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Some Problems in Reliability of Systems Composed of Unreliable Components

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Abstract

The thesis considers several connected problems in the study of reliability of systems with unreliable components. Initially it reviews the dichotomic case introducing a new characterization which clarifies the importance of the k -out-of- n structures. This characterization is applied to two problems, failure to operate and to idle and stress-strength modelling. Whilst most previous work has concerned itself with coherent structures, being thought of as reasonable systems, the latter section of the chapter considers the case of non-coherent structures. The following chapter moves from the dichotomic models to multilevel models, considering possible extensions and producing a hierarchy within the possible definitions. Chapter 3 considers the stochastic properties of such systems, again attention is paid to the k -out-of- n structures and non-coherent models. The final two chapters consider component performance. In chapter 4 the system is assumed to be hierarchical, so that a component is part of a subsystem, and it is assumed the whole subsystem may be replaced. The effect on the component is examined. The final chapter considers optimal age replacement of a component considering 3 possible alternative criteria.

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Introduction

Reliability Theory may be regarded as the study of systems which may not perform as desired. Such a broad definition immediately implies that the subject is not restricted to a single field of inquiry. The initial interest arose from attempts at quantification by engineers of the performance and safety of systems, using the word system to cover any type of manufactured article. From such a beginning there arose a collection of material which resembled a 'folk-lore' in various branches of Engineering, some of which was mathematical in form. Since the major concern was the uncertainty of performance the area needed statistical development. The development pursued work in other areas of statistics and probability theory, especially in stochastic processes with development of Renewal Theory, and the Analysis of life data. It is, however, unfortunate that the division between the practicing engineer and the statisticians has meant the development of almost two separate identities in Engineering and Statistics. Thus in each area the perception of the major problems differ and the only commonality is the study of systems. Even within statistics there has occurred the divergence between probabilistic modelling and inferential aspects. This has meant that the literature on the subject is very widely spread, from specific Engineering journals to general Statistical journals. There have been attempts to collate the material with the birth of specialized journals, IEEE Transactions on Reliability, Microelectronics and Reliability, and Reliability Engineering. Also several authors have attempted to summarize the major ideas both from the mathematical standpoint, Bain [1978], Barlow and Proschan [1965,1976], Schafer, Singpurwalla and Mann [1974], and Gnedenko, Belyayev and

Solovyev [1969], and from the Engineering viewpoint Green and Bourne [1972]. One minor effect of this fragmentation is differences in notation and definition which have arisen.

In a review paper Lomnicki [1973] suggested that the subject was born in the early sixties, though he does quote earlier major contributions by Von Neumann [1956] and Moore and Shannon [1956], and text books by Bazovsky [1961], Lloyd and Lipow [1962], Zelen [1963] had already appeared. The journals mentioned above appeared in the early sixties. However major papers and views on the modelling aspects of Reliability did arise during the early part of the decade, particularly Birnbaum, Esary and Saunders [1961] contribution to the study of coherent systems. In some senses the paper formed the basis for much of the development over the next decade. It marked the shift from emphasis on the component to the system, giving a definition on which to base further studies. This can be particularly seen in the change of emphasis in the writing of Barlow and Proschan. In their Mathematical Theory of Reliability the theme is heavily shaped towards the stochastic properties of components, whereas their Statistical Theory of Reliability, concentrate almost entirely on system studies.

This thesis initially reconsiders the importance of structure and reassesses the definition of the reasonable machine suggested in the early work of Birnbaum, Esary and Saunders [1961]. In doing so it presents a characterization of coherent systems which proves helpful in obtaining results for specific models. The models considered in the thesis are the 'failure to operate and idle' system and 'stress-strength' models. Emphasis is laid on the practical importance of non-coherent

structures as plausible description of system performance. This is particularly important since the advent of Fault Tree Analysis in the middle seventies. Hence throughout the thesis consideration is given to the extension of the derived results to the non-coherent structure.

Chapter 2 starts with an examination of the different possible extensions from dichotomic definitions of coherency to multilevel definitions. It concentrates on results concerning the hierarchy of the definitions in terms of structure rather than via the relevancy criteria which appear to be used by previous authors. The latter sections of this chapter concerns the enumeration of the structures of the various types.

Whilst chapter 2 is primarily concerned with deterministic questions concerning the development to multilevel systems chapter 3 considers the stochastic element, starting with a review of previous work and then considering the generalization of work given in chapter 1.

The last two chapters concentrate on the component rather than the system. In chapter 4 the performance of the component when it may be replaced is considered. Following on from this the last chapter considers optimal replacement of components.

Basic Defintions

Reliability, denoted by $R(t)$, may be defined as the probability a device is working at a specified time t . If the state of a component is described by random variable $X(t)$, such that $X(t)=1$ if device is working at time t and $X(t)=0$ if it is not then,

$$R(t) = E[X(t)].$$

For devices which have a time to failure, S , with probability density function $f(s)$, $s > 0$, then the reliability is,

$$R(t) = \int_t^{\infty} f(s) ds.$$

When a system or device can be repaired then the term availability is often used in place of reliability.

Availability may be defined as the probability that a component is available for use at a given time or the component is working at the given time.

Availability may be measured at a point of time or over a period, hence there are two possible measures, Availability $A(t)$ at time t and Average Availability $A_{AV}(T)$ in $[0, T]$.

Availability at time t is defined as,

$$A(t) = E[X(t)].$$

Average Availability in $[0, T]$ is defined as,

$$A_{AV}(T) = \frac{1}{T} \int_0^T A(t) dt .$$

Chapter 1

Dichotomic Description of Systems Performance

1.1 Introduction

Early work on Reliability naively assumed that components either worked or failed, (Von Neumann [1956], Shannon and Moore [1956], Birnbaum, Esary and Saunders [1961]), described this as dichotomic performance. A partial defence of this approach was to say that a component though subject to degradation on a continual scale the component is either capable of performing at an acceptable level or at an unacceptable level.

When authors considered systems the same dichotomic assumption was usually made, though notably Simon [1969,1970,1972] made the assumption the system composed of dichotomically performing components could itself be described as performing at various level. The assumption of dichotomy considerably eased the debate as will be illustrated in the next chapter on Multilevel Systems.

Given a desire to look generally at systems then one simple needs to describe a 'reasonable' machine. Obviously a 'reasonable' machine is one in which the improvement of a component does not mean the degradation of the system. It was subsequently assumed every component would have an effect on the systems. Such a system would be described as a Coherent System, a mathematical definition following Barlow and Proschan [1975] will be given in section 1.2. It is now appreciated that

coherency need not be a property of a real machine, mainly due to the advent of Fault Tree Analysis. (Fault Tree Analysis is a practical approach to evaluating systems performance initiated by Fussel [1970], further details will be given in section 1.5.) The technique still requires further development particularly from a stochastic standpoint, see Ansell and Bendell [1985], Barlow [1983] and Weber [1983].

This chapter initially sets out the formulation for the dichotomic model and then considers a characterization of coherent systems in terms of k -out-of- n systems. The characterization is particularly useful when considering systems of identical components and in this context two applications are given.

The remainder of the chapter considers more general approaches to systems performance, firstly by considering Fault Tree Analysis and then by examining Non-Coherent systems.

1.2 Formulation and Notation

It is assumed for convenience that 1 represents the working state for component and system and 0 represents the failed state in the dichotomic model.

Assume system has n components, X_1, X_2, \dots, X_n which take the states $\underline{x} = (x_1, x_2, \dots, x_n)$.

Let $(x_j = m, \underline{x})$ be any vector of states of the components in which the

jth component takes the value m. Let $(x_j=*, \underline{x})$ be any vector of states of components without the jth component.

Define $f(\underline{x})$ to be the structure function for a system. It is the description of the state of the system given the components X_1, X_2, \dots, X_n are at \underline{x} . Therefore in the dichotomic case $f(\underline{x})$ is a mapping from $\{0,1\}^n$ to $\{0,1\}$.

Obviously for most systems $f(\underline{1})$ should take the value 1, if all the components are working then system should work, similarly $f(\underline{0})=0$.

A coherent structure is a structure in which the improvement of a component does not degrade the state, which is easily translated as;

$$f((x_j=1, \underline{x})) \geq f((x_j=0, \underline{x})) \text{ for all } j=1, \dots, n \text{ and } (x_j=*, \underline{x}) \quad (1.1)$$

This obviously means the coherent structures are monotonic in \underline{x} .

The assumption of relevance of components is usually added to the above definition of coherency. A component is relevant if a change in its state while other component remain unchanged affects the state of the system. This may be taken as

$$f((x_j=1, \underline{x})) \neq f((x_j=0, \underline{x})) \text{ for at least one } (x_j=*, \underline{x}). \quad (1.2)$$

There are a number of special cases of coherent structures, particularly k-out-of-n, series and parallel systems. The k-out-of-n system functions if k out of the n components work, the series system is

a n -out-of- n system and the parallel system is a 1-out-of- n system.

A non-coherent structure is usually taken to be structure such that

$$f((x_j=1, \underline{x})) < f((x_j=0, \underline{x})) \text{ for at least one } j \text{ and } (x_j=*, \underline{x}). \quad (1.3)$$

A dual structure function f is defined for a structure function f_D as,

$$f_D(\underline{x}) = 1 - f(1 - \underline{x}).$$

A path is any vector \underline{x} such that $f(\underline{x}) = 1$, and is usually described by the set of components working. (1.4)

A cut is any vector \underline{x} such that $f(\underline{x}) = 0$, and is usually described by the set of failed components. (1.5)

A minimum path \underline{x} is a path for which there exists no \underline{y} such that

$$\underline{y} < \underline{x} \text{ and } f(\underline{y}) = 1. \quad (1.6)$$

A minimum cut \underline{x} is a cut for which there exists no \underline{y} such that

$$\underline{y} > \underline{x} \text{ and } f(\underline{y}) = 0. \quad (1.7)$$

The length of a path is the number of components which are working.

The length of system is the minimum length of its paths.

Define the following vectors as

$\underline{a} = (a_1, a_2, \dots, a_n)$ is vector of path numbers of lengths $=1, 2, \dots, n$.

$\underline{c} = (c_{k1}, c_{k2}, \dots, c_{kn})$ is vector of path numbers for a k 'out'of' n system.

Let \underline{a}^* be the vector of path numbers for the dual structure of the structure whose vector of path numbers is \underline{a} .

Comment

For considerable period it was assumed that non-coherent structures were implausible, however many safety systems work in such way that they are often non-coherent. A simple example of a non-coherent system is given in Bendell [1982] where an object is balanced by 4 springs, if any one fails the system fails, if two opposite fail then the system still works though possibly less safely. The main reason for studying coherent systems appears to be that their monotonic nature allows the development of results which cannot be established for general systems. The final section of this chapter examines the possible extension of the results to non-coherent systems.

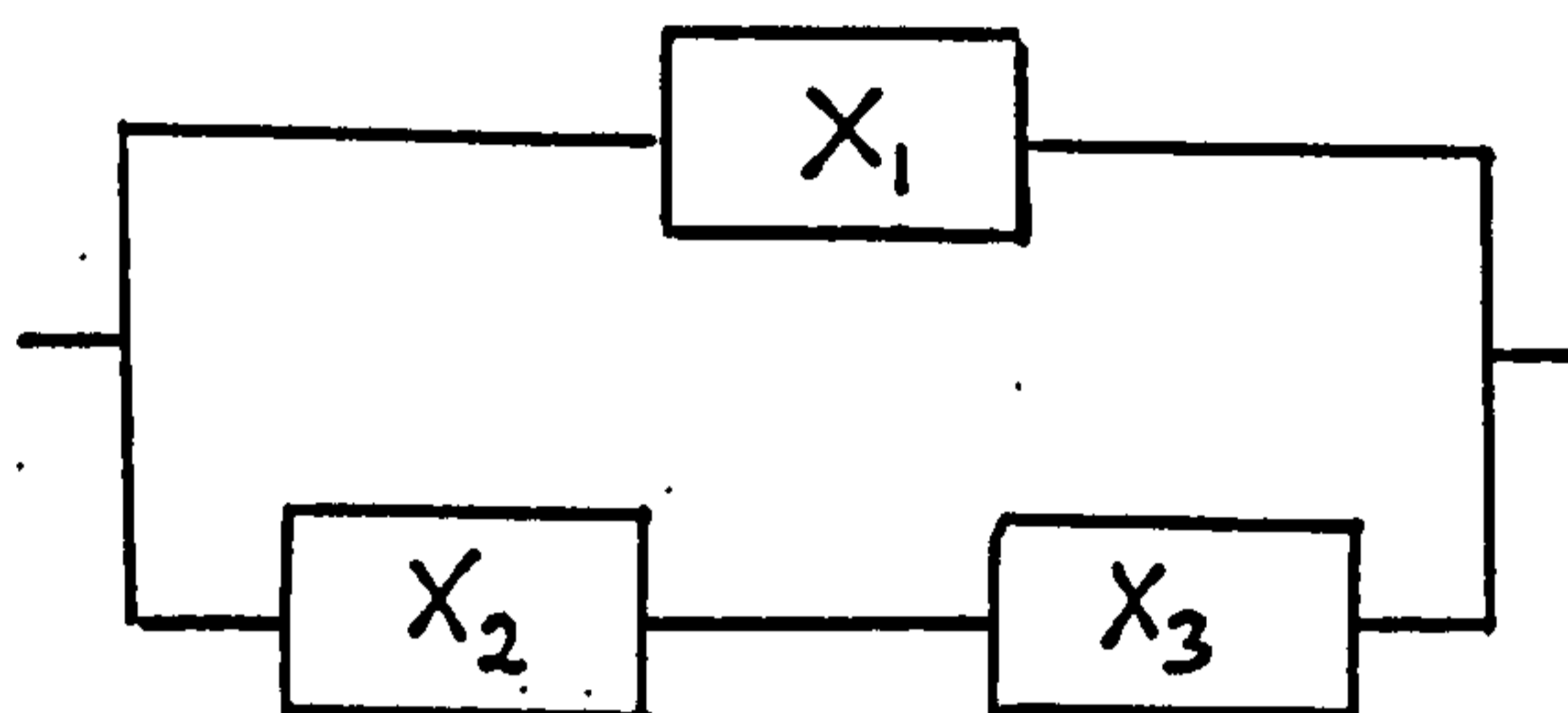
1.3 A characterization of coherent structures

Several representations of Coherent Structures are plausible. The most appealing visually is to describe the states of the components as a

lattice, then a coherent system will be a division of the nodes of the lattice into two connected regions. More usually the coherent structure is described in terms of its structure function which will be a polynomial of the states of the the components. As an example $f(x) = x + x x - x x x$ is the structure function for the system given in figure 1.1. The set of polynomials is restricted, but has so far not been totally delineated.

Figure 1.1

A two terminal coherent system.



An alternative description is via paths of the system or more compactly via minimum paths. If the system is assumed to be coherent then the set of minimum paths totally describe the performance of the system, since it will be possible to derive all the working states of the system. If the components were not labelled then it would be possible to reduce the description to the vectors of path numbers of lengths, $l=1,2,\dots,n$. Hence these vectors would yield all the basic structural forms. Birnbaum et al [1961] established some basic results about the vectors of path number of lengths, which are;

$$(1) \quad c_{kj} = \begin{cases} \binom{n}{j}, & \text{for } k \leq j \leq n \\ 0, & \text{otherwise.} \end{cases} \quad (1.8)$$

(2) For any \underline{a} , there is an integer ℓ (the length of the system),
 $1 \leq \ell \leq n$, such that

$$a_j = 0, \text{ for } j = 1, 2, \dots, \ell - 1. \quad (1.9)$$

$$0 < a_j \leq \binom{n}{j} \text{ for } j = \ell, \ell + 1, \dots, n$$

$$(3) \quad a_j + a_{n-j}^* = \binom{n}{j} \text{ for } j = 1, 2, \dots, n \quad (1.10)$$

$$(4) \quad \frac{a_j}{\binom{n}{j}} \geq \frac{a_{j-1}}{\binom{n}{j-1}} \text{ for } j = \ell, \ell + 1, \dots, n \quad (1.11)$$

They also established that

$$(5) \quad \text{The dual of a coherent system is also coherent.} \quad (1.12)$$

Whilst these results do restrict the class of structure functions quite significantly it does not totally define the class. The problem of delineation has had a long history. In Pure Mathematics the primary motivation has been the enumeration of the number of coherent structures.

Dedekind [1897] formulated the problem in terms of a distributive lattice. Several other authors have considered the problem since, (Church [1947,1965] and Hansel [1966]). In the next chapter a recursive bound is derived, which is an improvement on Hansel's bound.

Given that the basic structural forms can be derived from the vector of path number of lengths the following theorem gives further insight to the nature of coherent structures by describing their path structure in terms of the k -out-of- n structures.

Proposition 1.3.1

$$\underline{a} = \sum_{k=1}^n W_k \underline{c} = \sum_{k=\ell}^n W_k \underline{c} \quad (1.13)$$

where

$$W_k = \begin{cases} 0 & , \text{ for } k=1,2,\dots,\ell-1, \\ \frac{\binom{n}{k} - \binom{n}{k-1}}{\binom{n}{k}} & , \text{ for } k=\ell,\ell+1,\dots,n. \end{cases} \quad (1.14)$$

$$0 \leq W_k \leq 1, \text{ for all } k > \ell \text{ and } \sum_{k=1}^n W_k = 1.$$

Proof.

By elementary linear algebra since \underline{a} has the form in (1.7) and \underline{c} has the form in (1.8) \underline{a} can be expressed as a linear combination of \underline{c}

for $k=e, e+1, \dots, n$ as follows,

$$\underline{a} = \sum_{k=e}^n W_k c_k = \sum_{k=e}^n W_k c_k \quad (1.14)$$

where $W_k = 0$ for $k=1, 2, \dots, e-1$. It remains to establish that the properties for W_k , $k=1, \dots, n$

From (1.12),

$$a_e = \sum_{k=e}^n W_k c_{ke} = W_e \binom{n}{e}$$

since

$$a_{e-1} = 0, \quad W_e = \frac{a_e}{\binom{n}{e}} = \frac{a_{e-1}}{\binom{n}{e}} + \frac{a_{e-1}}{\binom{n}{e-1}} \quad (1.15)$$

Also

$$\begin{aligned} a_{e+1} &= \sum_{k=e}^n W_k c_{ke+1} = W_e c_{ee+1} + W_{e+1} c_{e+1e+1} \\ &= W_e \binom{n}{e+1} + W_{e+1} \binom{n}{e+1} = \frac{a_e}{\binom{n}{e+1}} + W_{e+1} \binom{n}{e+1} \end{aligned}$$

Therefore,

$$W_{e+1} = \frac{a_{e+1}}{\binom{n}{e+1}} - \frac{a_e}{\binom{n}{e}} \quad (1.16)$$

By induction proceed to prove that

$$W_k = \frac{a_k}{\binom{n}{k}} - \frac{a_{k-1}}{\binom{n}{k-1}}, \text{ for all } k=l, l+1, \dots, n. \quad (1.17)$$

Suppose it holds for $k=l, l+1, \dots, i$ where $i < n$. Then by (1.17)

$$\begin{aligned} a_{l+1} &= \sum_{k=l}^n W_k c_{k, l+1} = \sum_{k=l}^{l+1} W_k c_{k, l+1} \\ &= \sum_{k=l}^i W_k c_{k, l+1} + W_{l+1} + W_{l+1} \binom{n}{i+1} \\ &= \binom{n}{i+1} \sum_{k=l}^i \left[\frac{a_k}{\binom{n}{k}} - \frac{a_{k-1}}{\binom{n}{k-1}} \right] + W_{l+1} \binom{n}{i+1} \\ &= \binom{n}{i+1} \left[\frac{a_i}{\binom{n}{i}} - \frac{a_{l-1}}{\binom{n}{l-1}} + W_{l+1} \right] \\ &= \binom{n}{i+1} \left[\frac{a_i + W_{l+1}}{\binom{n}{i}} \right], \text{ since } a_{l-1} = 0. \end{aligned}$$

Therefore

$$W_{\ell+1} = \frac{a_{\ell+1}}{\binom{n}{\ell+1}} - \frac{a_{\ell}}{\binom{n}{\ell}}.$$

Thus if (1.17) holds for all $k=\ell, \ell+1, \dots, i$ it also holds for $k=i+1$.

Since by (1.15) and (1.16) it holds for $k=1, 1+1$ it holds for $k=1+2$, and so on for all $k=\ell, \ell+1, \dots, n$.

By (1.17) and (1.8) then for all $k=\ell, \ell+1, \dots, n$

$$W_k = \frac{a_k}{\binom{n}{k}} - \frac{a_{k-1}}{\binom{n}{k-1}} \leq 1$$

Similarly, by (1.17) and (1.10)

$$W_k = \frac{a_k}{\binom{n}{k}} - \frac{a_{k-1}}{\binom{n}{k-1}} \geq 0$$

Also,

$$\begin{aligned} \sum_{k=1}^n W_k &= \sum_{k=\ell}^n W_k = \sum_{k=\ell}^n \left[\frac{a_k}{\binom{n}{k}} - \frac{a_{k-1}}{\binom{n}{k-1}} \right] \\ &= a_n \frac{\binom{n}{n}}{\binom{n}{n}} - a_{\ell-1} \frac{\binom{n}{\ell-1}}{\binom{n}{\ell-1}} = a_n \frac{\binom{n}{n}}{\binom{n}{n}} = 1. \end{aligned}$$

A revised version of the proof appeared in Ansell and Bendell [1982] following the suggestion of a referee. The referee suggested writing

$$\sum_{k=1}^j w_k = \frac{a_j}{\binom{n}{j}}, \quad \text{for } 1 \leq j \leq n,$$

so that immediately

$$\sum_{k=1}^n w_k = 1 \quad \text{and hence } 0 \leq w_k \leq 1 \text{ for } 1 \leq k \leq n.$$

Then

$$\begin{aligned} \underline{a} &= (a_1, a_2, \dots, a_n) \\ &= \left(w_1 \binom{n}{1}, (w_1 + w_2) \binom{n}{2}, \dots, \sum_{k=1}^j w_k \binom{n}{j}, \dots, \sum_{k=1}^n w_k \binom{n}{n} \right) \\ &= \sum_{k=0}^n w_k \underline{c}_k = \sum_{k=1}^n w_k \underline{c}_k. \end{aligned}$$

The proof establishes that coherent structures are convex combinations of k -out-of- n systems.

The weights w_k prove as elusive to delineate as the structures themselves unfortunately. Fortunately the following corollary reduces the computation of possible structures slightly.

1.3.2 Corollary

$$\underline{a}^* = \sum_{k=0}^n w_k^* \underline{c}_k^* = \sum_{k=0}^n w_{n-k+1}^* \underline{c}_k^* \tag{1.18}$$

Proof

By Proposition 1.3.1, for $k=1, 1+1, \dots, n$

$$\begin{aligned}
 W_k^* &= \frac{a_k^*}{\binom{n}{k}} - \frac{a_{k-1}^*}{\binom{n}{k-1}} \\
 &= \frac{\left[\binom{n}{k} - a_{n-k} \right]}{\binom{n}{k}} - \frac{\left[\binom{n}{k-1} - a_{n-k-1} \right]}{\binom{n}{k-1}} \quad \text{by (1.9)} \\
 &= \frac{a_{n-k+1}}{\binom{n}{k-1}} - \frac{a_{n-k}}{\binom{n}{k}} = W_{n-k+1}
 \end{aligned}$$

Examples 1.3.1

Considering 4 component systems table 1.1 gives a list of the 4 k -out-of- n systems and 13 distinct coherent structures with their associated weights. One can immediately see that corollary 1.3.2 holds.

Obviously the weights associated with a system will indicate if a system might be coherent or not. A system of 5 components with path lengths $(1,10,5,5,1)$ has weights $(1/5, 4/5, -1/2, 1/2)$ which means that it is non-coherent. However, for the 4 component system with path lengths $(0,1,3,1)$ two systems might be suggested, given in table 1.2, the first

is coherent the second is not.

Table 1.1

Illustration of proposition 1.3.1 and corollary 1.3.2 for 4 identical components.

| Path numbers of size | | | | Weights | | | |
|------------------------|---|---|---|---------|-------|-------|-------|
| j = 1 | 2 | 3 | 4 | W_1 | W_2 | W_3 | W_4 |
| k-out-of-4 systems | | | | | | | |
| 4 | 6 | 4 | 1 | 1 | 0 | 0 | 0 |
| 0 | 6 | 4 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 4 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| other coherent systems | | | | | | | |
| 0 | 0 | 2 | 1 | 0 | 0 | 1/2 | 1/2 |
| 0 | 0 | 3 | 1 | 0 | 0 | 3/4 | 1/4 |
| 0 | 1 | 3 | 1 | 0 | 1/6 | 7/12 | 1/4 |
| 0 | 1 | 4 | 1 | 0 | 1/6 | 5/6 | 0 |
| 0 | 2 | 4 | 1 | 0 | 1/3 | 2/3 | 0 |
| 0 | 3 | 3 | 1 | 0 | 1/2 | 1/4 | 1/4 |
| 0 | 3 | 4 | 1 | 0 | 1/2 | 1/2 | 0 |
| 0 | 4 | 4 | 1 | 0 | 2/3 | 1/3 | 0 |
| 0 | 5 | 4 | 1 | 0 | 5/6 | 1/6 | 0 |
| 1 | 3 | 4 | 1 | 1/4 | 1/4 | 1/2 | 0 |
| 1 | 5 | 4 | 1 | 1/4 | 7/12 | 1/6 | 0 |
| 1 | 6 | 4 | 1 | 1/4 | 3/4 | 0 | 0 |
| 2 | 6 | 4 | 1 | 1/2 | 1/2 | 0 | 0 |

Table 1.2

Alternative structure functions for systems of 4 identical components
with vector of path numbers (0,1,3,1)

| Components | | | | Coherent System | Non ⁴ Coherent System |
|------------|---|---|---|-----------------|----------------------------------|
| A | B | C | D | | |
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

1.4 Identical Component Systems

Many authors have considered systems composed of identical

components, see Barlow and Proschan [1975]. Whilst it may seem very idealistic the examination of such a group might be used to yield either indications of possible results for non-identical systems or more helpfully to produce bounds for the reliability, availability or mean time to failure (MTTF) of such systems. It is assumed that identical components means stochastically identical that they have the same probabilities of failure, singularly and jointly. Hence if M is a vector of probabilities of failure for paths of lengths $j=1, \dots, n$ then the reliability of the system with path length a is given by

$$R(\underline{a}) = \underline{a} \underline{M}. \quad (1.19)$$

Hence the reliability of the system may be expressed as

$$R(\underline{a}) = \sum_{i=1}^n W_i \underline{c}_i \underline{M} = \sum_{i=1}^n W_i R(\underline{c}_i). \quad (1.20)$$

Similarly one can express availability at time t , $A(t)$, of coherent structures of identical components in terms of the availability of k out of n structures $A_k(t)$, and the mean time to failure, this gives,

$$A(T) = \sum_{k=1}^n W_k A_k(T) \quad (1.21)$$

If the probability of a component is working at T is $p(T)$ then,

$$A_k(T) = \binom{n}{k} \{p(T)\}^k \{1-p(T)\}^{n-k}$$

For the mean time to failure for system

$$MTTF = \sum_{k=1}^n W_k MTTF_k \quad (1.22)$$

where MTTF is the mean time to failure of system and MTTF_k is the mean time to failure for k-out-of-n system.

The following example is taken from Ansell [1984].

Example 1.4.1

The reliability of the systems, R_s , produced in figure 1.2, assuming identical components may be expressed in terms of the reliability of 2-out-of-5, 3-out-of-5 and 4-out-of-5 systems of identical components as

$$R_s = (1/5) R_{2/5} + (7/10) R_{3/5} + (1/10) R_{4/5}$$

The Availability of the system, $A_s(T)$, assuming the component lifetimes and repair times are independent and identical distributed exponential with rates λ and μ will be given by;

$$A_s(T) = (1/5) A_{2/5} + (7/10) A_{3/5} + (1/10) A_{4/5}$$

where $A_{k/n}$ is the availability of k-out-of-n system at time T which is

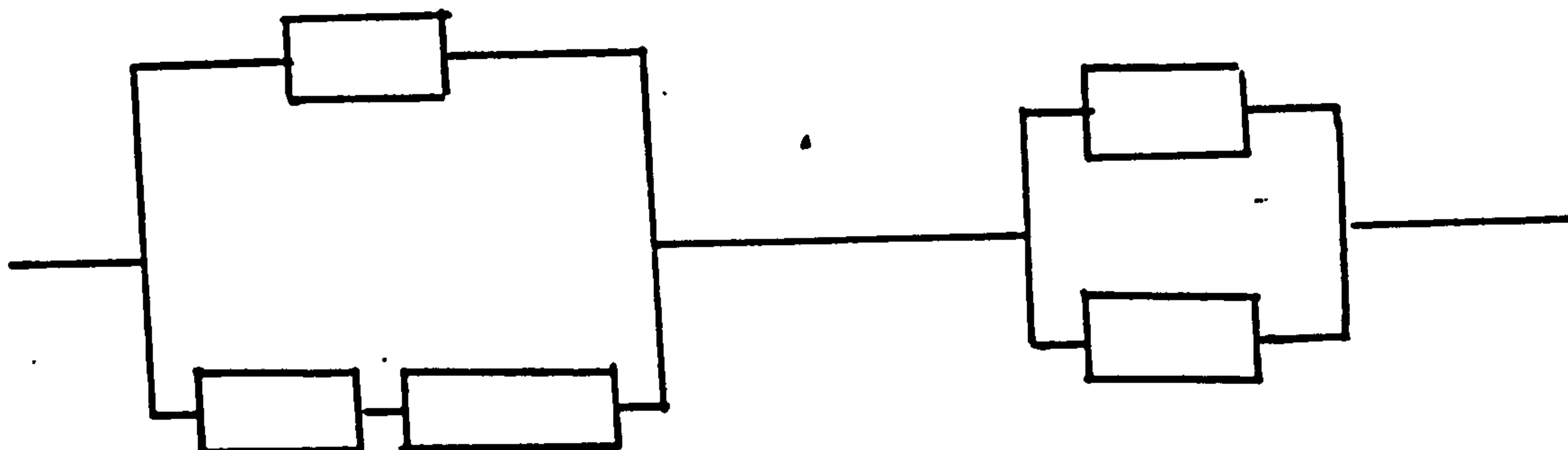
$$A_{k/n}(T) = \sum_{t=k}^n \binom{n}{t} [p(T)]^t [1-p(T)]^{n-t}$$

and

$$p(T) = \left(\frac{\mu}{\lambda + \mu} \right) + \left(\frac{\lambda}{\lambda + \mu} \right) e^{-(\lambda + \mu)T}$$

Figure 1.2

Illustration of the two terminal system used in example 1.4.1



For non-identical components it is not possible to express the reliabilities, or other measures, in such simplistic form. However, it is possible to produce bounds for the reliability. Assuming the components are independent define p_{\min} and p_{\max} to be

$$p_{\min} = \min_i \{ p_i \}$$

and

$$p_{\max} = \max_i \{ p_i \}$$

where p_i is the probability that the i th component works.

Then the reliability of the system is bounded by

$$\sum_{k=1}^n W_k R(p_{\min}) \leq R(p) \leq \sum_{k=1}^n W_k R(p_{\max}). \quad (1.23)$$

The following two sections are applications of proposition 1.3.1 in the case of identical components.

1.4.1 Failure to Operate and Failure to Idle

Proposition 1.3.1 yields a more incisive view of a result originally proved by Phillips [1980] when he establish the optimality of k -out-of- n systems composed of identical components which fail both to operate and to idle. When considering the reliability of alarms, switches and valves there is not only the possibility that they will fail to function when required but also the strong possibility of them failing to idle, (operate when not required). The spring evening is often rudely interrupted by burglar alarms, to such an extent now that the police are suggesting penalties on owners when the alarms are false. These failure to operate and failure to idle are often described as opposite failure modes.

Several authors have considered the problem of designing system which have maximum reliability in the case of opposite failure modes, Barlow and Hunter [1960], Barlow et al [1963], Lloyd and Lipow [1962], Lomnicki [1973,1977], Kopocinski [1974], Phillips [1976b] and Kaufmann et al [1977]. Phillips gave a solution to the problem, showing that k -out-of- n systems were optimal. Ben Dov [1980] presented method for finding which k -out-of- n system was optimal.

From Proposition 1.3.1 the proof that k -out-of- n systems have maximum reliability is immediately obvious, reducing to the observation that

$$R(\underline{a}) = \underline{a} (\underline{M}^k \underline{Q}) \tag{1.24}$$

$$= \sum_{k=1}^n W_k c_k (M^* Q) = \sum_{k=1}^n W_k R(c_k) \quad (1.24)$$

where \underline{Q} is the vector probabilities of failure to idle.

Hence $R(\underline{a})$ is a convex combination of $R(\underline{c}_k)$ then $R(\underline{c}_k)$ must be optimal, since $R(\underline{a})$ must be less than or equal to one of the $R(\underline{c}_k)$.

The result is more general than that in Phillips [1980] since it only requires identical components and not independent components.

1.4.2 Stress⁴Strength Models

Another application of Proposition 1.3.1 is to Stress⁴Strength Models. In such models the interest lies in whether a component or the system can survive the stress or stresses, applied across them. Stress is the force a component, or system, is subject to, its strength is its ability to withstand the stress. It is assumed in this section that both the stress and the strengths are stochastic variables. (In some engineering contexts they may be deterministic.) There is continued interest in stress⁴strength modelling in engineering, see Carter [1979], Moss [1977,1983], Sherwin and Lees [1980], at both the system and component level. In statistics there was considerable interest in the early seventies in the inferential aspects of stress⁴strength modelling of single components, see for example Church and Harris [1970], Mazumbar [1970], Downton [1973], and Windwood and Kelly [1977].

For the single component the reliability or the probability that

the component survives stress it suffers, is usually expressed as

$$R = \int_{-\infty}^{\infty} [1 - G(x)] dF(x), \quad (1.25)$$

where $G(x)$ is the distribution function for the strength of the component and $F(x)$ is the distribution function for the stress.

Obviously one can easily extend the problem to systems, with a stress being applied across the whole system. This is again a fairly old problem, an early author was Daniels [1945]. Other authors have followed considering the probabilistic modelling, notably Smith [1979,1983], Filius [1983] and Borges [1983] who obtained limiting results for the reliability of various systems, which are generically described as bundles of fibres or composite materials. These systems are assumed to consist of a series of parallel structures. The parallel structures are the bundles, the original problem referred to woollen fibres, subsequently extended to other fibres, including those of glass. The main interest lies in the breaking point of whole system. The inferential questions for simpler systems have been considered by Bhattacharyya and Johnson [1974,1977] and Draper and Guttman [1978], who restricted their interest to k -out-of- n systems.

Both types of structure above are fairly simplistic. More complex models can be contemplated but the likelihood of producing results becomes small. Essentially the problem revolves around the distribution of the stress over the components. If system is k -out-of- n the components should be treated to equal stresses hence the model should be

simple. For the fibre problems each bundle is considered separately and the stress on each bundle is assumed the same, within a bundle the distribution depends on the state of closest fibres. Here one allows a fibre to take extra strain if neighbouring fibres have failed.

In the following it is assumed that the stresses are equal over the components. Considering the k-out-of-n structures then the reliability of the system with a stress with distribution $F(x)$ and each component having strength with distribution function $G(x)$ would be

$$R_k = \sum_{t=k}^n \binom{n}{t} \int_{-\infty}^{\infty} [1-G(x)]^t [G(x)]^{n-t} dF(x). \quad -(1.26)$$

From proposition 1.3.1, for a general coherent system the reliability would be

$$R_S = \sum_{j=1}^n w_j \sum_{t=j}^n \binom{n}{t} \int_{-\infty}^{\infty} [1-G(x)]^t [G(x)]^{n-t} dF(x). \quad -(1.27)$$

From this one can suggest bounds for performance if the stress is not equally distributed. Ansell [1984] suggests a similar approach to that of non-identical components, so that it is possible to produce bounds at least for the reliability.

Define $G_{\max}(x) = \sup (G_i(x) : i = 1, \dots, n)$ and $G_{\min}(x) = \inf(G_i(x) : i=1, \dots, n)$, then the reliability of the system, R , of non-identical, but independent variables would be bounded as follows,

$$\sum_{j=1}^n w_j \sum_{t=j}^n \binom{n}{t} \int_{-\infty}^{\infty} [1-G_{\min}(x)]^t [G_{\min}(x)]^{n-t} dF(x) \leq R$$

$$R \leq \sum_{j=1}^n w_j \sum_{t=j}^n \binom{n}{t} \int_{-\infty}^{\infty} [1 - G_{\max}(x)]^t [G_{\max}(x)]^{n-t} dF(x)$$

Example 1.4.2.1

Assuming are the components have exponential strengths with parameters θ_j then

$$G_{\max} = G_v \text{ provided } \theta_v \leq \theta_j \text{ for all } v \neq j$$

$$G_{\min} = G_u \text{ provided } \theta_u \geq \theta_j \text{ for all } u \neq j$$

Hence if $F(x)$ is also exponential then

$$1^{\lambda} \sum_{j=1}^n w_j \prod_{d=1}^{\lambda} [t / (\lambda_1 + t)] \leq R_S \leq 1^{\lambda} \sum_{j=1}^n w_j \prod_{d=1}^{\lambda} [t / (\lambda_2 + t)]$$

where

$$\lambda_1 = \theta_F / \theta_u \text{ and } \lambda_2 = \theta_F / \theta_v .$$

As indicated above several authors have considered inferential aspect of stress-strength models for the k -out-of- n structures, because coherent structures can be characterized as convex combination of k -out-of- n structures their results can be immediately extended. The following examples illustrate this.

1.4.2.2 Examples

Assuming the distribution for both stress and strengths are exponential then the analysis employed for the case of coherent

structures of identical components will follow those previously employed in Bhattacharyya and Johnson [1974,1977] and Draper and Guttman [1978]. If it is assumed that there is a sample of m stresses, X_i , for $i=1, \dots, n$ and sample of d strengths, Y_j , for $j=1, \dots, d$, where X 's and Y 's have parameters θ_1 and θ_2 respectively then the estimators under the three methods of inference are presented below. Let $\lambda = \theta_1 / \theta_2$.

Parametric Inference

Since the reliability of a coherent function is a monotonic function of λ then one obtains the following minimum variance unbiased estimator for the reliability of the system,

$$1 - \tilde{R}_S = \sum_{j=1}^n \frac{W_j D(j)}{\beta [j, n-j+1]}$$

where

$$D(j) = \sum_{t=\Gamma_1}^{\Gamma_1} (-1)^t \binom{n-t}{t} (j+t)^{-1} \left\{ 1 - {}_2F_1 [1-d, 1; m; (j+t)V] \right\} \\ + \sum_{t=\Gamma_2}^{n-j} (-1)^t \binom{n-j}{t} (j+t)^{-1} {}_2F_1 [1-m, 1; d; (j+t)^{-1} V^{-1}]$$

and where

$F [a, b; c; x]$ is the hypergeometric function defined as

$$F [a, b; c; x] = \frac{(c-1)!}{(b-1)!(c-b-1)!} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a} dt$$

$$V = T_1 / T_2, \quad T_1 = \sum_{i=1}^n X_i, \quad T_2 = \sum_{j=1}^d Y_j$$

$$\Gamma_1 = \text{integer part } [\min (n-j, V - j)]$$

and

$$\Gamma_2 = \text{integer part } [\max (0, V - j)].$$

Some simplification may be possible in particular cases.

Fortunately it is found that the asymptotic estimator performs reasonably even for moderate m and d and this is given by;

$$1^{\wedge}R_S = \sum_{j=1}^n w_j \prod_{t=j}^n \{t/(t+\hat{\lambda})\}$$

where

$$\hat{\lambda} = \bar{Y}/\bar{X}$$

Non-parametric Inference

The minimum variance unbiased estimator is given by

$$R^* = \sum_{j=1}^n w_j \left[\binom{m}{n} \right]^{-1} \sum_{l=n-j+1}^{d-j+1} \binom{l-1}{n-j} \binom{d-1}{j-1} (S_{(i)}^{j-1})$$

where $S_{(i)}$, $i=1, \dots, d$ are the ordered Y ranks in combined sample.

Bayesian Inference

Using the following prior,

$$p(\theta_1, \theta_2) \propto \theta_1^{a_1-2} \theta_2^{a_2-2} \theta_1, \theta_2 > 0 \text{ and } a_1, \text{ and } a_2 \text{ integer,}$$

the estimator of R_S will be given by evaluating

$$E[R_S(\underline{x}, \underline{y})] = 1 \cdot \sum_{j=1}^n w_j \left[\prod_{t=j}^n t \right] E \left[\prod_{t=j}^n (\lambda + t)^{-1} \mid \underline{x}, \underline{y} \right]$$

where λ has the posterior distribution

$$\lambda \sim \hat{\lambda}^K F_{2(m+a_1-1), 2(d+a_2-1)}$$

and

$$K = (m+a_1-1) d / m (d+a_2-1) .$$

1.5 Fault Tree Analysis

Coherent structures are a 'reasonable' class of possible structures and their development has given considerable insight into system performance. They are not the only class of plausible structures which many authors still fail to realize, see chapter 2. Obviously when faced with a 'real' system the need is to quantify its performance whether it be coherent or not. The example of systems which fail to operate and idle indicate the likelihood that a component may appear in the description of a system both as itself and itself negated. Fault Tree Analysis underlines this.

Originally, Fussell [1970], considered only coherent systems, systems in which the failure of the component could only degrade the system. This is equivalent to saying the boolean expression representing the system does not contain negation. However, as the approach was applied to a larger number of systems, it was soon perceived that negation was need, see Lapp and Powers[1977] and Locke [1979].

This primarily arose in the context of safety systems when there is a use of 'exclusive or'. A logical example of the 'exclusive or' is A and not B, and not A and B. For a system this would mean a problem

would arise if a certain event occurs and a related event does not occur and vice versa. For example heating of a system, event A, may be acceptable if a heater is working, event B, otherwise not.

At present there are Fault Tree Packages which may cope not only with negation, but also other forms of complex trees, such as sequential trees and trees with looping. Sequential trees are trees in which the order of components failure is taken into account, hence if component A fails then component B the system fails but if B and then A the system does not. A tree which has looping is one in which information over a sequence of events feeds back into the tree in a recursive manner, important in the area of chemical process plants, see Lapp and Powers [1977] and Andow [1980].

The objective of a fault tree analysis is to deduce from a listing, or tree, of events the simplest description of the system. Hence the analysis hopes to reduce the description to the minimum set of components referred to as the Prime Implicants. If initial description contains no negation in any form then the Prime Implicants will be the minimal path set, which will be a unique representation of the system. Several algorithms exist for the elucidation of the Prime Implicants this based usually on Quine-McClusey theory, (Quine [1952,1955], McClusey [1956]), but alternatives have been produced see Weber [1983].

However if negation is allowed in whatever form then there is no unique representation. Also the concept of the minimum cut set is no longer is useful, since it is not immediately possible to derive the structure function from them. Algorithms have been developed to derive

the minimum description of the systems in these contexts, but none are ideal, see Kumamoto and Henley, [1978] and Locke [1979]. There also has been interest in non-dichotomic Fault Tree Analysis, see Caldorolla [1980b] and Ogunbiyi and Henley [1981].

In practice problems arise with the use of fault trees because of the size of trees and hence the expressions encountered. Any realistic system would swamp the ability of any of the packages available. In fact Rosenthal [1975] showed the problem of finding the complete minimal cut set family associated with any tree is a member of the class of NP-complete problems. This means it is not possible to place polynomial bounds, in the number of components, on the run time required to find the prime implicants. The practitioner therefore tends to reduce their system to manageable size. Also restrictions are placed on the size or probability associated with the set of minimal cuts.

The restriction imposed can lead to misleading results. Modarres and Dezfuli [1984] show that the reliability obtained by restrictions either of size of cut set or in the probability associated with the set of minimal cuts can be far from the true value. They suggest an alternative approach which they refer to as combined truncation which considers both criteria simultaneously. Their algorithm has the advantage of being reasonably fast and accurate, for the small fault tree they have considered.

There are a number of other practical problems such as the identification of components and their likely modes of failures. Further simplistic probabilistic modelling assumptions are usually made

in Fault Tree Analysis, which indicates that further work is required. For further details of such problems associated with fault trees, see Ansell and Bendell [1985].

In conclusion the fault tree approach has firmly established that whilst coherent structures are pleasant they do not encompass the set of all plausible systems. The last section of this chapter therefore considers extensions of previous results to non-coherent systems.

1.6 Characterizing Non-coherent Systems

The removal of the assumption of coherency from studies of structures leaves little to say generally about the systems. Of the results of Birnbaum, Esary and Saunders [1961] only (1) and (3) hold and (2) respecified as

$$0 \leq b_j \leq \binom{n}{j} \text{ for } j=1, \dots, n. \quad -(1.28)$$

Proposition 1.3.1 and corollary 1.3.2 can be modified, so that W is such that

$$-1 \leq W_k \leq 1$$

and $\sum_{k=1}^n W_k = 1$ if system operates when all components operates
 $\sum_{k=1}^n W_k = 0$ if system fails if all components operate.

Proof:

The proof follows that of proposition 1.3.1 and corollary 1.3.2 except that

$$1. \sum_{k=1}^n W_k = b_n = \begin{cases} 1, & \text{if the system operates if all the} \\ & \text{components operate} \\ 0, & \text{otherwise.} \end{cases}$$

2. Since by (1.28) b_k can be equal to zero for some $k > 0$, it follows that

$$W_k = \frac{b_k}{\binom{n}{k}} - \frac{b_{k-1}}{\binom{n}{k-1}} \text{ can be negative.}$$

However by (1.9) $-1 \leq W \leq 1$.

The result is that the useful property of the characterization is lost and hence the representation as a convex combination does not hold, which typifies the problem of dealing with this class of structures. Hence no longer can one use the proposition to establish the optimality of k-out-of-n structures for systems which fail to operate or to idle. In the case of stress-strength models the results fail again because the functions are no longer monotonic in λ . There seems little reason therefore to persevere with the generality of such structures but again consider specific structures.

Chapter 2

Multilevel Systems

2.1 Introduction

The term Multilevel in this chapter is used as opposed to Multistate to highlight the deterministic view of the system. Hence in this chapter the emphasis is primarily on definition of performance given components are at particular states or levels. The next chapter consider the stochastic elements.

The generalization from the dichotomic models for systems performance to multilevel models has brought with it a plethora of definitions. The main concern being the definition of coherent structures. The differences amongst these definitions of coherency relates to the generalization of the definition of relevancy of the components in the dichotomic case. (For definition of relevancy in the dichotomic case see Chapter 1, section 1.2.) The first section of the chapter reviews the definitions presented in the recent statistical literature. The chapter then proceeds to concentrate on the definitions of coherency which do not involve relevancy as these give a clear view of the structures considered. It might be argued following the last section of the previous chapter that concentrating again on coherent structures is very narrow. It should be stressed though that it is useful to consider a group which may give at least some insight into the performance of more general structures. In the later parts of the

chapter consideration is given to non-coherent models for system performance.

2.2 Definitions of Coherency in Multilevel Systems

The following notation is used throughout the remainder of the chapter.

Let the number of states of the system and the components be $M+1$, with the states being $M, M-1, \dots, 1, 0$. M will be the level of perfect operation of the component or system and 0 will be the failed state.

Define $\underline{x} = (x_1, x_2, \dots, x_n)$ be the state vector of the n -components X_1, X_2, \dots, X_n , where x_α , the level of the α th component, may be any one of the levels $0, 1, \dots, M$.

The state of system will be described in terms of the components by $f(\underline{x})$, the structure function of the system, which will be a mapping from $\{0, 1, \dots, M\}$ to $\{0, 1, \dots, M\}$.

Let $\underline{j} = (j, j, \dots, j)$ for $j=0, 1, \dots, M$.

Define the partial ordering $\underline{x} \leq \underline{y}$ as

$$\underline{x} \leq \underline{y} \text{ if and only if } x_i \leq y_i \text{ for } i=1, 2, \dots, n. \quad (2.1)$$

A system is monotonic if and only if $f(\underline{x})$ is non-decreasing in $\underline{x} > 0$. A system which is not monotonic in the above sense, is called non-monotonic. -(2.2)

The component binary image at minimum level j , $j=1, \dots, M$ is given

by

$$S_j(\underline{x}) = \{S_j(x_1), S_j(x_2), \dots, S_j(x_n)\} \quad (2.3)$$

where

$$S_j(x_i) = \begin{cases} 1, & \text{if } x_i \geq j \\ 0, & \text{if } x_i < j. \end{cases}$$

A state vector \underline{x} is a path at minimum level j , $j=1, 2, \dots, M$, if and only if

$$S_j[f(\underline{x})] = 1 \quad (2.4)$$

A state vector \underline{x} is a cut at minimum level j , $j=1, 2, \dots, M$, if and only if

$$S_j[f(\underline{x})] = 0 \quad (2.5)$$

Analogous to the dichotomic case, a path at minimum level j is a minimal path at minimum level j ($j=1, 2, \dots, n$) if and only if

$$S_j[f(\underline{y})] = 0 \text{ for all } \underline{y} \prec \underline{x}. \quad (2.6)$$

A path at minimum level j is of size l ($l = 1, 2, \dots, M$) if and only if

$$\sum_{\alpha=1}^n S_j(x_\alpha) = l. \quad (2.7)$$

For a coherent system with a well-defined binary image, denote the number of paths at minimum level j of size l , which have distinct binary

images at minimum level j , by $L_{\underline{j}}$.

The following set of definitions of multilevel coherency does not explore all possible generalizations of the dichotomic definition of coherency, but those which have received most attention in the recent statistical literature. For each given definition the author(s) are given in brackets.

Definition 2.1 (Ansell, Bendell and Humble [1981])

Let $f(\underline{x})$ be monotonic.

Define $f(\underline{x})$ to be a coherent system in the wide sense if and only if $f(\underline{0})=0$ and $f(\underline{M})=M$.

Definition 2.2 (Ansell, Bendell and Humble [1981])

Let $f(\underline{x})$ be monotonic.

Define $f(\underline{x})$ to be a coherent system in the narrow sense if and only if $f(\underline{j}) = j$, for all $j=0,1, \dots, M$.

Definition 2.3 (Ansell and Bendell [1986])

A wide sense coherent system $f(\underline{x})$ is said to have a well defined binary image if and only if

$$S_j(\underline{x}) = S_j(\underline{y}) \text{ for any } j$$

implies that

$$S_j[f(\underline{x})] = S_j[f(\underline{y})]$$

Definition 2.4 (Ansell and Bendell [1986])

A coherent system with structure function $f(\underline{x})$ and a well defined binary image has a constant binary image if and only if the binary image at minimum level j of any minimal path at minimum level j is also the binary image at minimum level i of an equivalent minimal path at minimum level i for all $i, j=1,2, \dots, M$.

Definition 2.5 (Barlow and Wu [1978])

$$f(\underline{x}) = \max_{1 \leq r \leq p} \min_{i \in P_r} x_i$$

or equivalently,

$$f(\underline{x}) = \min_{1 \leq s \leq k} \max_{i \in K_s} x_i,$$

where P_1, \dots, P_p are the minimum path sets and K_1, \dots, K_k the minimum cut sets of a corresponding (one to one) dichotomic coherent system of the Barlow and Proschan [1975] type. It follows that P_1, \dots, P_p are non-empty subsets of $\underline{C} = \{1, 2, \dots, n\}$ such that

$$\bigcup_{r=1}^p P_r = \underline{C}$$

and

$$P_i \cap P_j = \emptyset, i \neq j.$$

Also

$$\bigcup_{S=1}^k K_S = \underline{C}$$

and

$$K_i \not\subseteq K_j, i \neq j.$$

This definition will be referred to as the BW definition in the remainder of the chapter.

Definition 2.6 (El-Newehi, Proschan and Sethuraman [1978])

Let $f(\underline{x})$ be monotonic.

Define $f(\underline{x})$ to be a Multistate Coherent system if and only if

$$f(\underline{j}) = j \text{ for } j=0,1, \dots, M,$$

and for every level j of the component i , there exists a vector $(x_i=j, \underline{x})$ such that

$$f((x_i=j, \underline{x})) = j$$

while $f((x_i=l, \underline{x})) \neq j$ for $l \neq j$, $i=1, \dots, n$ and $j=0,1, \dots, M$.

[The definition of $(x_i=j, \underline{x})$ is the obvious extension of that given in section 1.2]

Definition 2.7, 2.8 and 2.9 (Griffiths [1980])

Let $f(x)$ be monotonic.

Define $f(x)$ to be a monotone multistate system if and only if

$$\min_i f(\underline{x}_i) \leq f(\underline{x}) \leq \max_i f(\underline{x}_i)$$

Let $f(\underline{x})$ also be a monotone multistate system.

- (a) Define $f(\underline{x})$ to be a strongly coherent system if and only if for any component i and state j , there exist \underline{x} such that

$$f((x_i=j, \underline{x})) = j$$

while $f((x_i=l, \underline{x})) \neq j$ for $l \neq j$.

- (b) Define $f(\underline{x})$ to be a coherent system if and only if for any component i and state $j > 1$, there exists \underline{x} such that

$$f((x_i=j-1, \underline{x})) < f((x_i=j, \underline{x}))$$

- (c) Define $f(\underline{x})$ to be a weakly coherent system if and only if for any component i and state j , there exists \underline{x} such that

$$f((x_i=j, \underline{x})) \neq f((x_i=l, \underline{x})) \text{ for some } l \neq j.$$

Definition 2.10 (Butler [1979])

Let $f(\underline{x})$ be monotonic.

Define $f(\underline{x})$ to be multistate coherent system if and only if for every $i = 1, 2, \dots, n$ there exists an \underline{x} such that

$$f((x_i=M, \underline{x})) > f((x_i=0, \underline{x})),$$

$$f(\underline{0}) = 0 \text{ and } f(\underline{M}) = M.$$

Definition 2.11 (Natvig [1982])

Let $f(\underline{x})$ be monotonic.

Define $f(\underline{x})$ to be a multistate coherent system of type 1 (denoted as MCS I) if and only if for all $i = 1, 2, \dots, n$ and $j = 0, 1, \dots, M$ there

exist (x_i^*, \underline{x}) such that

$$f((x_i=j, \underline{x})) \geq j,$$

$$f(x_i=j+1, \underline{x}) \leq j+1$$

and

$$f(\underline{j}) = j.$$

Definition 2.12 (Natvig [1982])

A system of n components is said to be a multistate coherent system of type 2 (denoted by MCS II) if and only if there exists a dichotomic structure function $f_j, j=1, 2, \dots, M$ such that the structure function

$$f(\underline{x}) \geq j \Leftrightarrow f_j(S_j(\underline{x})) = 1.$$

A number of other definitions primarily in the engineering literature exist referring usually to specific areas of application; for example in the nuclear industry Caldarola [1980a, 1980b], Fardis and Cornell [1981] and Amesz, Gorriba and Volta [1977], in production systems Virtanen [1977], in transportation systems and water systems Barlow [1978], and in electrical power systems, communication systems and computer systems Cavers [1975], Chou and Abraham [1980], Sui and Chan [1978], and Livini and Bar-Ness [1978]. There are also sets of definition which have the system and components with differing levels of performance. Hirsch, Meisner and Boll [1968] and Simon [1969, 1970, 1972] have the system performing at multilevel whilst the components are still dichotomic, and Hochberg [1973] and Fardis and Cornell [1981] more generally allow number of levels to vary between components and the system.

The definitions 2.1 to 2.4 are generated directly from Birnbaum, Esary and Saunders [1961], whilst the definitions 2.5 to 2.12 with their relevancy assumptions follow Barlow and Proschan [1975]. It is the emphasis on relevancy which seems to obscure the insight into the types of system being considered. Before delving deeply into the structural differences one ought to give a brief account of the equivalency of the definitions ignoring, of course, relevancy, the order will be from least restrictive to most. Wide sense is related directly to Hochberg [1973] and Butler [1979], and is similar to Block and Savitts [1981]. The set of narrow sense coherent systems contains all the Griffith's definitions. As Griffiths [1980] indicates the strongly coherent systems are identical to El-Newehi, Proschan and Sethuraman's multistate coherent systems. The set of well defined binary image coherent systems are equivalent to Natvig's MCS II. Whilst Barlow and Wu's and Natvig's MCS I are contained within the set of coherent systems with constant binary image.

In the following section the differences between the structural definitions will be considered and a hierarchy will be developed. This hierarchy will also be extended to non-coherent system.

2.3 Structural relationships

It is immediately obvious that wide sense coherent systems contain the narrow sense coherent systems, and those structures with well defined binary images contain those with constant binary image. The full ordering is as follows;

wide sense \subset narrow sense \subset well defined binary images \subset constant binary images

The orderings are formally established in the following propositions and examples. It should be noted that the well defined binary image does not necessarily imply narrow sense, this has to be established.

Proposition 2.3.1

For a wide sense coherent system with a well defined binary image the minimal paths at minimum level j , $j=1,2,\dots,M$ are of the form

$$\begin{cases} x_i = j, i \in J \\ x_i = 0, i \notin J. \end{cases} \quad (2.8)$$

where J is a non-empty subset \underline{C} .

Proof

Consider the state vector \underline{y} such that $f(\underline{y}) = j$. Then any vector \underline{x} which satisfies

$$S_j(\underline{x}) = S_j(\underline{y}) \quad (2.9)$$

will have $f(\underline{x}) \geq j$.

Let

$$\underline{x} = j S_j(\underline{y}),$$

which is equivalent to (2.8) and satisfies (2.9). Since $\underline{x} \leq \underline{y}$ it follows from the coherency of f that

$$f(\underline{x}) = j$$

and

$$S_j[f(\underline{x})] = 1.$$

Hence any $\underline{y} \neq \underline{x}$ is not a minimal path at level j .

Now consider \underline{z}_i which is identical to \underline{x} except $x_i = j - 1$ for an $i \in J$.

Denote $f(\underline{z}_i) = \ell_i$. By coherency $\ell_i \leq j$. Further, since the system has a well defined binary image.

$$S_{j-1}[f(\underline{z}_i)] = S_{j-1}[f(\underline{x})] = 1$$

and

(2.10)

$$S_{j+1}[f(\underline{z}_i)] = S_{j+1}[f(\underline{x})] = 0.$$

The equalities (2.10) will hold if and only if $\ell_i = j-1$ or j . Thus either $\ell_i = j$ for all $i \in J$, or $\ell_i = j-1$ for one or more i . In the latter case \underline{x} is a minimal path at minimum level j .

Proposition 2.3.2

All wide sense coherent systems with a well defined binary image are narrow sense coherent systems.

Proof

From Proposition 2.3.1, $f(\underline{j}) \geq j$. Also, since f has a well defined binary image and for any $j = 0, 1, \dots, M$,

$$S_{j+1}(\underline{j}) = S_{j+1}(\underline{0}) = 0,$$

it follows that

$$S_{j+1}[f(\underline{j})] = S_{j+1}[f(\underline{0})] = 0,$$

which implies $f(\underline{j}) < j+1$. Thus $f(\underline{j}) = j$ for all $j = 0, 1, \dots, M$.

Proposition 2.3.2 yields the result that the class of coherent systems with well defined binary images is contained within the narrow sense coherent structures, the following example shows that the classes are not identical.

Example 2.3.1

A narrow sense coherent system that does not have a well defined binary image.

Consider the following system composed of two three-level components given in table 2.1.

Table 2.1

| \underline{x} | $f(\underline{x})$ | $S_2(\underline{x})$ | $S_2[f(\underline{x})]$ | $S_1(\underline{x})$ | $S_1[f(\underline{x})]$ |
|-----------------|--------------------|----------------------|-------------------------|--|-------------------------|
| 0 0 | 0 | 0 0 | 0 | 0 0 | 0 |
| 0 1 | 0 | 0 0 | 0 | $\left\{ \begin{array}{l} 0 1 \\ 0 1 \end{array} \right\}^*$ | 0 |
| 0 2 | 2 | 0 1 | 1 | | 1 |
| 1 0 | 0 | 0 0 | 0 | 1 0 | 0 |
| 1 1 | 1 | 0 0 | 0 | 1 1 | 1 |
| 1 2 | 2 | 0 1 | 1 | 1 1 | 1 |
| 2 0 | 0 | 1 0 | 0 | 1 0 | 0 |
| 2 1 | 1 | 1 0 | 0 | 1 1 | 1 |
| 2 2 | 2 | 1 1 | 1 | 1 1 | 1 |

Entries '*' contradict definition 2.3, for $S_1(0 2) = S_1(0 1)$ but $S_1[f(0 2)] = S_1[f(0 1)]$. The first component could be said to be relevant only at level 1, since it only affects the state of system when it and the second component are at level 1.

To confirm that the narrow sense coherent systems with well defined binary image class is not identical with the constant binary image group the following example is given.

Example 2.3.2

A coherent system with a well defined, but not constant binary image. Again consider the following systems of two components each with three

levels in table 2.2.

Table 2.2

| \underline{x} | $f(\underline{x})$ | $S_2(\underline{x})$ | $S_2[f(\underline{x})]$ | $S_1(\underline{x})$ | $S_1[f(\underline{x})]$ |
|-----------------|--------------------|----------------------|-------------------------|----------------------|-------------------------|
| 0 0 | 0 | 0 0 | 0 | 0 0 | 0 |
| 0 1 | 0 | 0 0 | 0 | 0 1 | 0 |
| 0 2 | 0 | 0 1 | 0 | 0 1 | 0 |
| 1 0 | 1 | 0 0 | 0 | 1 0 | 1 |
| 1 1 | 1 | 0 0 | 0 | 1 1 | 1 |
| 1 2 | 1 | 0 1 | 0 | 1 1 | 1 |
| 2 0 | 1 | 1 0 | 0 | 1 0 | 1 |
| 2 1 | 1 | 1 0 | 0 | 1 1 | 1 |
| 2 2 | 2 | 1 1 | 1 | 1 1 | 1 |

The minimum path at minimum level 2 is (2,2) with binary image at minimum level 2 of (1,1), whilst the minimal path at minimum level 1 is (1,0) with binary image at minimum level 1 of (1,0). This contradicts Definition 2.4.

The following example indicates that the class of structures with constant binary images do exist.

Example 2.3.3

Again consider a two component three level system presented in table 2.3

Table 2.3

| \underline{x} | $f(\underline{x})$ | $S_2(\underline{x})$ | $S_2[f(\underline{x})]$ | $S_1(\underline{x})$ | $S_1[f(\underline{x})]$ |
|-----------------|--------------------|----------------------|-------------------------|----------------------|-------------------------|
| 0 0 | 0 | 0 0 | 0 | 0 0 | 0 |
| 0 1 | 1 | 0 0 | 0 | 0 1 | 1 |
| 0 2 | 2 | 0 1 | 1 | 0 1 | 1 |
| 1 0 | 1 | 0 0 | 0 | 1 0 | 1 |
| 1 1 | 1 | 0 0 | 0 | 1 1 | 1 |
| 1 2 | 2 | 0 1 | 1 | 1 1 | 1 |
| 2 0 | 2 | 1 0 | 1 | 1 0 | 1 |
| 2 1 | 2 | 1 0 | 1 | 1 1 | 1 |
| 2 2 | 2 | 1 1 | 1 | 1 1 | 1 |

This system could be described as two component parallel system, and would be included in the set of BW structures. It could have a parallel dichotomic representation.

Following from proposition 2.3.1 it is possible to establish that the coherent systems with well defined binary images must pass through all levels on any route through the structure from M to 0.

Corollary 2.3.3

In the absence of simultaneous component changes of state, a structure with a well defined binary image must pass through all levels $M, \dots, 1, 0$.

Proof.

From the proof of proposition (2.3.1) system changes of states are always of magnitude unity. This with (2.1) proves the corollary.

It is possible to formulate definition 2.4 in two alternative forms, one of which appears rather stronger (2.4') and the other weaker (2.4''). The proof of their equivalence is omitted.

Definition 2.4'

A coherent structure with structure function $f(\underline{x})$ and a well defined binary image has a constant binary image if and only if the binary image at minimum level i of an equivalent path at minimum level i for all $i, j = 1, 2, \dots, M$.

Definition 2.4''

A coherent system with structure function $f(\underline{x})$ and a well defined binary image, has a constant binary image if and only if

$$L_e^i = L_{\ell}^i, \text{ for all } i, j = 1, 2, \dots, M \quad (2.11)$$

and for all $\ell = 1, 2, \dots, n$.

2.4 Monotonic Systems.

By deleting requirement (2.1), that is $f(\underline{0}) = 0$ and $f(\underline{M}) = M$ the systems are no longer coherent but are monotonic, such a definition is considered by Ross [1978]. It follows from the definition that the class of monotonic systems with well defined binary images contain that of monotonic systems with constant binary images, and that the proof of proposition 2.3.1 is still valid. Monotonic systems which violate (2.1) may, or may not, have well defined and constant binary images, as the following examples illustrate.

Example 2.4.1

A monotonic system with a constant binary image is given in Table 2.4.

Table 2.4

| \underline{x} | $f(\underline{x})$ | $S_2(\underline{x})$ | $S_2[f(\underline{x})]$ | $S_1(\underline{x})$ | $S_1[f(\underline{x})]$ |
|-----------------|--------------------|----------------------|-------------------------|----------------------|-------------------------|
| 0 0 | 2 | 0 0 | 1 | 0 0 | 1 |
| 0 1 | 2 | 0 0 | 1 | 0 1 | 1 |
| 0 2 | 2 | 0 1 | 1 | 0 1 | 1 |
| 1 0 | 2 | 0 0 | 1 | 1 0 | 1 |
| 1 1 | 2 | 0 0 | 1 | 1 1 | 1 |
| 1 2 | 2 | 0 1 | 1 | 1 1 | 1 |
| 2 0 | 2 | 1 0 | 1 | 1 0 | 1 |
| 2 1 | 2 | 1 0 | 1 | 1 1 | 1 |
| 2 2 | 2 | 1 1 | 1 | 1 1 | 1 |

The system is constant, so that both components are irrelevant to the systems behaviour.

Example 2.4.2

A monotonic system with a well defined, but not constant, binary image is given in table 2.5.

Table 2.5

| \underline{x} | $f(\underline{x})$ | $S_2(\underline{x})$ | $S_2[f(\underline{x})]$ | $S_1(\underline{x})$ | $S_1[f(\underline{x})]$ |
|-----------------|--------------------|----------------------|-------------------------|----------------------|-------------------------|
| 0 0 | 1 | 0 0 | 0 | 0 0 | 1 |
| 0 1 | 1 | 0 0 | 0 | 0 1 | 1 |
| 0 2 | 2 | 0 1 | 1 | 0 1 | 1 |
| 1 0 | 1 | 0 0 | 0 | 1 0 | 1 |
| 1 1 | 1 | 0 0 | 0 | 1 1 | 1 |
| 1 2 | 2 | 0 1 | 1 | 1 1 | 1 |
| 2 0 | 2 | 1 0 | 1 | 1 0 | 1 |
| 2 1 | 2 | 1 0 | 1 | 1 1 | 1 |
| 2 2 | 2 | 1 1 | 1 | 1 1 | 1 |

Example 2.4.3

A monotonic system without a well defined binary image is given in Table 2.6.

Table 2.6

| \underline{x} | $f(\underline{x})$ | $S_2(\underline{x})$ | $S_2[f(\underline{x})]$ | $S_1(\underline{x})$ | $S_1[f(\underline{x})]$ |
|-----------------|--------------------|----------------------|-------------------------|----------------------|-------------------------|
| 0 0 | 1 | 0 0 | 0 | 0 0 | 1 |
| 0 1 | 1 | 0 0 | 0 | 0 1 | 1 |
| 0 2 | 2 | 0 1 | 1 | 0 1 | 1 |
| 1 0 | 1 | 0 0 | 0 | 1 0 | 1 |
| 1 1 | 1 | 0 0 | 0 | 1 1 | 1 |
| 1 2 | 2 | 0 1 | 1 | 1 1 | 1 |
| 2 0 | 1 | 1 0 | 0 | 1 0 | 1 |
| 2 1 | 2 | 1 0 | 1 | 1 1 | 1 |
| 2 2 | 2 | 1 1 | 1 | 1 1 | 1 |

2.5 Relationships with other definitions

In this section the connection between definitions 2.1 to 2.4 and others are considered in more detail. For those where the relationship is obvious depending solely on relevance are left to the reader and the comments at the end of section 2.2. The two groups of interest are therefore BW, definition 2.5, and Natvig's MCS II, definition 2.12.

The class of functions defined by BW are contained within the class of functions with constant binary images this follows from the next proposition.

Proposition 2.5.1

All BW coherent systems are contained within the class of narrow sense coherent systems possessing well defined binary images.

Proof

From definition 2.5, if $f(\underline{x})$ is BW coherent then $f(\underline{x})$ is monotonic and $f(j) = j$ for all $j = 0, 1, \dots, M$. Hence $f(\underline{x})$ is a coherent system in the narrow sense. Further, either there exists some dichotomic minimum path set P_r for which the level of all the components in the state vector exceeds $(j-1)$ or no such $P_r \in C$ exists. In the former case the binary image at the minimum level j of the state vector is the binary path P_r and

$$S_j[f(\underline{x})] = 1,$$

whilst in the latter case the binary image of the state vector at minimum level j is a binary cut and

$$S_j[f(\underline{x})] = 0.$$

It follows immediately that definition 2.3 holds.

Proposition 2.5.2

The minimal paths at minimum level j , $j = 2, 3, \dots, M$, of a BW

coherent system are of the form

$$\underline{x}_j = j \underline{x}_1 \quad (2.12)$$

where \underline{x}_1 is the corresponding minimal path at minimum level 1, and which is one-to-one with a dichotomic minimum path set.

Proof.

Let $f(\underline{x})$ be a BW coherent. Then by Proposition 2.3.4, (2.8) holds. Let \underline{x}_1 be a minimal path at minimum level 1. Then by definition 2.3 and (2.8) there exists in \underline{x}_1 some dichotomic minimum path set P_r , $1 \leq r \leq p$ for which

$$\min_{i \in P_r} x_i = 1 \quad (2.13)$$

but there is no such dichotomic minimum path set for any $\underline{y}_1 < \underline{x}_1$.

Consider $\underline{x}_j = j \underline{x}_1$, $j = 1, \dots, M$. Then there exist in \underline{x}_j the dichotomic minimum path set P_r for which

$$\min_{i \in P_r} x_i = j,$$

so that $f(\underline{x}_j) = j$, and thus $S_j[f(\underline{x}_j)] = 1$. Thus $\underline{x}_j = j \underline{x}_1$ is a path at minimum level j unless for some $k \in P_r$

$$S_j[f(\underline{y}_{jk})] = 1$$

where the i th entry of \underline{y}_{jk} is

$$y_{jki} = \begin{cases} j-1, & i = k \\ j, & i \in P_r, i \neq k \\ 0, & i \notin P \end{cases} \quad (2.14)$$

However, by definition 2.3, $f(\underline{y}_{jk}) = j-1$ so that

$$S_j[f(\underline{y}_{jk})] = 0 \quad \text{for all } k \in P_r$$

Thus $\underline{x} = j \underline{x}_1$ is a minimal path at minimum level j .

To prove that all minimal paths at minimum level j are of this form, suppose that \underline{x}_j is a minimal path at minimum level j (so that (2.8) holds), but $\underline{x}_j \neq j \underline{x}_1$. Then either the non-zero entries of \underline{x}_j include those of one or more $j \underline{x}_1$ in which case \underline{x}_j is not a minimal path at level j (contradiction), or the non-zero entries of \underline{x}_j do not include the non-zero entries for any such $j \underline{x}_1$. In the later case \underline{x} does not correspond to any binary path set, and by definition 2.3

$$f(\underline{x}_j) = 0$$

so that \underline{x}_j is not a minimal path at minimum level j (contradiction).

Corollary 2.5.3

All BW coherent systems are narrow sense coherent systems with a constant binary image.

Proof

$$S_j(j \underline{x}_1) = \underline{x}_1 \text{ for all } j.$$

The proof that Natvig's MCS II are equivalent to coherent systems with well defined binary images is given below, and is due to unknown referee.

Proof:

From definition 2.12 of MCS II, $f_j(\underline{x})$ provides the well defined binary images.

To show well defined binary images imply MCS II, construct $f_j(S_j(\underline{x}))$ by defining the following:

$$\begin{aligned} E &= \{ \underline{x} : f(\underline{x}) \geq j \} = \{ \underline{x} : S_j(f(\underline{x})) = 1 \} \\ B &= \{ S_j(\underline{x}) : \underline{x} \in E \} \end{aligned}$$

and

$$f_j(S_j(\underline{x})) = \begin{cases} 1 & \text{if } S_j(\underline{x}) \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then it is required to show that $f(\underline{x}) \geq j \Leftrightarrow f_j(S_j(\underline{x})) = 1$.

That $f(\underline{x}) > j \Leftrightarrow f_j(S_j(\underline{x})) = 1$ is obvious from the definition of $f_j(S_j(\underline{x}))$.

To prove that $f(\underline{y}) < j \Leftrightarrow f_j(S_j(\underline{y})) = 0$ or $S_j(\underline{y}) \notin B$, assume $f(\underline{y}) < j$ and $S_j(\underline{y}) \in B$. Then there exists a $\underline{z} \in E$ such that $S_j(\underline{z}) = S_j(\underline{y})$, but by the definition of well defined binary image then $S_j(f(\underline{z})) = S_j(f(\underline{y}))$. This would be contradiction to $f(\underline{y}) < j$.

Hence for any coherent system with a well defined binary image it is possible to construct a set of f_j which satisfy definition 2.12.

2.6 Decomposition Theorem

Birnbaum et al [1961], established the result for dichotomic systems that a monotonic system of $n+1$ components could be described in terms of a monotonic system of n components. The following theorem generalizes this result to monotonic multilevel systems.

Proposition 2.6.1

A function f of $n+1$ components is monotonic if and only if it can be represented as a linear combination of functions $g_i(s)$, of n components in the form

$$f(\underline{s}, s_{n+1}) = \sum_{i=0}^M H_i(s_{n+1}) g_i(\underline{s})$$

where $H_i(s_{n+1})$ are Lagrange interpolating polynomials such that

$$H_i(s_{n+1}) = \begin{cases} 1, & s_{n+1} = i, \\ 0, & s_{n+1} = j, \end{cases}$$

and $g_i(\underline{s})$ are monotonic functions, with

$$g_M(\underline{s}) \geq g_{M-1}(\underline{s}) \geq \dots \geq g_0(\underline{s}) \text{ for all } \underline{s}.$$

Proof

If f is monotonic then f can be written as a linear combination of monotonic functions.

$$g_i(\underline{s}) = f(\underline{s}, i) \quad i = 1, 2, \dots, M.$$

where $f(\underline{s}, M) \geq f(\underline{s}, M-1) \geq \dots \geq f(\underline{s}, 0)$

and hence $g_i(\underline{s}) \geq g_{M-1}(\underline{s}) \geq \dots \geq g_0(\underline{s})$.

Conversely if $f(\underline{s}, s_{n+1}) = \sum_{i=0}^M H_i(s_{n+1}) g_i(\underline{s})$

where (2.16) is satisfied, then for any two vectors $(\underline{s}, s_{n+1}), (\underline{t}, t_{n+1})$

with $(\underline{t}, t_{n+1}) > (\underline{s}, s_{n+1})$

$$\begin{aligned} f(\underline{t}, t_{n+1}) - f(\underline{s}, s_{n+1}) &= \sum_{i=0}^M [H_i(t_{n+1}) - H_i(s_{n+1})] g_i(\underline{t}) \\ &\quad + \sum_{i=0}^M H_i(s_{n+1}) [g_i(\underline{t}) - g_i(\underline{s})] \end{aligned}$$

$\geq 0,$

and hence $f(\underline{s}, s_{n+1})$ is monotonic.

A revised proof appeared in Ansell, Bendell and Humble [1981] following the suggestion of a referee.

This theorem will be used later to produce bounds for the number of coherent systems in this chapter and in next it will be used to consider some of the stochastic properties. Several related theorems has been proved in the literature, based on the authors definition of coherency, see Hatoyama [1979], Block and Savits [1981] and El-Newehi et al [1978].

2.7 Special Case k -out-of- n structures.

In chapter 1 these structures where shown to play a central role in the dichotomic case. For illustrative purposes El-Newehi et al [1978], and Hatoyama [1979], defined k -out-of- n structures, however they did not explore their structural properties. This section defines such structures and then proceeds to examine their properties. In the next chapter stochastic aspects are considered for these structures.

Obviously a multistate system is dichotomic if $M=1$, other definitions are required before proceeding.

Definition 2.13

A system is a (multistate) k-out-of-n system if and only if

$$f(\underline{x}) = x_{(n-k-1)} \quad (2.17)$$

where $x_{(1)} \leq \dots \leq x_{(n)}$ is a non-decreasing rearrangement of x_1, \dots, x_n . It is immediately obvious that a multistate k-out-of-n system is a coherent system with a well defined binary image.

If $M=1$, the system is a dichotomic k-out-of-n system.

Definition 2.14

For an arbitrary dichotomic coherent system $f(x)$ denote the number of paths of size ℓ by L_ℓ , $\ell = 1, \dots, n$, and write

$$\underline{L} = (L_1, L_2, \dots, L_n) \quad (2.18)$$

For a multistate coherent system similarly write

$$\underline{L}^{\dot{}} = (L_1^{\dot{}}, L_2^{\dot{}}, \dots, L_n^{\dot{}}) \quad (2.19)$$

Definition 2.15

For a dichotomic k-out-of-n system denote the number of paths of size ℓ by c_ℓ , $\ell = 1, \dots, n$, and write

$$\underline{c}_k = (c_{k1}, c_{k2}, \dots, c_{kn}) \quad (2.20)$$

Definition 2.15

The length of a dichotomic system with path numbers L is defined as

$$e^* = \min \{e \mid L_e \neq 0\} \quad (2.21)$$

The corresponding length of a system's binary image at minimum level j ($j=1, \dots, M$) is

$$e_j^* = \min \{e \mid L_e^j \neq 0\} \quad (2.22)$$

Proposition 2.7.1

For any j , the binary image at minimum level j of a multistate coherent system with a well defined binary image is a dichotomic coherent system.

Proof

For the binary image to be coherent it is required that the following three conditions hold for all $j = 1, \dots, M$.

- (i) If $S_j(\underline{x}) = 0$, then $S_j[f(\underline{x})] = 0$.
- (ii) If $S_j(\underline{x}) = 1$, then $S_j[f(\underline{x})] = 1$.
- (iii) If $S_j(\underline{x}) \geq S_j(\underline{y})$ then $S_j[f(\underline{x})] \geq S_j[f(\underline{y})]$

(i) holds as $S_j(\underline{x}) = S_j(\underline{0})$, so that $S_j[f(\underline{x})] = S_j[f(\underline{0})]$ for all j .

(ii) holds as $S_j(\underline{x}) = S_j(\underline{M})$, so that $S_j[f(\underline{x})] = S_j[f(\underline{M})] = 1$ for all j .

To show that (iii) holds let

$$\underline{z} = \sum_j S_j(\underline{x}) \quad (2.23)$$

$$\underline{w} = \sum_j S_j(\underline{y})$$

so that $\underline{z} \geq \underline{w}$ and $f(\underline{z}) > f(\underline{w})$. Then

$$S_j[f(\underline{x})] = S_j[f(\underline{z})] \geq S_j[f(\underline{w})] = S_j[f(\underline{y})]$$

Corollary 2.7.2

For a multistate coherent system with a well defined binary image

$$\underline{L}^j = \sum_{k=1}^n W_k^j \epsilon_k = \sum_{k \in \mathcal{L}_j^*} W_k^j \epsilon_k \quad (2.24)$$

where

$$W_k^j = \begin{cases} 0, & \text{for } k < 1 \\ \frac{L_k^j}{\binom{n}{k}} - \frac{L_{k-1}^j}{\binom{n}{k-1}}, & \text{otherwise} \end{cases}$$

$$0 \leq w_k^j \leq 1$$

and

$$\sum_{k=1}^n w_k^j = 1.$$

Proof

The proof is immediate from Proposition 1.3.1 and 2.7.1.

Proposition 2.7.3

For any j , the binary image at minimum level j of a multistate k -out-of- n system is a dichotomic k -out-of- n system.

Proof

For the binary image to be dichotomic k -out-of- n it is required that

$$S_j[f(\underline{x})] = \begin{cases} 1, & \text{if } \sum_{\alpha=1}^n S_j(x_\alpha) = \ell \geq k \\ 0, & \text{otherwise.} \end{cases} \quad (2.25)$$

Suppose that $x_{(n-k+1)} = 1$. Then if $i \geq j$, $\sum_{\alpha=1}^n S_j(x_\alpha) \geq k$ and $f(\underline{x}) \geq j$ so that $S_j[f(\underline{x})] = 1$. Conversely if $i < j$, $\sum_{\alpha=1}^n S_j(x_\alpha) < k$ and $f(\underline{x}) < j$ so that $S_j[f(\underline{x})] = 0$.

2.8 Extension to Non-Coherent Structures.

As in Chapter 1 it seems with the increasing realization of the plausibility of such structures that one should consider how the results and definition may be adapted. Therefore in this section the generalization of definition 2.1 to 2.4 are considered and the previous results examined to see if they still hold.

Definition 2.16

$f(\underline{x})$ is wide sense non-monotonic if and only if it is non-monotonic and (2.1) holds.

Definition 2.17

$f(\underline{x})$ is narrow sense non-monotonic if and only if it is non-monotonic and (2.2) holds.

Definition 2.18

A non-monotonic system has a well defined binary image if and only if

$$S_j(\underline{x}) = S_j(\underline{y}) \quad \text{for any } j$$

implies that

$$S_j[f(\underline{x})] = S_j[f(\underline{y})].$$

Definition 2.19

A non-monotonic system with structure function $f(\underline{x})$ and a well defined binary image has a constant binary image if and only if the binary image at minimum level j of any minimal path at minimum level j is also the binary image at minimum level i of an equivalent minimal path at minimum level i for all $i, j = 1, \dots, M$.

In fact the ordering established in the previous section for monotonic structures does hold for non-monotonic structures. Hence the following Propositions 2.8.1 and 2.8.3, and lemma 2.8.2 are direct analogues of those for coherent structures.

Proposition 2.8.1

For a non-monotonic system with a well defined binary image, the minimal paths at minimum level $j, j = 1, \dots, M$ are of the form (2.8).

Proof

Consider the state vector \underline{y} such that $f(\underline{y}) = j$. Let

$$\underline{x} = j S_j(\underline{y})$$

so that

$$\underline{x} \leq \underline{y}$$

and

$$f(\underline{x}) \geq j = f(\underline{y}).$$

It follows that \underline{y} is not a minimal path at minimum level j unless $\underline{y} = \underline{x}$ where \underline{x} has the form (2.3.1).

A lemma to Proposition 2.8.1, which provides some insight into the monotonicity component of the property of a well defined binary image, is the following.

Lemma 2.8.2

If \underline{x} is of the form (2.8) and is a path at minimum level j for a system with a well defined binary image, whilst \underline{z} is identical to \underline{x} except that $z_k = j - 1$ for some $k \in J$, then \underline{z} is a path at minimum level $(j+1)$.

Proof

Since the system has a well defined binary image

$$S_{j-1} [f(\underline{z})] = S_{j-1} [f(\underline{x})] = 1.$$

Proposition 2.8.3

All wide sense non-monotonic systems with a well defined binary image are narrow sense non-monotonic systems.

Proof

The proof is immediate from Proposition 2.8.1 and the proof of 2.3.2.

Hence the structural properties appear to carry directly over from the coherent to the non-coherent case. Unfortunately when considering the stochastic properties the same is not true, this will be seen in chapter 3.

2.9 Number of Multilevel Coherent Structures.

The problem of enumerating the number of dichotomic coherent structures, (as mentioned in chapter 1), or its equivalent (the cardinality of distributive free lattices, number of monotone Boolean functions or Dedekind's problem), has been of interest to Pure Mathematicians for nearly 90 years, see Dedekind [1897], Church [1947,1965] and Hansel [1966]. Direct enumeration of number of dichotomic coherent systems reached $n=7$ by 1965, (Church [1965] with the actual value being 2,414,682,040,996). The relevancy to reliability is tenuous, however, knowing the number of possible structures indicates the least amount of information required to specify a particular system, see Lomnicki [1973]. A similar defence may be made in the case of Multilevel System.

For the multilevel case, direct enumeration soon leads to problems with integer storage and whilst it is possible to circumvent these there seems little to gain by doing so. As an illustration of the size of the problem table 2.7 gives the numbers under various definition of coherency for moderate number of components and of levels.

The values obtained for 2 components and levels 2,3,4,5 in narrow sense coherent case lead on to speculate the number of systems is

given by

$$2^{M(M+1)}$$

where $M+1$ is the number of levels.

The result seems so simple but a proof has not yet emerged.

Table 2.7

The number of multilevel coherent structures
under various definitions

| n | (1) Narrow sense coherent | | | (2) Weakly coherent | | |
|----------------------------|------------------------------|------------|---------|--------------------------|------------|---------|
| | 1 | 2 | 3 | 1 | 2 | 3 |
| M+1 | | | | | | |
| 2 | 1 | 4 | 18 | 1 | 2 | 9 |
| 3 | 1 | 64 | 151,236 | 1 | 62 | 151,047 |
| 4 | 1 | 4,096 | ▲ | 1 | 4094 | ▲ |
| 5 | 1 | 1,048,576 | ▲ | 1 | 1,048,574 | ▲ |
| (3) Wide Sense coherent | | | | (4) Butler's coherent | | |
| 2 | 1 | 4 | 18 | 1 | 2 | 9 |
| 3 | 3 | 136 | 738,122 | 3 | 130 | 737,723 |
| 4 | 10 | 18,676 | ▲ | 10 | 18,656 | ▲ |
| 5 | 35 | 15,374,304 | ▲ | 35 | 15,374,234 | ▲ |

Given that there are numerical problems in calculating exactly the number of structures even for the dichotomic case several authors have sought upper bounds, Dedekind [1897], Gilbert [1954], Korobkov [1963], Hansel [1966], Kliezman [1969], Hanish et al [1969], Alekseev [1973], Kliezman and Markowsky [1975] and Kurshunov [1977]. The sharpest bound so far obtained, due to Hansel, who showed,

$${}^n S_2 < 3^{B_n}$$

where ${}^n S_2$ is the number of structures of n components with 2 levels and B_n is the middle Binomial coefficient, i.e.,

$$B_n = \begin{cases} \frac{n!}{(n/2)!(n/2)!} & \text{if } n \text{ is even,} \\ \frac{n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n+1}{2}\right)!} & \text{if } n \text{ is odd.} \end{cases}$$

From proposition 2.5.1 it is possible to derive a recursive bound for the number of structures both in the dichotomic and multilevel case. It follows from (2.15) that if ${}^n S_{M+1}$ were the number of possible n component and $M+1$ level monotonic functions and if it were possible to order these functions then by considering the ways in which these functions may be

identified with $g_i(s)$ in (2.15) and (2.16) the number of $n+1$ components with $M+1$ levels would be given by

$$\binom{n}{M+1} S_{M+1} + M \quad (2.26)$$

Unfortunately the $\binom{n}{M+1} S_{M+1}$ functions cannot be totally ordered and hence, (2.26) is an upper bound for the number of monotonic functions with $n+1$ components. Obviously if $\binom{n}{M+1} S_{M+1}$ is replaced by $\binom{n}{M+1} U_{M+1}$, its upper bound, then the value derived from it will still be an upper bound. Hence a recursive upper bound can be obtained $\binom{n+1}{M+1} S_{M+1}$ which is

$$\binom{n+1}{M+1} U_{M+1} = \binom{n}{M+1} U_{M+1} + M \quad (2.27)$$

In the dichotomic case $M=1$ and so the upper bound becomes,

$$\binom{n+1}{2} U_2 = \binom{n}{2} U_2 + (n+1)/2 \quad (2.28)$$

Based on the value obtained by Church [1965] for $\binom{7}{2} S_2$, (2.28) provides an upper bound which is sharper than those given by Hansel [1966] for $7 < n < 15$, see table 2.8. Beyond $n=15$, the bound may be used recursively with the Hansel bound to produce a systematic improvement on Hansel's for even values.

The procedure would be if n is even use Hansel's bound, if n is odd use

$${}^n U_2 = 2^{(\log_2 3)^n} B_n \quad (2.29)$$

with

$$B_n = \frac{n!}{\left(\frac{n+1}{2}\right)! \left(\frac{n+1}{2}\right)!}$$

So

$${}^{n+1} U_2 = 2^{(\log_2 3)^{n+1}} + 2^{(\log_2 3)^n} B_{n-1}$$

Table 2.8

Upper Bounds for the number of dichotomic
monotonic functions of n components

$$\log_{10} ({}^n U_2)$$

| n | Recursive Bound | Hansel's Bound |
|----|-----------------|----------------|
| 8 | 24.46468 | 33.39845 |
| 9 | 48.62831 | 60.11722 |
| 10 | 96.95558 | 120.23445 |
| 11 | 193.61012 | 220.42983 |
| 12 | 386.91919 | 440.85962 |
| 13 | 773.53735 | 818.73926 |
| 14 | 1546.77368 | 1637.47876 |
| 15 | 3093.24634 | 3070.27222 |

It is possible to deduce also lower bounds for monotonic systems, and these upper and lower bounds may be extended to the wide sense and narrow sense coherent structures, see section 6 of Ansell, Bendell and Humble [1981].

Proposition 2.9.1

In the special case of series or parallel networks the number of narrow sense coherent systems with n components and $M+1$ levels is equal to the number of coherent structures in the narrow sense with n components and M levels.

Proof

Let P be the set of narrow sense coherent functions for a parallel system with n components and $M+1$ levels. Hence if $f \in P$,

$$f(\underline{j}) = j \quad \text{for all } j = 0, 1, \dots, M$$

$$f(\underline{x}) = M \quad \text{if } x_{\alpha} = M \text{ for any } \alpha$$

$$f(\underline{x}) > f(\underline{y}) \quad \text{if } \underline{x} > \underline{y}$$

and

$$f(\underline{M}) = M, \quad f(\underline{0}) = 0.$$

It also follows that if $f(\underline{M}) = M$ then $x_{\alpha} = M$ for some α .

Hence if one considers only the M possible levels $0, \dots, M-1$

then $f(\underline{M-1}) = M-1$, and the system is then identical to system with M levels of performance.

For series narrow sense coherent system the proof is obtained by discounting the level 0 for which the series system must fail.

It should be noted that through this chapter have assumed that the levels of performance have been assumed to be $0, 1, \dots, M$ for ease of description. However the results will in the main hold for any ordered set of levels either on $(0, M)$ or $(0, 1)$.

Chapter 3

Multistate Systems

3.1 Introduction

Chapter 2 considered definitions and associated structural properties of multilevel systems. The emphasis was on the deterministic properties of systems composed of components which have multiple levels of performance. This chapter concentrates on the stochastic properties of such systems. This chapter is a review of previous work on multistate systems by other authors, i.e. Ross [1978], El-Newehi, Proschan and Sethuraman [1978], Griffiths [1980] and Natvig [1982]. However the sections on the special case of k -out-of- n is generalization of material presented in the first chapter.

The chapter opens with consideration of description of the state of the system at a given time point, when the probabilities of component states are not time dependent. Again the simplicity of the dichotomic systems is lost, since one has to consider the distribution over several states rather than simple $\{0,1\}$ case. However most of the results presented in this chapter are obtained by the use of binary splitting and then the application of results for the dichotomic case. The second half of the chapter then considers time dependent systems. These latter sections mainly concentrate on systems where components are not repaired or replaced, but the components degrade with time. In the next chapter the performance of components subject to replacement is considered.

3.2 Definitions

As in chapter 1 and 2, the structure function will be denoted as $f(x)$.

Let the probability of component i being in state j be denoted by p_{ij} so

$$p_{ij} = P\{X_i=j\} \quad i=1,\dots,n \text{ and } j=0,1, \dots, M.$$

Let the probability that component i is in a state greater than or equal to j be q_{ij} , so

$$q_{ij} = P\{X_i \geq j\} \quad i=1,\dots,n \text{ and } j=0,1, \dots, M-1.$$

Let the probability that the system is in state j be denoted by P_j so

$$P_j = P\{f(x)=j\} \quad j=0,1, \dots, M$$

Define the performance of the system, following El-Newehi et al [1978], to be the probability the system is in state j or higher. Denote it by $P(j)$, so that $P(j)$ is

$$P(j) = P\{f(x) \geq j\} \quad j=0,1,\dots,M.$$

3.3 Static Model

In this section systems performance at a single point of time is considered, or systems whose probability of being in given state is not dependent on time. The section starts with consideration of elementary

results, then reviews some of the main ideas such as Importance. The last part of the section considers again the centrality of k -out-of- n systems for identical components.

3.3.1 Elementary Results

In this section it will be assumed that the components are stochastically independent. Three measures of interest are the probabilities P_j and $P(j)$, definitions given above, and the expected state of the system, $E[f(\underline{x})]$.

It is obvious that they are multilinear in p_{ij} since they are obtained by applying the additive and multiplicative laws of probability, given the assumption of independence.

That $E[f(\underline{x})]$ is non-decreasing in p_{ij} was established initially by El-Newehi et al [1978], for their restricted class of multistate coherent systems, but the result is true for all multistate non-decreasing systems. Assuming their relevancy condition then $E[f(\underline{x})]$ is strictly increasing. To prove that $P(j)$ is non-decreasing requires the differentiation of $P(j)$ as expressed in the decomposition theorem Theorem 4.3 of El-Newehi et al. The proof is similar for P_j expressing P_j as

$$P_j = P(j) \cdot P(j+1) > 0, \quad j = 0, 1, \dots, M-1.$$

If the components are assumed to have the same distribution, i.e. $p_{ij} = p_j = p_j$, $k=1, \dots, n$ and $j=0, 1, \dots, M$, then the $E[f(\underline{x})]$ and $P(j)$

P_j are polynomials in p_j .

The dichotomic result that redundancy at component level yields higher system reliability than redundancy at the system level depends on definition of redundancy in the multistate case. Two possible measures of redundancy have been considered in the literature these are the truncated sum and maximum. The truncated sum, defined by minimum $[\sum_{i=1}^n x_i, M]$, is often applied in pipeline studies where flow may be allowed through redundant sections, the total flow being measured at some point finally where there is limited capacity, see Fardis and Cornell [1981] and Hudson and Kapur [1982]. Zijlstra [1980] applied this definition of redundancy to parallel lighting units. The maximum value is the more usual measure considered, see El-Newehi et al [1978]. Whilst the truncated sum and maximum will lead to the same value in the dichotomic case, in the multistate case they differ. In the case of the truncated sum the following illustration shows that redundancy at the system level can be superior.

Consider the system with structure function defined as follows;

$$f(\underline{1})=f(\underline{2})=1 \quad M > 2$$

then

$$f(\underline{1})+f(\underline{1})=2$$

whilst

$$f(\underline{1+\underline{1}})=1 .$$

If one simply considers maximum form of redundancy then redundancy at the component level has higher reliability than redundancy at the system level. This follows immediately from;

$$f(\max(x_1, y_1), \max(x_2, y_2), \dots, \max(x_n, y_n)) \geq \max(f(\underline{x}), f(\underline{y}))$$

for component state vectors \underline{x} and \underline{y} .

Hence

$$P[f(\max(x_1, y_1), \max(x_2, y_2), \dots, \max(x_n, y_n)) \geq s] \\ \geq P[\max(f(\underline{x}), f(\underline{y})) \geq s] \text{ for all } s = 0, \dots, M,$$

and

$$E[f(\max(x_1, y_1), \max(x_2, y_2), \dots, \max(x_n, y_n))] \\ \geq E[\max(f(\underline{x}), f(\underline{y}))] > \max\{E[f(\underline{x})], E[f(\underline{y})]\}.$$

3.3.2 Importance

Often it is desired to measure what contribution a component makes to the reliability of a system or how crucial or important the component is. In the dichotomic case at least three possible definitions have been given, Birnbaum's [1961], Barlow and Proschan's [1975] and Butler's [1979]. Birnbaum's definition of importance is the simplest and is defined for the i th component to be

$$I(i) = \frac{d E[f(\underline{x})]}{d p_i}$$

In the dichotomic case this becomes,

$$I(i) = E[f(X_1, \dots, 1_i, \dots, X_n)] - E[f(X_1, \dots, 0_i, \dots, X_n)]$$

There have been several generalizations to this definition. Griffiths [1980] suggests $I(i)$ the importance vector of component i , which under particular set of utilities becomes,

$$I(i) = (E[f(x_1, \dots, 1_i, \dots, x_n)] - E[f(x_1, \dots, 0_i, \dots, x_n)], \\ E[f(x_1, \dots, 2_i, \dots, x_n)] - E[f(x_1, \dots, 1_i, \dots, x_n)], \\ \dots \dots \dots \\ E[f(x_1, \dots, M_i, \dots, x_n)] - E[f(x_1, \dots, M-1_i, \dots, x_n)]).$$

Butler's definition is based on the number of occurrences of the component in the minimum cut-set and therefore be readily generalized to the multistate case. It is as follows.

For each component of a coherent system with t minimum cuts, let $d_{ij}^{(e)}$ denote the number of collections of i distinct minimum cuts such that the union of each collection contains j components containing e , ($1 \leq i \leq t$, $1 \leq j \leq n$). Let $b_j^{(e)} = \sum_{i=1}^t \binom{t}{i} d_{ij}^{(e)}$. Let $\underline{b}^{(e)} = (b_1^{(e)}, \dots, b_n^{(e)})$. Component e is more cut important than component k if and only if $\underline{b}^{(e)} > \underline{b}^{(k)}$, denoted $e >_c k$. Components e and k are equally cut important if and only if $\underline{b}^{(e)} = \underline{b}^{(k)}$, denoted $e =_c k$.

Whilst this seems a more contorted definition than those above it has the practical advantage of being easily defined for a system. It does not depend on the need to specify the probabilities of failure of components. This causes problems in many contexts when knowledge of

component probabilities of failure is limited. Bulter [1979] showed that when component reliabilities are high this definition is equivalent to Birnbaum's.

Given that this procedure is structural it must be related to the number of paths in the system hence there is an connection with the following section which considers a generalization of the characterization from chapter 1.

3.3.3 Special Case k-out-of-n systems.

In this section the generalization of the result in Chapter 1 concerning k-out-of-n systems is considered.

Definition 3.3.3.1

Denote the probability associated with any distinct binary image of paths at minimum level j of size Q by $P_Q(j)$ and so

$$\underline{P}(j) = (P_1(j), P_2(j), \dots, P_n(j))$$

If the components are independent, each with probability q of exceeding $(j-1)$ then $P(j)$ would be given by;

$$P_Q(j) = q_j^Q (1-q_j)^{n-Q}$$

3.3.3.1 Proposition

For any j , the binary image at minimum level j of a multistate k -out-of- n system is a dichotomic k -out-of- n system.

Proof. For the binary image to be dichotomic k -out-of- n system it is required that

$$S_j[f(x)] = \begin{cases} 1, & \text{if } \sum_{\alpha=1}^n S_j(x_\alpha) = Q \geq k, \\ 0, & \text{otherwise} \end{cases}$$

Suppose that $x_{(n-k+1)} = 1$. Then if $i > j$, $\sum_{\alpha=1}^n S_j(x_\alpha) \geq j$ and $f(x) \geq j$ so that $S_j[f(x)] = 1$. Conversely if $i < j$, $\sum_{\alpha=1}^n S_j(x_\alpha) < j$ and $f(x) < j$ so that $S_j[f(x)] = 0$.

3.3.3.2 Corollary

For a multistate coherent system of identical components with a well defined binary image

$$\begin{aligned} P(j) &= \sum_{k=1}^n W_k^j R_j(c_k) \\ &= \sum_{k=c_j^*}^n W_k^j R(c_k), \quad j = 1, \dots, M, \end{aligned}$$

where $R_j(c_k) = c_k P_{-}(j)$ is the probability that a multistate k -out-of- n system of identical components exceeds level $(j-1)$.

It may be interpreted as the reliability of a dichotomic k -out-of- n

system of identical components with a vector of path probabilities of size ℓ ($\ell=1, \dots, n$), $\underline{P}(j)$.

Proof.

Post-multiplying the $(1 \times n)$ vector on either side of (2.24) by the $(n \times 1)$ vector $\underline{P}(j)$ yields the result.

Proposition 3.3.3.4

For a multistate coherent system of identical components with a constant binary image

$$\begin{aligned} P_j &= \sum_{k=1}^n W_k r_j(c_k) \\ &= \sum_{k \in \mathcal{E}_j^*} W_k r_j(c_k) \end{aligned}$$

where $W_k^j = W_k$ and $\mathcal{E}_j^* = \mathcal{E}^*$ for all $j=1, \dots, M$ and

$r_j(c_k) = c_k [P(j) \cdot P(j+1)]$ is the probability that a multistate k -out-of- n system of identical components takes the value j .

Proof

By definition 2.4'' and the definition of W in corollary 2.6.2

$$W_k^j = W_k^i = W_k \text{ say, for all } i, j = 1, \dots, M \text{ and all } k.$$

Therefore

$$P_j = P(j) - P(j+1) = \sum_{k=1}^n W_k [R_j(c_k) - R_{j+1}(c_k)]$$

The above Corollaries 3.3.3.2 and proposition 3.3.3.1 provided the generalization of the characterization given in Chapter 1 for the classes of multistate coherent systems with well defined and constant binary images. It emphasises the importance of the k-out-of-n system in these special cases. There are possibly other properties which could be derived from this characterization, particularly bounds for systems performance in the non-identical case. The bounds would be similar to those obtained in the dichotomic case given in section 1.4 in Chapter 1. This would be an alternative approach to that others such as Butler [1979] and Natvig [1982] who either used the inclusion-exclusion principal or Bonferroni Inequality to obtain bounds for the probability of system being in given state. The above results are obtained without the assumption of independence.

Again as in the dichotomic case it is possible to use the properties of the weights as a procedure to enumerate the possible systems within these cases, and evaluating the state probabilities for such multistate systems. An illustration is provided in the following example.

Example 3.3.3.1

Consider the system of three three-level components shown in Figure 3.1, with structure function defined as the maximum of the minimum of the paths. This example was considered in Barlow and Wu [1978].

It is assumed that the components are identical and independent,

each having probability p of being in state $j, j = 0, 1, 2$. The system state probabilities are given in Table 3.1, together with expressions for the 2-out-of-3 and 3-out-of-3 three state systems.

Table 3.1

| j | System State probabilities | 2-out-of-3 | 3-out-of-3 |
|-----|---|--------------------------------|---------------------------------|
| | System Figure 3.1 | | |
| 0 | $p_0 + p_0^2 + p_0^3$ | $3p_0^2 + 2p_0^3$ | $p_0^3 + 3p_0^2 + p_0$ |
| 1 | $p_1(4p_0p_2 + p_2^2) + p_1^2(2 + p_2) + p_1^3$ | $3p_1^2 + 2p_1^3 + 6p_0p_1p_2$ | $p_1^3 + 3p_1^2p_2 + 3p_1p_2^2$ |
| 2 | $2p_2^2 + p_2^3$ | $3p_2^2 + 2p_2^3$ | p_2^3 |

Obviously one can write the state probabilities for system in figure 1 as

$$P[f(x)=j] = \frac{2}{3} r_j(\underline{c}_2) + \frac{1}{3} r_j(\underline{c}_3) \quad j=0,1,2.$$

3.4 Dynamic Model

In the previous sections of this chapter the interest has been focussed on the state of the system at a given point in time. This section reviews two results on the performance of system over time. The components are assumed to be non-repairable and are assumed also to be

degrading as time progresses. The two results were originally proved by Ross [1979] and several authors have repeated the work for more specific systems.

The results obtained are derived directly from the dichotomic case by the use of binary splitting, see Chapter 2 for more details.

3.4.1 IFRA

A component has an increasing failure rate, IFR, if its hazard function is increasing with time. It therefore positively ages in the sense that as time increases it is more likely to fail. It is fortunate that systems composed of such components need not have increasing failure rate, this can be simply established by considering a parallel system composed of components whose failure rate is increasing. Hence for coherent structures there is not a closure theorem for IFR distribution. (It is not possible to prove that a coherent system composed of IFR components has IFR distribution for its lifetime.) The desire to locate the types of distribution for which the systems distributions has the same properties as the components lead to defining increasing failure rate average, IFRA, distributions.

Definition 3.4.1.1

X has a IFRA distribution if and only if

$$R(\alpha t) > R^\alpha(t) \text{ for all } 0 < \alpha < 1 \text{ and } t \geq 0.$$

3.4.1.1 Proposition.

If a coherent system is composed of components which have IFRA distributions the lifetime of the system is, also IFRA.

Proof.

See Barlow and Proschan [1975].

Ross [1978] suggested a generalization for multivalued systems by relating them back to dichotomic systems. It requires the definition of IFRA processes.

Definition 3.4.1.2

A real-valued stochastic process $\{X(t), t > 0\}$ is an increasing failure rate process if T_a is an increasing failure rate average random variable for every a , where

$$T_a = \inf\{t: X(t) < a\}$$

is the first time the process reaches or goes below a .

Obviously if one regarded a as the failed state then simply is stating that the time to a , or beyond, has IFRA distribution.

Proposition 3.4.1.2

A monotone system of states composed of components whose performance is described by independent IFRA processes is also IFRA process.

Proof.

See Ross [1979].

Ross's definition of monotone systems contains all the definitions of coherency presented in chapter 2, hence the result applies to all coherent systems. Block and Savits [1981] generalized Ross's result to non-independent processes by extending the definition of IFRA processes to multidimensional processes.

3.4.2 NBU

A weaker property than IFRA is New Better than Used (NBU). This is defined as follows.

Definition 3.4.2.1

A stochastic process $\{X(t), t \geq 0\}$ is new better than used if with probability 1,

$$P\{T_a > s+t | X(u), 0 < u < s\} < P\{T_a > t\}$$

for all $s, t, a > 0$, where T_a denotes the first time the process hits or goes below a .

Proposition 3.4.2.1

If the component processes are independent new better than used processes, then a monotone system composed of such also has a performance which is a new better than used process.

Proof.

See Ross [1979].

There is considerable scope for more development in the modelling of systems subject to degradation. A number of other results about systems with partial operation are considered by Bendell and Humble [1985]. Baxter [1983] has considered continuous measures of availability for non-repairable systems. In the next Chapter a specific model is considered of component which degrade, but also may be replaced or repaired.

Chapter 4

Nested Renewal

4.1 Introduction

Many systems are hierarchical in design, with components being parts of subsystems which might also be parts of other subsystems with the system being the apex of the hierarchy. Most mainframe computers may be thought of in this manner. Murphy [1984] considers freight trains as hierarchical systems. The whole train is the total system, the train and wagons form subunits, and then constituent components form the lowest level. Later in the chapter road vehicles are presented as such a hierarchical system.

Hence in order to consider the performance of the system in terms of the performance of the components one must allow for the effect of the subsystem on the component. In this chapter the performance of the component in a system is examined when subsystems can be replaced. This model is more specific than the models considered in the previous chapter, where components could have been stochastically related. It is assumed that the component is degrading according to a renewal process. Ansell, Bendell and Humble [1980a,1980b] coined the phrase "Nested Renewal" for such processes, since the component's renewal process is nested within the renewal process of the subsystem. The model does however allow for component repair which may depend either on

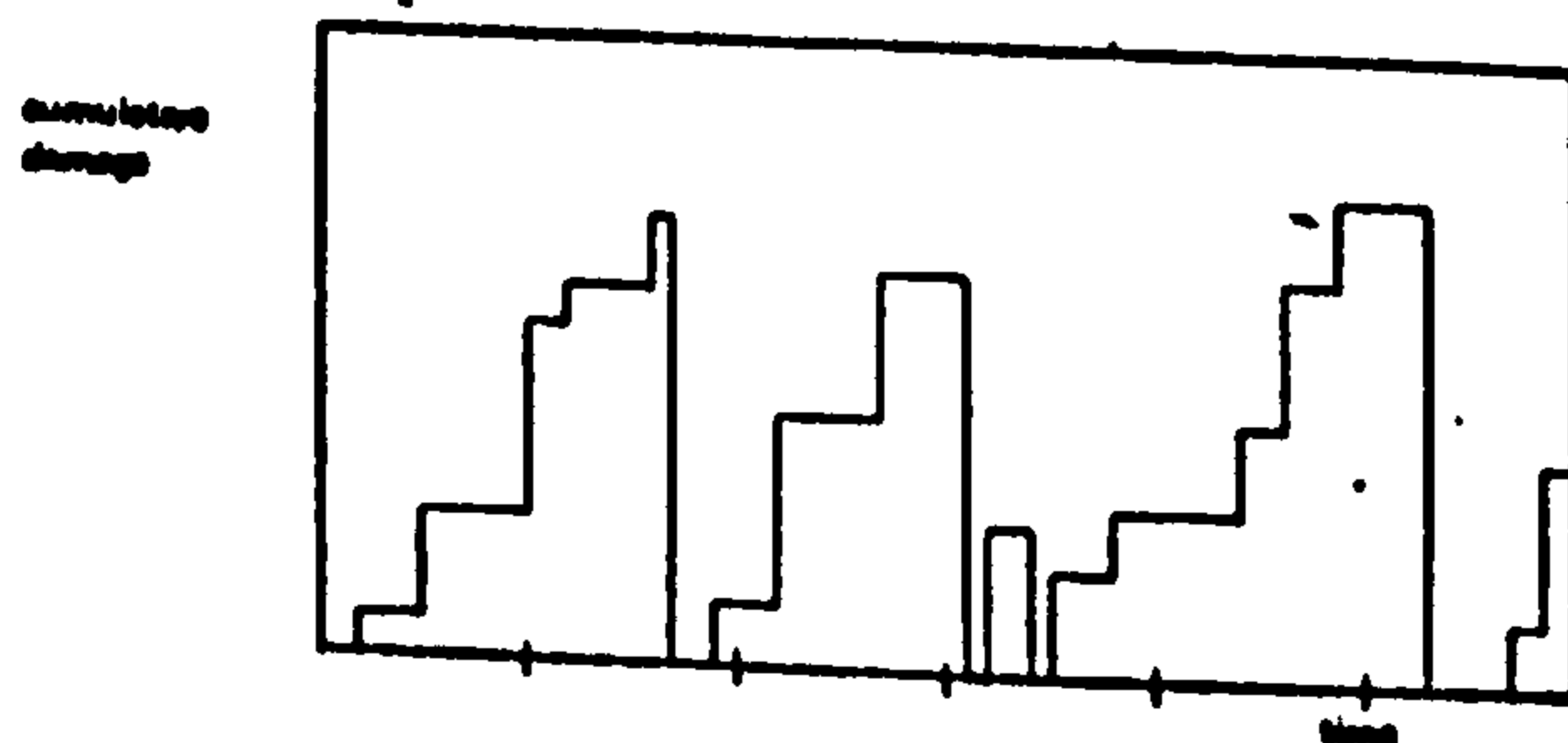
time or state of component.

Figure 4.1 illustrates a nested renewal process in which shocks occur to a component as a renewal process, each shock causing a random amount of damage. The amount of damage being independent of number of shocks. The component can then be replaced according to a second renewal process by a component which has suffered no damage.

Such processes are obviously related to, but not identical with, superimposed, alternating and cumulative renewal processes, see Cox [1962], and also the univariate and multivariate shock and damage processes of Mercer and Smith [1959], Marshall and Olkin [1967], and Esary, Marshall and Proschan [1974]. There are also similarities to Smith's original formulation of renewal processes which allowed for shocks of negative size, Smith [1955].

Figure 4.1

An illustration of Second Order Nested Renewal Process



4.2 Formulation and Notation

Let $X(t)$ be the number of blows accumulated by the component in place by time t , where $X(t)$ will be defined on the non-negative integers. The process $\{X(t), t > 0\}$ then describes the state of the component at time t , where $X(t)$ is random variable on the non-negative integers.

Let $\theta, (\theta > 0)$, be the time between shocks, or first order renewals, with probability density function $g(\theta)$, and τ be the time between replacements, or second order renewals, with probability density function $q(\tau)$. It is assumed that 0 returns to 0 as soon as shock occurs, and similarly θ and τ are returned to 0 following a replacement. The probability that k shocks have been accumulated by time t , $p\{X(t)=k\}$, can either be described by the differential equations (4.1) or in a renewal formulation by (4.2)

$$\frac{dp(X(t)=k)}{dt} = -[\gamma(t) + \lambda(k, t)] p(X(t)=k) + \lambda(k-1, t) p(X(t)=k-1)$$

$$k \neq 0, k = 1, 2, \dots$$

$$\frac{dp(X(t)=0)}{dt} = -\gamma(0, t) p(X(t)=0) + \sum_{k=0}^{\infty} \gamma(t) p(X(t)=k)$$

where $\gamma(t) = \sum_{i=1}^n q^i(s)$, $\lambda(k, t) = \int_0^t Q(s) \sum_{i=0}^{\infty} q^i(t-s) g^{k+1}(s) ds$ and $q^i(s)$ is the i th convolution of $q(s)$ and $Q(s)$ its survival function, similarly for $g^i(s)$ and $G(s)$.

$$p(x(t)=k) = \int_0^t \sum_{i=0}^k q^i(t-s) Q(s) \int_0^s g^k(s-u) G(u) du ds \quad (4.2)$$

The above process would be described as a Second Order Nested Renewal Process since there are two levels of renewal, the shock process and the replacement process. It is possible to define higher order processes, introducing further renewal processes which dominate the lower order processes. A third order process would be one where renewal of the third process would automatically mean renewal of both of the other processes.

The renewal formulation is far simpler to work with and so in the remainder of the chapter will concentrate on this formulation.

If $f(s)$ is the density for damage s resulting from a shock, then the density function for the accumulated damage $Y(t)$, ($t > 0$), at time t , $p(y(t))$, is given by

$$p(y(t)) = \sum_{k=0}^{\infty} p(X(t)=k) f^k(y(t)) \quad 4.3$$

where $f^k(s)$ is the k convolution of $f(s)$.

Given the convoluted nature of the formulation it is often helpful to apply Laplace transformations to (4.2). The Laplace transform of (4.2) may be written as

$$p^*(k,r) = L(k,r) / r \sum_{k=0}^{\infty} L(k,r) \quad 4.4$$

where

$$L(k,r) = \int \left\{ M(t) \int_0^t g^k(t-u) G(u) du \right\} \quad 4.5$$

Similarly the single and double transforms of $p(y(t))$ may be written

as

$$p^*(y(t), r) = \frac{L(y, r)}{r \int_0^{\infty} L(y, r) dy} = \frac{\sum_{k=0}^{\infty} L(k, r) f^k(y)}{r \sum_{k=0}^{\infty} L(k, r)} \quad 4.6$$

$$p^{**}(s, r) = \frac{L^*(s, r)}{r L^*(0, r)} = \frac{\sum_{k=0}^{\infty} L(k, r) \{f^*(s)\}^k}{r \sum_{k=0}^{\infty} L(k, r)} \quad 4.7$$

4.2.1 Example

Let $f(x)$, $g(\theta)$ and $q(\tau)$ each be exponential with parameter μ , λ and β respectively, then

$$p(X(t)=k) = \frac{\lambda^k e^{-(\lambda+\beta)t}}{(\lambda+\beta)^{k+1}} \left\{ \frac{\lambda(\lambda+\beta)^k t^k}{k!} + \beta \sum_{m=k}^{\infty} \frac{(\lambda+\beta)^m t^m}{m!} \right\}$$

and

$$p(y(t)) = \frac{\delta(x)}{(\lambda+\beta)} \left\{ \lambda e^{-(\lambda+\beta)t} + \beta \right\} + e^{-\mu x} \int_0^t e^{-(\lambda+\beta)s} \left(\frac{\lambda \mu s}{x} \right)^{1/2} I_1 \left([4 \lambda \mu s x]^{1/2} \right) \left\{ \beta + \delta(s-t) \right\} ds.$$

where I_1 is the Bessel function of imaginary arguments, see Gradshteyn and Ryzhik [1965].

The process is therefore a Poisson process subject to renewal itself.

4.2.2 Example

Let $q(r)$ be gamma (n, β) then for any $f(x)$ and $g(\theta)$,

$$p^*(k, r) = \frac{(\beta+r)^n}{((\beta+r)^n - \beta^n)} \sum_{m=0}^{n-1} \frac{(-\beta)^m}{m!} \left(\frac{d}{dv} \right)^m \left[\frac{g^*(v)^k \{1 - g^*(v)\}}{v} \right] \Big|_{v=\beta+r}$$

$$p^{**}(s, r) = \frac{(\beta+r)^n}{((\beta+r)^n - \beta^n)} \sum_{m=0}^{n-1} \frac{(-\beta)^m}{m!} \left(\frac{d}{dv} \right)^m \left[\frac{1 - g^*(v)}{v(1 - f^*(s)g^*(v))} \right] \Big|_{v=\beta+r}$$

where $g^*(v)$ is the Laplace transformation of $g(\theta)$.

Laplace transformations are particularly helpful in obtaining the limiting distributions which are;

$$\lim_{t \rightarrow \infty} p\{X(t)=k\} = \lim_{r \rightarrow 0} r p^*(k, r)$$

$$\lim_{t \rightarrow \infty} p^*(s, t) = \lim_{r \rightarrow 0} r p^{**}(s, r)$$

4.2.3 Examples

Recalling example 4.2.1

$$\lim_{t \rightarrow \infty} p\{X(t)=k\} = \{g^*(\beta)\} \{1 - g^*(\beta)\}$$

and

$$\lim_{t \rightarrow \infty} p^*(s, t) = \frac{\beta(\mu+s)}{\{(\beta+\lambda)(\mu+s) - \lambda\mu\}}$$

hence

$$\lim_{t \rightarrow \infty} p^*(y(t)) = \frac{\beta}{(\beta+\lambda)} \delta(\lambda) + \frac{\lambda\mu}{(\beta+\lambda)^2} \exp\left\{-\frac{\beta\mu x}{\lambda+\beta}\right\}$$

It is interesting to note the limit of the distribution of the number of shocks when $q(\tau)$ is exponential, with parameter, β , is geometric with parameter $g^*(\beta)$.

4.3 Characteristics of second-order processes

In this section some of the important measures for the processes are derived, following general renewal theory, see Cox [1962].

4.3.1 The number of shocks accumulated by time t .

The moments of $X(t)$ are most easily obtained from the appropriate generating function.

$$E\{X(t)\} = \left. \frac{\partial \pi(z,t)}{\partial z} \right|_{z=1} = \int_0^{\infty} \left\{ \left. \frac{\partial \pi^*(z,\tau)}{\partial z} \right|_{z=1} \right\} \quad 4.7$$

$$E\{X(t)^2\} = \int_0^{\infty} \left\{ \left. \frac{\partial^2 \pi^*(z,t)}{\partial z^2} \right|_{z=1} + \left. \frac{\partial \pi^*(z,t)}{\partial z} \right|_{z=1} \right\} \quad 4.8$$

where $\pi(z,t) = \sum_{k=0}^{\infty} z^k p(X(t)=k)$.

Hence

$$\text{Var}(X(t)) = \int_0^{\infty} \left\{ \left. \frac{\partial^2 \pi^*(z,t)}{\partial z^2} \right|_{z=1} + \left. \frac{\partial \pi^*(z,t)}{\partial z} \right|_{z=1} \right\} - \left\{ \int_0^{\infty} \left. \frac{\partial \pi^*(z,t)}{\partial z} \right|_{z=1} \right\}^2 \quad 4.9$$

Example 4.3.1.1

Let $g(\theta)$ be exponential (λ) and $q(\tau)$ be gamma ($2, \beta$) then

$$\pi^*(z, t) = \frac{(\beta+r)^2}{\{(\beta+r)^2 - r^2\}} \left\{ \frac{1}{\{\beta+r+\lambda(1-z)\}} + \frac{\beta}{\{\beta+r+\lambda(1-z)\}^2} \right\}$$

$$\mathcal{L}\{E[X(t)]\} = \frac{\lambda}{r(2\beta+r)} + \frac{2\lambda\beta}{r(2\beta+r)(\beta+r)}$$

and so

$$E[X(t)] = \frac{\lambda}{2\beta} \left[3 - 4e^{-\beta t} + e^{-2\beta t} \right]$$

Higher order moments can be obtained in a similar fashion. It is interesting to note that unlike ordinary renewal processes $E[X(t)]$ need no longer be non-decreasing in t . In an example Ansell, Bendell and Humble where $g(\theta)$ was exponential and $q(\tau)$ gamma ($3, \beta$), it was claimed incorrectly that an oscillatory pattern results. Bendell and Scott [1984] gave examples in which a non-monotonic behaviour was established for $g(\theta)$ exponential and $q(\tau)$ gamma ($7, \beta$).

Bendell and Scott [1984] also gave further details on limits for moments in the special case of Erlangian family of distributions.

4.3.2 The damage accumulated by t .

By conditional expectations it is possible to derive the moments of

the damage accumulated by time t , and the first moments are;

$$E\{Y(t)\} = E\{X(t)\} \mu_0 \quad 4.13$$

$$\text{Var} (Y(t)) = \text{Var} (X(t)) \mu_0^2 + E\{X(t)\} \sigma_0^2 \quad 4.14$$

$$\text{Cov} (X(t), Y(t)) = \text{Var} (Y(t)) \mu_0 \quad 4.15$$

where μ_0 and σ_0^2 are the mean and variance of the distribution for amount of damage from a shock.

4.3.2 Time since the last event

It is often desired to know the distribution of the time to or since the last event. If one was considering the replacement process this would simply be the usual forward and backward recurrence-time distributions. When considering both replacement and shock then the distribution for the time since the last event, t' , at time t has the following form;

$$P(\text{No shock or replacement}) = G(t) Q(t)$$

$$p(t' | t) = G(t') \left[Q(t) \sum_{k=1}^{\infty} g^k(t-t') + Q(t') \sum_{l=1}^{\infty} q^l(t-t') + \int_{t'}^t Q(\tau) \left\{ \sum_{l=1}^{\infty} q^l(t-\tau) \right\} \sum_{k=1}^{\infty} g^k(\tau-t') d\tau \right] \quad 4.16$$

A related property is the intensity with which the k th shock since a replacement is realized at time t ;

$$f_k(t) = \int_0^{t_\infty} \sum_{l=0}^{\infty} g^k(\tau) q^l(t+\tau) Q(\tau) d\tau \quad 4.17$$

4.3.3.1 Example

For any $g(\theta)$, let $q(\tau)$ be gamma (n, β) , then the Laplace transform of $\pi_k(r, z)$, (the generating function of $f_k(t)$) is

$$\pi_k^*(r, s) = \frac{(\beta+r)^n}{\{(\beta+r)^n - \beta^n\}} \sum_{m=0}^{n-1} \frac{(-\beta)^m}{m!} \left(\frac{d}{dv} \right)^m \left[\frac{1}{1 - q^*(v)z} \right] \Big|_{v=\beta+r}$$

The density function of the first time to k th accumulated shocks is

$$g_k(t) = \int_0^t \sum_{l=0}^{\infty} g^k(\tau) Q(\tau) z^l(t+\tau) d\tau \quad 4.18$$

where

$$z^i(t) = \int_0^t \int_0^s z^{i-1}(t'+s) q(s) \sum_{j=0}^{\infty} g^j(s+u) G(u) du ds \quad i=1, 2, \dots$$

$$\text{and } z^0(t') = \delta(t')$$

4.4 Renewals dependent on the accumulation of shocks and damage

In previous sections the assumption has been made that the second order renewal (replacement) process was purely time dependent, for many types of component replacement will also depend on the state of the component. Hence there are two replacement mechanisms, a replacement of the subassembly and replacement of the component due to the number of shocks or accumulated damage. If the following simplifying assumptions

are made, that the two replacement processes are independent and competing and also that the probability distribution function for number of shocks at replacement is $b(k)$ then,

$$p(X(t)=k) = B(k) \int_0^t \left\{ \int_0^\tau g^k(\tau-s) G(s) ds \right\} Q(\tau) \sum_{i=0}^{\infty} Z^i(t-\tau) d\tau \quad -4.19$$

$$\text{where } Z^i(t) = \int_0^t Z^{i-1}(t-s) \left\{ q(s) \sum_{m=0}^{\infty} \int_0^s g^m(s-u) G(u) B(m) du + \sum_{k=1}^{\infty} g^k(s) b(k) Q(s) \right\} ds$$

$i=1, 2, \dots$

$$Z^0(t) = \delta(t) \text{ and } B(k) = \sum_{i=k+1}^{\infty} b(i)$$

Similar expressions may be derived for the damage model, with $b(x)$ now the probability density function for the accumulated damage at replacement in which case

$$p(Y(t)) = \sum_{k=0}^{\infty} f^k(x) B(x) \int_0^t \left\{ \int_0^\tau g^k(\tau-s) G(s) ds \right\} Q(\tau) \sum_{i=0}^{\infty} W^i(t-\tau) d\tau \quad -4.20$$

where

$$W^i(t) = \int_0^t W^{i-1}(t-s) \left\{ q(s) \sum_{n=0}^{\infty} \int_0^s \int_0^s g^m(s-u) G(u) B(x) f^m(x) dudx + \sum_{k=1}^{\infty} g^k(s) \left[\int_0^s f^{k-1}(x-y) f(y) B(x-y) - B(x) \right] Q(s) dy dx \right\} ds$$

$$W^0(t) = \delta(t) \text{ and } B(x) = \int_x^{\infty} b(y) dy$$

Expressions for the Laplace transforms can be written as

$$p(k, r) = \frac{\tilde{L}(k, r)}{r \sum_{k=0}^{\infty} \tilde{L}(k, r)} \quad -4.21$$

$$p^*(y, r) = \frac{\tilde{L}(y, r)}{r \sum_{k=0}^{\infty} \tilde{L}(y, r) dy}$$

*4.22

where $\tilde{L}(k, r) = B(k)L(k, r)$

and $\tilde{L}(y, r) = B(y)L(y, r)$.

Example 4.4.1

From example 4.2.1

$$\tilde{L}(k, r) = \frac{\lambda^k}{(\lambda + \beta + r)^{k+1}} B(k)$$

$$\tilde{L}(y, r) = \frac{\lambda B(y)}{(\lambda + \beta + r)^2} \exp\left\{-\frac{\mu(\beta + r)y}{\lambda + \beta + r}\right\} + \frac{\delta(y) B(y)}{(\lambda + \beta + r)}$$

Hence

$$\lim_{t \rightarrow \infty} p(X(t)=k) = \lim_{r \rightarrow 0} \frac{\tilde{L}(k, r) B(k)}{\tilde{L}(k, r) B(k)} = \frac{B(x) \left[\delta(x) + \mu \exp\left\{-\frac{\mu \beta x}{\lambda + \beta}\right\} \right]}{1 + \mu \int_0^{\infty} B(x) \exp\left\{-\frac{\mu \beta x}{\lambda + \beta}\right\} dx}$$

4.5 Modified Second Order Processes

It has been assumed for ease of description that the renewal processes have been ordinary renewal processes, however for many practical models it is more appropriate to consider modified processes. Obviously one may modify any of the processes involved the replacement, shock or damage. This section considers modification of the damage process, the model developed will be applied in the next section to wear of tyres. It will be assumed that the density for the damage from the first is $f_1(x)$ and for subsequent shocks $f_2(x)$. By substituting the

following expression for the convolution $f^k(x)$,

$$f^k(x) = \int_0^x f_1(x-y) f_2^{k-1}(y) dy, \quad k=2,3,\dots \quad \text{4.23}$$

$$f^0(x) = f_1^0(x) = \delta(x), \quad f^1(x) = f_1^1(x) = f_1(x)$$

where $f_2^j(x)$ is the j th convolution of f_2 , then the density for the accumulative damage by time t will again be given by 4.3

4.5.1 Example

Recalling example 4.2.2 then the equivalent double Laplace transformation will be

$$p^{**}(s,r) = \frac{(\beta+r)^n}{(\beta+r)^n - r^n} \sum_{m=0}^{n-1} \frac{(-\beta)^m}{m!} \left(\frac{d}{dv} \right)^m \left[\frac{\{1-g^*(v)\} \{1 - (f_1^*(s) - f_2^*(s))g^*(v)\}}{v \{1 - f_2^*(s)g^*(v)\}} \right] \Big|_{v=\beta+r}$$

where $f_1^*(s)$ and $f_2^*(s)$ are the Laplace transforms of $f_1(x)$ and $f_2(x)$.

The moments of the modified process are

$$E\{Y(t)\} = \mu_{01} + (E\{X(t)\} - 1) \mu_{02} + (\mu_{01} - \mu_{02}) p \quad \text{4.24}$$

$$\begin{aligned} \text{Var}\{Y(t)\} &= \sigma_{01}^2 (1-p) + \sigma_{02}^2 (E\{X(t)\} - 1 + p) + \text{Var}\{X(t)\} \mu_{02}^2 \\ &\quad + p (\mu_{02} - \mu_{01}) [p(\mu_{02} - \mu_{01}) + 2(E\{X(t)\} - 1) \mu_{02} + \mu_{01}] \end{aligned} \quad \text{4.25}$$

$$\text{Cov}\{X(t), Y(t)\} = \text{Var}\{X(t)\} \mu_{02} - (\mu_{02} - \mu_{01}) p E\{X(t)\} \quad \text{4.26}$$

where $p = p(X(t)=0)$ and μ_{0i}, σ_{0i}^2 are the means and variances of $f_i(x)$.

Example 4.5.2

Let $g(\theta)$, $q(\tau)$ be exponential with parameters λ and β respectively, and let $f_1(x)$ and $f_2(x)$ be respectively gamma $(3, \mu)$ and gamma $(2, \mu)$ then

$$p^{**}(s, r) = \frac{\beta \{(\lambda + \beta + r)(\mu + s)^3 - s\mu^2\lambda\}}{r(\lambda + \beta + r)(\mu + s) \{(\mu + s)^2(\lambda + \beta + r) - \mu^2\lambda\}}$$

Again the limits may be obtained from the Laplace transform and hence for example 4.5.1 one has

$$\lim_{t \rightarrow \infty} p(y(t)) = \frac{\delta(x) + \mu e^{-\mu x} [\cosh \{ \mu x (\frac{\lambda}{\lambda + \beta})^{1/2} \} - 1]}{1 + \int_0^{\infty} \mu e^{-\mu x} [\cosh \{ \mu x (\frac{\lambda}{\lambda + \beta})^{1/2} \} - 1] dx}$$

$$\lim_{t \rightarrow \infty} E\{Y(t)\} = \frac{\lambda(3\beta + 2\lambda)}{\mu(\lambda + \beta)\beta}$$

$$\lim_{t \rightarrow \infty} \text{Var}\{Y(t)\} = \frac{\lambda(2\lambda^3 + 8\lambda^2\beta + 15\beta^2\lambda + 10\beta^3)}{\mu^2(\lambda + \beta)^2\beta^2}$$

$$\lim_{t \rightarrow \infty} \text{Cov}\{X(t), Y(t)\} = \frac{\lambda(2\lambda^2 + 4\beta\lambda + 3\beta^2)}{\mu(\lambda + \beta)\beta^2}$$

4.6 Application To Tyre Wear Data.

From a survey of parked cars by Grogan and Watson [1974] for the Dunlop Tyre Company, data on car tyre wear was obtained, and this is presented in Figure 4.2. It is assumed that each tyre has initially a tread of 9mm, and that Figure 2 represents the steady-state distribution of tyre wear. It is believed that initially a disproportionately large amount of tread is lost as 'the corners are knocked off', but subsequently the rate of loss is fairly constant. Hence it seems

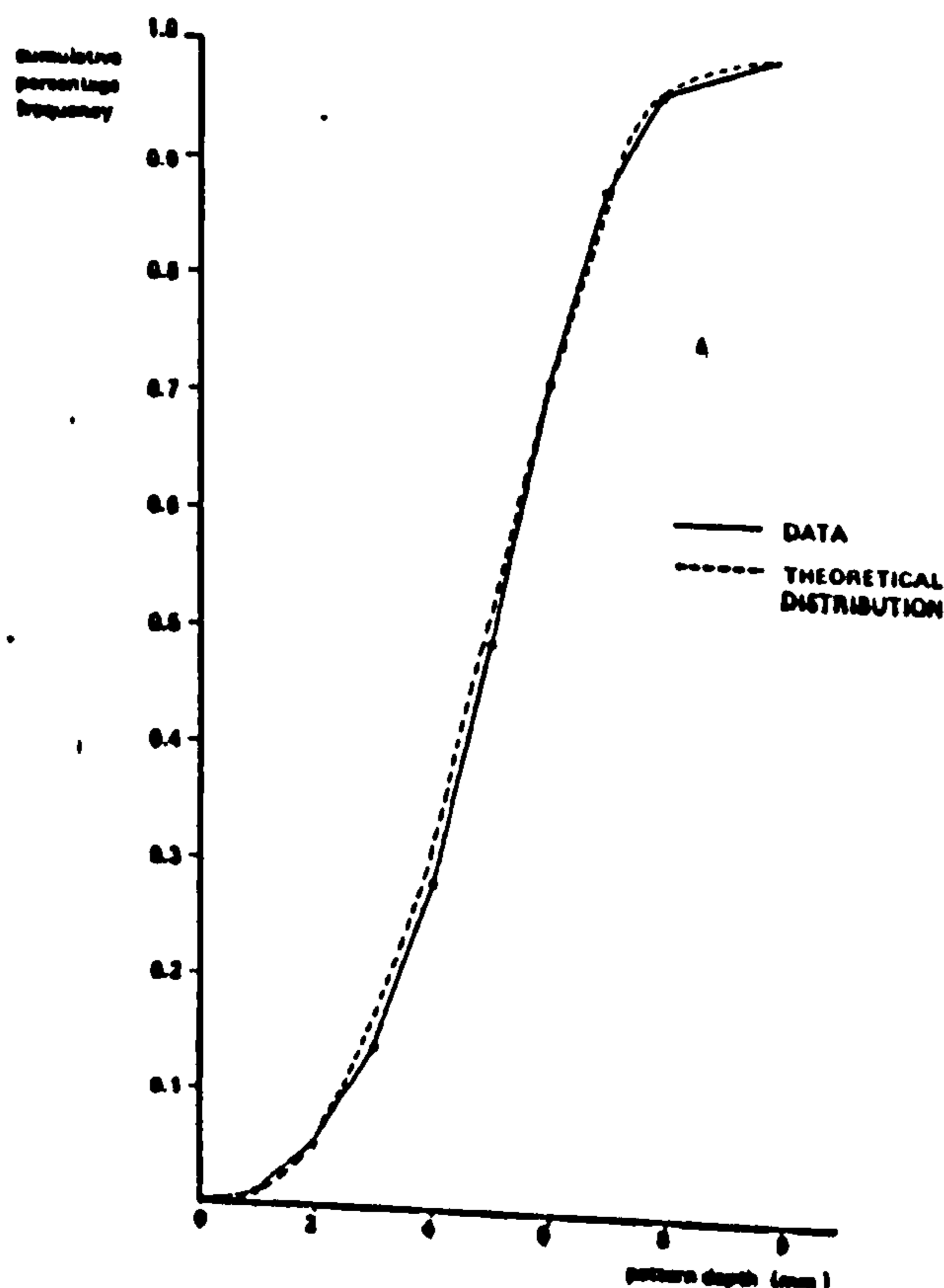
sensible to model the process via the modified process, given in section 4.5. Replacement is due not only to mileage, time t , but also to the amount of wear x , or tread $9^A x$. Hence it seems reasonable to apply a model of type given in section 4.4.

Given the above description the following model seems appropriate. The initial loss is characterized by $f_1(x)$ and $f_2(x)$ which are gamma $(3, \beta)$ and $(2, \beta)$ and $g(\tau)$ and $q(\nu)$ are exponential with parameters λ, μ with $b(x)$ having a truncated density

$$b(x) = \begin{cases} \frac{1 - \exp[-\beta(9^A x)^2]}{\int_0^9 1 - \exp[-\beta(9^A x)^2] dx} & x \leq 9 \\ 0 & x > 9 \end{cases}$$

Figure 4.2

Cumulative tread distribution



This model is a simple extension of that of 4.5.2 and it leads to the distribution for the tread of tyre on road as

$$\lim_{t \rightarrow \infty} p(y(t)) = \frac{e^{-\mu x} [\cosh\{x\mu(\lambda/\lambda+\beta)\} + 1][1 + e^{-\beta(q-x)^2}]}{\int_0^q e^{-\mu x} [\cosh\{x\mu(\lambda/\lambda+\beta)\} + 1][1 + e^{-\beta(q-x)^2}] dx}$$

The fit to the observed values of wear, shown in Figure 2, was performed by Least Squares using a Conjugate Gradient Method. Whilst the fit is reasonable it might be improved by the introduction of a threshold parameter representing the number of shocks accumulated before the initial shock damage distribution $f_1(x)$ is superseded by $f_2(x)$.

4.7 Higher Order Processes

As indicated in section 4.2 it is possible to construct higher order processes, allowing further nesting of renewal processes. These would be attempts at modelling more complex systems involving several layers of subsystems. The models are simple extension of the Second Order Processes, for example in the case of Third Order Processes one would have;

$$p^{(3)}(X(t)) = \int_0^t \sum_{\ell=0}^{\infty} q_2^\ell(t-s) Q_2(s) p(X(s)) ds \quad \text{4.27}$$

$$p^{(3)}(y(t)) = \int_0^t \sum_{\ell=0}^{\infty} q_2^\ell(t-s) Q_2(s) p(Y(t)) ds \quad \text{4.28}$$

where $Q_2(t) = \int_t^\infty q_2(s) ds$ and $q_2(t)$ is the density for second replacement.

In general for the u th order process one would have

$$p^{(u)}(X(t)) = \int_0^t \sum_{\ell=0}^{\infty} q_{u-1}^{\ell}(t-s) Q_{u-1}(s) p^{(u-1)}(X(t)) ds \quad -4.29$$

$$p^{(u)}(y(t)) = \int_0^t \sum_{\ell=0}^{\infty} q_{u-1}^{\ell}(t-s) Q_{u-1}(s) p^{(u-1)}(y(t)) ds \quad -4.30$$

where $Q_u(t) = \int_t^{\infty} q_u(s) ds$

and $p^{(2)}(X(t)) = p(X(t)), p^{(2)}(y(t)) = p(y(t)).$

The Laplace Transforms of $p(X(t))$ and $p(y(t))$ would be

$$p^{(u)*}(k, r) = \frac{L^{(u)}(k, r)}{r \int_0^{\infty} L^{(u)}(k, r)} \quad -4.31$$

$$p^{(u)*}(y, r) = \frac{L^{(u)}(y, r)}{r \int_0^{\infty} L^{(u)}(y, r) dy} \quad -4.32$$

where $L^{(u)}(k, r) = \int \{ Q_{u-1}(t) p^{(u-1)}(X(t)) \}$

and $L^{(u)}(x, r) = \int \{ Q_{u-1}(t) p^{(u-1)}(Y(t)) \}$

Example 4.7.1

Let $q_2(\tau)$ be gamma (m, δ) then

$$L^{(3)}(k, r) = \sum_{\ell=0}^{m-1} \frac{(-\delta)^{\ell}}{\ell!} \left(\frac{\partial}{\partial u} \right)^{\ell} p^{(2)*}(k, u) \Big|_{u=\delta+r}$$

Note from example 4.7.1 if $q_2(\tau)$ and $q(\tau)$ are both exponential then the third order process reduces to a second order process. In general if r of the replacements are one-parameter exponential then a u th order process will reduce to $(u-r)$ th order process. This follows from the constant hazard or lack of memory of the exponential distribution. If $q_r(t)$ is exponential and $q_{r-1}(t)$ is not, the resultant effect is identical to that of a pair of superimposed processes, whilst if $q_r(t)$ is non-exponential and $q_{r-1}(t)$ is exponential they form together a sub-process subject to competing risks.

4.8 Limiting Distributions

As has been shown throughout, Laplace transforms generally provide a simple procedure for obtaining the limiting distribution as $t \rightarrow \infty$. For the special case of the Erlangian distributions Bendell and Scott [1984] employed the approach to obtain a further limiting result. An alternative approach which in certain cases provides an easier route would be to consider the structure of the processes. The second order renewal (replacement) process may be treated as an ordinary renewal processes so in the limit as $t \rightarrow \infty$ the processes will reach its equilibrium distribution, see Cox [1962], its density function will then be given by

$$\frac{Q(t)}{E(t)} \qquad -4.33$$

where $E(t)$ is the mean of the distribution with probability density function $q(\tau)$.

Hence the time since the last replacement has the distribution given above, thus in the limit the probability of k shocks would be

$$\lim_{t \rightarrow \infty} p(X(t)=k) = \int_0^{\infty} \frac{Q(u)}{E(t)} \int_0^u g^k(y) G(u-y) dy du \quad -4.34$$

Example 4.8.1

Taking $q(t)$ as exponential parameter β

$$\begin{aligned} \lim_{t \rightarrow \infty} p(X(t)=k) &= \int_0^{\infty} e^{-\beta u} \int_0^u g^k(y) G(u-y) dy du \\ &= \{ g^*(\beta) \}^k \{ 1 - g^*(\beta) \} \end{aligned}$$

Again, as in example 4.2.3, $X(t)$ has limiting distribution which is geometric with parameter $g^*(\beta)$, the Laplace transformation of g evaluated at β , as $t \rightarrow \infty$.

Example 4.8.2

Take $g(\theta)$ to be exponential (λ) then

$$\begin{aligned} \lim_{t \rightarrow \infty} p(X(t)) &= \int_0^{\infty} \frac{Q(u)}{E(t)} \frac{(\lambda u)^r e^{-\lambda u}}{r!} du \\ &= \frac{1}{Q^*(0)} \frac{(-\lambda)^r}{r!} \frac{d^r Q^*(\lambda)}{d\lambda^r} \end{aligned}$$

The general probability generating function would be in the limit

$$\pi(z) = \lim_{t \rightarrow \infty} \pi(z, t) = \int_0^{\infty} \frac{Q(u)}{Q^*(0)} \int_{\tau_0}^u g^k(x) G(u-x) z^k dx du \quad 4.35$$

Example 4.8.3 If $g(\theta)$ exponential (λ) then

$$\pi(z) = \lim_{t \rightarrow \infty} \pi(z, t) = \frac{Q^*(\lambda(z-1))}{Q^*(0)}$$

The moments can again be obtained from the Probability Generating Function.

In the case of $g(\theta)$ exponential

$$\begin{aligned} E\{X(t)\} &= \left. \frac{d\pi(z)}{dz} \right|_{z=1} = \frac{1}{Q^*(0)} \left. \frac{dQ^*(v)}{dv} \right|_{v=0} \lambda \\ &= \frac{\lambda \int_0^{\infty} t Q(t) dt}{E(t)} = \frac{\lambda E(t^2)}{2 E(t)} = \lambda \left\{ \frac{\sigma_w^2}{\mu_w} + \mu_w \right\} \end{aligned}$$

where σ_w^2, μ_w are the variance and mean of the $q(\tau)$ distribution.

4.9 Systems Models

The chapter has concentrated on the performance of the component rather than the performance of the system. By supposing that each subassembly consists of one component then following chapter 3 it is

possible to describe the performance of a system in which the components are replaced. However this is not a traditional repair model, see Singh and Billington [1977], since components do not fail and wait for repair. Obviously one could designate some of the states as failed states of the component. For example all the states such that $X(t) \geq k$, for some integer k , could be deemed failed states of the component. Time spent in these states would then be the repair time of the component.

Chapter 5

Optimal Age Replacement

5.1 Introduction

Following on from chapter 4 this chapter considers the cost of replacement of a component of a system. There is assumed to be a sequence of components which are replaced either on their failure or preventatively. The components are assumed to suffer ageing and that there are costs associated with their replacement. Given that failure is likely to be more costly than preventative replacement the desire is to construct an optimal policy for preventative replacement which minimizes the overall cost. The policies considered in this chapter have the following form, if a component fails before time T it is replaced by a new component, if it is still working at T it is replaced. Hence the strategy is to choose T to minimize costs. The policy will depend on the cost criteria considered, this chapter examines three alternative cost criteria, previously considered by Barlow and Proschan [1962,1965], Derman and Sacks [1960], and Christer [1978]. Initially the work considers the simple case of non-discounted costs and establishes a time ordering for replacement with constant costs and then considers variable costs. The later section examine the case of discounted costs. For the cost criterion in Derman and Sacks [1960] more than one generalization is considered. The time ordering established in the non-discounted case does not hold for discounted case.

5.2 Constant Cost Case

The simplest model is to assume that the cost per component are fixed for both cost on failure, c_2 , and cost on preventative replacement, c_1 , with $c_2 > c_1$. This can be interpreted as a cost for replacement plus a cost for unplanned stoppage in process. If the replacement time is T and the lifetime of i th component is t_i then cost associated with component i in the sequence, X_i , will be c_2 if $t_i < T$ and c_1 if $t_i \geq T$.

5.2.1 Alternative Criteria

Barlow and Proschan [1962,1965] derived both optimal periodic and sequential policies to minimize the accumulated expected cost of replacements and failures up to a time horizon t , $c(t)$, or equivalently the average cost per unit time $c(t)/t$. The periodic policy assumes a fixed replacement time T throughout, whereas the sequential policy allows the replacement time T to vary dependent on the remaining time to reach t . The latter objective function can be written in the following form;

$$c(t)/t = \left\{ \sum_{i=1}^{N(t)} E \{ X_i \} \right\} / t \quad \text{---(5.1)}$$

where $N(t)$ is the renewal function representing the number of renewals in time t .

Unfortunately, whilst the policy is optimal for a finite time

horizon, the analytic identification and practical implementation of the policy can be only obtained through dynamic programming. In practice an asymptotic approximation to (5.1) is used, (Barlow and Proschan [1965], Berg [1976], Bergman [1980a,b], Cox [1962], Glasser [1967] and Hjorth [1978]).

By Wald's relationship for Renewal Reward Processes assuming the lifetimes are independently and identically distributed one can write $c(t)$ as follows;

$$c(t) = \{H(t)+1\} E[X] - E[X_{N(t)+1}] \quad - (5.2)$$

where $H(t)$ is the renewal function. (The renewal function is the expected number of renewals, replacements.) From Cox [1962] $H(t)$ can be expressed as follows;

$$H(t) = (t/\mu) + (\sigma^2 + \mu^2) / 2\mu^2 + o(1). \quad - (5.3)$$

where μ and σ^2 are the mean and variance of the lifetime of a component.

The limit as $t \rightarrow \infty$ for $c(t)/t$ then may be written as

$$\psi(T) = \lim_{t \rightarrow \infty} c(t)/t = E[X]/\mu = \left\{ c_2 + \frac{(c_1 - c_2) R(T)}{\int_0^T R(s) ds} \right\} \quad - (5.4)$$

where $R(t) = \int_t^\infty f(s) ds$ is the survivor function, or reliability of the component, and $f(t)$ is probability density function for lifetime of the component.

Christer's refined criterion is obtained by taking the second term in the asymptotic expansion which leads after some manipulation to

$$\int_t^{\infty} \psi(T) = \psi(T) \left[1 - \frac{T}{t} + \frac{\int_0^T R(s) ds}{t \int_0^T R(s) ds} \right] + \frac{c_2}{t} \left[\frac{T}{\int_0^T R(s) ds} - 1 \right] \quad -5.5$$

As $t \rightarrow \infty$ Christer's criterion (5.5) converges to (5.4), referred throughout the chapter as the usual criterion, whilst for small t it is not valid. Thus the advantage of Christer's criterion may have over the usual is when approximating $c(t)/t$ for large but finite t . If this is so then it will be achieved at the expense of added complexity of solution. However Barlow and Proschan [1962,1965] suggested in such circumstances of fixed time horizons it might be more appropriate to consider a sequential policy. The sequential policy suggested by them was as follows, at each replacement point calculate a new optimal replacement time which minimizes the expected cost over the remaining time. They also indicate for the example of Erlangian Distribution with parameter 2 that sequential policy is closely approximated by the asymptotic approximation, see Figure 5.2. It therefore appears that Christer's criterion offers limited advantage.

The third criterion considered is the selection of a sequential policy which is based on minimizing the average cost per time per unit which may be written as;

$$\eta(t) = \frac{1}{N(t)} \sum_{i=1}^{N(t)} E\{X_i/t_i\} \quad - (5.6)$$

In the limit as $t \rightarrow \infty$ the (X_i/t_i) are independent random variables, so that

$$\lim_{t \rightarrow \infty} \eta(t) = \phi(t) = E\{X/t\} \quad - (5.7)$$

exists and is independent of i provided $t > 0$ with probability 1. (This is assumed since $R(0)=1$). Thus the appropriate asymptotic criterion in this case is to choose the preventative replacement time to minimize the expected cost per unit time within a replacement cycle or any number of complete cycles (including an infinite number),

$$\phi(t) = c_1 \frac{R(T)}{T} + c_2 \int_0^T \frac{f(x)}{x} dx \quad - (5.8)$$

This criterion was first suggested by Derman and Sacks, [1960], and subsequently in Ansell, Bendell and Humble [1980], and Ansell and Bendell [1983]. In subsequent sections it will be referred to as the DS criterion. The criterion would be appropriate in the situation in which it is known a priori that the process were to be run for a fixed number of cycles rather than a fixed time period.

5.2.2 Numerical Solution.

The mean lifetime of component will be denoted by μ and the hazard function will be denoted by $h(t)$, ($h(t) = f(t)/R(t)$).

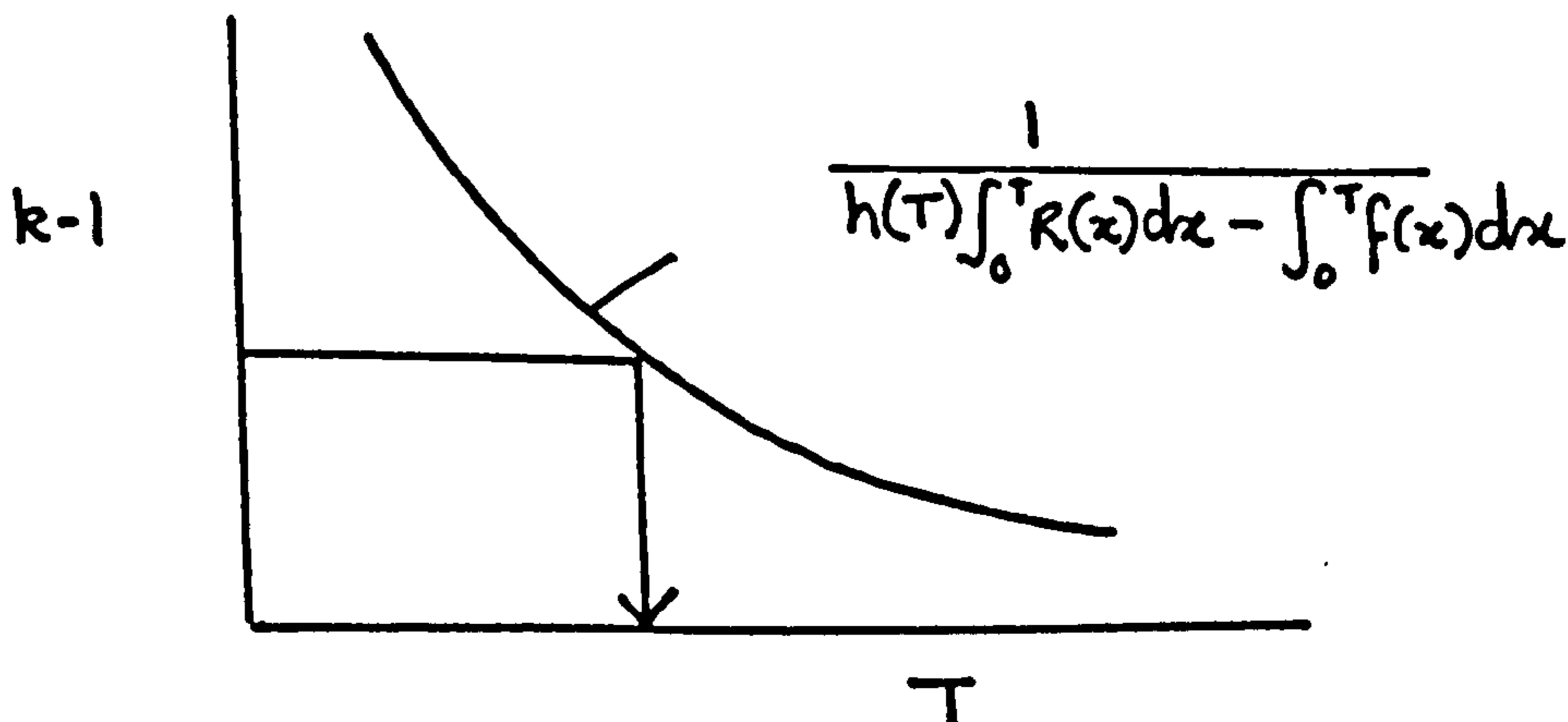
The usual asymptotic criterion has an optimal time replacement which satisfies the following equation, if it exists,

$$\frac{1}{k-1} = h(T) \int_0^T R(x) dx - \int_0^T f(x) dx \quad (5.9)$$

where $k = c_2/c_1 > 1$, see e.g. Berg [1980]. For most distributions considered equation (5.9), would require numerical solution, however for Weibull, gamma and truncated normal Glasser [1967] produced graphical solutions for the optimal times, see also Tadikamalla [1980]. By plotting the inverse the RHS of (5.9) against T it is then possible to find the optimal value of T by drawing a horizontal line at k^{-1} .

Diagram 5.1

Graphical Solution for the Optimal Time for the Usual Criterion



Results on existence and uniqueness of the time solution for this criterion are well known, e.g. Barlow and Proschan [1962,1985], and Berg [1980]. Following Cox, [1962], preventative replacement is worthwhile for an increasing hazard rate distribution if

$$k > [1 + (1/\mu h(\infty))] \quad (5.10)$$

Christer's refined criterion is more complex and requires greater numerical effort, the solution is T which satisfies the following equation,

$$\frac{1}{k-1} = \frac{c_1 h(T)}{\psi(T)} + \frac{[h(T) \int_0^T sR(s)ds + \int_0^T sf(s) ds] \int_0^T R(x)dx}{(t-T) \int_0^T R(s)ds + 2 \int_0^T sR(s) ds} \quad (5.11)$$

Graphical solutions can be obtained in similar manner to those of usual criterion since it is since (5.11) may be written as,

$$\frac{1}{k-1} = \frac{h(T) \int_0^T R(s)ds + F(T) + \int_0^T sf(s)ds \int_0^T R(s)ds + F(T) \int_0^T R(s)s ds}{(t-T) \int_0^T R(s)ds + 2 \int_0^T sR(s) ds} \quad (5.12)$$

Hence by plotting the inverse of (5.12) against T again optimal value may be read directly from the graph. Ansell and Bendell [1983] produce such graphs for Gamma (2,1).

Christer [1978] does not consider the existence and uniqueness of the optimal solution for the refined criterion, two theorems in this chapter gives further insight into the conditions under which the results might hold.

By contrast criterion (5.6) leads to considerably easier solutions which are the solution for T in the following,

$$\frac{1}{k-1} = T h(T) \quad (5.13)$$

A solution will exist if $k > 1$ and $h(\infty) \neq 0$. Provided the distribution has an increasing failure rate (IFR, see chapter 3) then the solution will be unique.

5.2.3 Time Orderings

In studying these criteria two possible questions arise concerning orderings, which is the 'cheapest' policy and secondly is there an ordering of the optimal time solutions for each criterion, T_ψ for the usual criterion, T_ϕ for DS criterion and T_ξ for Christer's criterion? There does not appear an immediate solution in general to the first since 'cheapness' depends on the time horizon as will be illustrated in an example later. There is however an answer to the second question, it is possible to establish the following ordering of the optimal times which is given in the following proposition.

Proposition 5.2.3.1

If the distribution has an increasing failure rate then

$$T_{\phi} < T_{\psi} < T_{\xi}$$

where T_{ψ} , T_{ξ} and T_{ϕ} are the optimal solutions for Usual (5.9), Refined (5.11) and DS (5.13) criterion.

Proof:

$$T_{\phi} \leq T_{\psi}$$

Consider RHS of (5.9) and (5.13), both are increasing in T if the distribution is IFR and $T_{\psi} = T_{\phi} = 0$ at $t=0$.

The optimal solution occurs when RHS equals $1/k^{\ast} > 0$.

Since $T \geq \int_0^T R(x) dx$ then RHS of (5.13) is greater than or equal to RHS of (5.9).

Proof is then immediate.

$$T_{\psi} \leq T_{\xi}$$

$$\lim_{t \rightarrow \infty} \int_t^{\infty} R(x) dx = \psi(T)$$

and consequently that

$$\lim_{t \rightarrow \infty} T_{\xi}(t) = T_{\psi}(t)$$

follows immediately from their definitions.

For all finite time t desire to prove $T_{\xi}(t) > T$, provided T_{ψ} and $T(t)$ are not both infinite.

A necessary condition for this is (5.10).

Since $h(T)$ is non-decreasing and

$$\int_0^T sR(s)[h(T)-h(s)] ds > 0$$

it follows that

$$h(T) \int_0^T sR(s) ds \geq \int_0^T sf(s) ds > 0$$

Thus the RHS of (5.11) is greater than or equal to the RHS of (5.9).

This completes the proof.

A earlier form of the proof appears in Ansell, Bendell and Humble [1984]. The proposition confirms the result found numerically by Christer for a uniform distribution.

Corollary 5.2.3.2

A necessary condition for a finite solution for the refined criterion is that the usual criterion has a finite solution.

Proof

Immediate from theorem 5.3.1

Corollary 5.2.3.3

If there is a unique solution for Christer's refined criteria for t then the solutions are monotonically decreasing as time horizon increases.

Proof.

Let RHS of (5.11) be $\int(T, t)$.

Then $\int(T, t)$ is increasing in t since it may be written as (5.12).

$\lim_{t \rightarrow \infty} \int(T, t)$ is RHS of (5.9).

If there is a unique solution then it results from $\int(T, t)$ crossing $1/k^A$ from below. Since $\int(T, t_2) > \int(T, t_1)$ for $t_2 > t_1$ and any T , then the solution for t_2 must be less than the solution for t_1 , and obviously solution for the usual criterion forms a lower bound.

Comment:

For there to be unique solution one simply requires that $\int(T, t)$ is increasing in T for $T < t$. Whilst this seems reasonable, the proof is not obvious.

5.2.4 Optimal solution for Gamma Distribution

In order to obtain further insight into the policy and their

expected costs the following example is considered on the Gamma distribution with shape parameter 2, (for ease the scale parameter is taken to be 1). This example has been frequently taken because of its simplicity, Barlow and Proschan [1962,1965].

The optimal solutions for the differing criteria are;

Usual : solution of

$$(k+1)e^{-T} = k+(2+k)T \quad (5.14)$$

Refined : solution of

$$\frac{1}{k+1} = \frac{1}{1+T} \{ (T+1) + e^{-T} \} \quad (5.15)$$

$$\frac{(2+(2+T)e^{-T}) \{ (T+2) + (2+T)e^{-T} \}}{t(2+(2+T)e^{-T}) + 2T+6+6e^{-T} + 4Te^{-2T}}$$

The DS criterion has the explicit solution

$$T = \{1 + (4k+3)^{1/2}\} / 2(k+1) \quad (5.16)$$

Note for the last criterion the solution exist for $k > 1$, whilst for usual criterion a solution is obtained only if $k > 2$, and this latter condition is necessary condition for a solution for the refined criterion.

Table 5.1 gives optimal replacement times for $k=2.5, 5, 10, 20$ and $t=2, 6, 10, 20$, for each of the criteria. Note Christer's solution depends on the time horizon. As expected the optimal times decrease as

k increase. However Table 5.1 does not show how close an approximation is achieved to the optimal sequential policy, this is conveyed in the Figure 5.2 in which the optimal replacement times for all the criterion including the sequential policy is given. The discontinuities of the optimal sequential policy are due to the increasing number of renewals allowed for as t increases. It is noticeable that Christer's refinement performs best just prior to a discontinuities otherwise the usual criterion might be described as best. The DS criterion performs badly as one might expect since it is designed to yield the optimal solution for a fixed number of cycles.

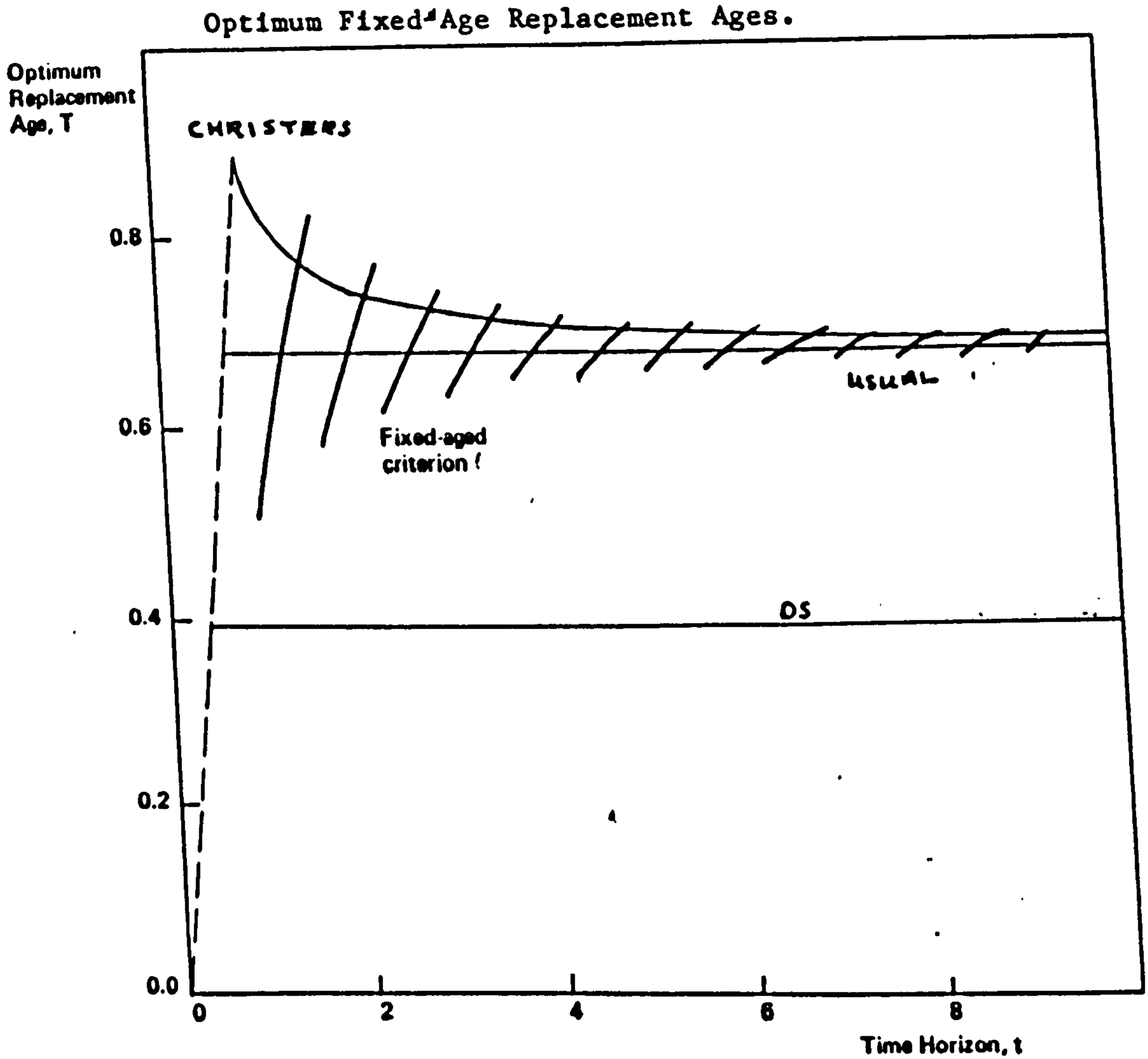
Table 5.1

Optimal Replacement Times for Erlangian Distribution (2,1)

| k | $T_{\xi}(2)$ | $T_{\xi}(6)$ | $T_{\xi}(10)$ | $T_{\xi}(20)$ | $T_{\psi} = T_{\xi}(\infty)$ | T_{ϕ} |
|------|--------------|--------------|---------------|---------------|------------------------------|------------|
| 2.5 | 2.000 | 6.000 | 10.000 | 5.625 | 4.979 | 1.215 |
| 5.0 | 1.705 | 1.386 | 1.351 | 1.327 | 1.305 | 0.640 |
| 10.0 | 0.740 | 0.698 | 0.690 | 0.685 | 0.680 | 0.394 |
| 20.0 | 0.431 | 0.417 | 0.415 | 0.414 | 0.412 | 0.257 |

In these cases preventative replacement under Christer's criterion is not worthwhile, so that preventative replacement is suppose to take place at the time horizon.

Figure 5.2



As an indication of the performance of these policies in practice the expected cost for the various criterion have been calculated assuming the distribution of lifetimes of component are Gamma (2,1) and these are shown graphically in Figure 5.3. For all criterion except Christer's the expected cost at time t , $L(t)$, was calculated as follows;

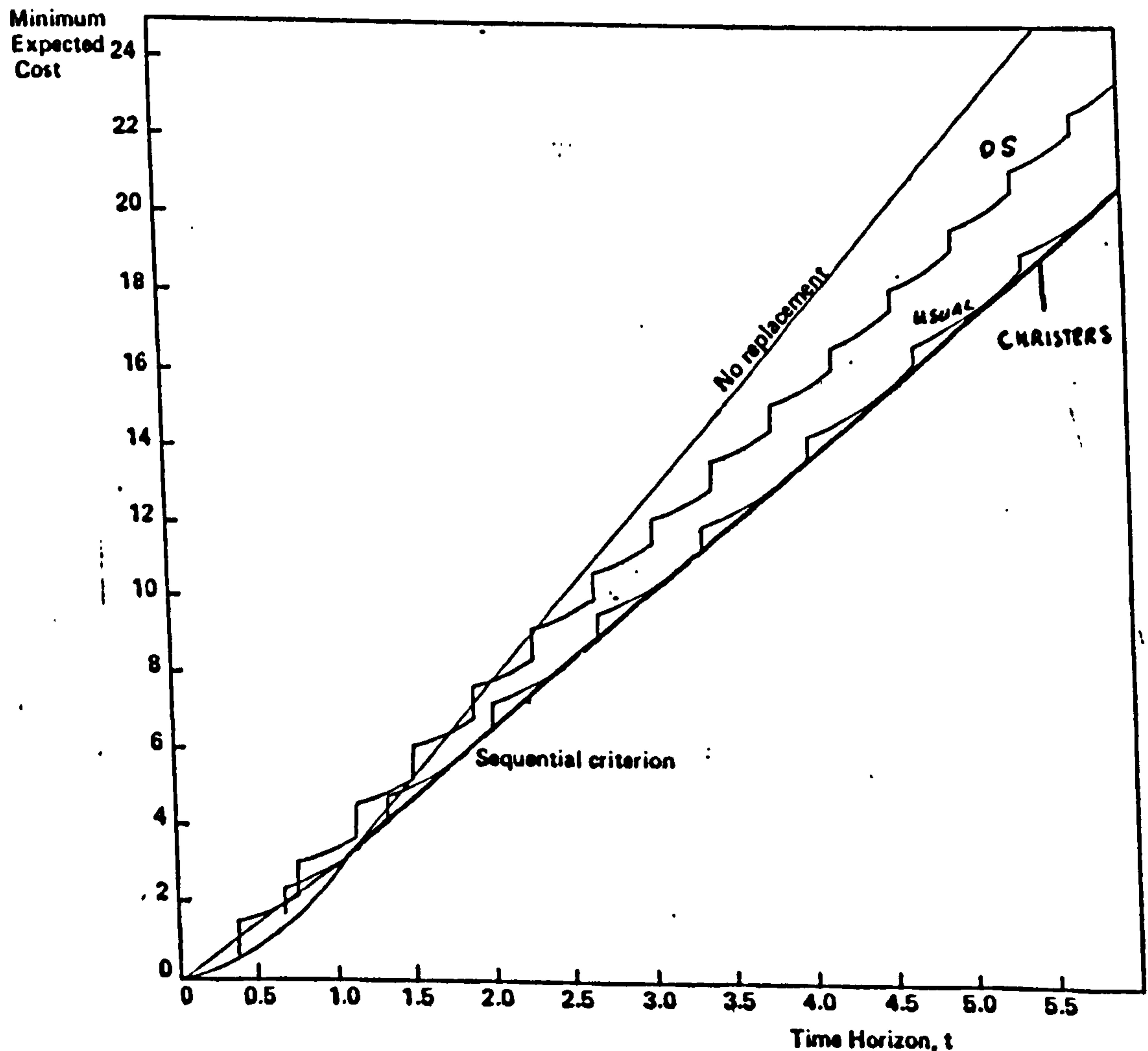
$$L(t) = \begin{cases} kN(t) & t < T \\ \int_0^t [k+L(t-s)]f(s)ds + [1+L(t-T)]R(t), & t > T \end{cases} \quad 5.17$$

where T is the optimum replacement age.

For Christer's criterion since the optimum time T depends on the horizon time t , it seems uncharitable to use the optimum time for a given horizon. Hence the expected cost is obtained by substituting the optimal solution back into (5.11) to get the expected cost. It can be seen that the usual asymptotic criterion performs reasonably well, compared to Christer's, optimum sequential and periodic criterion. It should be stressed that Christer's criterion is being shown in a favourable light in the Figure 5.3.

Figure 5.3

Expected Costs under the various Policies compared to the Sequential Policy.



5.3 Variable Cost case

In the previous section the costs were assumed fixed, however in practical problems it is often the case that costs vary with time so that the cost of failure is $A_1(T)$ and cost of preventative replacement $A_2(T)$. Such models have been employed in studies on forklift trucks, (Eilon et al [1966]) and air heaters in boiler plant, (Davidson [1970]). In Eilon et al [1966] the model for cost was of the form $(c + \int_0^t f(x) dx)/t$, where c was a constant and $f(x)$ a function dependent on previous history of maintenance costs, and Davidson uses $c + (\int_0^t f(x) dx)/t$. It will be assumed that the cost functions are analytic and differentiable. Given the comments made in the last section concerning the refined criterion this and subsequent sections will only consider criteria (5.4,5.6). The resulting expected cost are;

Usual Criterion:

$$W_u(T) = \frac{\int_0^T A_1(x) f(x) dx + A_2(T) R(T)}{\int_0^T R(x) dx} \quad - (5.19)$$

DS Criterion:

$$W_D(T) = \int_0^T \frac{A_1(x) f(x)}{x} dx + \frac{A_2(T) R(T)}{T} \quad - (5.20)$$

The optimal solutions are found from the following equations;

Usual Criterion:

$$0 = W_u(T) + (A_1(T) - A_2(T)) h(T) + A_2'(T) \quad - (5.21)$$

DS Criterion:

$$0 = (A_1(T) - A_2(T)) T h(T) + T A_2'(T) - A_2(T) \quad - (5.21)$$

For the DS criterion a graphical procedure is immediate, plotting $1/Th(T)$ against $V(T)=(A_1(T)-A_2(T))/(A_2(T)-TA'_2(T))$. The value of T at the intersection is the optimal replacement time. Certain forms for $A_1(t)$ and $A_2(t)$ lead to simple expressions for $V(t)$, for example if $A_i(T) = a_i T$ where a_i are constant then $V(t)$ is the constant $(a_1 - a_2)/(1-n) a_2$, whilst if $A_i(T)$ are linear of the form $(a_i T + c_i)$, with a_i and c_i constant, then $V(T)$ is a linear expression itself, being $V(t) = ((a_1 - a_2)T + (c_1 + c_2))/c_2$. A distinct advantage of the graphical form is that the sensitivity of the departures from the optimum solution is easily seen.

The extension of the fixed costs model to variable costs for the usual criterion is not so straightforward, see Shaeffer [1971] who considered the cost $A_i(T) = a_i T^n + c_i$.

To illustrate the time ordering in the case of variable cost, the linear cost model, $A_i(T) = a_i T + c_i$ (a_i and c_i are constants), is considered. Taking such a model leads to the optimal replacement times as the solution of the following equations

Usual criterion:

$$c_2 = (a_1 - a_2) \left(T h(T) \int_0^T R(x) dx - \int_0^T x f(x) dx \right) - 5.23 \\ + (c_1 - c_2) \left(h(T) \int_0^T R(x) dx - F(T) \right)$$

DS criterion:

$$c_2 = [(a_1 - a_2)T + (c_1 - c_2)] Th(T) \quad -5.24$$

Since $T \geq \int_0^T R(x) dx$ it follows that the RHS (5.99) of exceeds RHS of

(5.98), if the distribution is IFR and if $a_1 > a_2$ then the previous ordering will be preserved. However if $a_1 < a_2$ then the ordering will depend on the cost and the underlying failure distribution. The addition of higher power terms in t to the costs will obviously lead to further terms on RHS of (5.23) and (5.24).

5.4 Discounting

The simple constant cost model is unrealistic since it does not take into account that costs in the future will be affected by inflation. To make the model more realistic therefore introduce a discounting term to the costs. For simplicity the discounting will be taken as exponential. If the discount factor is α then a cost, c_i , discounted back to time zero will be multiplied by $c_i e^{-\alpha t}$.

Under the usual criterion the case of discounting has been considered by many author, recently Berg [1980]. The objective function can shown to be

$$\psi(T, \alpha) = \frac{c_2 \int_0^T e^{-\alpha x} f(x) dx + c_1 R(T) e^{-\alpha T}}{\alpha \int_0^T R(x) e^{-\alpha x} dx} \quad - (5.25)$$

The optimal solution if it exists is the solution of

$$\frac{1}{k-1} = h(T) \int_0^T e^{-\alpha x} R(x) dx - \int_0^T e^{-\alpha x} f(x) dx \quad -(5.26)$$

A straightforward extension of the simple non-discounting case. Again the numerical solution may be obtained by graphical means, plotting the inverse of the RHS of (5.26) against T . Note however the curves will also depend on α .

For the criterion given in (5.7) the extension to the discounting case is not easy and depends interpretation of the criterion, hence in this section two possible extensions are considered. In the previous sections the criteria was derived either as the cost per unit time within a cycle, or as the limit of sum of the cost per unit time over an infinite number of cycles, which were equivalent. In the discounting case these lead to differing objective functions. Both objective functions are derived and the equations for the optimal solutions are presented. However the second objective function as the discount factor tends to 0 yield an objective function different from (5.5), so the main attention is restricted to the single cycle case.

5.4.1 Single Cycle

Taking (5.5) to be the objective function for discounted costs in a single cycle gives,

$$\phi(T, \alpha) = c_2 \int_0^T \frac{e^{-\alpha x} f(x)}{x} dx + c_1 \frac{R(T)e^{-\alpha T}}{T} \quad -(5.27)$$

The optimal solution is therefore the solution of

$$\frac{1}{k-1} = \frac{Th(T)}{1+\alpha T} \quad - (5.28)$$

A numerical solution may generally be obtained by plotting the inverse of RHS (5.27) against T . As in usual case it should be noted that the solution depends on α .

5.4.2 Infinite number of cycles

The derivation is slightly more complex.

The cost of j th replacement is given by

$$c'_j = \begin{cases} c_1 e^{-\sum_{i=1}^j t_i \alpha} = c_1 e^{-t_j \alpha} e^{-\sum_{i=1}^{j-1} t_i \alpha} & t_i < T_0 \\ c_2 e^{-\sum_{i=1}^j t_i \alpha} = c_2 e^{-t_j \alpha} e^{-\sum_{i=1}^{j-1} t_i \alpha} & t_i \geq T \end{cases} \quad -5.28$$

where T_0 is the optimum replacement time.

If one assumes the j renewals are independent then

$$E \begin{bmatrix} c'_j \\ t_j \end{bmatrix} = \prod_{l=1}^{j-1} E \left[e^{-t_l \alpha} \right] E \begin{bmatrix} c' e^{-t_j \alpha} \\ t_j \end{bmatrix} \quad -5.29$$

If each of the lifetime distributions are identical then

$$E \left[\frac{c_j'}{t_j} \right] = E \left[e^{-t\alpha} \right]^k E \left[\frac{c_j' e^{-t_j \alpha}}{t_j} \right] \quad -5.30$$

Hence

$$\phi_2(T, \alpha) = \lim_{k \rightarrow \infty} \sum_{j=1}^k E \left[\frac{c_j'}{t_j} \right] = \lim_{k \rightarrow \infty} E \left[\frac{c' e^{-t\alpha}}{t} \right] \frac{\{1 - (E[e^{-t\alpha}])^k\}}{\{1 - E[e^{-t\alpha}]\}} \quad -5.31$$

$$= \frac{E \left[\frac{c' e^{-t\alpha}}{t} \right]}{[1 - E(e^{-t\alpha})]} \quad -5.32$$

Since $E[e^{-\alpha t}] < 1$ provided $t, \alpha > 0$.

This yields

$$\phi_2(T, \alpha) = \frac{c_1 \int_0^T e^{-\alpha x} f(x) dx + c_2 \frac{R(T) e^{-\alpha T}}{T}}{\alpha \int_0^T R(x) e^{-\alpha x} dx} \quad -5.33$$

5.4.3 Comparison of the two extensions

If one considers the limit as $\alpha \rightarrow 0$, then it is obvious that the infinite cycle does not lead to the same objective function as (5.5), hence one must consider whether it is a genuine extension.

For single cycle one obtains

$$\lim_{\alpha \rightarrow 0} \phi_1(T, \alpha) = \phi(T)$$

For the infinite cycle one obtains

$$\lim_{\alpha \rightarrow 0} \dot{\phi}_2(T, \alpha) = \frac{\dot{\phi}(T)}{\int_0^T R(x) dx}$$

5.4.4 Ordering of the Objective functions

In considering the use of the differing policies it is useful to note which is the 'cheapest'. As in the constant cost case this proves difficult to establish, hence this section gives details of an ordering of the objective function and the next consider the ordering of the optimal times.

Since $\alpha \int_0^T R(x) e^{-\alpha x} dx < 1$ then

$$\phi_2(T, \alpha) > \phi_1(T, \alpha) \quad -5.34$$

Also since $T \geq x$, $x \in (0, T)$ then

$$T \phi_2(T, \alpha) > \psi(T, \alpha). \quad -5.35$$

Unfortunately it is not obvious if $\phi_1(T, \alpha)$ and $\psi(T, \alpha)$ are ordered.

5.4.5 Ordering of Optimal Times

RHS of (5.22) and (5.24) are monotonically decreasing in α and if the distribution is IFR then they are increasing in t . So as expected the optimal replacement times will increase as α increases for IFR

distributions. However the previous time ordering is no longer preserved, since

$$\frac{Th(T)}{1+\alpha T} > h(T) \int_0^T e^{-\alpha x} R(x) dx - \int_0^T e^{-\alpha x} f(x) dx \quad (5.36)$$

does not hold for all T for all IFR distributions. This is illustrated in the following example.

Example 5.4.4.1

For gamma (10,1) with $\alpha=0.1$ the values of LHS and RHS of (5.36) are presented in table 5.5. It can be seen that for values of T from about 4 to 8 the RHS exceeds the LHS.

Table 5.5

Tabulation of the values of LHS and RHS of (5.36) for gamma (10,1) and discount rate $\alpha=0.1$

| Time | LHS | RHS |
|------|------------------------|------------------------|
| 1 | 0.092×10^{-5} | 0.083×10^{-5} |
| 2 | 0.318×10^{-3} | 0.307×10^{-3} |
| 3 | 0.624×10^{-2} | 0.616×10^{-2} |
| 4 | 0.381×10^{-1} | 0.382×10^{-1} |
| 5 | 0.125 | 0.126 |
| 6 | 0.282 | 0.285 |
| 7 | 0.503 | 0.506 |
| 8 | 0.770 | 0.776 |
| 9 | 1.062 | 1.045 |
| 10 | 1.366 | 1.323 |

It might be noted however for gamma (n,1) with n<10 a=0.1 then the optimum time solution for T for DS criterion discounted is always smaller than usual criterion discounted for all k, see Figure 5.6. Also it may be noted that the limit of LHS of (5.36) is always greater than (5.36) as $T \rightarrow \infty$. The limits are;

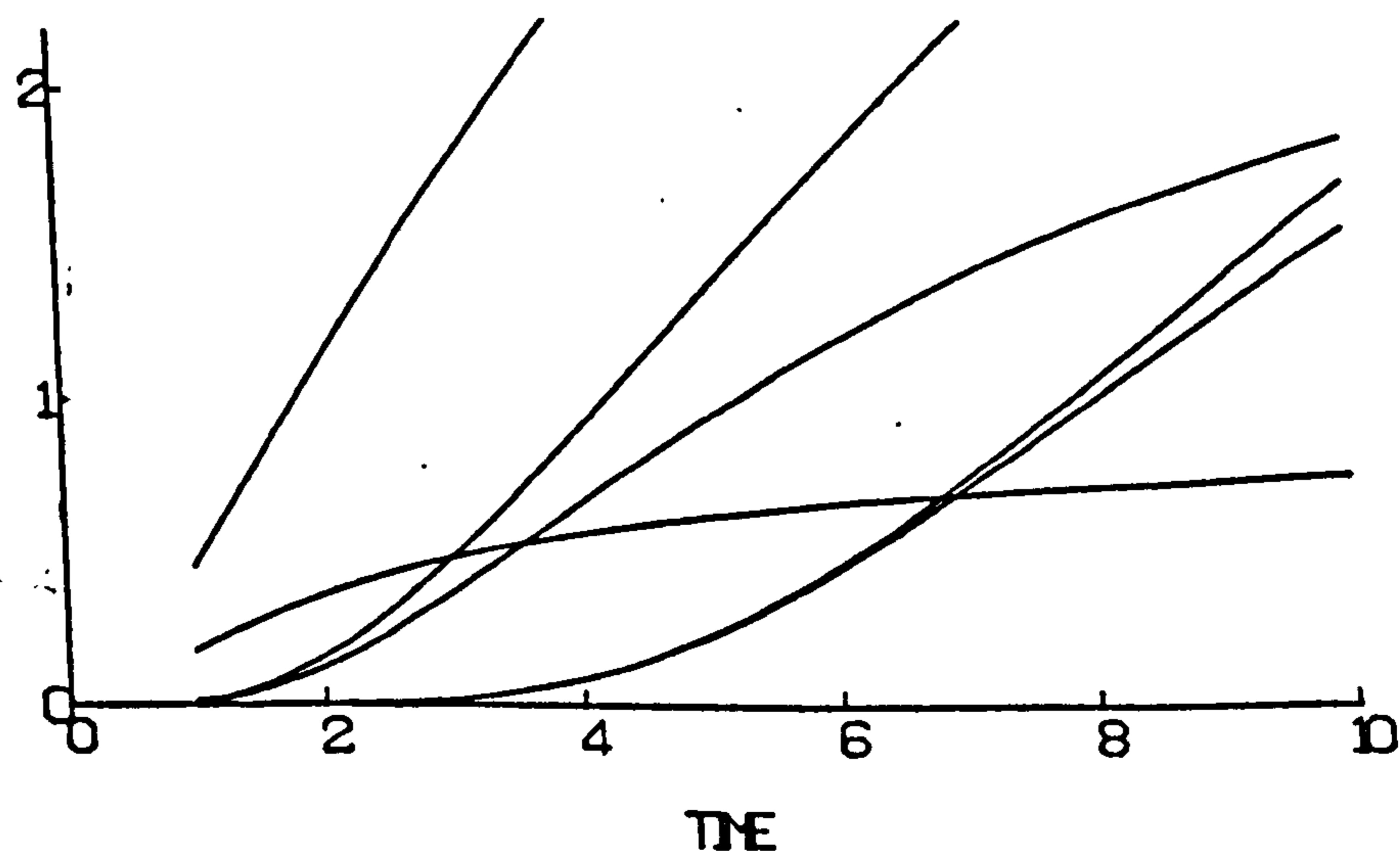
$$\lim_{T \rightarrow \infty} Th(T)/(1+\alpha T) = 1/\alpha$$

$$\lim_{T \rightarrow \infty} h(T) \int_0^T R(x) e^{-\alpha x} dx - \int_0^T f(x) e^{-\alpha x} dx = (1-(1/\alpha)^n)/\alpha - (1/\alpha)^n$$

Hence for the gamma distribution it appears that the period in which an optimum solution for T under the usual criterion is smaller than that for DS criterion is limited to values of k where $r(n, \alpha) \leq k \leq r(n, \alpha)$. The set of values will depend both on α and n. One possible advantage of these results is that the DS criterion may lead to optimum solutions which are a closer approximation to those under usual criterion in the discounted case.

Figure 5.6

Graph of LHS and RHS of (5.36) for Gamma (n,1), n=2,5,9, $\alpha=0.1$



Chapter 6

Further Work

6.1 Introduction

The aim of this chapter is to consider extensions of the work contained within this thesis and also to explore areas in Reliability which have not been covered already. The first section considers each Chapter indentifying further work which might be carried out. The second section explores other areas of work in Reliability.

6.2 Extensions

The section progresses through the Chapters in turn suggesting areas which might be explored by the author or other research workers. There are numerous problems.

6.2.1 Chapter 1 : Dichotomic Description of Systems Performance.

The main thrust of the chapter is the characterization of coherent and non-coherent structures in terms of k-out-of-n systems. The applications given are immediate from the characterization and there are probably many other applications worthy of study. Of particular

interest would be further consideration of stress-strength modelling, and the effect of component failure. Obvious clarification of the the relationship between k-out-of-n and k-out-of-(n-1) is needed in such a context.

The other main feature of the chapter is the concentration on Fault Tree Analysis and hence non-coherent systems. For too long the simplest models have been considered and there is need to try and find characterizations which would be helpful in describing realistic structures. In the area of Fault Trees itself there are numerous problems even in the dichotomic case. There have been improvements in the algorithms to find the set of Prime Implicants but this has not been met by more sophistication of the stochastic modelling. Most models are still based on Markovian assumptions or on asymptotic results. Both of these can be very misleading see Ansell [1983] and Ansell and Bendell [1985]. There have been some simulation studies of large systems which have indicated the poverty of the simplistic approach Ansell [1983] and Windebank [1982,1983].

6.2.2 Chapter 2 : Multilevel System

Further definitions of coherency would only add to the present mire. The elucidation of further structure would be useful but the interest should not be restricted to coherent systems alone but a more general class, the plausible systems. In the class of multilevel systems this does not purely mean the non-coherent structures, but those structures in which components and the system may take a different number of levels.

Further study is also required on Multilevel Fault Tree Analysis, to provide sensible models and efficient algorithms. Cardorolla [1980] and Ogunbiyi and Henley [1981] are initial steps in these directions.

6.2.3 Chapter 3 : Multistate Systems

Following directly on from Chapter 2 there is need for the consideration of more general systems than have so far been considered. Again Cardorolla provides an initial starting point for studies related to Fault Trees.

The relationship between Butler's structural definition of Importance and the characterizations of coherent system, (non-coherent systems), in terms of k -out-of- n systems is worthy of further exploration.

The chapter primarily considered non-repairable systems. There has been work on repairable multistate systems, see Ansell et al [1980] and Baxter [1983]. There would seem scope to be for further work in this area, particularly developing results out of the structural properties given in Chapter 2. It would be of interest to explore the behaviour of systems based on types of components suggested at the end of chapter 4.

6.2.4 Chapter 4 : Nested Renewal Processes.

Although many of the properties of the processes have been

studied in chapter 4, there are several interesting properties which have not been fully explored. Most of the detail has been limited for ease of manipulation to the Erlangian Family of Distribution and it might be worthwhile to explore other distributions.

The oscillating nature of the expected value for the process is also worthy of further study. Alongside this one could investigate the behaviour of higher order moments. This would give insight to limiting behaviour of the processes which have already been partially studied by Bendell and Scott [1984] and in the latter part of Chapter 4.

6.2.5 Chapter 5 : Optimal Replacement

The chapter primarily concerned itself with exploring fixed age replacement policies. There has been considerable work on variable age replacement policies and optimal inspection policies see Taylor [1977] and Thomas [1985].

The main practical problem is the numerical complexity of obtaining the optimal solutions for the various policies. The DS criteria leads to simpler solutions, but there may be a large discrepancy between the solution it yields and that of the periodic finite horizon policy. Further work to delineate the possible appropriateness of using the optimal solution of the DS criteria would be useful.

One reason for discounting Christer's criterion is the numerical complexity involved. Given that the solution is monotonically

decreasing as the time horizon increases for distributions with an increasing failure rate, see corollary 5.2.3.3, there might be some possibility of obtaining a close approximation to the optimal solution. A suggestion would be use of exponential function, which asymptotes to the usual optimal solution as $t \rightarrow \infty$.

It would also be of interest to investigate the possibility of a more appropriate approximation to the optimal solution of the periodic finite horizon policy.

6.3 Other Areas of Reliability

Several major areas of interest in reliability have not been covered in this thesis. The thesis has restricted itself to the modelling of systems and not to concerned itself with aspects of practical Engineering, such as modelling of failure due to fatigue or degradation. It has also assumed that there already exist information on the statistical properties of the components, that the distribution of the component failure is known as well as any parameters.

There are still many problems in the area of estimation in Reliability which require attention. Both for known distribution such as the three parameter Weibull, see Smith and Weisman [1985], and also when underlying distribution is unknown such as Proportional Hazard Modelling, see Cox [1972,1974], Kalbfliesch and Prentice [1980] and Ascher and Feingold [1984].

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