hilbert cube (§0, (ix)).

Following Bressler and Sion ((5)), we shall say that a topological space is <u>analytic</u> if there exists a topological space Q and a double sequence (K_{ij}) of closed compact sets in Q such that X is the image of the subspace $\bigcap_i \bigcup_j K_{ij}$ under a continuous mapping. The theory of sets and spaces which are analytic in this or a similar sense has been developed by G. Choquet, M. Sion and others; a discussion of the theory may be found in (5), (46) or (48), where further references are given.

THEOREM 8.6. Let S be a measurable space which admits the Souslin operation, and X a regular analytic space. Then if $\Gamma : S \to X$ is a measurable multifunction with non-empty closed values, there exists a measurable multifunction $\Delta : S \to X$ with non-empty compact values such that $\Delta(t) \subseteq \Gamma(t)$ for every $t \in S$.

Proof. There exists a topological space Q and a double sequence (K_{ij}) of closed compact subsets of Q such that

$$X = \varphi(\cap_i \cup_j K_{ij}) ,$$

φ being a continuous mapping. Let $Y = \cap_i \cup_j K_{ij}$. Then, by Lemma 8.5, the multifunction

$$\Omega : t \to \varphi^{-1}(\Gamma(t))$$

is closed-valued and measurable, since for any closed set F in Y,

$$\Omega^{-}(F) = \Gamma^{-}(\varphi(F)) .$$

We now apply Theorem 8.3, taking $\mathcal{L}$ to be the σ-algebra of measurable sets in S; there is a measurable multifunction $\Phi : S \to Y$ with non-empty compact values such that $\Phi(t) \subseteq \Omega(t)$ for each $t \in S$. We take therefore $\Delta = \varphi \circ \Phi$, which is compact-valued and measurable and satisfies the condition that $\Delta(t) \subseteq \Gamma(t)$ for each t.

If, in Theorem 8.6, X is such that every compact-valued measurable multifunction with non-empty values has a measurable selector, then Γ must have a measurable selector. In particular we have:

COROLLARY. Let S be a measurable space which admits the Souslin operation, and X a regular analytic space which satisfies condition (S) or (B) of §7. Then every measurable multifunction from S into X with non-empty closed values has a measurable selector.

This holds if X is the regular continuous image of a Polish space, which gives us Theorem 2 of (41). We close this section with a theorem on approximate selectors.

THEOREM 8.7. *Let* S *be a space with an algebra* $\mathcal{L}$ *of subsets,* (X,d) *a separable metric space, and* $\Gamma : S \to X$ *a multifunction with non-empty values such that* $\Gamma^{-}(G) \in \mathcal{L}_{\sigma}$ *for every open set* G *in* X. *Then, given any real number* $\delta > 0$, *there is a function* $\gamma : S \to X$ *such that* $\gamma^{-1}(G) \in \mathcal{L}_{\sigma}$ *for every open set* G *in* X, *and such that* $d(\gamma(t), \Gamma(t)) < \delta$ *for every* $t \in S$.

Proof. Let $\{x_i\}$ be a countable dense subset of X, and δ a positive real number. Then for each i we define

$$B_i = \{x \in X : d(x, x_i) < \delta\} .$$

Now the sets $\Gamma^{-}(B_i)$ are in $\mathcal{L}_{\sigma}$ and cover S. Therefore by Lemma 8.2, there is a partition of S by a sequence (A_i) of sets in $\mathcal{L}_{\sigma}$ such that $A_i \subseteq \Gamma^{-}(B_i)$ for each i. We define

$$\gamma(t) = x_i$$

for $t \in A_i$. Then $d(\gamma(t), \Gamma(t)) < \delta$ for every t, and if G is an open set

$$\gamma^{-1}(G) = \cup \{A_i : x_i \in G\} ,$$

which clearly belongs to $\mathcal{L}_{\sigma}$.

9. *Selectors in normed spaces*. In certain normed spaces a closed or compact convex set has a unique point which is nearest to a fixed point in the space. In this section we investigate the properties of the selector obtained from a convex-valued multifunction Γ by taking for each argument t the point of $\Gamma(t)$ nearest the origin.

LEMMA 9.1. *Let* S *be a space with an algebra* $\mathcal{L}$ *of subsets,* X *a topological space,* f *a lower semicontinuous real-valued function on* X *and* g *a real-valued funtion on* S *such that* $\{t : g(t) < c\}$ *is an* $\mathcal{L}_{\sigma}$ *set for every real number* c. *Then if* $\Gamma : S \to X$ *is a compact-valued*

<u>multifunction such that</u> $\Gamma^+(G) \in \mathcal{L}_\sigma$ <u>for every open set</u> G , <u>the multifunction</u>

$$\Delta : t \to \{x \in \Gamma(t) : f(x) \leqslant g(t)\}$$

<u>has the same property</u>.

<u>Proof</u>. We define, for each $t \in S$,

$$\Omega(t) = \{x \in X : f(x) \leqslant g(t)\} ,$$

and so, for each t , $\Delta(t) = \Gamma(t) \cap \Omega(t)$. Now let G be any open set in X . Then

$$\begin{aligned} \Delta^+(G) &= \{t \in S : \Gamma(t) \cap \Omega(t) \subseteq G\} \\ &= \{t \in S : \Gamma(t) \cap \Omega(t) \cap G' = \phi\} . \end{aligned}$$

The set $\Gamma(t) \cap G'$ is compact and so the function f attains a minimum on it, which for $t \in \Delta^+(G)$ is strictly greater than $g(t)$. There is therefore a rational number r such that $g(t) < r$ and $\Gamma(t) \cap G' \subseteq C_r$, where, for each real number c , C_c denotes the open set $\{x \in X : f(x) > c\}$. Conversely, if $g(t) < r$ and $\Gamma(t) \cap G' \subseteq C_r$, then $t \in \Delta^+(G)$. Now $\Gamma(t) \cap G' \subseteq C_r$ if and only if $\Gamma(t) \subseteq C_r \cup G$. Therefore

$$\Delta^+(G) = \bigcup_r [\{t \in S : g(t) < r\} \cap \Gamma^+(C_r \cup G)] ,$$

the union being taken over all rational numbers r . It follows from this that $\Delta^+(G) \in \mathcal{L}_\sigma$ as required.

LEMMA 9.2. <u>Let</u> T <u>and</u> X <u>be topological spaces</u>, f <u>a lower semicontinuous real-valued function on</u> X , <u>and</u> g <u>an upper semicontinuous real-valued function on</u> T . <u>Then if</u> $\Gamma : S \to X$ <u>is a u.s.c. compact-valued multifunction, so is the correspondence</u>

$$\Delta : t \to \{x \in \Gamma(t) : f(x) \leqslant g(t)\} .$$

Proof. If G is an open set in X, the argument of Lemma 9.1 shows that

$$\Delta^+(G) = \bigcup_r [\{t \in T : g(t) < r\} \cap \Gamma^+(C_r \cup G)] ,$$

the union being taken over all rational numbers r. Thus $\Delta^+(G)$ is an open set. $\Delta(t)$ is a closed subset of $\Gamma(t)$ for each t and so is compact.

We now apply these results. A normed linear space E is said to be rotund (or strictly convex) if, for any two distinct elements x,y in E such that $||x|| = ||y|| = 1$, $||\frac{1}{2}(x + y)|| < 1$.

A normed linear space is said to be strictly normable if it has an equivalent strictly convex norm (for a discussion of strict convexity, see (27) p. 342 et seq.). In particular it is shown that every reflexive Banach space is strictly normable; this is proved in (32).

If E is a strictly convex space, a compact convex set K in E has a unique point nearest the origin; if not, the function $x \to ||x||$ attains a minimum on K, at x_1 and x_2, say. If $x_1 \neq x_2$, then $\frac{1}{2}(x_1 + x_2) \in K$ and $||\frac{1}{2}(x_1 + x_2)|| < ||x_1||$, which contradicts our supposition that x_1 is at minimum distance from the origin.

THEOREM 9.3. Let S be a space with an algebra $\mathcal{L}$ of subsets and E a strictly normable space. Then if $\Gamma : S \to E$ is a multifunction with non-empty compact convex values such that $\Gamma^-(F) \in \mathcal{L}$ for every closed set F in E (or alternatively $\Gamma^-(G) \in \mathcal{L}$ for every open set G in E), there exists a selector γ for Γ such that $\gamma^{-1}(G) \in \mathcal{L}_\sigma$ for every open set G in E.

Proof. We assume that E has a strictly convex norm. Then we apply Lemma 9.1 taking $f(x) = ||x||$ $(x \in E)$, and

$$g(t) = \min \{||x|| : x \in \Gamma(t)\} \quad (t \in S) .$$

In this case

$$\Delta(t) = \{x \in \Gamma(t) : f(x) = g(t)\} ,$$

and, for each t , $\Delta(t)$ contains just one point, $\gamma(t)$ say. This function γ is the required selector. It remains to show that Γ and g satisfy the hypotheses of Lemma 9.1.

Suppose first that $\Gamma^-(F) \in \mathcal{L}$ for every closed set F . It follows at once that $\Gamma^+(G) = (\Gamma^-(G'))' \in \mathcal{L}$. If c is a real number,

$$\{t \in S : g(t) < c\} = \Gamma^-(\{x \in E : ||x|| < c\}) ,$$

which belongs to $\mathcal{L}_\sigma$, as the set $\{x \in E : ||x|| < c\}$ is clearly an $\mathcal{F}_\sigma$ set.

Suppose now that $\Gamma^-(G) \in \mathcal{L}$ for every open set G . If c is a real number,

$$\{t \in S : g(t) < c\} = \Gamma^-(\{x \in E : ||x|| < c\}) ,$$

which belongs to $\mathcal{L}$. If G is any open set in E , then $\Gamma^+(G) = (\Gamma^-(G'))'$, and the set $\Gamma^-(G')$ is in $\mathcal{L}_\delta$, by Theorem 1.6. Hence its complement, $\Gamma^+(G)$, is in $\mathcal{L}_\sigma$, and so the hypotheses of Lemma 9.1 are satisfied in each case.

COROLLARY. <u>If</u> T <u>is a topological space in which every open set is an</u> $\mathcal{F}_\sigma$ <u>and</u> E <u>is a strictly normable space, then every u.s.c. or l.s.c. multifunction</u> $\Gamma : T \to E$ <u>with non-empty compact convex values has a selector of the first class</u>.

<u>Proof</u>. This follows from Theorem 9.3 on taking $S = T$, and $\mathcal{L}$ to be the algebra of subsets of T which are both $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$.

THEOREM 9.4. <u>Let</u> T <u>be a topological space</u>, E <u>a strictly normable space, and</u> $\Gamma : T \to E$ <u>a continuous multifunction with non-empty compact convex values. Then</u> Γ <u>has a continuous selector</u>.

Proof. As before we assume that E has a strictly convex norm, and take $f(x) = ||x|| \ (x \in E)$, and

$$g(t) = \min \{||x|| : x \in \Gamma(t)\} \quad (t \in T) .$$

We take, for $t \in T$,

$$\Delta(t) = \{x \in \Gamma(t) : f(x) = g(t)\} ,$$

and this is u.s.c., by Lemma 9.2. For each t , $\Delta(t)$ contains just one point, $\gamma(t)$ say; the function γ is therefore continuous.

Now let $\mathcal{C}(E)$ denote the space of non-empty compact convex subsets of the normed space E with the Hausdorff topology, which is in this case the same as the Vietoris topology (§0, (xiii), (xiv)) . From Theorem 9.4 we deduce:

COROLLARY. *If* E *is a rotund space, then the choice function* $f : \mathcal{C}(E) \to E$, *where, for each* $K \in \mathcal{C}(E)$, $f(K)$ *is the unique point of* K *nearest to* 0 , *is continuous*.

Theorem 9.4 could have been obtained with the aid of the nearest-point property from the "Maximum Theorem" of C. Berge ((2), pp. 116 - 117), which holds for compact-valued multifunctions. However, we go on to extend these results to the case of multifunctions which are not necessarily compact-valued.

A normed linear space is said to be *uniformly convex* if it always follows from $||x_n|| \leq 1$, $||y_n|| \leq 1$ and $\lim_{n\to\infty} ||\frac{1}{2}(x_n + y_n)|| = 1$ that $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

A normed linear space is said to be *uniformly normable* if it has an equivalent uniformly convex norm. Uniform convexity is discussed in (27), §26. In particular the standard Banach spaces l^p and L^p are uniformly convex for $1 < p < \infty$.

LEMMA 9.5. <u>If</u> C <u>is a non-empty closed convex subset of a uniformly convex Banach space</u> E, <u>then there is a unique point of</u> C <u>which is nearest the origin.</u>

<u>Proof</u>. This result is trivial if $0 \in C$. Suppose therefore that $d(0,C) = c > 0$. Let (c_n) be a decreasing sequence of real numbers such that $c_n \to c$, and let $C_n = \{x \in E : ||x|| \leq c_n\}$. Then

$$\text{diam}\,(C_n \cap C) \to 0$$

as $n \to \infty$; for if this were not so, there would exist sequences (x_n), (y_n) of points such that, for each n, x_n and y_n are both in $C_n \cap C$, and a positive real number ϵ such that $||x_n - y_n|| > \epsilon$, for all n. Now for each n, as $\frac{1}{2}(x_n + y_n) \in C_n \cap C$,

$$c \leq ||\tfrac{1}{2}(x_n + y_n)|| \leq \tfrac{1}{2}(c_n + c_n) = c_n .$$

Hence, by uniform convexity, $\frac{1}{c_n}||x_n - y_n|| \to 0$, and as $c_n \to c(> 0)$, $||x_n - y_n|| \to 0$ as $n \to \infty$, which contradicts our supposition. Therefore, as E is a complete metric space, the sets $(C_n \cap C)$ intersect in one and only one point, which is at least distance from 0.

LEMMA 9.6. <u>If</u> C <u>is a non-empty closed convex set in a uniformly convex space, and if</u> F <u>is any closed set which does not contain the point of</u> C <u>nearest to</u> 0, <u>then</u> $d(0, C \cap F) > d(0,C)$.

<u>Proof</u>. Clearly $d(0, C \cap F) \geq d(0,C)$. Suppose that there is equality here. Let x_0 be the point of C nearest to 0. There is a sequence (y_n) of points in $C \cap F$ such that $||y_n|| \to d(0,C) = ||x_0||$. We may assume that $x_0 \neq 0$, as the conclusion is trivial when $x_0 = 0$. Since C is convex,

$$||x_0|| \leq ||\tfrac{1}{2}(x_0 + y_n)|| \leq \tfrac{1}{2}(||x_0|| + ||y_n||) ,$$

for each n, and hence

$$||\tfrac{1}{2}(x_0 + y_n)|| \to ||x_0|| .$$

Applying the definition of uniform convexity to the two sequences $(x_0/\ ||y_n||)$ and $(y_n/\ ||y_n||)$, we see that $y_n \to x_0$, which contradicts our assumption that $x_0 \notin F$. Therefore $d(0, C \cap F)$ is strictly greater than $d(0,C)$.

These two lemmas enable us to extend Theorems 9.3 and 9.4 to the case of multifunctions with closed convex values in a uniformly normable space.

THEOREM 9.7. Let S be a space with an algebra $\mathcal{L}$ of subsets and E a uniformly normable space. Then if $\Gamma : S \to E$ is a multifunction with non-empty closed convex values such that $\Gamma^-(F) \in \mathcal{L}$ for every closed F in E (or alternatively $\Gamma^-(G) \in \mathcal{L}$ for every open G in E), there exists a selector γ for Γ such that $\gamma^{-1}(G) \in \mathcal{L}_\sigma$ for every open set G in E.

Proof. We assume that the norm on E is uniformly convex. Then, for each $t \in S$, we define $\gamma(t)$ to be the unique point of $\Gamma(t)$ which is closest to the origin. If G is any open set in E, it follows from Lemma 9.6 that if $\gamma(t) \in G$ then

$$d(0,\ \Gamma(t) \cap G') > d(0,\ \Gamma(t)) .$$

Conversely, if this inequality holds, $\gamma(t) \in G$. For each real number c let $B_c = \{x \in E : ||x|| < c\}$ and let $C_c = \{x \in E : ||x|| > c\}$. Then

$$\gamma^{-1}(G) = \bigcup_r (\Gamma^-(B_r) \cap \Gamma^+(C_r \cup G)) , \qquad \ldots(i)$$

the union being taken over all rational numbers r. We now prove this fact. If $\gamma(t) \in G$, then there exist rational numbers r_1 and r_2 such that

$$d(0,\ \Gamma(t) \cap G') > r_1 > r_2 > d(0,\ \Gamma(t)) .$$

Then $\Gamma(t)$ meets B_{r_2} and $\Gamma(t) \cap G' \subseteq C_{r_2}$, i.e. $\Gamma(t) \subseteq C_{r_2} \cup G$.

Conversely, if $\Gamma(t)$ meets B_r and $\Gamma(t) \subseteq C_r \cup G$, then $d(0, \Gamma(t)) < r$ and $d(0, \Gamma(t) \cap G') \geqslant r$, whence $\gamma(t) \in G$.

The set B_r is an $\mathcal{F}_\sigma$, and so, if $\Gamma^-(F) \in \mathcal{L}$ for every closed set F, $\Gamma^-(B_r) \in \mathcal{L}_\sigma$. Therefore it follows from statement (i) that $\gamma^{-1}(G) \in \mathcal{L}_\sigma$ for every open set G.

Consider now the case where we are given that $\Gamma^-(G) \in \mathcal{L}$ for every open set G. Let F be a closed set in E. Then $F = \cap_n G_n$, where, for each n,

$$G_n = \{x \in E : d(x,F) < 1/n\} .$$

Clearly $\gamma(t) \notin F$ if and only if $\gamma(t) \notin \overline{G}_n$ for some n. Therefore $\gamma(t) \notin F$ if and only if

$$d(0, \Gamma(t) \cap G_n) > d(0, \Gamma(t))$$

for some n. This holds if and only if there is a rational number r such that $\Gamma(t) \cap G_n \subset B'_r$ and $\Gamma(t) \cap B_r \neq \phi$, and so we have

$$\gamma^{-1}(F') = \bigcup_{n=1}^{\infty} \bigcup_r (\Gamma^-(B_r) \cap \Gamma^+(B'_r \cup G'_n)) ,$$

where r runs through the rational numbers. Hence $\gamma^{-1}(F')$ belongs to $\mathcal{L}_\sigma$, and so $\gamma^{-1}(G) \in \mathcal{L}_\sigma$ for every open set G.

COROLLARY. <u>If</u> T <u>is a topological space in which every open set is an</u> $\mathcal{F}_\sigma$, <u>and</u> E <u>is a uniformly normable space, then every u.s.c. or l.s.c. multifunction from</u> T <u>into</u> E <u>with non-empty closed convex values has a selector of the first class.</u>

THEOREM 9.8. <u>Let</u> T <u>be a topological space,</u> E <u>a uniformly normable space, and</u> $\Gamma : T \to E$ <u>a continuous multifunction with non-empty closed convex values. Then</u> Γ <u>has a continuous selector</u>.

<u>Proof</u>. If we use the same notation as in Theorem 9.7, then, for

any open set G in E, the relation

$$\gamma^{-1}(G) = \bigcup_r (\Gamma^-(B_r) \cap \Gamma^+(C_r \cup G))$$

still holds, the union being taken over all rational numbers r. As B_r and $C_r \cup G$ are open sets and Γ is continuous, $\gamma^{-1}(G)$ is open, and so γ is continuous.

Let now $\mathcal{CH}(E)$ denote the space of non-empty closed convex sets in E with the Vietoris topology. Then we have:

COROLLARY. _If_ E _is a uniformly convex space, then the choice function_ $f : \mathcal{CH}(E) \to E$ _where, for each_ $F \in \mathcal{CH}(E)$, $f(F)$ _is the unique point of_ F _nearest to_ 0, _is continuous_.

Lastly we suppose that E is a reflexive Banach space with its equivalent strictly convex norm and its weak topology $\sigma(E,E')$.

LEMMA 9.9. _If_ C _is a non-empty closed convex subset of_ E, _then there is a unique point of_ C _which is nearest the origin_.

Proof. If $d(0,C) = r_o$, and (r_n) is a descending sequence of real numbers such that $r_n \to r_o$, then the set of nearest points is

$$\bigcap_{n=1}^{\infty} (C \cap \{x \in E : ||x|| \leqslant r_n\}) ,$$

which is non-empty since the sets $\{x : ||x|| \leqslant r_n\}$ are weakly compact, by the Banach-Alaoglu theorem. The uniqueness of the nearest point follows from the strict convexity of the norm.

We also have in this case an analogue of Lemma 9.6:

LEMMA 9.10. _If_ C _is a non-empty closed convex set in_ E, _and if_ F _is any weakly closed set which does not contain the point of_ C _nearest to_ 0, _then_ $d(0, C \cap F) > d(0,C)$.

Proof. Using the argument of Lemma 9.9 we see that $C \cap F$ has a point which is nearest the origin. If $d(0, C \cap F) = d(0,C)$, then this point must be the same as the nearest point of C, which contradicts our hypotheses.

THEOREM 9.11. Let S be a space with a field $\mathcal{L}$ of subsets and E a reflexive Banach space. Then if $\Gamma : S \to E$ is a multifunction with non-empty closed convex values such that $\Gamma^{-}(F) \in \mathcal{L}$ for every weakly closed set F in E, there exists a selector γ for Γ such that $\gamma^{-1}(G) \in \mathcal{L}_{\sigma}$ for every weakly open set G in E.

Proof. We assume that the norm on E is strictly convex. Then, for each $t \in S$, we define $\gamma(t)$ to be the unique point of $\Gamma(t)$ nearest the origin. If we define $B_c = \{x : ||x|| < c\}$ and $C_c = \{x \in E : ||x|| > c\}$ for each real number c, then B_c is a weak $\mathcal{F}_{\sigma}$ and C_c is weakly open. As in the proof of Theorem 9.7 we have

$$\gamma^{-1}(G) = \bigcup_r (\Gamma^{-}(B_r) \cap \Gamma^{+}(C_r \cup G)) ,$$

the union being taken over all rational numbers r, which clearly belongs to $\mathcal{L}_{\sigma}$. As this relation holds for every weakly open set G, the proof is complete.

COROLLARY 1. If T is a topological space in which every open set is an $\mathcal{F}_{\sigma}$ and E is a reflexive Banach space, then every u.s.c.$(\sigma(E,E'))$ multifunction $\Gamma : S \to E$ with non-empty closed convex values has a selector of the first class.

COROLLARY 2. If E is a reflexive Banach space with the weak topology, and $\mathcal{CF}(E)$ the space of non-empty closed convex subsets of E with the weak Vietoris topology, then there exists a choice function $f : \mathcal{CF}(E) \to E$ of the first class.

In this section we have obtained selectors by making use of the fact that, in the cases we have considered, one point of $\Gamma(t)$ is in some sense "better" than the others. A different approach to continuous selectors is taken by E. Michael in (35) - (38) (cf. especially (37), Lemma 8.3, p. 389).

10. Application to measurable multifunctions. We collect together here the results of the previous three sections as applied to the case of a measurable multifunction and we give sufficient conditions for the existence of measurable selectors.

THEOREM 10.1. *Let S be any measurable space, X a topological space, and $\Gamma : S \to X$ a measurable multifunction with non-empty values. Then the existence of a measurable selector for Γ is implied by any one of the following conditions:*

(i) *Γ is compact-valued and X satisfies either condition (B) or (S) of §7 ;*

(ii) *Γ is compact-valued and, assuming the Continuum Hypothesis, X is the continuous image of a 2C-space;*

(iii) *Γ is closed-valued and X is either a Polish space or the union of countably many closed compact subsets, each of which satisfies either condition (B) or (S) ;*

(iv) *X is a strictly normable space and Γ has compact convex values;*

(v) *X is a uniformly normable space and Γ has closed convex values;*

(vi) *X is a reflexive Banach space with the weak topology, and Γ has closed convex values;*

(vii) *S admits the Souslin operation, X is a regular analytic space which satisfies condition (B) or (S), and Γ is closed-valued.*

Proof. Result (i) follows from Theorems 7.6 and 7.7. Result

(ii) follows from Corollary 2 to Theorem 7.7. The second part of (iii) is a corollary of part (ii) (cf. Theorem 8.1), and the first part of (iii) follows from Theorem 8.4. Results (iv), (v) and (vi) follow respectively from Theorems 9.3, 9.7 and 9.11. Part (vii) is the same as the Corollary to Theorem 8.6.

Parts (iv) and (v) of Theorem 10.1 have been presented by the Author in (31), with a different method of proof. Condition (B) is satisfied by any 2C-space, or the regular continuous image of a 2C-space (cf. Lemma 3.10). Condition (S) is satisfied by any Hausdorff continuous image of a 2C-space (Proposition 7.4).

THEOREM 10.2. *Let* S *be a measurable space,* X *a topological space and* $\Gamma : S \to X$ *a* $\mathcal{G}$*-measurable multifunction with non-empty values. Then the existence of a measurable selector for* Γ *is implied by any one of the following conditions:*

(i) Γ *is compact-valued and* X *is a separable metrisable space;*

(ii) Γ *is closed-valued and* X *is a Polish space;*

(iii) X *is a strictly normable space and* Γ *has compact convex values;*

(iv) X *is a uniformly normable space and* Γ *has closed convex values;*

(v) S *admits the Souslin operation,* Γ *is closed-valued, and* X *is a Souslin space.*

Proof. Result (i) follows from Theorems 7.1 and 1.6. Results (ii), (iii) and (iv) follow respectively from Theorems 8.4, 9.3 and 9.7. Result (v) follows from Theorem 10.1 (vii) and Theorem 3.8, owing to the fact that a Souslin space is an analytic space, being the continuous image of a $K_{\sigma\delta}$ subset of the Hilbert cube (§0, (ix)).

THEOREM 10.3. *Let* S *be a measurable space,* X *a topological space and* $\Gamma : S \to X$ *a multifunction with non-empty values. Then if any one of the following conditions is satisfied,* Γ *has a countable family* (γ_n) *of measurable selectors such that, for each* $t \in S$, *the set* $\{\gamma_i(t) : i = 1,2, \ldots\}$ *is dense in* $\Gamma(t)$:

(i) Γ *is a compact-valued measurable multifunction, and* X *is the regular continuous image of a* 2C-*space*;

(ii) Γ *is closed-valued and* $\mathcal{G}$-*measurable, and* X *is a Polish space*;

(iii) S *admits the Souslin operation,* Γ *is closed-valued and measurable, and* X *is the regular continuous image of a Polish space*.

Proof. If condition (i) is satisfied, then the result follows from Theorem 7.3. Result (iii) follows from the Corollary to Theorem 8.6 in the sam way that Theorem 7.3 follows from Theorem 7.2. It remains to prove the theorem in the case where condition (ii) is satisfied. Let (U_i) be a base of open sets for the Polish space X. For each i let Γ_i be the refinement of Γ by the open set U_i. This is a $\mathcal{G}$-measurable multifunction, and hence so is $\overline{\Gamma}_i$, by Proposition 1.1. $\overline{\Gamma}_i$ has a measurable selector γ_i, which is also a selector for Γ, since, for each t, $\Gamma_i(t) \subseteq \Gamma(t)$. The family $\{\gamma_i\}$ of selectors has the required property, since, if $x \in \Gamma(t)$, then any neighbourhood of x contains $\gamma_i(t)$, for some i, because of the construction we have used.

Theorem 10.3 has already been proved under more restrictive hypotheses by C. Castaing ((8), Théorèmes 5.3 and 5.4) and our deduction of Theorem 7.3 from Theorem 7.2 was essentially by Castaing's method.

In the case of the "nearest-point" selectors, it is possible to relax the conditions on X at the expense of placing stricter conditions on

S . In (19), the authors have relaxed the uniform convexity condition by restricting S to be a locally compact Hausdorff space in which every compact subspace is metrisable, and also by modifying the definition of measurability.

Further results on selectors will be given in the following chapters.

III. MULTIFUNCTIONS OF SOUSLIN TYPE

11. Definition and basic properties. We now take a different approach to the measurable selection problem; we consider a class of multifunctions which are by definition well supplied with measurable selectors, and we examine its stability properties with respect to the usual set-theoretic and algebraic operations. Since this theory has parallels with the classical theory of Souslin sets (which is set out in (3) and (28)), we shall say that these multifunctions are of Souslin type.

DEFINITION. *Let S be a measurable space and X a topological space. Then a multifunction $\Gamma : S \to X$ is said to be of Souslin type if there exists a Polish space P, a measurable closed-valued multifunction $\Omega : S \to P$ and a continuous mapping $\varphi : P \to X$ such that $\Gamma = \varphi \circ \Omega$.*

It follows immediately from this definition that every multifunction of Souslin type is measurable. Moreover, if Γ is of Souslin type and has non-empty values, then Γ has a measurable selector. In fact we have:

THEOREM 11.1. *If S is a measurable space, X a topological space, and $\Gamma : S \to X$ a multifunction of Souslin type with non-empty values, then there exists a sequence (γ_n) of measurable selectors of Γ such that for each t the set $\{\gamma_n(t) : n = 1,2, \ldots\}$ is dense in $\Gamma(t)$.*

Proof. There exists a Polish space P, a continuous mapping $\varphi : P \to X$, and a closed-valued measurable multifunction $\Omega : S \to P$ such that $\Gamma(t) = \varphi(\Omega(t))$ for each t. Since $\Gamma(t)$ is non-empty for each t, so is $\Omega(t)$. By Theorem 10.3, Ω has a countable dense family (ω_n) of measurable selectors. The result follows on setting $\gamma_n = \varphi \circ \omega_n$ for each n.

The following proposition follows immediately from the definition:

PROPOSITION 11.2. *Let* X, Y *be topological spaces, and* $\psi : X \to Y$ *a continuous mapping. Then, if* S *is a measurable space and* $\Gamma : S \to X$ *a multifunction of Souslin type, so is the multifunction* $\psi \circ \Gamma$.

THEOREM 11.3. *If* (Γ_n) *is a sequence of multifunctions of Souslin type from any measurable space* S *into the topological space* X, *then so is the multifunction* $\bigcup_n \Gamma_n$.

Proof. Let the corresponding closed-valued multifunctions and mappings be

$$\Omega_n : S \to P_n \,, \quad \varphi_n : P_n \to X \,, \quad n = 1, 2, \ldots \,.$$

Let P be the Polish space $\Sigma_n P_n$ (§0, (ii)), and let φ be the continuous mapping from P to X defined by

$$\varphi(t) = \varphi_i(t) \quad \text{for} \quad t \in P_i \quad .$$

We then define

$$\Omega(t) = \Sigma_n \Omega_n(t)$$

for each t. This is a closed-valued measurable multifunction, and clearly $\Gamma = \varphi \circ \Omega$.

THEOREM 11.4. *If* S *is a measurable space which admits the Souslin operation,* (X_i) *a sequence of topological spaces and* (Γ_i) *a sequence of multifunctions of Souslin type from* S *into* X_i *respectively, then the multifunction* $\Pi_i \Gamma_i$ *is also of Souslin type*.

Proof. Let the corresponding closed-valued multifunctions and mappings be

$$\Omega_i : S \to P_i \,, \quad \varphi_i : P_i \to X_i \,, \quad i = 1, 2, \ldots \,.$$

We then define $\Omega = \Pi_i \Omega_i$ and $P = \Pi_i P_i$. P is a Polish space (§0, (ii)), and Ω is a closed-valued measurable multifunction by Theorem 4.10. We define a continuous mapping φ from P into $\Pi_i X_i$ by

$$\varphi(p_1, p_2, \ldots) = (\varphi_1(p_1), \varphi_2(p_2), \ldots), \quad p_i \in P_i .$$

Then $\Pi_i \Gamma_i = \varphi \circ \Omega$, and so is of Souslin type.

THEOREM 11.5. If S is a measurable space which admits the Souslin operation, and (Γ_n) a sequence of multifunctions of Souslin type into a Hausdorff space X, then so is the multifunction $\bigcap_n \Gamma_n$.

Proof. Let the corresponding closed-valued multifunctions and mappings be

$$\Omega_n : S \to P_n , \quad \varphi_n : P_n \to X , \quad n = 1, 2, \ldots .$$

Then the multifunction $\Pi_n \Omega_n$ is measurable, by Theorem 4.10. We define a mapping $\varphi : \Pi_n P_n \to X^N$ by

$$\varphi(p_1, p_2, \ldots) = (\varphi_1(p_1), \varphi_2(p_2), \ldots) \quad (p_i \in P_i) .$$

This is a continuous mapping. Let D be the diagonal of X^N, and let $\psi : D \to X$ be the natural homeomorphism between D and X. D is a closed subspace of X^N, since X is Hausdorff. Then, if $\Omega = \Pi_n \Omega_n$, the correspondence

$$\Omega_* : t \to \Omega(t) \cap \varphi^{-1}(D)$$

is a closed-valued measurable multifunction from S into the Polish space $\varphi^{-1}(D)$. Moreover, for each t,

$$\Gamma(t) = \psi(\varphi(\Omega_*(t))) = (\psi \circ \varphi)(\Omega_*(t)) .$$

Hence Γ is of Souslin type.

12. Characterisation by means of the Souslin operation, and examples. In what follows, if $(S, \mathcal{M})$ is a measurable space and X a topological space, $\mathcal{R}$ shall denote the class of sets $A \times B$, where

$A \in \mathcal{M}$ and B is closed in X. Then multifunctions of Souslin type can be characterized by means of the Souslin operation as follows:

THEOREM 12.1. Let S be a measurable space, X a Hausdorff space and $\Gamma : S \to X$ a multifunction of Souslin type. Then the graph of Γ is Souslin-$\mathcal{R}$

Proof. Let P be a Polish space, $\varphi : P \to X$ a continuous mapping, and $\Omega : S \to X$ a closed-valued measurable multifunction such that $\Gamma = \varphi \circ \Omega$. Let $(U_{\sigma|n})$ be a countable family of closed sets of P forming a sifting. For each finite sequence $\sigma|n$ of positive integers we define

$$A_{\sigma|n} = \Omega^{-}(U_{\sigma|n}) , \quad \text{and} \quad B_{\sigma|n} = \overline{\varphi(U_{\sigma|n})} .$$

Then we have

$$G(\Gamma) = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (A_{\sigma|n} \times B_{\sigma|n}) , \qquad \ldots(i)$$

which is clearly Souslin-$\mathcal{R}$. It remains to prove that this statement holds. Suppose that $(t,x) \in G(\Gamma)$. Then there exists $y \in P$ such that $x = \varphi(y)$ and $y \in \Omega(t)$. There also exists a sequence σ of integers such that $\{y\} = \bigcap_n U_{\sigma|n}$. Therefore $x \in \varphi(U_{\sigma|n}) \subseteq B_{\sigma|n}$, all n, and $\Omega(t)$ meets $U_{\sigma|n}$ for all n, i.e. $t \in \Omega^{-}(U_{\sigma|n})$. Hence (t,x) belongs to the right-hand side of statement (i).

Conversely, suppose that, for some sequence σ,

$$(t,x) \in A_{\sigma|n} \times B_{\sigma|n} , \quad \text{for all } n .$$

Then $\Omega(t)$ meets $U_{\sigma|n}$ for all n. Therefore $\Omega(t)$ contains y_σ, where y_σ is the unique element of the set $\bigcap_n U_{\sigma|n}$. Moreover $x = \varphi(y_\sigma)$; for if not, there is a closed neighbourhood V of $\varphi(y_\sigma)$ such that $x \notin V$; since φ is continuous, $\varphi(U_{\sigma|n}) \subseteq V$ for some n, and so $B_{\sigma|n} \subseteq V$, which contradicts our supposition that $x \in B_{\sigma|n}$ for all n.

Therefore $x \in \varphi(\Omega(t)) = \Gamma(t)$, as required.

The converse also holds under slightly different hypotheses:

THEOREM 12.2. Let S be a measurable space which admits the Souslin operation, X a space which is the continuous image of a Polish space, and $\Gamma : S \to X$ a multifunction with Souslin-$\mathcal{R}$ graph. Then Γ is of Souslin type.

Proof. Suppose that $X = \varphi(P)$ where P is a Polish space, and

$$G(\Gamma) = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} (A_{\sigma|n} \times B_{\sigma|n}) ,$$

where each $A_{\sigma|n}$ is measurable and each $B_{\sigma|n}$ is closed. Let N be the space of positive integers with the discrete topology, and let N^N denote the Polish space which is the product of N with itself countably many times. Consider now the multifunction $\Omega : S \to P \times N^N$ defined by

$$G(\Omega) = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} [A_{\sigma|n} \times (B^*_{\sigma|n} \times C_{\sigma|n})] ,$$

where, for each $\sigma|n$, $B^*_{\sigma|n} = \varphi^{-1}(B_{\sigma|n})$ and

$$C_{\sigma|n} = \{\tau \in N^N : \tau_1 = \sigma_1, \ldots, \tau_n = \sigma_n\} .$$

Clearly the sets $C_{\sigma|n}$ are closed. Ω is a measurable multifunction, by the Corollary to Lemma 3.3.

Moreover, Ω is closed-valued; suppose that $\Omega(t) \neq \phi$ and that $(x,\kappa) \notin \Omega(t)$. Then, for every σ for which $\bigcap_n A_{\sigma|n}$ is non-void,

$$(x,\kappa) \notin \bigcap_n (B^*_{\sigma|n} \times C_{\sigma|n}) = \bigcap_n B_{\sigma|n} \times \{\sigma\} .$$

There are two cases: if $\bigcap_n A_{\kappa|n} = \phi$, then the neighbourhood $X \times \{\kappa\}$ of (x,κ) does not meet $\Omega(t)$. On the other hand, if $\bigcap_n A_{\kappa|n} \neq \phi$, then $x \notin \bigcap_n B^*_{\kappa|n}$, and since the latter is a closed set, there exists a neighbourhood U of x which does not intersect it. Then the neighbourhood $U \times \{\kappa\}$ of (x,κ) does not meet $\Omega(t)$.

Finally, let ψ be the mapping $\varphi \circ \pi_1$ where π_1 is the natural projection $P \times N^N \to P$. Then, for each t, $\Gamma(t) = \psi(\Omega(t))$, and so Γ is of Souslin type.

This last step needs to be proved: if $(t,x) \in G(\Gamma)$, then, for some σ, $(t,x) \in A_{\sigma|n} \times B_{\sigma|n}$ for all n. Suppose also that $x = \varphi(y)$, where $y \in P$. Then $y \in B^*_{\sigma|n}$ for all n, and so $(t,y,\sigma) \in G(\Omega)$.

COROLLARY. <u>If</u> S <u>is a measurable space which admits the Souslin operation and</u> X <u>is a Hausdorff space, then the class of multifunctions of Souslin type from</u> S <u>into</u> X <u>is closed with respect to the Souslin operation</u>.

<u>Proof</u>. Let $(\Gamma_{\sigma|n})$ be a countable family of multifunctions of Souslin type from S into X. Let the corresponding Polish spaces and continuous mappings be $P_{\sigma|n}$ and $\varphi_{\sigma|n}$. Then without any loss of generality we may replace X by the space $\varphi(P)$ where P is the topological sum of the spaces $(P_{\sigma|n})$ and $\varphi : P \to X$ is the continuous mapping which coincides with $\varphi_{\sigma|n}$ on $P_{\sigma|n}$. Let

$$\Gamma(t) = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} \Gamma_{\sigma|n}(t) \quad \text{for each } t .$$

Now each $G(\Gamma_{\sigma|n})$ is a Souslin-$\mathcal{R}$ subset of $S \times \varphi(P)$, by Theorem 12.1. Hence $G(\Gamma)$ is Souslin-$\mathcal{R}$, by Proposition 2.1, and so Γ is of Souslin type, by Theorem 12.2.

As the operation of forming countable intersections is a special case of the Souslin operation, this Corollary gives an alternative proof of Theorem 11.5. We now consider some particular examples of multifunctions which are of Souslin type.

<u>Example</u> (i). If $\Gamma : S \to X$ is a measurable closed-valued multifunction, and X is a Polish space, then Γ is of Souslin type.

Example (ii). If $\Gamma : S \to X$ is a compact-valued measurable multifunction, X being a separable metrisable space, and $i : X \to \hat{X}$ is the embedding of X in its completion $\hat{X}$ with respect to some suitable metric, then the multifunction $t \to i(\Gamma(t))$ is of Souslin type.

Example (iii). Similarly, if (X,d) is a separable metric space, and $\Gamma : S \to X$ is a multifunction with values which are complete subsets of X, then Γ is of Souslin type if X is regarded as a subset of its completion (multifunctions of this type are studied in (19)).

Example (iv). If S admits the Souslin operation and X is a regular continuous image of a Polish space P, then any measurable closed-valued multifunction $\Gamma : S \to X$ is of Souslin type. This follows from the fact that if $X = \varphi(P)$, φ being a continuous mapping, then the correspondence $t \to \varphi^{-1}(\Gamma(t))$ is also a measurable closed-valued multifunction ((41), Lemma 1).

In each of the four examples considered so far, $G(\Gamma)$ has belonged to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$, where $\mathcal{M}$ is the σ-algebra of measurable sets in the domain S of Γ (this follows from Theorem 3.7). We have the following generalization of this:

THEOREM 12.3. Let $(S,\mathcal{M})$ be a measurable space which admits the Souslin operation, X a space which is the continuous image of a Polish space, and $\Gamma : S \to X$ a multifunction such that $G(\Gamma)$ belongs to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$. Then Γ is of Souslin type.

Proof. $G(\Gamma)$ is Souslin-$\mathcal{R}$, by Lemma 3.4; hence Γ is of Souslin type, by Theorem 12.2.

We deduce from this and from Theorem 11.1, the following somewhat stronger version of Théorème 1 of (45):

COROLLARY. _If_ S, X, Γ _satisfy the hypotheses of Theorem_ 12.3, _and_ $\Gamma(t)$ _is non-empty for each_ t, _there exists a sequence_ (γ_n) _of measurable selectors of_ Γ _such that for each_ $t \in S$, _the set_ $\{\gamma_n(t) : n = 1, 2, \ldots\}$ _is dense in_ $\Gamma(t)$.

Example (v). In Theorem 12.3, we cannot in general relax the requirement that S admit the Souslin operation. For instance, take both S and X to be the interval $[0,1]$, with the usual topology. We take $\mathcal{M}$ to be the Borel σ-algebra on S. Then $\mathcal{M} \otimes \mathcal{B}_X$ is just the Borel σ-algebra on $S \times X$. Now S contains a set A which is Souslin-$\mathcal{F}$, but not a Borel set ((17), §33, Theorem I). Moreover there is a Borel set B in $S \times X$ such that $\pi_1(B) = A$ ((28), p 458, Theorem). Now let Γ be the multifunction with graph B. Then $\Gamma^-(X) = A$, and hence Γ is not measurable with respect to the Borel σ-algebra on S. Therefore Γ is not of Souslin type.

Example (vi). It is elementary (Proposition 1.4) that, if S is a measurable space, then a measurable multifunction Γ from S into a metrisable space X is $\mathcal{G}$-measurable. The converse of this statement is not true; we shall show that there exists a measurable space S (in fact, $[0,1]$ with the σ-algebra of Borel sets), a Polish space X, and a closed-valued $\mathcal{G}$-measurable multifunction $\Gamma : S \to X$ which is not measurable. In order to do this, we consider the following alternative proof of Theorem 12.3:

Let S be any measurable space which admits the Souslin operation, and let X be the continuous image of a Polish space. If A is a subset of $S \times X$, let Γ_A denote (for now) the unique multifunction with graph A. We shall denote by $\mathcal{S}$ the class of all sets A in $S \times X$ such that Γ_A is of Souslin type. It follows at once from Theorems 11.3 and 11.5 that $\mathcal{S}$ is closed with respect to the formation of countable unions and intersections.

Now let $A = B \times C$, where B is measurable and C is closed. Now $X = \varphi(P)$, say, where φ is a continuous mapping and P is a Polish space. If we define a multifunction $\Omega : S \to P$ by $\Omega(t) = \varphi^{-1}(C)$ $(t \in B)$ and $\Omega(t) = \phi$ $(t \notin B)$, then $\Gamma_A = \varphi \circ \Omega$ and so is of Souslin type. Consider the complement of A. This is the set

$$A' = (B' \times X) \cup (S \times C') .$$

The multifunction $\Gamma_{B' \times X}$ is of Souslin type, by the argument we have just used. The set $\varphi^{-1}(C')$ is open, and so is a Polish subspace of P. Hence, if $P_1 = \varphi^{-1}(C')$ and $\Omega(t) = P_1$, all t, then $\Gamma_{S \times C'} = \varphi \circ \Omega$, so that $\Gamma_{S \times C'}$ is of Souslin type. It follows from Theorem 11.3 that $\Gamma_{A'}$ is of Souslin type.

Let $\mathcal{S}_1$ be the class of all sets A in $\mathcal{S}$ such that A' is also in $\mathcal{S}$. This is clearly a σ-algebra and contains the sets $B \times C$ where B is measurable and C is closed. It therefore contains the class $\mathcal{M} \otimes \mathcal{B}_X$, which completes this proof of Theorem 12.3.

Let $(S,\mathcal{M})$ be the interval $[0,1]$ with the σ-algebra of Borel sets. We shall assume that every $\mathcal{G}$-measurable closed-valued multifunction from S into a Polish space is measurable, and we shall show that this assumption leads to a contradiction. The first consequence of this assumption is that, if (X_i) is a sequence of Polish spaces and (Γ_i) a sequence of measurable closed-valued multifunctions from S into X_i respectively, then the multifunction $\Pi_i \Gamma_i$ is measurable, by Proposition 4.7. The conclusion of Theorem 11.5 would then hold without the hypothesis that S admit the Souslin operation; if we apply this to the above alternative proof of Theorem 12.3, we conclude that, if X is a Polish space and $\Gamma : S \to X$ a multifunction with Borel measurable graph, then Γ is of Souslin type; this is in contradiction to what we have established in Example (v).

Therefore, there exists a Polish space X and a closed-valued $\mathcal{G}$-measurable multifunction $\Gamma : S \to X$ which is not measurable (however, Γ is measurable with respect to the σ-algebra of Lebesgue measurable sets in S, by Theorem 3.8).

Example (vii). Several authors (for instance Castaing (8) and Himmelberg and Van Vleck (21)) have studied the case of a multifunction whose graph is the continuous image of a Polish space. In the two papers cited here the multifunctions considered are of Souslin type, as is shown by the theorem below. We shall need the following lemma:

LEMMA 12.4. *Let* Y *be a Hausdorff space,* X *a Polish space and* $\varphi : X \to Y$ *a continuous mapping. Then* $\varphi(X)$ *is a Souslin set in* Y.

Proof. This follows from Theorem 12.1 on taking S to be the measurable space consisting of one point, s say, and $\Gamma(s) = \varphi(X)$.

THEOREM 12.5. *Let* S *be a locally compact Hausdorff space with a Radon measure* μ, X *a topological space, and* $\Gamma : S \to X$ *a multifunction such that* $G(\Gamma)$ *is the continuous image of a Polish space. Then* Γ *is of Souslin type*.

Proof. There is a Polish space P and a continuous mapping φ from P into $S \times X$ such that $\varphi(P) = G(\Gamma)$. We define a multifunction Ω from S into P by

$$\Omega(t) = \varphi^{-1}(\{t\} \times \Gamma(t)) , \text{ for } t \in S .$$

Since the set $\{t\} \times \Gamma(t)$ is closed relative to $G(\Gamma)$ and φ is continuous, Ω is closed-valued. The proof will be complete when we have shown that Ω is measurable, since, for each t,

$$\Gamma(t) = \pi_2(\varphi(\Omega(t))) ,$$

where $\pi_2 : S \times X \to X$ is the natural projection. Thus Γ is of Souslin type.

Let F be any closed set in P. Then

$$\Omega^{-}(F) = \{t \in S : \varphi^{-1}(\{t\} \times \Gamma(t)) \cap F \neq \phi\}$$

$$= \{t \in S : (\{t\} \times \Gamma(t)) \cap \varphi(F) \neq \phi\} .$$

Now $$\{t\} \times \Gamma(t) = (\{t\} \times X) \cap G(\Gamma) = (\{t\} \times X) \cap \varphi(P) .$$

Therefore

$$\Omega^{-}(F) = \{t \in S : (\{t\} \times X) \cap \varphi(F) \neq \phi\}$$

$$= \pi_1(\varphi(F)) ,$$

where $\pi_1 : S \times X \to S$ is the natural projection. The set F is a Polish subspace of P, and so $\Omega^{-}(F)$ is a Souslin set, by Lemma 12.4. It is therefore a μ-measurable set (cf. remarks on p. 11), and so Ω is measurable, as required.

The key idea in the above proof is taken from the proof of Theorem 1 of Himmelberg and Van Vleck's paper (21); however they make the additional assumption that every compact subspace of S is metrisable, or alternatively that S is the union of countably many compact metrisable subspaces.

<u>Example</u> (viii). We close this section with an example which shows that a multifunction may be of Souslin type and yet have a graph which is not the continuous image of a Polish space. Let S be the interval $[0,1]$ with the σ-algebra of Lebesgue measurable sets, and let X be the same interval, with the usual topology. The set S contains a set A which is a Souslin set, but not a Borel set ((17), §33, Theorem I). The complement A' of A cannot then be a Souslin set ((28), p. 486). Consider the multifunction whose graph is the set $A' \times X$. This graph is Souslin-$\mathcal{R}$, and so the multifunction is of Souslin type, by Theorem 12.2. However, $A' \times X$ is not the continuous image of a Polish space, for that would imply that A' is Souslin (Lemma 12.4).

The values which are taken by a multifunction of Souslin type are always continuous images of some Polish space, from the definition.

13. Further properties of multifunctions of Souslin type.

THEOREM 13.1. *Let* S *be a measurable space which admits the Souslin operation,* H *a topological semi-group and* Γ_1, Γ_2 *multifunctions of Souslin type from* S *into* H. *Then the multifunction* $\Gamma_1 \cdot \Gamma_2$ *is of Souslin type, where, for each* t,

$$(\Gamma_1 \cdot \Gamma_2)(t) = \Gamma_1(t) \cdot \Gamma_2(t) = \{xy : x \in \Gamma_1(t) , y \in \Gamma_2(t)\} .$$

Proof. From Theorem 11.4, the multifunction $t \to \Gamma_1(t) \times \Gamma_2(t)$ from S into $H \times H$ is of Souslin type. Now the mapping $\varphi : H \times H \to H$ defined by $\varphi(x,y) = x \cdot y$ $(x,y \in H)$ is continuous. Hence $\Gamma_1 \cdot \Gamma_2$ is of Souslin type, by Proposition 11.2, since, for each t,

$$\Gamma_1 \cdot \Gamma_2(t) = \varphi(\Gamma_1(t) \times \Gamma_2(t)) .$$

The following theorem extends some results of C. Castaing ((8), Théorème 4.4, Corollaries 1 and 2) to the case of multifunctions of Souslin type.

THEOREM 13.2. *Let* S *be a measurable space which admits the Souslin operation,* E *a topological vector space, and* Γ_1, Γ_2 *multifunctions of Souslin type from* S *into* E. *Then the multifunction*

$$t \to \Gamma_1(t) + \Gamma_2(t) \quad (t \in S)$$

is of Souslin type. Moreover, if α *is any measurable scalar-valued function on* S, *the multifunction*

$$t \to \alpha(t)\, \Gamma_1(t) \qquad (t \in S) ,$$

is also of Souslin type.

Proof. The first part of this theorem follows from Theorem 13.1. To prove the second part, we use the fact that the multifunction

$$t \to \{\alpha(t)\} \times \Gamma_1(t) \quad (t \in S)$$

is of Souslin type, from Theorem 11.4. The mapping $\varphi : (\lambda, x) \to \lambda x$, where λ belongs to the field of scalars and x belongs to E, is continuous, and for each $t \in S$ we have:

$$\alpha(t)\, \Gamma_1(t) = \varphi(\{\alpha(t)\} \times \Gamma_1(t)) .$$

The multifunction $t \to \alpha(t)\, \Gamma_1(t)$ is therefore of Souslin type, by Proposition 11.2.

The above conclusion still holds if α is taken to be a multifunction of Souslin type from S into the field of scalars. The proof is the same.

THEOREM 13.3: Let S be a measurable space which admits the Souslin operation, E a topological vector space, and $\Gamma : S \to E$ a multifunction of Souslin type. Then the multifunction $\hat{\Gamma}$, where, for each t, $\hat{\Gamma}(t)$ is the convex hull of $\Gamma(t)$, is also of Souslin type.

Proof. Let Λ_n be the simplex in R^n defined by:

$$\underset{\sim}{\lambda} \in \Lambda_n \text{ if and only if } \lambda_i \geqslant 0 \text{ all } i \text{ , and } \sum_{i=1}^{n} \lambda_i = 1 .$$

We then define a sequence (Γ_n) of multifunctions by taking, for each n and each t,

$$\Gamma_n(t) = \{\lambda_1 x_1 + \ldots + \lambda_n x_n : \underset{\sim}{\lambda} \in \Lambda_n , x_i \in \Gamma(t) \text{ all } i\} .$$

Let E^n denote the product of E with itself, with n factors. We define a mapping $\varphi_n : \Lambda_n \times E^n \to E$ by the formula:

$$\varphi_n(\underset{\sim}{\lambda}, x_1, \ldots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n .$$

This mapping is continuous, since E is a topological vector space. By Theorem 11.4, the multifunction

$$\Delta_n : t \to \Lambda_n \times \Gamma(t) \times \ldots \times \Gamma(t) ,$$

where the factor $\Gamma(t)$ occurs n times, is of Souslin type. Now, for each t, $\Gamma_n(t) = \varphi_n(\Delta_n(t))$, and so Γ_n is of Souslin type, by Proposition

11.2. Therefore, since $\hat{\Gamma} = \cup_n \Gamma_n$, $\hat{\Gamma}$ is of Souslin type, by Theorem 11.3.

THEOREM 13.4. Let S be a measurable space which admits the Souslin operation, and X a closed convex Souslin subspace of a topological vector space. Then, if $\Gamma : S \to X$ is a multifunction of Souslin type, the multifunction $\tilde{\Gamma}$ is also of Souslin type, where, for each t, $\tilde{\Gamma}(t)$ is the closed convex hull of $\Gamma(t)$.

Proof. For each $t \in S$, $\tilde{\Gamma}(t)$ is the closure of $\hat{\Gamma}(t)$. $\hat{\Gamma}$ is of Souslin type, and so is $\mathcal{G}$-measurable; therefore $\tilde{\Gamma}$ is $\mathcal{G}$-measurable, by Proposition 1.1. It follows from Theorem 3.8 that $\tilde{\Gamma}$ is measurable, and so is of Souslin type, from Example (iv) of §12.

14. Applications of the theory. Let S be a measurable space and X, Y topological spaces. A function $f : S \times X \to Y$ will be said to satisfy condition (C) (C stands for Carathéodory) if for each $t \in S$ the function $x \to f(t,x)$ from X to Y is continuous, and if for each $x \in X$ the function $t \to f(t,x)$ from S to Y is measurable.

LEMMA 14.1. Let $(S,\mathcal{M})$ be any measurable space, X the regular continuous image of a 2C-space, and Y a perfectly normal space. Then, if f satisfies condition (C), f is measurable with respect to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$ on $S \times X$.

Proof. Let $X = \varphi(Z)$, where Z is a 2C-space, and φ is a continuous mapping. Let (U_i) be a countable base of open sets for Z, and let $V_i = \overline{\varphi(U_i)}$ for each i.

Let $\{x_i^n : n = 1,2, \ldots\}$ denote a countable dense subset of V_i, for each i.

If F is a closed subset of Y, there exists a sequence (G_k) of open sets in Y such that $F = \cap_k G_k$ and

$$G_1 \supset \overline{G}_2 \supset G_2 \supset \overline{G}_3 \supset \ldots .$$

Suppose that $f(t,x) \in F$. Then $f(t,x) \in G_k$, all k, and moreover there is a closed neighbourhood V of x such that $f(t,V) \subseteq G_k$, since X is a regular space. Hence

$$f(t, V_i) \subseteq G_k, \quad \text{where} \quad x \in V_i,$$

for some i, as φ is continuous. Therefore $f(t, x_i^n) \in G_k$, all n.

Conversely, if $f(t, x_i^n) \in G_k$ for all n, then $f(t, V_i) \subseteq \overline{G}_k$, taking closures, and so $f(t,x) \in \overline{G}_k$ if $x \in V_i$. We have, therefore,

$$f^{-1}(F) = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{n=1}^{\infty} [\{t \in S : f(t, x_i^n) \in \overline{G}_k\} \times V_i],$$

which clearly belongs to $\mathcal{M} \otimes \mathcal{B}_X$.

LEMMA 14.2. <u>Let</u> $(S,\mathcal{M})$ <u>be a measurable space,</u> X <u>and</u> Y <u>topological spaces and</u> $f : S \times X \to Y$ <u>a function which is measurable with respect to the</u> σ-<u>algebra</u> $\mathcal{M} \otimes \mathcal{B}_X$ <u>on</u> $S \times X$. <u>Then, if</u> $u : S \to X$ <u>is a measurable function, the function</u> $t \to f(t, u(t))$ <u>from</u> S <u>into</u> Y <u>is measurable</u>.

<u>Proof</u>. Let $h(t) = f(t, u(t))$, for each t. We define a function $g : S \to S \times X$ by

$$g(t) = (t, u(t)), \quad \text{for} \quad t \in S.$$

Then $h = f \circ g$. Since f is measurable, it is sufficient to show that if $M \in \mathcal{M} \otimes \mathcal{B}_X$, then $g^{-1}(M) \in \mathcal{M}$. Now $\mathcal{M} \otimes \mathcal{B}_X$ is generated by the sets $A \times B$ where $A \in \mathcal{M}$ and B is a closed subset of X. For one of these sets

$$g^{-1}(A \times B) = A \times u^{-1}(B),$$

which is clearly measurable. Therefore g is measurable, as required.

PROPOSITION 14.3. <u>Let</u> S <u>be a measurable space,</u> X <u>the regular continuous image of a</u> 2C-<u>space, and</u> $\Gamma : S \to X$ <u>a multifunction of</u>

Souslin type, with non-empty values. Then, if f is a real-valued function on $S \times X$ which satisfies condition (C), the function

$$w : t \to \sup\{f(t,u) : u \in \Gamma(t)\}$$

is measurable.

Proof. By Theorem 11.1, there is a countable family $\{\gamma_i\}$ of selectors for Γ such that, for each t, the set $\{\gamma_i(t) : i = 1,2, \ldots\}$ is dense in $\Gamma(t)$. Then, for each t,

$$w(t) = \sup\{f(t, \gamma_i(t)) : i = 1,2, \ldots\} .$$

Each of the functions $t \to f(t, \gamma_i(t))$ is measurable, by Lemmas 14.1 and 14.2, and so w is measurable.

The above conclusion still holds if X is any topological space and f is any measurable real-valued function on $(S \times X, \mathcal{M} \otimes \mathcal{B}_X)$.

A type of implicit function theorem useful in the theory of optimal control is that which is known as "Filippov's Lemma" from its appearance in a paper of A.F. Filippov (15). We now prove another version of this result:

THEOREM 14.4. Let $(S,\mathcal{M})$ be a measurable space which admits the Souslin operation, X a Hausdorff space, and Y a separable metrisable space. Let $f : S \times X \to Y$ be a function which is measurable with respect to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$, and let $g : S \to Y$ be a measurable function. Suppose that $\Gamma : S \to X$ is a multifunction of Souslin type such that, for each t, $g(t) \in f(t, \Gamma(t))$. Then Γ has a measurable selector γ such that $g(t) = f(t, \gamma(t))$ for each t.

Proof. There exists a Polish space P, a measurable closed-valued multifunction Ω and a continuous mapping $\varphi : P \to X$ such that $\Gamma = \varphi \circ \Omega$. We may therefore replace X by the subspace $\varphi(P)$.

In order to prove that the selector γ exists, it is sufficient to show that the multifunction

$$\Omega : t \to \{u \in \Gamma(t) : f(t,u) = g(t)\}$$

is of Souslin type. Let

$$\Delta(t) = \{u \in \varphi(P) : f(t,u) = g(t)\} \quad (t \in S) .$$

Then $\Omega(t) = \Gamma(t) \cap \Delta(t)$ for each $t \in S$, and so, by virtue of Theorem 11.5, it is sufficient to prove that Δ is of Souslin type. Now

$$G(\Delta) = \{(t,u) \in S \times \varphi(P) : f(t,u) = g(t)\} .$$

Y has a countable base of open sets, (U_i) say. Then $(t,u) \notin G(\Delta)$ if and only if there exists an integer i such that $g(t) \in U_i$ and $f(t,u) \in U_i'$. Thus

$$G(\Delta)' = \bigcup_{i=1}^{\infty} [f^{-1}(U_i') \cap (g^{-1}(U_i) \times \varphi(P))] ,$$

and so $G(\Delta)$ belongs to $\mathcal{M} \times \mathcal{B}_{\varphi(P)}$. Therefore Δ is of Souslin type, by Theorem 12.3; it follows that Ω is of Souslin type, and so the measurable selector γ exists, by Theorem 11.1.

In the previous theorem, Y could have been any space which satisfies what we called condition (S) in §7. We now prove another version of Filippov's lemma, in a different setting. Let S be a locally compact Hausdorff space with a Radon measure μ. If Y is a topological space, a function $g : S \to Y$ is said to be μ-<u>measurable</u> if, given any compact set $K \subseteq S$ and any positive real number ϵ, there is a compact subset K_ϵ of K such that $|\mu|(K \setminus K_\epsilon) \leq \epsilon$ and such that the restriction of g to K_ϵ is continuous.

THEOREM 14.5. <u>Let</u> S <u>be a locally compact Hausdorff space with a Radon measure</u> μ. <u>Let</u> X,Y <u>be Hausdorff spaces,</u> $f : S \times X \to Y$ <u>a continuous function,</u> $\Gamma : S \to X$ <u>a multifunction of Souslin type, and</u>

$g : S \to Y$ <u>a</u> μ-<u>measurable function such that</u> $g(t) \in f(t, \Gamma(t))$ <u>for all</u> t. <u>Then</u> Γ <u>has a measurable selector</u> γ <u>such that</u> $f(t, \gamma(t)) = g(t)$ <u>for all</u> $t \in S$.

<u>Proof</u>. There exists a Polish space P, a measurable multifunction $\Omega : S \to P$ with closed values, and a continuous mapping $\varphi : P \to X$ such that $\Gamma = \varphi \circ \Omega$. Therefore, without any loss of generality, we may suppose that $X = \varphi(P)$. For each $t \in S$, let

$$\Delta(t) = \{u \in \varphi(P) : f(t,u) = g(t)\} .$$

This multifunction has non-empty closed values. Let K be any compact subset of S; then for any integer n there exists a compact set $K_n \subseteq K$ such that $|\mu|(K \setminus K_n) \leq 1/n$ and such that the restriction of g to the set K_n is continuous. Then the restriction of Δ to the set K_n has a closed graph. Consider the multifunction $\Psi : S \to P$ defined by

$$\Psi(t) = \varphi^{-1}(\Delta(t)) \quad (t \in S) .$$

This is closed-valued, and its restriction to the set K_n has a closed graph. Now the union $\cup_n K_n$ differs from K by a set of measure zero. Therefore if F is a closed subset of P,

$$\Psi^{-}(F) \cap K = \pi_1[(\cup_n K_n \times F) \cap G(\Psi)] \cup A ,$$

where A is a set of measure zero. The set inside the square brackets is an $\mathcal{F}_\sigma$ subset of $K \times P$ and so belongs to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_P$. This is true because, if (U_i) is a countable base of open sets for P, every open set in $K \times P$ is of the form $\cup_i(A_i \times U_i)$ where the A_i's are open subsets of K; hence any closed or $\mathcal{F}_\sigma$ set in $K \times P$ belongs to $\mathcal{M} \otimes \mathcal{B}_P$. Therefore, by Proposition 3.5, $\Psi^{-}(F) \cap K$ is a measurable set. Since the intersection of $\Psi^{-}(F)$ with any compact subset of S is measurable, $\Psi^{-}(F)$ is itself a measurable set ((4), ch. IV, §5.1, Proposition 3).

Therefore Ψ is a measurable closed-valued multifunction. As $\Delta = \varphi \circ \Psi$, Δ is of Souslin type. The multifunction $t \to \Gamma(t) \cap \Delta(t)$ is of Souslin type, by Theorem 11.5, and it has non-empty values. It has a measurable selector γ, by Theorem 11.1, which satisfies the equation $f(t, \gamma(t)) = g(t)$ $(t \in S)$.

From Theorems 14.4 and 14.5, together with Lemma 14.1, we may deduce the various versions of Filippov's Lemma which are given in (8), (19), (21), (24) and (25). A slight difficulty arises when the multifunction Γ is a compact-valued measurable multifunction with values in a separable metrisable space, which may not be the continuous image of a Polish space. We overcome this as follows:

THEOREM 14.6. *Let* $(S,\mathcal{M})$ *be a measurable space which admits the Souslin operation and* X,Y *separable metrisable spaces. Let* $f : S \times X \to Y$ *be a function which is measurable with respect to the* σ-*algebra* $\mathcal{M} \otimes \mathcal{B}_X$, *and let* $g : S \to Y$ *be a measurable function. Suppose that* $\Gamma : S \to X$ *is a measurable compact-valued multifunction such that, for each* t, $g(t) \in f(t, \Gamma(t))$. *Then* Γ *has a measurable selector* γ *such that* $g(t) = f(t, \gamma(t))$ *for each* t.

Proof. Consider the multifunction $\Delta : S \to X$ defined by

$$\Delta(t) = \{u \in \Gamma(t) : f(t,u) = g(t)\} .$$

Now $G(\Gamma) \in \mathcal{M} \otimes \mathcal{B}_X$ by Theorem 3.6, and the set

$$\{(t,u) \in S \times X : f(t,u) = g(t)\}$$

also belongs to $\mathcal{M} \otimes \mathcal{B}_X$ as in the proof of Theorem 14.4. We may regard X as a subset of its completion $\hat{X}$ with respect to some compatible metric. Then $G(\Delta)$ belongs to $\mathcal{M} \otimes \mathcal{B}_{\hat{X}}$, and so Δ is of Souslin type, by Theorem 12.3. Therefore Δ has a measurable selector γ.

Here is another example of an implicit function theorem ((8), Corollaire 5.1).

THEOREM 14.7. *Let* S *be a locally compact Hausdorff space with a Radon measure* μ. *Let* X *be a separable metrisable space,* Y *a Hausdorff space,* $f : S \times X \to Y$ *a continuous function, and* $g : S \to Y$ *a* μ*-measurable function. Suppose that* $\Gamma : S \to X$ *is a measurable compact-valued multifunction such that, for each* t, $g(t) \in f(t, \Gamma(t))$. *Then* Γ *has a measurable selector* γ *such that* $g(t) = f(t, \gamma(t))$ *for each* t.

Proof. Let $\mathcal{M}$ denote the σ-algebra of μ-measurable sets in S. For each $t \in S$, we define

$$\Delta(t) = \{u \in X : f(t,u) = g(t)\} .$$

This multifunction has non-empty closed values. Let K be any compact subset of S; then for any integer n there exists a compact set $K_n \subseteq K$ such that $|\mu|(K \setminus K_n) \leq 1/n$ and such that the restriction of g to the set K_n is continuous. The restriction of Δ to the set K_n has a closed graph. Consider the multifunction $\Omega : S \to X$ defined by

$$\Omega(t) = \Gamma(t) \cap \Delta(t) \quad (t \in S) .$$

This has non-empty closed values. Now the union $\cup_n K_n$ differs from K by a set of measure zero. Therefore if F is a closed subset of X,

$$\Omega^-(F) \cap K = \pi_1[(\cup_n K_n \times F) \cap G(\Omega)] \cup A ,$$

where A is a set of measure zero. Therefore,

$$\Omega^-(F) \cap K = A \cup \bigcup_{n=1}^{\infty} \pi_1[(K_n \times F) \cap G(\Gamma) \cap G(\Delta)] .$$

Now $G(\Gamma) \in \mathcal{M} \times \mathcal{B}_X$, by Theorem 3.6, and the set $(K_n \times F) \cap G(\Delta)$ is closed. Therefore, if we regard X as a subset of $\hat{X}$, its completion with respect to some suitable metric, the set $(K_n \times F) \cap G(\Gamma) \cap G(\Delta)$ belongs to $\mathcal{M} \times \mathcal{B}_{\hat{X}}$. Hence $\Omega^-(F) \cap K$ is measurable, by Proposition 3.5. It follows that $\Omega^-(F)$ is measurable ((4), ch. IV, §5.1, Proposition 3), and so Ω is a measurable compact-valued multifunction, which has a measurable selector

γ satisfying the equation

$$g(t) = f(t, \gamma(t)) \quad (t \in S) .$$

In Theorems 14.6 and 14.7 we could instead have taken Γ to be a measurable multifunction whose values are non-empty complete subsets of a separable metric space; the proofs remain unaltered.

As a further application we give two propositions which have some applicability to the theory of optimal control, and which can be proved in a similar manner to Theorem 14.4.

PROPOSITION 14.8. Let $(S,\mathcal{M})$ be a measurable space which admits the Souslin operation, X a Hausdorff space, and f a real-valued function on $S \times X$ which is measurable with respect to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$. Then if $\Gamma : S \to X$ is a multifunction of Souslin type with non-empty values, and if, for each $t \in S$,

$$g(t) = \sup\{f(t,u) : u \in \Gamma(t)\} ,$$

the multifunction

$$\Omega : t \to \{u \in \Gamma(t) : f(t,u) = g(t)\}$$

is also of Souslin type.

Proof. The function g is measurable, by Proposition 14.3. The rest follows exactly as in the proof of Theorem 14.4.

This last proposition is a more general form of a result of G. Debreu ((14), result 4.5) which is said there to be "basic to the theory of economic equilibrium".

PROPOSITION 14.9 (cf. (8), Théorèmes 4.6 and 4.6'). Let S,X,f,Γ be as in Proposition 14.8, and let w be any measurable real-valued function on S. Then the multifunction

$$t \to \{u \in \Gamma(t) : f(t,u) \leq w(t)\}$$

is of Souslin type.

Proof. The set $\{(t,u) : f(t,u) \leq w(t)\}$ is in $\mathcal{M} \otimes \mathcal{B}_X$, and so its intersection with $G(\Gamma)$ determines a multifunction of Souslin type.

We now consider an important special case of Theorem 14.4. This is an example of a "lifting theorem"; such theorems, with applications, have been given by McShane and Warfield (33) and H.J. Kushner (30). The present result is a more general version of a theorem of Himmelberg and Van Vleck (21).

THEOREM 14.10. *Let S be a measurable space which admits the Souslin operation, X the continuous image of a Polish space, and Y a separable metrisable space. Let $f : X \to Y$ be a Borel measurable function, and $g : S \to Y$ a measurable function such that $g(S) \subset f(X)$. Then there exists a measurable function $h : S \to X$ such that $f \circ h = g$.*

Proof. Let $X = \varphi(P)$ where P is a Polish space, and φ is a continuous mapping. It follows from Theorem 14.4, by taking $\Gamma(t)$ to be constant and equal to P, that there exists a measurable function $\gamma : S \to P$ such that $f(\varphi(\gamma(t))) = g(t)$ for all $t \in S$. Then $h = \varphi \circ \gamma$ is the required lifting of g.

In Theorem 14.10, Y can be any topological space which satisfies condition (S) (§7); more generally, it could be any topological space with a countable family of Borel sets which separate points. If we identify the spaces S and Y, we obtain a theorem on inverse functions:

THEOREM 14.11. *Let X be the continuous image of a Polish space, Y a separable metrisable space, and $f : X \to Y$ a Borel measurable function. Suppose also that Y has a σ-algebra $\mathcal{M}$ of subsets which admits the Souslin operation and which contains the Borel sets. Then there is a function $h : f(X) \to X$ which is measurable with respect to $\mathcal{M}$ and which is such that $h(f(x)) = x$ for all x in X.*

<u>Proof</u>. This follows from Theorem 14.10 on taking $S = f(X)$ and taking $g : f(X) \to Y$ to be the inclusion mapping.

Again, Y could be any topological space with a countable family of Borel sets which separate points. Hence if f is a continuous mapping, we need only assume that Y is a Hausdorff space. This gives us the measurable choice theorem of J. von Neumann ((39), p. 448, Lemma 5), which has been used by R.J. Aumann to prove another measurable selection theorem ((1), Proposition 2.1).

IV. MULTIFUNCTIONS IN LINEAR SPACES

15. <u>The continuity and semicontinuity of convex-valued multifunctions</u>. Let T be some space, E a locally convex Hausdorff topological vector space over the real numbers, and $\Gamma : T \to E$ a multifunction with non-empty compact convex values; then it is possible to relate the behaviour of Γ to the behaviour of the real-valued functions

$$M_{x'} : t \to \max \{\langle x,x'\rangle : x \in \Gamma(t)\}$$

where x' belongs to the dual space E' of E (the work of L. Hörmander (23) on the support function of a convex set is of interest in this connection). If T is a measurable space and if each of the functions $M_{x'}$ is measurable, then Γ is said to be <u>scalarly measurable</u> (Valadier (50)).

THEOREM 15.1. <u>Let</u> S <u>be a measurable space</u>, E <u>a separable metrisable locally convex space over the reals, and</u> $\Gamma : S \to E$ <u>a multifunction with non-empty compact convex values. Then</u> Γ <u>is measurable if and only if it is scalarly measurable</u>.

Versions of this theorem are given by C. Castaing ((8), p. 119), G. Debreu ((14), result 5.10), and M. Valadier ((50), p. 272, Remarque). In this section we prove similar theorems for continuous or semicontinuous multifunctions. Throughout the section, unless stated otherwise, T will denote a topological space, E a real locally convex Hausdorff space, and E' the dual of E.

For our purposes, a <u>polyhedron</u> in E is a set of the form

$$\{u \in E : \langle u,x_i'\rangle < \alpha_i, \text{ all } i \in I\}$$

where $\{x_i' : i \in I\}$ is a finite family of points in E' and the α_i are real numbers.

LEMMA 15.2. <u>Let</u> $K \subset E$ <u>be a convex compact set and</u> G <u>a set containing</u> K <u>which is open for the weak topology on</u> E . <u>Then there exists a polyhedron</u> P <u>such that</u> $K \subset P \subset G$.

<u>Proof</u>. There exist convex open neighbourhoods $U_1, \ldots, U_k$ of points in K such that

$$K \subset U_1 \cup \ldots \cup U_k \subset G$$

We may suppose that for each j $(1 \leqslant j \leqslant k)$,

$$U_j = U_{j1} \cap \ldots \cap U_{jm_j} ,$$

where

$$U_{j\ell} = \{u \in E : \langle u, x'_{j\ell} \rangle < \alpha_{j\ell}\} \ (1 \leqslant \ell \leqslant m_j) ,$$

each m_j $(1 \leqslant j \leqslant k)$ being a positive integer and each $x'_{j\ell}$ $(1 \leqslant j \leqslant k , 1 \leqslant \ell \leqslant m_j)$ an element of E' . The $\alpha_{j\ell}$ are real. Then

$$(U_1 \cup \ldots \cup U_k)' = U_1' \cap \ldots \cap U_k'$$

$$= (U'_{11} \cup \ldots \cup U'_{1m_1}) \cap \ldots \cap (U'_{k1} \cup \ldots \cup U'_{km_k}) .$$

By the distributive law, this is equal to the union of all sets of the form

$$U'_{1p_1} \cap \ldots \cap U'_{kp_k} \quad (1 \leqslant p_j \leqslant m_j , \text{all } j).$$

These sets are closed, convex and disjoint from K . We shall call them $V_1, \ldots, V_n$, where n is some integer. By the Hahn-Banach theorem there exist points $x_1', \ldots, x_n'$ in E' such that, for each i $(1 \leqslant i \leqslant n)$, $\langle u, x_i' \rangle < \alpha_i$ for all u in K and $\langle u, x_i' \rangle > \alpha_i$ for all u in V_i, where the α_i are real numbers.

Therefore, since G contains the set

$$(V_1 \cup \ldots \cup V_n)' = V_1' \cap \ldots \cap V_n' ,$$

we have

$$K \subseteq \{u \in E : \langle u, x_i' \rangle < \alpha_i, \text{ all } i\} \subseteq G ,$$

which is the required result.

THEOREM 15.3. *Let* $\Gamma : T \to E$ *be a multifunction with non-empty compact convex values. Then* Γ *is weakly u.s.c. at a point* t_o *in* T *if and only if each of the functions* $M_{x'}$ *is upper semicontinuous there, where*

$$M_{x'}(t) = \max \{\langle x,x'\rangle : x \in \Gamma(t)\}$$

for $t \in T$ *and* $x' \in E'$.

Proof. If $\Gamma(t)$ is weakly u.s.c. at t_o , then for any $x' \in E'$ and any $\epsilon > 0$,

$$\Gamma(t_o) \subset \{u \in E : \langle u,x'\rangle < M_{x'}(t_o) + \epsilon\} = G \quad \text{say.}$$

The set G is open and so there exists a neighbourhood U of t_o such that $t \in U$ implies that $\Gamma(t) \subseteq G$, whence

$$M_{x'}(t) < M_{x'}(t_o) + \epsilon .$$

Thus $M_{x'}$ is upper semicontinuous at t_o .

Conversely, suppose that G is open in the weak topology on E , and that $\Gamma(t_o) \subset G$. Then, by Lemma 15.2, there are points $x_1', \ldots, x_n'$ in E' and real numbers α_i such that

$$\Gamma(t_o) \subset \{u \in E : \langle u,x_i'\rangle < \alpha_i,\ 1 \leqslant i \leqslant n\} \subset G .$$

Since $M_{x_i'}$, is upper semicontinuous for each i , there is a neighbourhood U of t_o such that $M_{x_i'}(t) < \alpha_i$ for each t in U and for each i . Therefore $\Gamma(t) \subset G$ for each t in U , and so Γ is weakly u.s.c. at t_o ..

This result depends entirely on the duality (E,E') ; it does not hold in general for a topology on E other than the weak topology. Any point-valued function into E which is weakly continuous but not continuous will provide a counterexample.

The next two lemmas lead to a theorem which is analogous to Theorem 15.3 for the case of lower semicontinuity.

LEMMA 15.4. *If* E *is a locally convex Hausdorff space, then a base of neighbourhoods of the origin for the weak topology on* E *is given by the sets*

$$U = \{x \in E : |\langle x, x_i' \rangle| < 1 , \quad i = 1, \ldots, n\}$$

where each $\{x_1' , \ldots, x_n'\}$ *is a finite linearly independent subset of* E' .

Proof. Suppose that U is a weak neighbourhood of the origin, where

$$U = \{x \in E : |\langle x, x_i' \rangle| < 1 , \quad i = 0, \ldots, m\} ,$$

and

$$x_0' = \lambda_1 x_1' + \ldots + \lambda_m x_m' ,$$

the λ's being real numbers. Then the weak neighbourhood

$$U_1 = \{x \in E : |\langle x, \mu_i x_i' \rangle| < 1 , \quad i = 1, \ldots, m\} ,$$

where $\mu_i = \max \{1, m|\lambda_i|\}$ for each i $(1 \leq i \leq m)$, is contained in U , since, for x in E ,

$$|\langle x, x_0' \rangle| \leq |\langle x, \lambda_1 x_1' \rangle| + \ldots + |\langle x, \lambda_m x_m' \rangle| .$$

Hence we have effectively removed the dependent vector x_0' . We repeat this process until we obtain a neighbourhood defined by a set of linearly independent vectors in E' .

LEMMA 15.5. *If* $\{x_1', \ldots, x_n'\}$ *is a linearly independent subset of* E' , *where* E *is a real locally convex Hausdorff space, then the mapping* f *defined by*

$$f(x) = (\langle x, x_1' \rangle , \ldots, \langle x, x_n' \rangle) \quad (x \in E) ,$$

maps E *onto* R^n .

Proof. Suppose that $f(E)$ is a proper subspace of R^n . Then there is a non-zero linear functional on R^n which vanishes on $f(E)$.

That is, there exists a non-zero n-tuple $(\lambda_1, \ldots, \lambda_n)$ in R^n such that, for all $x \in E$,

$$\lambda_1 \langle x, x_1' \rangle + \ldots + \lambda_n \langle x, x_n' \rangle = 0 .$$

Therefore

$$\lambda_1 x_1' + \ldots + \lambda_n x_n' = 0 ,$$

which contradicts our initial assumption that the x_i' are linearly independent. Therefore $f(E) = R^n$.

THEOREM 15.6. Let Q be a bounded convex subset of E, and $\Gamma : T \to Q$ a multifunction with non-empty compact convex values. Then Γ is weakly l.s.c. at a point t_o of T if and only if each of the functions $M_{x'}$ $(x' \in E')$ is lower semicontinuous there.

Proof. If $x' \in E'$ and c is a real number, $M_{x'}(t_o) > c$ if and only if $\Gamma(t_o)$ meets the set $\{u : \langle u, x' \rangle > c\}$, which is open in the weak topology. Therefore, if Γ is l.s.c. at t_o, $M_{x'}$ is lower semicontinuous at t_o.

Conversely, suppose that Γ is not weakly l.s.c. at t_o. Then there exists a set G which is open in the weak topology and which meets $\Gamma(t_o)$, and a net (t_β) such that $t_\beta \to t_o$ and $\Gamma(t_\beta) \cap G = \phi$ for all β.

Without loss of generality we may take G to be the set $y + U$, where $y \in \Gamma(t_o)$ and U is the neighbourhood

$$\{x \in E : |\langle x, x_i' \rangle| < 1 , \quad i = 1, \ldots, n\} ,$$

the set $\{x_1', \ldots, x_n'\}$ being a linearly independent subset of E' (Lemma 15.4). For each $x \in E$, we define

$$p(x) = \max \{|\langle x, x_i' \rangle| : i = 1, \ldots, n\} .$$

Then p is a continuous seminorm. By the Hahn-Banach theorem there exists for each index β a linear functional $x'_\beta \in E'$ such that

$$\langle x - y, x'_\beta \rangle \leq -1 \quad \text{for} \quad x \in \Gamma(t_\beta) \qquad \ldots(i)$$

and

$$\langle x, x'_\beta \rangle > -1 \qquad \text{for} \quad x \in U .$$

Since U is absolutely convex, it follows that, for each β, $|\langle x, x'_\beta \rangle| < 1$ whenever $p(x) < 1$. Hence

$$|\langle x, x'_\beta \rangle| \leq p(x) \quad (x \in E) .$$

It follows from the Banach-Alaoglu theorem that there is a subnet of (x'_β) which is convergent in the weak* topology on E'. We shall retain the notation (x'_β) for this subnet, and we shall use (t_β) to denote the corresponding subnet of (t_β). Let x'_o be the weak* limit of (x'_β). Then

$$|\langle x, x'_o \rangle| \leq p(x) \quad \text{for all} \quad x \in E . \qquad \ldots(ii)$$

Clearly, as $y \in \Gamma(t_o)$,

$$M_{x'_o}(t_o) \geq \langle y, x'_o \rangle .$$

The proof will be complete when we have shown that $M_{x'_o}$ is not lower semicontinuous at t_o. For each β, there exists a point x_β in $\Gamma(t_\beta)$ such that

$$\langle x_\beta, x'_o \rangle = M_{x'_o}(t_\beta) .$$

Then

$$\begin{aligned} M_{x'_o}(t_\beta) &= \langle x_\beta, x'_\beta \rangle + \langle x_\beta, x'_o - x'_\beta \rangle \\ &\leq M_{x'_\beta}(t_\beta) + \langle x_\beta, x'_o - x'_\beta \rangle , \quad \text{for each } \beta . \end{aligned}$$

Therefore

$$M_{x'_o}(t_\beta) \leq \langle y, x'_\beta \rangle - 1 + \langle x_\beta, x'_o - x'_\beta \rangle , \qquad \ldots(**)$$

from statement (i). Since the set Q is bounded, the set $\{\langle x_\beta, x'_i \rangle\}$ is bounded for each i. In particular, the set

$$\{(\langle x_\beta, x_1'\rangle, \ldots, \langle x_\beta, x_n'\rangle)\}$$

is a bounded subset of R^n. Hence (x_β) has a subnet such that, for some n-tuple $(k_1, \ldots, k_n) \in R^n$,

$$\langle x_\beta, x_i'\rangle \to k_i \ , \quad \text{each } i .$$

By Lemma 15.5, there exists a point x_o in E such that $\langle x_o, x_i'\rangle = k_i$ $(i = 1, \ldots, n)$. Therefore $p(x_o - x_\beta) \to 0$, from the definition of p.

Summarizing, we have nets (t_β), (x_β), (x_β') with the same index set such that

$$t_\beta \to t_o ,$$

$$x_\beta' \to x_o' \quad \text{in the weak* topology,}$$

$$p(x_\beta - x_o) \to 0 , \quad \text{where } x_\beta \in \Gamma(t_o) ,$$

and

$$M_{x_o'}(t_\beta) \leq \langle y, x_\beta'\rangle - 1 + \langle x_\beta, x_o' - x_\beta'\rangle .$$

The right-hand side of this last expression is equal to

$$\langle y, x_\beta'\rangle - 1 + \langle x_\beta - x_o, x_o' - x_\beta'\rangle + \langle x_o, x_o' - x_\beta'\rangle ,$$

which, by statement (ii), is not greater than

$$\langle y, x_\beta'\rangle - 1 + 2p(x_o - x_\beta) + \langle x_o, x_o' - x_\beta'\rangle ,$$

and this tends to $\langle y, x_o'\rangle - 1$ as $\beta \to$.
However $M_{x_o'}(t) \geq \langle y, x_o'\rangle$, so that $M_{x_o'}$ is not lower semicontinuous at t_o.

Example. We cannot relax the condition that Q be bounded in Theorem 15.6, even when E is finite-dimensional. Take T to be the interval $[0, \pi/2]$ and let $E = R^2$. For $t > 0$, let $\Gamma(t)$ be the line segment

$$[(-\cot t , -1), (\cot t , 1)] ;$$

the ends of the segment lie on the lines $y = \pm 1$, and it makes an angle t with the x-axis. Let $\Gamma(0)$ be the line segment whose end points are $(0, -1)$ and $(0, 1)$. Γ clearly fails to be lower (and upper) semi-continuous at $t = 0$. This may be seen by drawing a diagram. Now let $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. Then, for $t > 0$,

$$M_{\underset{\sim}{\lambda}}(t) = \max \{- \lambda_1 \cot t - \lambda_2, \lambda_1 \cot t + \lambda_2\} ,$$

and

$$M_{\underset{\sim}{\lambda}}(0) = |\lambda_2| .$$

This function is always lower semicontinuous at $t = 0$.

<u>Example</u>. The boundedness condition is not needed if $E = \mathbb{R}^1$. If Γ is l.s.c. and has non-empty compact convex values in $\mathbb{R}$, then there is an upper semicontinuous function f and a lower semicontinuous function g such that $f(t) \leq g(t)$ for all t, and

$$\Gamma(t) = \text{ the interval } [f(t), g(t)] \quad (t \in T) .$$

If (a,b) is an open interval, $\Gamma(t) \cap (a,b) = \phi$ if and only if $g(t) \leq a$ or $f(t) \geq b$.

We now combine Theorems 15.3 and 15.6 to obtain a result which does not need any boundedness condition.

THEOREM 15.7. <u>If E is a locally convex Hausdorff space, and $\Gamma : T \to E$ a multifunction with non-empty compact convex values, then Γ is weakly continuous at a point t_0 of T if and only if $M_{x'}$ is continuous there for every</u> $x' \in E'$.

<u>Proof</u>. If Γ is weakly continuous at t_0, it follows from Theorems 15.3 and 15.6 that, for each $x' \in E'$, $M_{x'}$ is both upper and lower semicontinuous at t_0, and so it is continuous there.

Conversely, suppose that $M_{x'}$ is continuous at t_0 for each x'. It follows from Theorem 15.3 that Γ is weakly u.s.c. there.

Suppose that Γ is not l.s.c. at t_o ; then we proceed exactly as in the proof of Theorem 15.3, down to the line (**) ; we keep to the same notation. Let

$$G_1 = \{x \in E : p(x - z) < 1/4 \text{ for some } z \in \Gamma(t_o)\} .$$

The set G_1 is open for the weak topology on E , and as Γ is u.s.c. at t_o , there is an index β_o such that $\Gamma(t_\beta) \subset G_1$ for $\beta \geq \beta_o$. In particular, there is for each $\beta \geq \beta_o$ an element y_β of $\Gamma(t_o)$ such that $p(y_\beta - x_\beta) < 1/4$. Since $\Gamma(t_o)$ is compact, the net (y_β) has a subnet which converges to a point y_o , say, of $\Gamma(t_o)$. Taking the corresponding subnets, we have $t_\beta \to t_o$, $x'_\beta \to x'_o$ as before, $y_\beta \to y_o$ and $p(y_\beta - x_\beta) < 1/4$. We substitute all these into line (**) of the previous proof and we obtain:

$$\begin{aligned} M_{x'_o}(t_\beta) &\leq - 1 + \langle y, x'_\beta \rangle + \langle x_\beta - y_\beta, x'_o - x'_\beta \rangle + \langle y_\beta, x'_o - x'_\beta \rangle , \\ &\leq - 1 + \langle y, x'_\beta \rangle + 2p(x_\beta - y_\beta) + \langle y_\beta - y_o, x'_o - x'_\beta \rangle + \langle y_o, x'_o - x'_\beta \rangle , \\ &\leq - \frac{1}{2} + \langle y, x'_\beta \rangle + 2p(y_\beta - y_o) + \langle y_o, x'_o - x'_\beta \rangle . \end{aligned}$$

This expression has the limit $- \frac{1}{2} + \langle y, x'_o \rangle$ as $\beta \to$.

However, as $y \in \Gamma(t_o)$, $M_{x'_o}(t_o) \geq \langle y, x'_o \rangle$.

This is in contradiction of the hypothesis that $M_{x'_o}$ be continuous, and so Γ is l.s.c. after all.

This result does not hold if $\Gamma(t_o)$ is not convex; take for instance, $T = R$, $E = R$, $\Gamma(0) = \{- 1, 1\}$ and $\Gamma(t) = [- 1, 1]$ for $t \neq 0$. Then $M_{x'}$ is constant for all x' , but Γ is not u.s.c. at $t = 0$.

We end this section with some results on the continuity of the convex hull of a multifunction. The following lemma is the local version of a theorem of C. Berge ((2), p. 114, Theorem 4).

LEMMA 15.8. <u>Let</u> $T, X_1, \ldots, X_n$ <u>be topological spaces and</u> $\Gamma_i : T \to X_i$ $(i = 1, \ldots, n)$ <u>multifunctions which are l.s.c. at the point</u>

t_o of T. Then the multifunction

$$\Gamma : t \to \Gamma_1(t) \times \dots \times \Gamma_n(t) \quad (t \in T),$$

is also l.s.c. at t_o.

Proof. Let A_i be an open subset of X_i for each i, such that $\Gamma(t_o)$ meets $A_1 \times \dots \times A_n$, a basic open set in $X_1 \times \dots \times X_n$. Then there exist neighbourhoods $U_1, \dots, U_n$ of t_o such that, for $t \in U_i$, $\Gamma_i(t)$ meets A_i $(i = 1, \dots, n)$. Thus for $t \in U_1 \cap \dots \cap U_n$, $\Gamma(t)$ meets $A_1 \times \dots \times A_n$. Hence Γ is l.s.c. at t_o.

THEOREM 15.9. Let T be a topological space, and E any topological vector space. Then, if $\Gamma : T \to E$ is a multifunction which is l.s.c. at the point $t_o \in T$, the multifunctions $\hat{\Gamma}$ and $\tilde{\Gamma}$ are also l.s.c. at t_o, where, for each t in T, $\hat{\Gamma}(t)$ is the convex hull of $\Gamma(t)$ and $\tilde{\Gamma}(t)$ is the closed convex hull of $\Gamma(t)$.

Proof. For each positive integer n let Λ_n be the simplex in R^n consisting of all points $\underset{\sim}{\lambda}$ where $0 \leq \lambda_i \leq 1$ $(i = 1, \dots, n)$ and $\lambda_1 + \dots + \lambda_n = 1$. For each t, we define

$$\Omega_n(t) = \Lambda_n \times \Gamma(t) \times \dots \times \Gamma(t),$$

the factor $\Gamma(t)$ being taken n times. The multifunction Ω_n is l.s.c. at t_o, by Lemma 15.8. Let E^n denote the product space of E with itself, taken n times in all. Then the mapping φ_n from $\Lambda_n \times E^n$ into E defined by

$$\varphi_n(\underset{\sim}{\lambda} ; x_1, \dots, x_n) = \lambda_1 x_1 + \dots + \lambda_n x_n$$

is continuous. For each n, t we define

$$\Gamma_n(t) = \varphi_n(\Omega_n(t)).$$

Since φ_n is continuous, Γ_n is l.s.c. at t_o, by Proposition 1.3. Now, for each t,

$$\hat{\Gamma}(t) = \bigcup_{n=1}^{\infty} \Gamma_n(t).$$

Therefore $\hat{\Gamma}$ is l.s.c., by Proposition 1.4; applying Proposition 1.1, we see that $\tilde{\Gamma}$ is l.s.c. as well.

THEOREM 15.10. _Let_ T _be a topological space, and_ E _a locally convex Hausdorff space. Then if_ $\Gamma : T \to E$ _is a multifunction which is u.s.c. at a point_ t_o _of_ T _and if_ $\tilde{\Gamma}(t_o)$ _is compact,_ $\tilde{\Gamma}$ _is u.s.c. at_ t_o .

Proof. Suppose that $\tilde{\Gamma}(t_o) \subset G$, where G is an open set in E . Then there exists an open neighbourhood U of the origin such that $\tilde{\Gamma}(t_o) + U \subset G$, and moreover $(\tilde{\Gamma}(t_o) + U)^{-} \subset G$.

There is a neighbourhood V of t_o in T such that, for $t \in V$, $\Gamma(t) \subset \tilde{\Gamma}(t_o) + U$. Since $\tilde{\Gamma}(t_o) + U$ is a convex set, this last statement implies that $\hat{\Gamma}(t) \subset \tilde{\Gamma}(t_o) + U$, and hence $\tilde{\Gamma}(t) \subset G$, as required.

If the space E is complete, and Γ is a compact-valued multifunction, then $\tilde{\Gamma}$ is also compact-valued (for a proof of this, see G. Köthe (27)).

16. _Extreme points of multifunctions_. If A is a non-empty subset of a vector space over the reals, then a point $x \in A$ is an _extreme point_ of A if and only if x is not an interior point of any line segment whose end-points are contained in A . If A is convex, this is equivalent to saying that there is no pair u,v of distinct points in A such that $x = \frac{1}{2}(u + v)$.

If Γ is a multifunction with values which are convex subsets of a real topological vector space, then, for each argument t , $\Gamma^{o}(t)$ shall denote the set of extreme points of $\Gamma(t)$. The next few results are extensions of a result of Himmelberg and Van Vleck ((22), Proposition 1). We shall need to consider dual pairs (E,E') of vector spaces, where E'

contains a countable subset which separates the points of E. This property may be characterized as follows:

PROPOSITION 16.1. *Let* (E,E') *be a dual pair of vector spaces. Then* E' *has a countable subset separating the points of* E *if and only if it is weak*-separable*.

LEMMA 16.2 ((9), Théorème 5). *Let* E *be a locally convex Hausdorff space over the reals with a weak*-separable dual* E'. *Then* E' *has a countable subset* A *with the following property:*

if K *is a non-empty compact convex subset of* E *and if* $y \notin K$, *then there exists* $x' \in A$ *such that*

$$\max \{\langle x,x'\rangle : x \in K\} < \langle y,x'\rangle .$$

COROLLARY. *If* E *is a locally convex Hausdorff space over the reals such that* E' *is weak*-separable, then every compact convex set in* E *is a* $\mathcal{G}_\delta$ *set*.

PROPOSITION 16.3. *Let* T *be any topological space,* E *a locally convex Hausdorff space with a weak*-separable dual, and* $\Gamma : T \to E$ *a u.s.c. multifunction with compact convex values. Then* $G(\Gamma^o)$ *is a* $\mathcal{G}_\delta$ *subset of* $G(\Gamma)$.

Proof. Let (x_n') be a sequence of points in E' which separates the points of E (Proposition 16.1). For each pair (i,j) of positive integers and each argument t, let

$$\Gamma_{ij}(t) = \{\tfrac{1}{2}(y + z) : y,z \in \Gamma(t) \text{ and } |\langle y - z, x_i'\rangle| \geq 1/j\} .$$

Then, for each t,

$$\Gamma^o(t) = \Gamma(t) \setminus \bigcup_{i,j} \Gamma_{ij}(t) .$$

Hence

$$G(\Gamma^o) = G(\Gamma) \setminus \bigcup_{i,j} G(\Gamma_{ij}) . \qquad \ldots(i)$$

We now prove that $G(\Gamma_{ij})$ is closed for each i,j. We define a multifunction $\Omega_{ij} : T \to E \times E$ by taking, for each $t \in T$,

$$\Omega_{ij}(t) = (\Gamma(t) \times \Gamma(t)) \cap \{(y,z) \in E \times E : |\langle y - z, x_i' \rangle| \geq 1/j\} .$$

Then Ω_{ij} is compact-valued and u.s.c. ((2), p. 114, Theorem 4'). Let $\varphi : E \times E \to E$ be the continuous mapping defined by

$$\varphi(y,z) = \frac{1}{2}(y + z) \quad (y, z \in E) .$$

Then $\Gamma_{ij} = \varphi \circ \Omega_{ij}$ and so Γ_{ij} is u.s.c.. It is also compact-valued, and so has a closed graph ((2), p. 112, Theorem 6). Hence $G(\Gamma^o)$ is a $\mathcal{G}_\delta$ subset of $G(\Gamma)$, from statement (i).

PROPOSITION 16.4. Let T be any topological space, X a metrisable subspace of a topological vector space E, and $\Gamma : T \to X$ a u.s.c. multifunction with compact convex values. Then $G(\Gamma^o)$ is a $\mathcal{G}_\delta$ subset of $G(\Gamma)$.

Proof. Let d be a compatible metric for X. Then, for each positive integer m and each $t \in T$, we define

$$\Gamma_m(t) = \{\tfrac{1}{2}(y + z) : y, z \in \Gamma(t) \text{ and } d(y,z) \geq 1/m\} .$$

Then, for each t,

$$\Gamma^o(t) = \Gamma(t) \setminus \cup_m \Gamma_m(t) ,$$

and so

$$G(\Gamma^o) = G(\Gamma) \setminus \cup_m G(\Gamma_m) .$$

To complete the proof, we show that $G(\Gamma_m)$ is a closed set for each m, as in Proposition 16.3. We define a multifunction Ω_m from T into $X \times X$ by taking

$$\Omega_m(t) = (\Gamma(t) \times \Gamma(t)) \cap \{(y,z) : d(y,z) \geq 1/m\} .$$

Then Ω_m is compact-valued and u.s.c. for each m ((2), p. 114, Theorem 4'). Let φ be the continuous mapping from $E \times E$ into E

defined by

$$\varphi(y,z) = \frac{1}{2}(y + z) \qquad (y,\ z \in E)\ .$$

Then $\Gamma_m = \varphi \circ \Gamma_m$ for each m. Hence Γ_m is compact-valued and u.s.c.. Therefore $G(\Gamma_m)$ is closed for each m, which completes the proof.

LEMMA 16.5. Let T be any topological space, E a locally convex Hausdorff space with a weak*-separable dual space, and $\Gamma : T \to E$ an l.s.c. multifunction with compact convex values. Then $G(\Gamma)$ is a $\mathcal{G}_\delta$ set in $T \times E$.

Proof. Let $\{x_n'\}$ be the countable subset of E' which has the property described in Lemma 16.2. As in §15, we define

$$M_{x_n'}(t) = \max \{\langle u,x_n'\rangle : u \in \Gamma(t)\}\ ,$$

for each x_n' and each $t \in T$. Then, by Lemma 16.2, $(t,x) \notin G(\Gamma)$ if and only if there is a positive integer n and a rational number α such that

$$M_{x_n'}(t) \leqslant \alpha < \langle x,x_n'\rangle \quad .$$

Therefore

$$G(\Gamma)' = \bigcup_{n,\alpha} \{t : M_{x_n'}(t) \leqslant \alpha\} \times \{x \in E : \langle x,x_n'\rangle > \alpha\}\ ,$$

the union being taken over all positive integers n and all rationals α. The set $\{t : M_{x_n'}(t) \leqslant \alpha\}$ is closed, since $M_{x_n'}$ is lower semicontinuous (cf. Theorem 15.6), and the set $\{x \in E : \langle x,x_n'\rangle > \alpha\}$ is clearly an $\mathcal{F}_\sigma$. Hence $G(\Gamma)'$ is an $\mathcal{F}_\sigma$ set, and so $G(\Gamma)$ is a $\mathcal{G}_\delta$.

LEMMA 16.6. Let T be any topological spaces, X a separable metrisable space, and $\Gamma : T \to X$ a closed-valued l.s.c. multifunction. Then $G(\Gamma)$ is a $\mathcal{G}_\delta$ set in $T \times X$.

Proof. Let (U_n) be a countable base of open sets for X. Then $(t,x) \notin G(\Gamma)$ if and only if there exists a positive integer m such that $\Gamma(t) \cap U_n = \phi$ and $x \in U_n$. Thus

$$G(\Gamma)' = \bigcup_{n=1}^{\infty} (\Gamma^+(U_n') \times U_n)\ .$$

Each $\Gamma^+(U'_n)$ is closed, since Γ is l.s.c., and each U_n is an $\mathcal{F}_\sigma$ set. Hence $G(\Gamma)$ is a $\mathcal{G}_\delta$ set.

PROPOSITION 16.7. Let T be any topological space, E a locally convex Hausdorff space with a weak*-separable dual, and $\Gamma : T \to E$ a continuous multifunction with compact convex values. Then $G(\Gamma^O)$ is a $\mathcal{G}_\delta$ set in $T \times E$.

Proof. This follows immediately from Proposition 16.3 and Lemma 16.5.

PROPOSITION 16.8. Let T be any topological space, X a separable metrisable subspace of a topological vector space, and $\Gamma : T \to X$ a continuous multifunction with convex compact values. Then $G(\Gamma^O)$ is a $\mathcal{G}_\delta$ set in $T \times X$.

Proof. This result follows from Proposition 16.4 and Lemma 16.6.

Himmelberg and Van Vleck ((22), p. 723) ask whether it is possible to obtain a result for l.s.c. multifunctions which is analogous to their result for u.s.c. multifunctions (Ibid., Proposition 1). The following result is in answer to this question.

PROPOSITION 16.9. Let T be any topological space, X a separable metrisable subspace of a topological vector space E, and $\Gamma : T \to X$ an l.s.c. multifunction with compact convex values. Then $G(\Gamma^O)$ is an $\mathcal{F}_{\sigma\delta}$ subset of $G(\Gamma)$.

Proof. For each positive integer m we define

$$\Gamma_m(t) = \{\tfrac{1}{2}(y + z) : y,\ z \in \Gamma(t) \text{ and } d(y,z) > 1/m\} ,$$

where d is a compatible metric for X. We also define, for each m and t,

$$\Omega_m(t) = (\Gamma(t) \times \Gamma(t) \cap \{(y,z) : d(y,z) > 1/m\} .$$

Then, for each m , Ω_m is an l.s.c. multifunction from T into $X \times X$, by Lemma 15.8. Let $\varphi : E \times E \to E$ be the continuous mapping defined by

$$\varphi(y,z) = \frac{1}{2}(y + z) \qquad (y,\ z \in E) .$$

Then $\Gamma_m = \varphi \circ \Omega_m$, and so Γ_m is l.s.c.. By Proposition 1.1, $\overline{\Gamma}_m$ is also l.s.c., $G(\overline{\Gamma}_m)$ is therefore a $\mathcal{G}_\delta$ set in $T \times X$, for all m , by Lemma 16.6.

For each m, t , we have:

$$\Gamma_m(t) \subseteq \{\tfrac{1}{2}(y + z) : y,\ z \in \Gamma(t) \text{ and } d(y,z) \geq 1/m\} ,$$

and so, as the set on the right is compact,

$$\overline{\Gamma}_m(t) \subseteq \{\tfrac{1}{2}(y + z) : y,\ z \in \Gamma(t) \text{ and } d(y,z) \geq 1/m\} \subseteq \Gamma_{m+1}(t) .$$

Therefore

$$\begin{aligned} G(\Gamma^o) &= G(\Gamma) \setminus \cup_m G(\Gamma_m) , \\ &= G(\Gamma) \setminus \cup_m G(\overline{\Gamma}_m) . \end{aligned}$$

Since $G(\overline{\Gamma}_m)$ is a $\mathcal{G}_\delta$ set for each m , the result follows.

If T is a topological space in which every open set is an $\mathcal{F}_\sigma$, then, in the above Proposition, $G(\Gamma^o)$ is an $\mathcal{F}_{\sigma\delta}$ subset of $T \times X$.

We now consider extreme points of measurable multifunctions.

The following Theorem has been proved in the finite-dimensional case by Himmelberg and Van Vleck ((22), Theorem 4).

THEOREM 16.10. Let $(S,\mathcal{M})$ be any measurable space, X a separable metrisable subspace of a topological vector space and $\Gamma : S \to X$ a measurable multifunction with compact convex values. Then $G(\Gamma^o)$ belongs to the σ-algebra $\mathcal{M} \otimes \mathcal{B}_X$.

Proof. Let d be a compatible metric for X . Then, as before, we define, for each $t \in S$ and each positive integer m ,

$$\Gamma_m(t) = \{\tfrac{1}{2}(y + z) : y,\ z \in \Gamma(t) \text{ and } d(y,z) \geq 1/m\} ,$$

and so

$$G(\Gamma^o) = G(\Gamma) \setminus \cup_m G(\Gamma_m) . \qquad \ldots(i)$$

We also define, for each m, a multifunction

$$\Omega_m : T \to (\Gamma(t) \times \Gamma(t)) \cap \{(y,z) : d(y,z) \geq 1/m\} ,$$

from S into $X \times X$. Each Ω_m is measurable, by Theorem 4.8. Then $\Gamma_m = \varphi \circ \Omega_m$ for each m, where

$$\varphi(y,z) = \tfrac{1}{2}(y + z) \qquad (y, z \in X) .$$

This mapping φ is continuous, and so Γ_m is measurable for each m; $G(\Gamma_m)$ therefore belongs to $\mathcal{M} \otimes \mathcal{B}_X$, by Theorem 3.6. $G(\Gamma)$ belongs to $\mathcal{M} \otimes \mathcal{B}_X$ similarly, and therefore so does $G(\Gamma^o)$, from line (i).

If, in Theorem 16.10, $(S,\mathcal{M})$ also admits the Souslin operation and X is a Souslin space, then Γ^o is also a measurable multifunction, by the Corollary to Lemma 3.3. The following proposition enables us to prove a converse result to Theorem 16.10.

PROPOSITION 16.11. *If* S *is a measurable space,* E *a separable metrisable topological vector space and* $\Gamma : S \to E$ *any* $\mathcal{G}$-*measurable multifunction, then the multifunction* $\hat{\Gamma}$ *is also* $\mathcal{G}$-*measurable, where, for each* $t \in S$, $\hat{\Gamma}(t)$ *is the convex hull of* $\Gamma(t)$.

Proof. The proof of this result is similar to that of the corresponding result for multifunctions of Souslin type (Theorem 13.3). For each positive integer n, let Λ_n be the simplex of points $\underset{\sim}{\lambda}$ in R^n such that $0 \leq \lambda_i \leq 1$ $(i = 1, \ldots, n)$ and $\lambda_1 + \ldots + \lambda_n = 1$. Then for each n we define

$$\Omega_n(t) = \Lambda_n \times \Gamma(t) \times \ldots \times \Gamma(t) ,$$

the factor $\Gamma(t)$ occurring n times. Then the multifunction Ω_n is $\mathcal{G}$-measurable, by Proposition 4.7. We define a continuous mapping

$$\varphi_n : \Lambda_n \times E \times \ldots \times E \to E$$

by the formula

$$\varphi_n(\underset{\sim}{\lambda}\ ;\ x_1, \ldots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n \ .$$

Now

$$\hat{\Gamma} = \cup_n (\varphi \circ \Omega_n) \ ,$$

and so $\hat{\Gamma}$ is also $\mathcal{G}$-measurable.

COROLLARY. Let S, E, Γ be as in Proposition 16.11. Then the multifunction $\tilde{\Gamma}$ is $\mathcal{G}$-measurable where, for each $t \in S$, $\tilde{\Gamma}(t)$ is the closed convex hull of $\Gamma(t)$.

Proof. $\tilde{\Gamma}(t)$ is for each t the closure of the set $\hat{\Gamma}(t)$. The multifunction $\hat{\Gamma}$ is $\mathcal{G}$-measurable, and therefore so is $\tilde{\Gamma}$, by Proposition 1.1.

Proposition 16.11 generalizes a result of C. Castaing ((8), p. 107, Corollaire 2), and it enables us to prove the following partial converse of Theorem 16.10:

THEOREM 16.12. Let S be a measurable space, E a separable metrisable locally convex Hausdorff space and $\Gamma : S \to E$ a multifunction with values which are non-empty weakly compact convex subsets of E. Then if Γ^o is $\mathcal{G}$-measurable, so is Γ.

Proof. This follows from the Corollary to Proposition 16.11, since, for each t, $\Gamma(t)$ is the closed convex hull of $\Gamma^o(t)$, by the Krein-Milman theorem.

The rest of this section is devoted to a study of selection theorems involving extreme-valued selectors. Theorem 16.10 yields the following corollary:

THEOREM 16.13. Let S be a measurable space which admits the Souslin operation, X a Souslin subspace of a locally convex Hausdorff space and $\Gamma : S \to X$ a measurable multifunction with non-empty compact convex values. Then Γ has a countable family $\{\gamma_n\}$ of measurable selectors such that:

(i) for each n and each t, $\gamma_n(t)$ is an extreme point of $\Gamma(t)$;

(ii) for each t, $\Gamma(t)$ is the closed convex hull of the set $\{\gamma_n(t) : n = 1,2, \ldots\}$.

Proof. Since $\Gamma(t)$ is compact and convex for each t, $\Gamma^o(t)$ is non-empty, by the Krein-Milman theorem. $G(\Gamma^o)$ is a measurable subset of $S \times X$, by Theorem 16.10. By Theorem 12.3 (Corollary), Γ^o has a countable family (γ_n) of measurable selectors such that the set $\{\gamma_n(t) : n = 1,2, \ldots\}$ is dense in $\Gamma^o(t)$ for each t. Therefore

$$\Gamma^o(t) \subseteq \{\gamma_n(t)\}^- \subseteq \Gamma(t)$$

for each t. Now $\Gamma(t)$ is the closed convex hull of $\Gamma^o(t)$, by the Krein-Milman theorem; therefore it is also the closed convex hull of the set $\{\gamma_n(t) : n = 1,2, \ldots\}$.

The following result is obtained from Theorem 16.10 in the same way:

THEOREM 16.4. Let S be a measurable space which admits the Souslin operation, X a Souslin subspace of a topological vector space, and $\Gamma : S \to X$ a measurable multifunction with compact convex values. Then provided $\Gamma^o(t)$ is non-empty for each $t \in S$, Γ has a measurable selector γ such that $\gamma(t)$ is an extreme point of $\Gamma(t)$ for each t.

Now let X be a topological space on which there exists a sequence (f_n) of upper semicontinuous real-valued functions which separate the points of X. Let K be a non-empty compact set in X. We define a descending sequence (K_n) of subsets of K as follows: we take $K_o = K$; then if we assume that K_{n-1} is defined $(n \geq 1)$, we take

$$K_n = \{u \in K_{n-1} : f_n(u) \geq a_n\} ,$$

where

$$a_n = \max \{f_n(u) : u \in K_{n-1}\} .$$

Then the intersection $\cap_n K_n$ is non-empty and contains just one point. We call this point the <u>lexicographic maximum</u> of K with respect to the sequence (f_n) , and denote it by the abbreviation lex. max. K.

If E is a locally convex Hausdorff space over the reals with a sequence (x_n') in E' which separates the points of E , and K is a non-empty compact subset of E , then the lexicographic maximum of K with respect to (x_n') is an extreme point of K ; this follows from the definition of an extreme point given at the beginning of this section.

THEOREM 16.15. <u>Let</u> S <u>be a measurable space, and</u> X <u>a topological space on which is defined a sequence</u> (f_n) <u>of upper semi-continuous real-valued functions which separate the points of</u> X . <u>Then if</u> $\Gamma : S \to X$ <u>is a compact-valued measurable multifunction with non-empty values, the function</u>

$$t \to \text{lex. max. } \Gamma(t) \qquad (t \in S)$$

<u>is a measurable selector for</u> Γ .

<u>Proof</u>. Let $\Gamma_o = \Gamma$; we define a sequence (Γ_n) of measurable multifunctions from S into X inductively as follows. Suppose that $n \geq 1$ and that Γ_{n-1} is defined; let g_{n-1} be the function

$$t \to \max \{f_n(u) : u \in \Gamma_{n-1}(t)\} .$$

Then we define for each t

$$\Gamma_n(t) = \{x \in \Gamma_{n-1}(t) : f_n(x) = g_{n-1}(t)\} .$$

Each multifunction Γ_n is compact-valued, and the sequence (Γ_n) is descending. Moreover, Γ_n is measurable, for each n ; this follows from Lemma 9.1, by induction on n . Therefore the multifunction $\cap_n \Gamma_n$ is also measurable, by Proposition 1.5. Now, for each t , $\cap_n \Gamma_n(t)$ contains just one point, namely lex. max. $\Gamma(t)$; hence the function

$$t \to \text{lex. max. } \Gamma(t)$$

is measurable, as required.

C. Castaing (9) has obtained a result similar to this, but in a different setting; in his paper, S is a locally compact Hausdorff space with a Radon measure, and Γ is required to satisfy a Lusin-type condition. We now use Theorem 16.15 to prove another theorem on extreme-valued selectors.

THEOREM 16.16. Let S be any measurable space, E a locally convex Hausdorff space over the reals, with a weak*-separable dual, and $\Gamma : S \to E$ a measurable multifunction with non-empty compact convex values. Then Γ has a countable family $\{\gamma_n\}$ of measurable selectors such that:

(i) for each n , $\gamma_n(t)$ is an extreme point of $\Gamma(t)$;

(ii) for each t , $\Gamma(t)$ is the closed convex hull of the set $\{\gamma_n(t) : n = 1,2, \ldots\}$.

Proof. Let (x'_n) be a sequence of elements of E' which has the property of Lemma 16.2. For each n , we take

$$M_{x'_n}(t) = \max \{\langle u, x'_n \rangle : u \in \Gamma(t)\} \quad (t \in S) ,$$

and we define, for each $t \in S$ and each n ,

$$\Gamma_n(t) = \{u \in \Gamma(t) : \langle u, x'_n \rangle = M_{x'_n}(t)\} .$$

Then, for each n , Γ_n is measurable and has non-empty compact values, by Lemma 9.1; it therefore has a measurable selector γ_n , by Theorem 16.15; since $\gamma_n(t)$ is for each t a lexicographic maximum, it is an extreme point of $\Gamma(t)$.

In order to prove part (ii), we consider the multifunction Ω, where $\Omega(t)$ is for each t the closed convex hull of the set $\{\gamma_n(t) : n = 1,2, \ldots\}$. Ω has compact convex values and $\Omega(t) \subseteq \Gamma(t)$ for each t. Suppose that for some t there is a point x in $\Gamma(t) \setminus \Omega(t)$. Then, by Lemma 16.2, there exists an integer n such that

$$\max \{\langle u, x'_n \rangle : u \in \Omega(t)\} < \langle x, x'_n \rangle .$$

In particular,

$$\langle \gamma_n(t) , x'_n \rangle < \langle x, x'_n \rangle ,$$

which contradicts the fact that

$$\langle \gamma_n(t) , x'_n \rangle = M_{x'_n}(t) .$$

Hence $\Omega(t) = \Gamma(t)$ for all t, which proves the second part of the theorem.

M. Valadier ((50), p. 271), has proved a similar result for multifunctions which are "scalarly measurable".

LEMMA 16.17. Let S be a measurable space and E a separable metrisable locally convex space over the reals. Then a function from S into E is measurable if and only if it is measurable with respect to the weak topology on E.

Proof. Let $f : S \to E$ be a measurable function. Then it is clearly measurable with respect to the weak topology on E.

Conversely, let (U_n) be any countable base of open sets for the topology on E; then the sets (V_n) also form a base for the topology, where, for each n, V_n is the closed convex hull of U_n. This follows from the fact that E has a base of neighbourhoods of the origin which consists of closed convex sets. Then if

$$G = \cup \{V_i : i \in I\}$$

is an open set in E , where I is some subset of N , and f is a weakly measurable function,

$$f^{-1}(G) = \bigcup_{i \in I} f^{-1}(V_i) .$$

Since each V_i is closed and convex, it is closed in the weak topology on E . Hence $f^{-1}(V_i)$ is measurable for each i . Therefore $f^{-1}(G)$ is measurable for every open set G in E , and so f is measurable.

THEOREM 16.18. Let S be any measurable space, E a separable metrisable locally convex space over the reals, and $\Gamma : S \to E$ a measurable multifunction with non-empty weakly compact convex values. Then Γ has a countable family $\{\gamma_n\}$ of measurable selectors such that:

(i) for each n , $\gamma_n(t)$ is an extreme point of $\Gamma(t)$,

(ii) for each t , $\Gamma(t)$ is the closed convex hull of the set $\{\gamma_n(t) : n = 1, 2, \ldots\}$.

Proof. E has a countable base (U_n) of open sets consisting of convex open sets. For each disjoint pair $\{U_m, U_n\}$ of these, there is a functional $x'_{mn} \in E'$ which separates them, by the Hahn-Banach theorem. Hence the family $\{x'_{mn}\}$ separates the points of E , and so (Proposition 16.1), E with its weak topology and Γ satisfy the hypotheses of Theorem 16.16.

Hence Γ has a collection $\{\gamma_n\}$ of weakly measurable selectors which satisfies conditions (i) and (ii). It follows from Lemma 16.17 that each of the selectors $\{\gamma_n\}$ is measurable, which completes the proof.

We now consider the algebraic properties of lexicographic maxima.

THEOREM 16.19. Let E be a locally convex Hausdorff space over the reals such that E' contains a sequence (x'_n) separating the points of E . Let $\mathcal{K}^*$ be the space of non-empty compact subsets of E

with the Vietoris topology. Then the mapping $f : K^* \to E$ defined by $f(K) = \text{lex. max. } K$ $(K \in K^*)$ is Borel measurable. Moreover, for any pair K_1, K_2 in $\mathcal{K}^*$ and for any positive scalar λ,

$$\text{(i)} \quad f(K_1 + K_2) = f(K_1) + f(K_2) ,$$

and

$$\text{(ii)} \quad f(\lambda K_1) = \lambda f(K_1) .$$

Proof. The measurability of f follows at once from Theorem 16.15, on taking $\Gamma(K) = K$ for each K.

We now prove that condition (i) is satisfied. Let $x_1 = f(K_1)$ and $x_2 = f(K_2)$. Let also $(K_{1,n})$, $(K_{2,n})$, $((K_1 + K_2)_n)$ be the descending sequences of compact sets formed as in our definition of lexicographic maximum. Then it is enough to show that

$$x_1 + x_2 \in (K_1 + K_2)_n$$

for all n. This follows from the fact that for each n,

$$K_{1,n} + K_{2,n} = (K_1 + K_2)_n ,$$

which we prove by induction on n. It clearly holds for $n = 0$, and if it holds for $n = k$,

$$\begin{aligned}(K_1 + K_2)_{k+1} &= \{u_1 + u_2 : \langle u_1 + u_2, x'_{k+1}\rangle = \max., \ u_1 \in (K_1)_n, \ u_2 \in (K_2)_n\} \\ &= \{u_1 \in (K_1)_k : \langle u_1, x'_{k+1}\rangle = \max.\} + \{u_2 \in (K_2)_k : \langle u_2, x'_{k+1}\rangle = \max.\} \\ &= (K_1)_{k+1} + (K_2)_{k+1} .\end{aligned}$$

The proof of part (i) is therefore complete, and the proof of (ii) is similar.

In Theorem 16.19, the choice function f is clearly Borel measurable for any topology on $\mathcal{K}^*$ for which the sets

$$\{K \in \mathcal{K}^* : K \cap F \neq \phi\} , \ F \text{ closed},$$

are all closed.

We now extend the idea of lexicographic maxima to the case where the space E' has no countable sequence which separates the points of E. Let E be a locally convex Hausdorff space and $\{x_i' : i \in I\}$ a family in E' which separates the points of E. We suppose that the index set I is well-ordered; let 0 denote its first element. If K is a non-empty compact set in E, we define $K_0 = K$, and for $i > 0$,

$$K_i = \{u \in \bigcap_{j<i} K_j : \langle u, x_i'\rangle = \text{maximum}\} .$$

Then we define

$$\text{lex. max. } K = \bigcap_{i \in I} K_i .$$

The function $K \to \text{lex. max. } K$ may not be well-behaved topologically. However, it still has properties (i) and (ii) of Theorem 16.19. In order to prove that it has property (i), it is sufficient to prove that

$$K_{1,i} + K_{1,i} = (K_1 + K_2)_i$$

for each $i \in I$. We do this by transfinite induction; we assume that this statement is true for $j < i$. Then

$$(K_1 + K_2)_i = \{u \in \bigcap_{j<i} (K_{1,j} + K_{2,j}) : \langle u, x_i'\rangle = \text{max.}\}$$

$$= \{u_1 \in \bigcap_{j<i} K_{1,j} : \langle u_1, x_i'\rangle = \text{max.}\} + \{u_2 \in \bigcap_{j<i} K_j : \langle u_2, x_i'\rangle = \text{max.}\}, \quad \ldots(*)$$

provided that

$$\bigcap_{j<i} K_{1,j} + \bigcap_{j<i} K_{2,j} = \bigcap_{j<i} (K_{1,j} + K_{2,j}) . \quad \ldots(**)$$

If statement $(**)$ holds, then the right-hand side of $(*)$ is equal to $(K_{1,i} + K_{2,i})$. Therefore, for all i,

$$\text{lex. max. } K_1 + \text{lex. max. } K_2 \in (K_1 + K_2)_i ,$$

and hence

$$\text{lex. max. } K_1 + \text{lex. max. } K_2 = \text{lex. max. } (K_1 + K_2) .$$

In order to verify (**), it is sufficient to prove that the right-hand side is contained in the left-hand side. This is clearly true if i has an immediate predecessor; otherwise, suppose that, for each $j < i$,

$$x = a_j + b_j ,$$

where $a_j \in K_{1,j}$ and $b_j \in K_{2,j}$; then the sequence (a_j) has a cluster point a_0 in $\bigcap_{j<i} K_{1,j}$ and (b_j) has a cluster point b_0 in $\bigcap_{j<i} K_{2,j}$. Then $x = a_0 + b_0$, and so x belongs to the left-hand side.

Property (ii) may be verified similarly.

We conclude this section by considering the relationship between the extreme-valued measurable selectors of a multifunction and the class of all measurable selectors. We shall need the following lemma:

LEMMA 16.20. <u>Let</u> S <u>be a measurable space,</u> E <u>a separable metrisable topological vector space, and</u> Γ_1, Γ_2 <u>measurable compact-valued multifunctions from</u> S <u>into</u> E; <u>then the multifunction</u>

$$t \to \Gamma_1(t) + \Gamma_2(t)$$

<u>is also compact-valued and measurable. If</u> α <u>is a measurable scalar-valued function on</u> S, <u>then the multifunction</u>

$$t \to \alpha(t)\, \Gamma_1(t)$$

<u>is also measurable</u>.

<u>Proof</u>. Let $\varphi : E \times E \to E$ be the continuous mapping defined by

$$\varphi(x,y) = x + y \qquad (x,y \in E) .$$

Then, for each t,

$$\Gamma_1(t) + \Gamma_2(t) = \varphi(\Gamma_1(t) \times \Gamma_2(t)) ,$$

and since the multifunction

$$t \to \Gamma_1(t) \times \Gamma_2(t)$$

is measurable and compact-valued (Theorem 4.8), so is the multifunction

$$t \to \Gamma_1(t) + \Gamma_2(t) .$$

The second part of this lemma is proved similarly.

As in Lemma 16.20, let S be any measurable space, E a separable metrisable topological vector space, and Γ a measurable multifunction from S into E with values which are non-empty, convex and compact. The class of all measurable functions from S into E is clearly a vector space under the usual operations of addition and scalar multiplication, and the set M_Γ of measurable selectors of Γ is clearly a convex subset of this space. We have the following generalization of a result of Aumann ((1), p. 10):

THEOREM 16.21. <u>The set of extreme points of</u> M_Γ <u>is</u> M_{Γ^o}, <u>where</u> M_{Γ^o} <u>denotes the set of measurable selectors of</u> Γ^o.

<u>Proof</u>. Suppose that $f \in M_\Gamma$ is not an extreme point. Then there exist functions g,h in M_Γ such that $f = \frac{1}{2}(g + h)$, and which are not equal. Therefore there is a point $t \in S$ such that $f(t) = \frac{1}{2}(g(t) + h(t))$ and $g(t) \neq h(t)$. Hence $f(t) \notin \Gamma^o(t)$ and so f does not belong to M_{Γ^o}. Therefore if ext. M_Γ denotes the set of extreme points of M_Γ,

$$M_{\Gamma^o} \subseteq \text{ext. } M_\Gamma .$$

Conversely, let f be a measurable selector for Γ such that $f(t) \notin \Gamma^o(t)$ for some t. Then there exists a non-zero element x_o of $\Gamma(t)$ such that $f(t) \pm x_o \in \Gamma(t)$.

Consider the multifunction

$$t \to (\Gamma(t) - f(t)) \cap (f(t) - \Gamma(t)) \cap \{x \in E : d(x,0) \geq d(x_o,0)\} .$$

This multifunction is measurable, by Lemma 16.20 and Theorem 4.3. It is non-empty for at least one value of t, and, on the set A (which is measurable) on which it is non-empty, it has a measurable selector g (Theorem 10.1).

We extend g to the whole of S by taking $g(t) = 0$ for $t \notin A$. Then, for all $t \in S$,

$$f(t) \pm g(t) \in \Gamma(t) ,$$

and, as g is not everywhere zero, f cannot be an extreme point of M_Γ. Hence

$$\text{ext.}\, M_\Gamma \subseteq M_{\Gamma^o} ,$$

which completes the proof.

The above conclusion is valid if S is a measurable space which admits the Souslin operation and Γ is a multifunction of Souslin type with convex (but not necessarily closed or compact) values.

Lastly, we present Theorem 16.21 again, in a form which is more useful from the point of view of applications. Let $(S, \mathcal{M}, \mu)$ be a measure space, μ being a positive measure, E a separable metrisable topological vector space, and $\Gamma : S \to E$ a measurable multifunction with values which are non-empty, convex and compact. We consider again the vector space of all measurable functions from S into E, but we do not distinguish between functions which differ only on a set of measure zero. We define M_Γ to be the set of measurable functions $f : S \to E$ such that

$$f(t) \in \Gamma(t) \qquad \text{a.e.},$$

and M_{Γ^o} is defined similarly. Then, if ext. M_Γ denotes the set of extreme points of M_Γ, we have:

THEOREM 16.22. ext. $M_\Gamma = M_{\Gamma^o}$.

Proof. Suppose that $f \in M_\Gamma$ is not an extreme point. Then there exist functions g, h in M_Γ, which differ on a set of positive measure, such that $f = \frac{1}{2}(g + h)$. Therefore there is a set of positive measure on which

$$f(t) = \frac{1}{2}(g(t) + h(t))$$

and $g(t) \neq h(t)$. Hence $f(t) \notin \Gamma^o(t)$ on a set of positive measure, and so f does not belong to M_{Γ^o}. Thus

$$M_{\Gamma^o} \subseteq \text{ext. } M_\Gamma .$$

Conversely, let f be a measurable selector for Γ such that $f(t) \notin \Gamma^o(t)$ for $t \in A$, where A is of positive measure. Consider the multifunctions

$$\Gamma_n : t \to (\Gamma(t) + f(t)) \cap (f(t) - \Gamma(t)) \cap \{x \in E : d(x,0) \geq 1/n\} ,$$

for $n = 1,2, \ldots$. Each of these is measurable, by Lemma 16.20 and Theorem 4.3. Moreover, there exists an integer m such that $\Gamma_m(t)$ is non-empty on a set of positive measure; if this were not so, $\cup_n \Gamma_n$ would be empty almost everywhere, and so the set

$$(\Gamma(t) + f(t)) \cap (f(t) - \Gamma(t)) \cap \{x \in E : x \neq 0\}$$

would be empty a.e.. However, this set is empty if and only if $f(t) \in \Gamma^o(t)$, and we have assumed that $f(t) \notin \Gamma^o(t)$ on a set of positive measure - a contradiction.

So the multifunction Γ_m is non-empty on a set B of positive measure; let g be a measurable selector for Γ_m on B. We extend g to the whole of S by taking $g(t) = 0$ for $t \notin B$. Then, for all $t \in S$,

$$f(t) \pm g(t) \in \Gamma(t) ,$$

and, as g is not almost everywhere zero, f cannot be an extreme point of M_Γ. Hence

$$\text{ext. } M_\Gamma \subseteq M_{\Gamma^o} ,$$

as required.

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