CIRCUITS OF EDGE-COLOURED COMPLETE GRAPHS

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by

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PREFACE

This thesis presents the results of research carried out by the author at the University of Keele, 1977 - 1981. Except where acknowledged otherwise the work reported here is claimed as original, and has not previously been submitted for a higher degree of this or any other University.

I wish to thank

Mr. Keith Walker, my supervisor, for his help over the years,
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ABSTRACT

This thesis investigates the circuits of edge-coloured complete graphs. There are various kinds of edge-coloured circuits. Amongst the most interesting are polychromatic circuits (each edge is differently coloured), alternating circuits (adjacent edges are differently coloured), and monochromatic circuits (every edge is the same colour). In the case of triangles, there are monochromatic, bichromatic (2-edge-coloured), and polychromatic triangles. This thesis is an investigation into edge-coloured complete graphs which do not contain one of the above kinds of circuits. Special emphasis is given to those edge-coloured complete graphs which do not contain one type of triangle, and those in which two types of triangle are forbidden. Where possible, the structure of these graphs is determined and a method of construction given; various types of extremal results are obtained.

CONTENTS

Chapter 1

INTRODUCTION

1. Terminology and Notation

Terminology and notation in graph theory is by no means standard. In some cases, different words have the same meaning, so that for instance a vertex can also be called a point or a node. More serious is when one word has different meanings, which usually occurs when an author studying a specific field in graph theory finds it convenient to make his definitions more specialised. A particularly pertinent example of this is the phrase 'edge-coloured graph', which to many people has the highly specific meaning of a graph whose adjacent edges have different colours. In this thesis, a specialised meaning will be given to the word 'graph'.

Because of these ambiguities, it has become customary at the start of a graph theoretical work to define the main terms used, especially those given an unusual meaning; this is the aim of this section. Terms and notation not defined here are either defined at the appropriate point in the text, or have standard definitions which can be found in either Bondy and Murty [B11] or Behzad, Chartrand, and Lesniak-Foster [B2]. Those terms defined in an unusual way in this section are given numbered definitions.

This thesis is concerned solely with edge-coloured graphs. Rather than take a particular graph and study the many possible edge-colourings of it, it is more convenient to consider its edge-colouring as an integral part of the graph, and then study the edge-coloured graphs themselves. With this aim in mind, we define the edge-coloured graphs below, and for the remainder of the thesis the phrase 'edge-coloured graph' will be shortened to 'graph'. Thus whenever the term 'graph' is encountered, it should be remembered that the graph is edge-coloured.

Definition 1.1

An edge-coloured graph G is a finite non-empty set V(G) of vertices, a finite set C(G) of colours, a finite set E(G) of unordered pairs of distinct elements of V(G) called edges, and a surjective function $f_G:E(G)\to C(G)$.

If an edge e of E(G) is formed from the subset {u,v} of V(G), e is denoted (u,v) (or equivalently (v,u) since the subset {u,v} is unordered). The edge e is said to be incident with the vertices u and v, and u and v are called adjacent vertices. Two edges incident with the same vertex are called adjacent edges. If e = (u,v) and $f_G(e) = c$, then e is said to be c-coloured, and the colour c is said to be incident with the vertices u and v.

The cardinality of a set is denoted $|S|$. $|V(G)|$ is the order of G, and if $|C(G)| = k$ G is said to be k-edge-coloured. If A_1 and A_2 are subsets of V(G), the A_1A_2-edges are those edges in E(G) incident with a vertex in A_1 and a (different) vertex in A_2. $\lfloor x \rfloor$ is the largest integer not greater than x, and $\lceil x \rceil$ is the least integer not less than x for any number x.

Definition 1.2

Two graphs G and H are isomorphic if there exist two bijections $\phi_1:V(G)\to V(H)$ and $\phi_2:C(G)\to C(H)$ such that an edge $(\phi_1(u),\phi_2(v))$ exists in H if and only if an edge (u,v) exists in G, and such that $f_G[(u,v)] = c$ if and only if $f_H[(\phi_1(u),\phi_1(v))] = \phi_2(c)$.

We shall deal only with isomorphism classes of graphs, called unlabelled graphs. It will on occasion be convenient to give a label to a vertex, edge, or colour of a graph. This should not be taken to imply that this element is pre-determined by its labelling, and such labels will be changed as necessary.

Definition 1.3

A graph H is a subgraph of G if V(H), C(H), and E(H) are subgraphs of V(G), C(G) and E(G) respectively, and if f_H is the restriction of f_G to E(H). G is then a supergraph of H. If V(G) = V(H), H is a spanning subgraph of G. If every V(H)V(H)-edge of E(G) is also in E(H), then H is the subgraph induced in G by V(H).

Definition 1.4

Let G be a graph, containing the colour c. The c-coloured subgraph of G is that subgraph with vertex set V(G), colour set {c}, and an edge set consisting of those edges of G which are c-coloured. H is a monochromatic subgraph of G if H is a c-coloured subgraph of G for some colour c contained in C(G).

Let u and v be (not necessarily distinct) vertices of a graph G. A u-v walk of G is a finite, alternating sequence of vertices and edges of G, starting with u and ending with v, such that every edge is immediately preceded and succeeded by the two vertices with which it is incident. The number of occurrences of edges in a walk is called its length. A u-v path is a u-v walk in which no vertex is repeated. A circuit is a u-v walk of length at least three in which no vertex is repeated except that the first and last vertices are the same. If the vertices in a u-v path are successively u, x_1, x_2, ..., x_n, and v, then the path can be denoted $ux_1x_2...x_nv$. If the vertices in a circuit are successively u, x_1, x_2, ..., x_n and u, then it can be denoted $ux_1x_2...x_nu$, or $ux_1x_2...x_n$ if it is clear that it is a circuit rather than a path. A circuit of length n is also often denoted a C_n. A circuit of length 3 is called a triangle.

Two distinct vertices u and v of G are connected in G if G contains a u-v path. A graph is connected if every pair of distinct vertices in V(G) are connected. A maximal connected subgraph of G is a connected component of G. A graph which is not connected is disconnected.

Definition 1.5

A graph G is connected in k colours if exactly k of the monochromatic subgraphs of G are connected.

A graph G is complete if every two distinct vertices of G are adjacent. A graph is trivial if it contains only one vertex. A graph is monochromatic if it contains only one colour, and polychromatic if every edge in the graph is differently coloured. A circuit is an alternating circuit if adjacent edges in it are differently coloured, and a triangle is bichromatic if it contains exactly two colours.

2. The Scope of the Thesis

There are three main areas of study of edge-coloured graphs. For many years, studying edge-coloured graphs meant studying graphs in which adjacent edges are differently coloured. Alternatively, this can be said to be the study of graphs G not containing a monochromatic path of length 2. Although it seems to have declined in importance recently, this work continues, and a good summary can be found in Fiorini and Wilson [F4].

In recent years, most articles written on edge-colourings have been concerned with Ramsey theory. Essentially, Ramsey theory asks the following question: given a set S of k monochromatic subgraphs, what is the order of the largest k-edge-coloured complete graph G not containing a subgraph isomorphic to a member of S? Ramsey theory has undergone a tremendous expansion over the last decade, and no single survey can hope to do it justice; a necessarily scanty survey can be found in Beineke and Wilson [B4].

The third main area of study of edge-coloured graphs is very rarely put in terms of edge-colourings. This is decomposition, usually studied in the form of decomposition of complete graphs. Essentially, a decomposition problem asks the following question: given a set S of monochromatic graphs of the same order, does there exist a (complete) graph G whose set of monochromatic subgraphs is S? Sometimes, S is replaced by a family of graphs, such as the monochromatic forests.

Each of the above problems is discussed here in the case where G is complete. However, this thesis is an attempt to deal with a specific case of a more general problem. The more general problem is: given a set S of graphs (which may or may not be monochromatic), what can be said about a graph G which contains no subgraph isomorphic to a graph in S? The specific case of the problem dealt with here is where G must be a complete graph, and S consists of a single circuit, or exceptionally a set of two or more circuits. Special emphasis is given to circuits of length 3, called triangles.

Chapters 2, 3 and 4 deal with those complete graphs which do not contain the various types of triangle. Chapter 2, concerned with complete graphs which do not contain a polychromatic triangle, was written in response to two questions asked by Chen and Daykin [C9]. The structure of these graphs is determined, and a method of construction derived. These graphs are in fact closely related to the 1- and 2-edge-coloured complete graphs, and can be constructed from the 1- and 2-edge-coloured complete graphs by means of an operation defined in chapter 2. Various kinds of extremal results are derived in the second section of the chapter.

Chapter 3 is concerned with complete graphs which do not contain bichromatic triangles. The structure of these graphs is given in terms of their monochromatic subgraphs. Relationships are derived between these graphs, affine planes, and orthogonal partial Latin rectangles. A method to construct some of these graphs is given which utilises orthogonal Latin squares.

Chapter 4 deals with complete graphs which do not contain monochromatic triangles, and so is concerned with a classical Ramsey problem. The Ramsey numbers $r_k(3)$ are defined, and most of the chapter is devoted to attempts to determine the values of these numbers; an extensive review of the literature on these numbers is given. In the final section of the chapter, some properties of the extremal graphs are derived.

Chapters 5 and 6 are devoted to complete graphs with two types of triangle missing. Since there are only three types of triangle - monochromatic, bichromatic, and polychromatic - these graphs contain only one type of triangle. It is easily checked that complete graphs in which every triangle is monochromatic must be 1-edge-coloured. These graphs are uninteresting, and are not studied.

Chapter 5 deals with complete graphs, all of whose triangles are polychromatic. It is determined that these are the 'properly' edge-coloured complete graphs mentioned at the start of this section. A relationship between these graphs and Latin squares is noted, and a method of construction using transversals is given. Those graphs which cannot be enlarged without adding more colours or creating a polychromatic triangle are studied. In the final section, other types of extremal results concerning these graphs are derived.

Chapter 6 is concerned with complete graphs, all of whose triangles are bichromatic. The structure of these graphs is determined with the aid of many of the results of chapter 2, and a method of construction found. Again, various types of extremal results are obtained in the final section.

The final three chapters study complete graphs in which various types of circuit are missing. Chapter 7 investigates complete graphs which have polychromatic circuits missing. Complete graphs with no polychromatic circuits at all are found to be exactly those complete graphs which have no polychromatic triangles. The existence of polychromatic circuits of various lengths in complete graphs is related. The complete graphs with no polychromatic C_4 are studied, with special reference to those which contain a polychromatic Hamiltonian circuit.

Chapter 8 considers alternating circuits in complete graphs. The first section studies 2-edge-coloured complete graphs: those with no alternating circuit at all are found to be exactly those with no alternating C_4, and the structure of these graphs is given. The 2-edge-coloured complete graphs with no alternating Hamiltonian circuits are also investigated. In the second section, the number of colours in the complete graphs is increased, and those complete graphs without small alternating circuits, or without any alternating circuits at all, are studied. In the final section, the existence of alternating circuits in a complete graph is related to the number of edges of any colour incident with each vertex.

In the final chapter, two main topics are discussed. The complete graphs with no monochromatic circuits at all are those in which all the monochromatic subgraphs are forests. The first two sections are therefore devoted to a decomposition problem. In the first section, those complete graphs in which every monochromatic subgraph is a tree are investigated. In the second section, those complete graphs whose monochromatic subgraphs are isomorphic forests are considered. This involves an extensive review of the literature, both of isomorphic decompositions and the related subject of graceful trees. The final section is concerned with Ramsey theory for circuits - complete graphs with no monochromatic C_n. Again, an extensive review of the literature is given.

Chapter 2

COMPLETE GRAPHS WITHOUT POLYCHROMATIC TRIANGLES

1. Structure and Construction

A polychromatic triangle is a circuit of length 3 in which each edge is a different colour. A $\overline{PC}_3$-graph is a complete graph with no polychromatic triangle. In this section, we shall characterise the $\overline{PC}_3$-graphs.

To begin with, we present some examples of $\overline{PC}_3$-graphs. A polychromatic triangle contains three different colours, so an infinite set of examples is the set of 1- and 2-edge-coloured complete graphs. The range of examples can be increased using the operation defined below.

Definition 2.1

Let G_1 and G_2 be two graphs with disjoint vertex sets. The join of G_1 and G_2 in colour c is formed by connecting each vertex of G_1 to each vertex of G_2 by a c-coloured edge, and is denoted $G_1 \overset{c}{+} G_2$.

Lemma 2.2

Let c be an arbitrary colour, and let G_1 and G_2 be two graphs with disjoint vertex sets.

i) If neither G_1 nor G_2 contains a polychromatic triangle, $G_1 \overset{c}{+} G_2$ does not contain a polychromatic triangle.

ii) If both G_1 and G_2 are $\overline{PC}_3$-graphs, $G_1 \overset{c}{+} G_2$ is a $\overline{PC}_3$-graph.

Proof

i) Let uvw be a polychromatic triangle in $G_1 \overset{c}{+} G_2$. As only one of the edges of uvw can be c-coloured, either u, v, and w are in $V(G_1)$ or u, v, and w are in $V(G_2)$, so that either G_1 or G_2 contains a polychromatic triangle.

ii) Let $G_1 \overset{c}{+} G_2$ be non-complete, so that there exist two non-adjacent vertices u and v. As each vertex of G_1 is connected to each vertex of G_2 by an edge in $G_1 \overset{c}{+} G_2$, either u and v are in $V(G_1)$ or u and v are in $V(G_2)$, so that either G_1 or G_2 is also non-complete. The result now follows from i).

If G_1 and G_2 are $\overline{PC}_3$-graphs coloured in red and blue, the join of G_1 and G_2 in green is a 3-edge-coloured $\overline{PC}_3$-graph. In general, by starting with k-edge-coloured graphs G_1 and G_2, and forming the join of G_1 and G_2 in a new colour, a (k+1)-edge-coloured graph can be constructed, which by lemma 2.2 is a $\overline{PC}_3$-graph if G_1 and G_2 are. Clearly as there are infinitely many 1-edge-coloured graphs which are complete, for each integer k, $k > 0$, there exist infinitely many k-edge-coloured $\overline{PC}_3$-graphs.

The following fact may be observed in the examples found so far. If G is a $\overline{PC}_3$-graph coloured in blue and red such that its blue subgraph is disconnected, the edges in G between vertices in different blue components are all the same colour, red. Similarly, if both G_1 and G_2 are connected in blue, and G_3 is the join of G_1 and G_2 in red, the edges between vertices in different blue components of G_3 are again all the same colour, red. In both cases, the edges between vertices in different components of a disconnected monochromatic subgraph are all the same colour. In fact, this is always the case.

<u>Lemma 2.3</u>

Let G_1 and G_2 be graphs with disjoint vertex sets containing no polychromatic triangles, and for $i = 1, 2$ let G_i be connected in colour c_i (c_1 and c_2 are not necessarily distinct). Define the graph G_3 by

$$V(G_3) = V(G_1) \cup V(G_2)$$

$$E(G_3) = E(G_1) \cup E(G_2) \cup E_o$$

where $E_o = \{(u,v) : u \in V(G_1),\ v \in V(G_2)\}$, and where no member of E_o is c_1- or c_2-edge-coloured. Then G_3 contains no polychromatic triangles if and only if each member of E_o is the same colour.

Proof

If each member of E_o is the same colour, then G_3 is the join of G_1 and G_2 in that colour, and G_3 contains no polychromatic triangle by lemma 2.2.

Now let G_3 contain no polychromatic triangle, and suppose that some member (u,v) of E_o is c_3-coloured, where $c_1 \neq c_3 \neq c_2$, $u \in V(G_1)$, and $v \in V(G_2)$. It is enough to show that if (x,y) is any other member of E_o, (x,y) is also c_3-coloured. G_1 is connected in c_1, so there exists a c_1-coloured path $w_o w_1 \ldots w_n$ in G_1, where $w_o = u$ and $w_n = x$. No edge in E_o is c_1-coloured and $vw_i w_{i+1}$ cannot be polychromatic for any i, so if (w_i,v) is c_3-coloured then (w_{i+1},v) is also c_3-coloured. As (w_o,v) is c_3-coloured, by repeated application $(w_n,v) = (x,v)$ is also c_3-coloured. Similarly, as G_2 is connected in c_2, there exists a c_2-coloured path $z_o z_1 \ldots z_m$ in G_2 where $z_o = v$ and $z_m = y$. As before, since (x,z_o) is c_3-coloured, $(x,z_m) = (x,y)$ is also c_3-coloured, giving the result.

Corollary 2.4

Let the c-coloured subgraph of the $\overline{PC}_3$-graph G be disconnected, and let the vertex sets of its connected components be $A_1, A_2, \ldots, A_n$, $n > 1$. Then for each i and j, $1 \leq i < j \leq n$, all the $A_i A_j$-edges in G are the same colour.

Proof

Put $A_i = V(G_1)$ and $A_j = V(G_2)$ in lemma 2.3, with $c_1 = c_2 = c$.

An important prerequisite to applying corollary 2.4 is that the $\overline{\overline{PC}}_3$-graph contain a disconnected monochromatic subgraph. Fortunately, all $\overline{\overline{PC}}_3$-graphs with more than two colours satisfy this condition.

Lemma 2.5

Let G be a k-edge-coloured $\overline{\overline{PC}}_3$-graph, $k > 2$. Then G contains a disconnected monochromatic subgraph.

Proof

By induction on the order p of G. Suppose that G is connected in each colour. A connected graph of order p has at least p-1 edges, and a complete graph of order p has $\frac{1}{2}p(p-1)$ edges. If G has at least three connected monochromatic subgraphs, then $\frac{1}{2}p(p-1) \geq 3(p-1)$, so that $p \geq 6$. Figure 2.1 shows that such a graph of order 6 does exist.

Let G be a complete graph of order 6, with at least three monochromatic subgraphs each of which is connected. As each vertex is incident with an edge of each colour, it is easily checked that there are at most 3 elbows at a vertex (an elbow is a pair of adjacent edges of the same colour). G contains $\binom{6}{3} = 20$ triangles, so some triangle contains no elbow and must be polychromatic.

Assume the lemma true for $\overline{\overline{PC}}_3$-graphs of order $6 \leq p < q$, and for some $k > 2$ let G be a k-edge-coloured $\overline{\overline{PC}}_3$-graph of order q with all of its monochromatic subgraphs connected. To prove the lemma, it is enough to derive a contradiction.

Take any vertex u in V(G), and let H be the graph obtained from G by removing u together with its incident edges. H must also be k-edge-coloured as each colour in G colours at least q-1 edges, and the

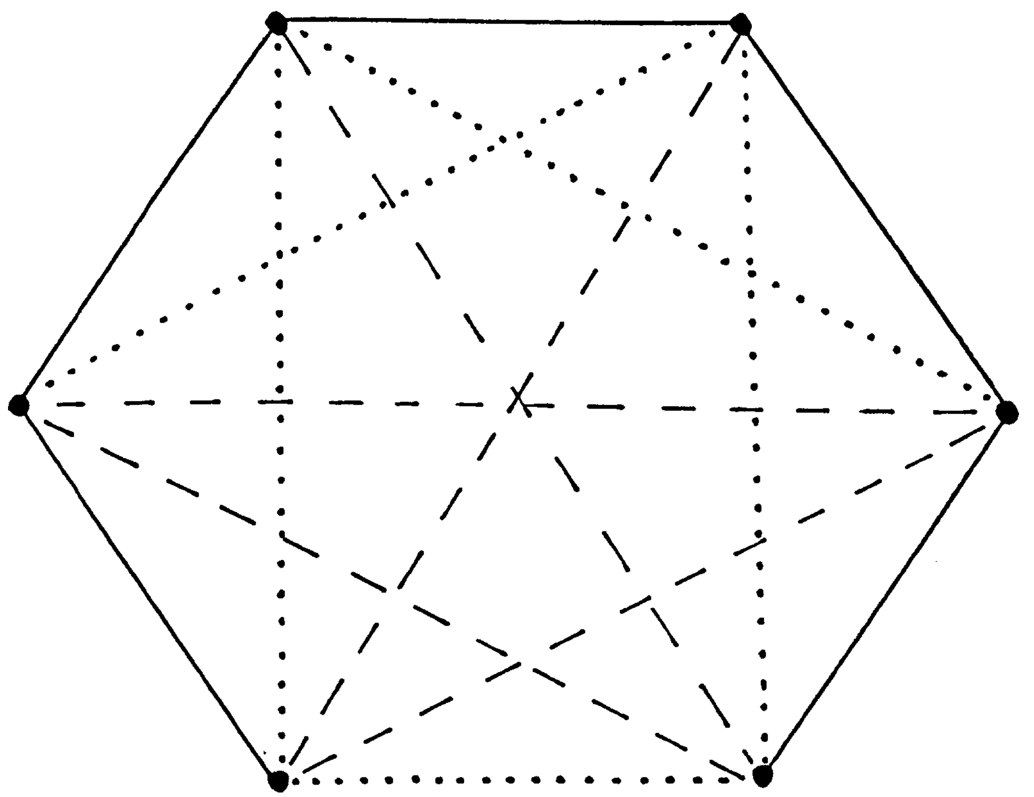


Figure 2.1

$q-1$ edges incident with u cannot be the same colour. Since $k > 2$, the induction assumption can be applied, hence H has a disconnected monochromatic subgraph, in colour c_1 say.

Let the vertex sets of the c_1-coloured connected components in H be $A_1, A_2, .., A_n$, $n > 1$. The c_1-coloured subgraph of G is connected, and no A_iA_j-edge can be c_1-coloured for $i \neq j$; therefore, for each i, $1 \leq i \leq n$, a vertex v_i can be found in A_i such that (u, v_i) is c_1-coloured. Now let c_2 be any other colour present in G. Since G is connected in c_2, u must be incident with a c_2-coloured edge (u,w), where w is in A_1 say. For $1 < i \leq n$, (w, v_i) must be c_2-coloured since no A_1A_i-edge can be c_1-coloured, and uv_iw cannot be polychromatic. Then by corollary 2.4 for $i = 2,3,..,n$ all A_1A_i-edges must be c_2-coloured in H, and therefore in G also.

As $k > 2$, there must be another colour c_3 present in G. The c_3-coloured subgraph of G is connected, so there must be a c_3-coloured edge from A_1 to the rest of the graph; this can only be (u,x) for some x in A_1. But then uxv_2 is a polychromatic triangle, a contradiction since G is a $\overline{PC}_3$-graph.

So let G be any $\overline{PC}_3$-graph containing more than two colours. Lemma 2.5 states that G contains a disconnected monochromatic subgraph. Let the vertex sets of the connected components of this monochromatic subgraph be $A_1, .., A_n$, $n > 1$, where a component need only consist of a single vertex. Corollary 2.4 states that for $i \neq j$, all A_iA_j-edges are the same colour.

For $i = 1,2,..,n$, let B_i be the subgraph of G induced by A_i, so that B_i is a $\overline{PC}_3$-graph. G can be thought of as a set of $\overline{PC}_3$-graphs $B_1, .., B_n$, any two of these graphs being joined by edges of one colour only. If the graphs B_i are each collapsed to a single vertex, and any two such vertices joined by an edge in the same colour as the edges joining the corresponding subgraphs of G, the resultant graph H has the same fundamental structure as G, but is simpler and therefore easier to deal with. H can formally be related to G by means of an operation defined in the next lemma.

Lemma 2.6

Let the c-coloured subgraph of the $\overline{PC}_3$-graph G be disconnected with non-empty connected components $B_1,\ldots,B_n$, $n > 1$, and for $i = 1,\ldots,n$ let $V(B_i) = A_i$. Define as follows a homomorphism θ_c taking G to a complete graph H with vertex set$\{v_1,\ldots,v_n\}$:

i) Suppose u is in V(G), so that u is in A_i for some i, $1 \leq i \leq n$; then $\theta_c(u) = v_i$.

ii) Suppose (u_1,u_2) is in E(G), where u_1 is in A_i and u_2 is in A_j, $i \neq j$; then the edge (v_i,v_j) in H is the same colour as (u_1,u_2) in G, where $\theta_c(u_1) = v_i$ and $\theta_c(u_2) = v_j$.

Then the homomorphism θ_c is well-defined, and the graph H has the following properties:

a) H is a $\overline{PC}_3$-graph;

b) H contains fewer colours and is of smaller order than G;

c) The c_r-coloured subgraph of H is connected if and only if the c_r-coloured subgraph of G is connected.

Proof

Corollary 2.4 shows that the homomorphism is well-defined. The proof of each of the other parts is outlined separately.

a) Let $v_r v_s v_t$ be a triangle in H. Then for any vertices x_1 in A_r, x_2 in A_s and x_3 in A_t, the triangle $x_1x_2x_3$ in G is in the same colours as $v_r v_s v_t$ in H by ii). As G is a $\overline{PC}_3$-graph, H must also be a $\overline{PC}_3$-graph.

b) There are no c-coloured edges in H, and by ii) every colour in H is in G. Secondly, if (x,y) is a c-coloured edge in G, then $\theta_c(x) = \theta_c(y)$

and H has smaller order than G.

c) First assume that G is connected in colour c_r. To prove that H is also connected in colour c_r, it is enough to find for any two distinct vertices v_i and v_j in H a c_r-coloured walk between them. Suppose x is in A_i and y is in A_j in G; as G is connected in c_r, there exists a c_r-coloured path $xu_1u_2\ldots u_my$ in G between x and y. Then by ii) $\theta_c(x)\theta_c(u_1)\ldots\theta_c(u_m)\theta_c(y)$ contains the required c_r-coloured walk in H. Now let H be connected in colour c_r, and suppose x and y are two distinct vertices in G. If $\theta_c(x) = \theta_c(y) = v_i$, then there exists a c_r-coloured edge (v_i,v_j) incident with v_i, and for any vertex z in A_j, xzy is a c_r-coloured path in G by ii). If $\theta_c(x) = v_i$ and $\theta_c(y) = v_j$ where $i \neq j$, then there exists a c_r-coloured path $v_iv_{t_1}v_{t_2}\ldots v_{t_m}v_j$ in H. Then for any z_s in A_{t_s}, $s = 1,..,m$, $xz_1z_2\ldots z_my$ is a c_r-coloured path in G.

We are now able to present a characterisation of the $\overline{\overline{PC}}_3$-graphs.

Theorem 2.7

Let G be a $\overline{\overline{PC}}_3$-graph. It is connected in either one or two colours, and if the edges in these colours are removed from G, n connected components with vertex sets $A_1,A_2,..,A_n$ say remain, $n > 1$. If G is connected in one colour only, then for $i \neq j$ every A_iA_j-edge is in that colour. If G is connected in two colours, then $n \geqslant 4$ and for $i \neq j$ every A_iA_j-edge is in one of the connected colours, which colour being dependent only on i and j.

Proof

By induction on the number k of colours in G. If $k = 1$, or $k = 2$ and G is connected in both colours, the theorem is trivial. If $k = 2$ and one of the monochromatic subgraphs, in blue say, is disconnected, then just as either a graph or its complement is connected the other monochromatic subgraph, in red say, must be connected. Since the blue subgraph is disconnected, it has non-empty connected components with vertex sets $A_1,..,A_n$, $n > 1$. Clearly, if $i \neq j$ all of the A_iA_j-edges must be red, which completes the proof for $k = 2$.

Now assume the theorem true for $k < k_o$, and let G be a k_o-edge-coloured $\overline{\overline{PC}}_3$-graph, where $k_o > 2$. From lemma 2.5, G must contain a disconnected monochromatic subgraph, in colour c say. Applying the homomorphism θ_c to G produces a k_1-edge-coloured $\overline{\overline{PC}}_3$-graph H, where $k_1 < k_o$ by lemma 2.6b). By the induction assumption H is connected in either one or two colours, and by lemma 2.6 so is G.

Take the case where H and G are connected in two colours, blue and red say (the case where H and G are connected in one colour only is proved similarly). When the blue and red edges are removed from H, by the induction assumption connected components with vertex sets $B_1,B_2,..,B_m$ remain, $m \geq 4$, and for any choice of i and j, $1 \leq i < j \leq m$, the B_iB_j-edges are either all red or all blue. Define $D_i = \{u \in V(G): \theta_c(u) \in B_i\}$ for $i = 1,..,m$, so that $D_i,..,D_m$ partition V(G). By lemma 2.6, all D_iD_j-edges are the same colour as the B_iB_j-edges, either red or blue. If all the red and blue edges were removed from G, and the subgraph induced in G by D_i was disconnected for some i, then D_i could be partitioned into two non-empty sets F_1 and F_2 such that all F_1F_2-edges were blue or red. As no F_1F_2-edge would be c-coloured, $\theta_c(F_1)$ and $\theta_c(F_2)$ are disjoint non-empty sets in H with only blue and red edges between them by lemma 2.6. But $\theta_c(F_1)$ and $\theta_c(F_2)$ is a partition of B_i which is connected when the blue and red edges are removed from H, a contradiction. Hence if all the blue and red edges are removed from G, m connected components remain, $m \geq 4$, and the edges in G between two of the connected components are either all red or all blue. The proof now follows by induction.

Let G be a $\overline{\overline{PC}}_3$-graph, and assume that G is disconnected in colour c. The homomorphism θ_c was defined to produce a graph H similar in structure to G but simpler. If H is disconnected in any other colour, then as H is a $\overline{\overline{PC}}_3$-graph another suitable homomorphism can be applied to create a $\overline{\overline{PC}}_3$-graph H_1, again similar in structure to H (and to G) but simpler. After repeated application of this technique, a $\overline{\overline{PC}}_3$-graph is obtained which is connected in each of its colours, but which has a similar structure to G; call this graph R(G).

Let $A_1, A_2, \ldots, A_n$ be the vertex sets of the connected components remaining after the removal from G of the edges contained in its connected monochromatic subgraphs, $n > 1$. As R(G) is connected in each of its colours, and by lemma 2.6 the c_r-coloured subgraph of $\theta_c(G)$ is connected if and only if the c_r-coloured subgraph of G is connected, the colours in R(G) are the colours whose monochromatic subgraphs in G are connected. As the subgraph of G induced by each A_i is connected in the union of the colours which do not appear in R(G), each A_i of G must be represented by at most one vertex of R(G). For any two distinct sets A_i and A_j, all of the A_iA_j-edges are in a colour with a connected monochromatic subgraph, so there can be no homomorphism θ_c which maps A_i and A_j onto the same vertex. Each A_i of G is therefore represented by exactly one vertex in R(G). Any two vertices of R(G) are joined by an edge in a colour in which G is connected, this colour being dependent on the colour of the edges between the corresponding sets A_i and A_j in G. As these edges are in a single colour, there is no ambiguity in the construction of R(G).

Hence a unique graph R(G) (up to isomorphism) can be derived from any $\overline{\overline{PC}}_3$-graph G (R(G) may be G itself); call R(G) the related graph of G.

Definition 2.8

Let G be a $\overline{\overline{PC}}_3$-graph, and let $A_1, A_2, .., A_n$ be the vertex sets of the connected components remaining after the removal from G of the edges contained in its connected monochromatic subgraphs, $n > 1$. The related graph R(G) of G is the complete graph with vertex set $\{v_1, v_2, .., v_n\}$, and where the edge (v_i, v_j) in E(R(G)) is the same colour as the A_iA_j-edges in G, $i \neq j$.

Theorem 2.9

H is the related graph of a $\overline{\overline{PC}}_3$-graph if and only if H is a 1- or 2-edge-coloured complete graph connected in each colour. Further, the related graph R(G) of a $\overline{\overline{PC}}_3$-graph G is connected in the same colours as G.

Proof

Let H be a 1- or 2-edge-coloured complete graph connected in each of its colours. H is a $\overline{\overline{PC}}_3$-graph, and since the maximal connected components remaining after the removal from H of the edges contained in the connected monochromatic subgraphs are single vertices, H is its own related graph.

Now let G be a $\overline{\overline{PC}}_3$-graph with H its related graph. Let $A_1, A_2, .., A_n$ be the vertex sets of the maximal connected components remaining after the edges contained in the connected monochromatic subgraphs of G are removed, $n > 1$. The colours of the A_iA_j-edges in G (and therefore of the edges contained in H) are exactly the colours in which G is connected. H is therefore 1- or 2-edge-coloured by theorem 2.7, and by definition is complete. If G is connected in one colour only, H is 1-edge-coloured, and must be connected in that colour. If G is connected in two colours, H is 2-edge-coloured. Assume H is disconnected in one of those colours, blue say, so that the indicial set $\{1,2,..,n\}$ can be be partitioned into two non-empty sets I and J such that for any i in I and j in J, (v_i, v_j) is not blue. But then for any i in I and j in J, no A_iA_j-edge in G can be blue, and so G is disconnected in blue, a contradiction. Hence H is connected in all of its colours, which are the colours in which G is connected.

Related graphs can be used to completely specify a graph as well as to give a general description of it. To obtain R(G) from G, certain complete subgraphs of G are reduced to single vertices. Conversely, G can be obtained from R(G) by expanding vertices of R(G) into complete graphs. Instead of immediately producing these complete graphs, the vertices can be expanded in stages, using only complete related graphs.

Definition 2.10

Let G_1 and G_2 be complete graphs with v any vertex of G_2. Define the following operation as substituting G_1 for v in G_2: for each vertex u in G_2 other than v, join u to each of the vertices in G_1 by edges in the same colour as (u,v) in G_2; then remove the vertex v together with all of its incident edges.

Lemma 2.11

Let H be the graph obtained by substituting G_1 for v in G_2, where G_1 and G_2 are complete graphs and v is a vertex in G_2. Then

i) H is complete;

ii) H is a $\overline{PC}_3$-graph if and only if G_1 and G_2 are both $\overline{PC}_3$-graphs;

iii) H is connected in colour c if and only if G_2 is connected in c;

iv) the colours in H are the colours in G_1 and G_2.

Proof

i) Straight from definition 2.10.

ii) G_1 is a subgraph of H, and if z is any vertex in G_1, the subgraph of H obtained by removing from H all the vertices of G_1 except for z, together with their incident edges, is isomorphic to G_2 and in the same colours as G_2. Hence if either G_1 or G_2 contains a polychromatic triangle, so does H.

Now let T = xyz be a polychromatic triangle in H. If one vertex of T, x say, is in G_1 and the others in G_2, then vyz is a polychromatic triangle in G_2. If x is in G_2 and the other vertices of T in G_1, then (x,y) and (x,z) are both the same colour as (x,v) and T could not be polychromatic. Otherwise, T is wholly in G_1 or wholly in G_2, and so one of G_1 and G_2 contains a polychromatic triangle.

iii) Let H be connected in colour c. To prove that G_2 is connected in colour c it suffices to show that for any vertex in x_o in G_2 other than v, there is a c-coloured path from v to x_o in G_2. Let y be a vertex in G_1; since H is connected in colour c, there exists a c-coloured path $x_o x_1 .. x_n$ in H where $x_n = y$. Let i be the least integer such that x_i is in G_1, so that $0 < i \leqslant n$; then $x_o x_1 .. x_{i-1} v$ is a c-coloured path in G_2 from x_o to v.

Now let G_2 be connected in colour c. To prove that H is also connected in c, it suffices to show that for some z in G_1, there is a c-coloured path in H between z and any other vertex y in H. If y is in G_2, then there is a c-coloured path $vx_1 x_2 .. x_n y$ in G_2 between v and y; hence $zx_1 x_2 .. x_n y$ is a c-coloured path in H between z and y. If y is in G_1, then since v is incident with some c-coloured edge (v,x) in G_2, zxy is a c-coloured path in H between z and y.

iv) As noted above, G_1 is a subgraph of H, and there is a subgraph in H with the same colours as G_2, so any colour in G_1 or G_2 is also in H. From its construction, any colour in H is in either G_1 or G_2, so the lemma is proved.

Since the graph H in lemma 2.11 is complete, any other complete graph G_3 can be substituted for any vertex u in H; clearly, the substitution process can go on ad infinitum. If u is a vertex in G_2 other than v, for brevity we say that G_1 and G_3 are successively substituted for v and u in G_2.

Theorem 2.12

Let G be a $\overline{\overline{PC}}_3$-graph. G can be obtained from a single vertex by performing a finite series of substitutions of related graphs.

Proof

By induction on the order p of G. The theorem is trivial for $p = 2$, so assume the theorem true for $p < p_o$, and let G be a $\overline{\overline{PC}}_3$-graph of order p_o. If G is a related graph, then substituting G itself for a single vertex produces G in the required manner. Otherwise, let $A_1, A_2, .. A_n$ be the vertex sets of the maximal connected subgraphs remaining after the removal from G of the edges in its connected monochromatic subgraphs, where $n > 1$ and $|A_1| > 1$. If B_1 is the subgraph induced in G by A_1, then B_1 is itself a $\overline{\overline{PC}}_3$-graph and has order less than p_o. By the induction assumption, B_1 can be obtained from a single vertex z by a finite series $S_1, S_2, .., S_m$ of substitutions of related graphs.

Now let H be the graph obtained from G by removing all the vertices of A_1 except one, together with their incident edges. Label the remaining vertex of A_1 z. Clearly H is a $\overline{\overline{PC}}_3$-graph, and it is easily checked that G can be obtained by substituting B_1 for z in H. Since $|A_1| > 1$, H has order less than p_o, so the induction assumption can be applied. H can therefore be obtained from a single vertex by a finite series of substitutions $T_1, T_2, .., T_r$ of related graphs. Extending this series by the substitutions $S_1, S_2, .., S_m$ for z in H gives G in the required manner.

From lemma 2.11 any graph obtained from a single vertex by a finite series of substitutions of related graphs is a $\overline{\overline{PC}}_3$-graph. Hence the set of $\overline{\overline{PC}}_3$-graphs is exactly the set of graphs generated from the related graphs using the operation of substitution.

2. Other Results

In this section, we apply the results already obtained to deriving inequalities linking some of the characteristics of $\overline{\overline{PC}}_3$-graphs. The particular characteristics we are concerned with here are the order of the graph, the number of colours contained in it, the number of edges of each colour, and the number of edges of any one colour incident with each vertex. Even with such a limited choice of characteristics, the list of possible problems is almost endless. We shall therefore concentrate on those we consider the most important, including problems posed and solved elsewhere.

The first result is due to Erdos, Simonovits, and Sos [E6].

Theorem 2.13

There exists a k-edge-coloured $\overline{\overline{PC}}_3$-graph of order p if and only if $p > k$.

Proof

By induction on k. The theorem is trivial for $k = 1,2$, so assume it true for $k < k_o$, where $k_o > 2$. If G is a k_o-edge-coloured $\overline{\overline{PC}}_3$-graph, by theorem 2.7 it is connected in m colours where $m = 1$ or 2. Let $A_1, A_2, .., A_n$ be the vertex sets of the connected components remaining after the removal from G of the edges contained in its connected monochromatic subgraphs; $n \geq m + 1$ by theorem 2.7. For $i = 1,2,..,n$, let

B_i be the subgraph induced in G by A_i, and let D_i be the graph obtained from B_i by recolouring in colour c the edges of B_i contained in the connected monochromatic subgraphs of G, where c is the colour of a disconnected monochromatic subgraph of G (such a colour exists by lemma 2.5, since $k_o \geq 3$). It is easily checked that each D_i is a $\overline{\overline{PC}}_3$-graph, coloured in fewer colours than G.

Now let K_i be the number of colours contained in D_i, $i = 1,2,..,n$. By the induction assumption, $K_i < |V(D_i| = |A_i|$. Since the colours with disconnected monochromatic subgraphs in G are present in some D_i, $k_o \leq \sum_{i=1}^{n} K_i + m$. Therefore,

$$\begin{aligned} p &= \sum_{i=1}^{n} |A_i| \\ &\geq \sum_{i=1}^{n} K_i + n \\ &> \sum_{i=1}^{n} K_i + m \\ &\geq k_o \end{aligned}$$

Next, let p be an integer satisfying $p > k_o$. By the induction assumption, there exists a (k_o-1)-edge-coloured $\overline{\overline{PC}}_3$-graph H of order p-1. Let c be a colour not contained in H, and let z be a vertex not in H. By lemma 2.2, $H \overset{c}{+} z$ is a k_o-edge-coloured $\overline{\overline{PC}}_3$-graph of order p, and this completes the proof.

The second result concerns the limits on the number of edges of a single colour relative to the order of a $\overline{\overline{PC}}_3$-graph. Denote by Q(p) the largest number of edges in any monochromatic subgraph of a complete graph G of order p, and denote by q(p) the least number of edges. Since any 1-edge-coloured complete graph is a $\overline{\overline{PC}}_3$-graph, both q(p) and Q(p) can achieve an upper bound of $\frac{1}{2}p(p-1)$ in $\overline{PC}_3$-graphs. Any 2-edge-coloured complete graph with a single blue edge is a $\overline{PC}_3$-graph also, so for $\overline{PC}_3$-graphs this gives $1 \leq q(p) \leq \frac{1}{2}p(p-1)$ with the bounds sharp.

A lower limit of $p - 1$ on $Q(p)$ for $\overline{PC}_3$-graphs was given by Schwenk (amongst others) [S3] in response to a problem set by Galvin [G1].

Theorem 2.14

Let G be a $\overline{PC}_3$-graph of order p. Then G contains $p - 1$ edges in some colour, but need not contain p edges in any colour.

Proof

By theorem 2.7, G contains at least one connected monochromatic subgraph, in blue say. Any connected graph of order p contains at least $p - 1$ edges, so G must have at least $p - 1$ blue edges.

Define H_p as follows: put $V(H_p) = \{v_1, v_2, .., v_p\}$, and if $1 \leq i < j \leq p$, colour (v_i, v_j) in colour c_j, where $c_2, c_3, .., c_p$ are distinct colours. H_p is the join in colour c_p of H_{p-1} and v_p. Since H_3 is a $\overline{PC}_3$-graph, repeated application of lemma 2.2 gives that H_p is a $\overline{PC}_3$-graph. H_p has $j-1$ edges in colour c_j for $j = 2,3,..p$, giving a maximum of $p-1$ edges in any colour, has order p, and so is the required graph.

Galvin in fact posed the problem in the following terms: colour the edges of a complete graph G of order p such that no colour is used for more than Q edges; what is the least integer p for which G must contain a polychromatic triangle? This problem can be generalised (see Hahn [H2]) to the concept of the anti-Ramsey number of a graph.

Definition 2.15

Let H be a graph with every edge differently coloured. The anti-Ramsey number ar(H,Q) is the least integer p such that every complete graph of order p with no more than Q edges in any one colour contains a subgraph isomorphic to H.

Theorem 2.14 now gives ar(H,Q) = Q + 2 when H is a triangle. (For the definition of the Ramsey number of a graph see chapter 4.)

The third result of this section concerns the limits on the number of edges of a single colour relative to the number of colours in a $\overline{\overline{PC}}_3$-graph graph. Denote by $Q'(k)$ the largest number of edges in any monochromatic subgraph of a k-edge-coloured complete graph G, and denote by $q'(k)$ the least number of edges. Since there are k-edge-coloured $\overline{\overline{PC}}_3$-graphs of arbitrarily large order, neither $Q'(k)$ nor $q'(k)$ can be bounded above for the set of $\overline{\overline{PC}}_3$-graphs (the case $q'(3)$ was given by Chen, Daykin, and Erdos [C10]). By substituting a single blue edge for a vertex in a (k-1)-edge-coloured $\overline{\overline{PC}}_3$-graph with no blue edge, a k-edge-coloured $\overline{\overline{PC}}_3$-graph is created with a single blue edge, so the attainable lower bound on $q'(k)$ for $\overline{\overline{PC}}_3$-graphs is 1. It remains to give a lower bound on $Q'(k)$ for $\overline{\overline{PC}}_3$-graphs.

Theorem 2.16

If G is a k-edge-coloured $\overline{\overline{PC}}_3$-graph, then G contains k edges in some colour, but need not contain k + 1 edges in any colour.

Proof

If G is k-edge-coloured, by theorem 2.13 G must have order at least k + 1. But then by theorem 2.14, G contains at least k edges in some colour.

The graph H_{k+1} as defined in the proof of theorem 2.14 is a k-edge-coloured $\overline{\overline{PC}}_3$-graph with no more than k edges in any colour.

Our final results are concerned with maximum and minimum degrees in the monochromatic subgraphs of a $\overline{\overline{PC}}_3$-graph. Chen and Daykin [C9] set two problems relating these to the order of the $\overline{\overline{PC}}_3$-graph. Firstly, they asked for values of p for which there existed a $\overline{\overline{PC}}_3$-graph of order p with maximum degree at most Δ for each monochromatic subgraph, so that no vertex is incident with more than Δ edges in any colour. They had already shown [C8] that no $p \geq 17\Delta$ would do; Busolini [B16] greatly improved this, in particular showing that no $p \geq 3\Delta$ would do. The next theorem gives all possible values of p.

Theorem 2.17

There exists a $\overline{\overline{PC}}_3$-graph G of order p with no more than Δ edges of any colour incident with each vertex if and only if

$$1 \leq p \leq \begin{cases} 2 & \Delta = 1 \\ 5.\tfrac{1}{2}\Delta & \Delta \text{ even} \\ 5.\tfrac{1}{2}\Delta - 1\tfrac{1}{2} & \text{otherwise} \end{cases} \tag{2A}$$

Proof

First we prove the existence of G if equation (2A) is satisfied. If $\Delta = 1$, G is just a single edge or a single vertex. Otherwise, let p satisfy equation (2A), and let H be a complete graph with vertex set $\{v_1, v_2, .., v_5\}$, and coloured in red and blue such that each monochromatic subgraph of H is a circuit. The graph G is constructed by successively substituting graphs $G_1, G_2, .., G_5$ for vertices $v_1, v_2, .., v_5$ in H (if G_i contains no vertices, this is achieved by removing vertex v_i from H). If the graphs $G_1, G_2, .., G_5$ are complete graphs coloured in green, by lemma 2.11 G is a $\overline{\overline{PC}}_3$-graph.

We now define the orders $|V(G_i)|$ of the graphs G_i to ensure that the order of G is p ($|V(G)| = |V(G_1)| + |V(G_2)| +..+ |V(G_5)|$) and that no vertex of G is incident with more than Δ edges of any one colour. Four cases are distinguished:

Case 1: $p = 5r$ for some integer r. Put $|V(G_i)| = r$ for $i = 1,2,..,5$. Now $\Delta \geq 2r$ since $p = 5r$ satisfies equation (2A). As $|V(G_i)| + |V(G_j)| = 2r$ for any $i \neq j$, no vertex of G is incident with more than Δ edges of any colour.

Case 2: $p = 5r + s$ for some integers r,s, $0 < s < 5$, and Δ is even. Put $|V(G_i)| = r + 1$ for $i = 1,2,..,s$, and $|V(G_i)| = r$ for $i = s + 1, s + 2,..,5$. Then $5r + s \leq 5.\frac{1}{2}\Delta$ by equation (2A), so $2r + \frac{2}{5}s \leq \Delta$, giving $2r + 2 \leq \Delta$ since Δ is even. As $|V(G_i)| + |V(G_j)| \leq 2r + 2$ for $i \neq j$, G is the required graph.

Case 3: $p = 5r + 1$ for some integer r, and Δ is odd. Put $|V(G_1)| = r + 1$, and $|V(G_i)| = r$ for $i = 2,3,4,5$. Then $5r + 1 \leq 5.\frac{1}{2}\Delta - 1\frac{1}{2}$ by equation (2A), giving $2r + 1 \leq \Delta$. As $|V(G_i)| + |V(G_j)| \leq 2r + 1$ for $i \neq j$, G is the required graph.

Case 4: $p = 5r + s$ for some integers r,s, $2 \leq s < 5$, and Δ is odd. Put $|V(G_i)| = r + 1$ for $i = 1,2,..,s$, and $|V(G_i)| = r$ for $i = s + 1, s + 2,..,5$. Then $5r + s \leq 5.\frac{1}{2}\Delta - 1\frac{1}{2}$ by equation (2A), so $2r + \frac{2s+3}{5} \leq \Delta$, giving $2r + 2 \leq \Delta$ as $s \geq 2$. Since $|V(G_i)| + |V(G_j)| \leq 2r + 2$ for $i \neq j$, G is the required graph.

We now prove that if G is a $\overline{PC}_3$-graph of order p with no more than Δ edges of each colour incident with any vertex, then p satisfies equation (2A). If $\Delta = 1$, any triangle would be polychromatic, so $p \leq 2$. Otherwise, put $\Delta \geq 2$ and take first the case where G is connected in one colour only, say blue. By theorem 2.7, V(G) can be partitioned into two non-empty sets A_1 and A_2 such that every A_1A_2-edge is blue. As each vA_2-edge is blue if v is in A_1, $|A_2| \leq \Delta$. Similarly, $|A_1| \leq \Delta$ so $p = |A_1| + |A_2| \leq 2\Delta$, which satisfies equation (2A).

If G is not connected in one colour only, then by theorem 2.7 it is connected in two colours, say red and blue, and V(G) can be partitioned into a maximum of n non-empty sets $A_1,A_2,..,A_n$, $n \geq 4$, such that for

$1 \leq i < j \leq n$ every A_iA_j-edge is blue or red. If $n = 4$, the related graph $R(G)$ has order 4, and is connected in both blue and red. Each monochromatic subgraph of $R(G)$ must therefore be a path, and as G has the same structure, without loss of generality we can say that the A_1A_2-, A_2A_3-, and A_3A_4-edges in G are blue. Considering the blue edges incident with a vertex firstly in A_2, and then in A_3, this gives respectively $|A_1| + |A_3| \leq \Delta$ and $|A_2| + |A_4| \leq \Delta$, so that $p = |A_1| + |A_3| + |A_2| + |A_4| \leq 2\Delta$, and p satisfies equation (2A).

Now assume that $n \geq 5$, and consider a vertex v in A_i. Every vA_j-edge is either blue or red for $i \neq j$, so that

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |A_j| \leq 2\Delta \qquad \text{for } i = 1,2,..,n \tag{2B}$$

Summing over i,

$$(n-1) \sum_{j=1}^{n} |A_j| \leq 2n\Delta$$

Hence

$$\begin{aligned} p &\leq \frac{2n\Delta}{n-1} \\ &\leq 5.\tfrac{1}{2}\Delta \qquad \text{if } n \geq 5 \end{aligned} \tag{2C}$$

So equation (2A) is satisfied unless Δ is odd and $p = 5.\frac{1}{2}\Delta - \frac{1}{2}$, in which case equation (2C) can be written

$$\begin{aligned} \frac{2n\Delta}{n-1} &\geq 5.\tfrac{1}{2}\Delta - \tfrac{1}{2} \\ 1 &\geq \frac{n-5}{n-1}\Delta \\ &\geq \frac{3(n-5)}{n-1} \qquad \text{as } \Delta \neq 1 \text{ and } \Delta \text{ odd} \\ n &\leq 7 \end{aligned} \tag{2D}$$

If $n = 7$, equation (2D) gives $\Delta = 3$, so $p = 5.\frac{1}{2}\Delta - \frac{1}{2} = 7$. Each A_i consists of a single vertex, so red and blue are the only colours present in G. Each vertex must have degree 3 in each colour, but this means that each monochromatic subgraph contains an odd number of vertices of odd degree. This is impossible, so $n \leq 6$.

If $n = 6$, equation (2D) gives $\Delta \leq 5$. As the related graph R(G) has order 6, each vertex of R(G) has degree at least 3 in some colour. Therefore, in G the A_1A_i-edges are blue (say) for three values of i, $i \neq 1$. As a vertex in A_1 can be incident with no more than 5 blue edges, $|A_m| = 1$ for some m, $2 \leq m \leq 6$. This together with equation (2B) for $i = m$ gives $p \leq 2\Delta + 1$. As $p = 5.\frac{1}{2}\Delta - \frac{1}{2}$, $\Delta = 3$ and $p = 7$, so that some set A_j contains 2 vertices and the rest contain a single vertex.

Any vertex in G which is not in A_j is incident with red and blue edges only. As $p = 7$ and $\Delta = 3$, such a vertex is incident with 3 vertices of each colour. The two vertices u_1 and u_2 in A_j cannot be joined by a red or blue edge, as this gives the case $n = 7$ again; so let (u_1,u_2) be green. Both u_1 and u_2 are incident with 5 red and blue edges, 3 in one colour and 2 in the other. If u_1 is incident with 3 (say) blue edges, then so is u_2 as every A_iA_j-edge is the same colour for $i \neq j$. But this means that the blue subgraph of G contains an odd number of vertices of odd degree. This is impossible, so $n = 5$.

Let the vertices of the related graph R(G) be $v_1,v_2,\ldots,v_5$. R(G) is connected in blue and red, so each vertex must be incident with both blue and red edges. Take the case where a vertex is incident with a single edge of some colour, so that for instance (v_1,v_2) is blue and (v_1,v_3), (v_1,v_4), and (v_1,v_5) are red. As v_2 must be incident with a red edge, without loss of generality take (v_2,v_3) to be red. Then in G, the A_1A_3-, A_1A_4-, A_1A_5-, and A_2A_3-edges are all red. Considering the red edges incident with a vertex firstly in A_3 and then in A_1 gives respectively $|A_1| + |A_2| \leqslant \Delta$ and $|A_3| + |A_4| + |A_5| \leqslant \Delta$. Hence $p \leqslant 2\Delta$, and equation (2A) is satisfied.

Otherwise, each vertex of R(G) is incident with two edges of each colour, so each monochromatic subgraph of R(G) is a circuit. The structure of G is similar, so if A_i and A_j are any two sets, $1 \leqslant i < j \leqslant 5$, and the A_iA_j-edges are blue say, then there exists a set A_r, $i \neq r \neq j$ such that the A_iA_r-edges and the A_jA_r-edges are all red. Considering the red edges incident with any vertex in A_r gives $|A_i| + |A_j| \leqslant \Delta$. Suppose that $|A_i| \leqslant \frac{1}{2}(\Delta - 1)$ for $i = 1,2,..,5$. Then $p = |A_1| + |A_2| + ..+|A_5| \leqslant 5.\frac{1}{2} - 2\frac{1}{2}$, a contradiction. Hence for some value j, $1 \leqslant j \leqslant 5$, $|A_j| = \frac{1}{2}(\Delta - 1) + s$, where $s \geqslant 1$. Then for $i \neq j$, $1 \leqslant i \leqslant 5$, $|A_i| + |A_j| \leqslant \Delta$ giving $|A_i| \leqslant \frac{1}{2}(\Delta + 1) - s$. But then $p \leqslant \frac{1}{2}(\Delta - 1) + s + 2(\Delta + 1) - 4s < 5.\frac{1}{2}\Delta - \frac{1}{2}$. This is again a contradiction, so equation (2A) must be satisfied.

For their second problem, Chen and Daykin imposed a minimum degree condition on the monochromatic subgraphs. The order of the $\overline{\overline{PC}}_3$-graph then depends on the number of colours contained in it, so they asked for values of δ, p, and k for which there existed a k-edge-coloured $\overline{\overline{PC}}_3$-graph of order p with minimum degree at least δ in each monochromatic subgraph.

<u>Theorem 2.18</u>

For each $k \geqslant 1$, there exists a k-edge-coloured $\overline{\overline{PC}}_3$-graph of order p with minimum degree at least δ in each monochromatic subgraph if and only if

$$p \geqslant \begin{cases} 2^{k-1}(\delta+1) & \delta \text{ odd} \\ 2^{k-2}(2\delta+1) & \delta \text{ even} \end{cases} \qquad (2E)$$

<u>Proof</u>

By induction on k. The theorem is trivial for $k = 1$, so assume it true for $k < K$, where $K \geq 2$. For $K = 2$, $p = 2\delta + 1$ and δ even, a complete graph G of order p can be recoloured in two colours as follows: each vertex of G has even degree, so G contains an Eulerian trail T; G has an even number $\delta(2\delta + 1)$ of edges, so the edges of T can be alternately coloured blue and red according to the sequence in which they appear in T. It is easily checked that each vertex in G is incident with the same number δ of edges of each colour.

For $K > 2$ and p satisfying equation (2E), or $K = 2$ and $p \geq 2\delta + 2$, by the induction assumption there exist $(K - 1)$-edge-coloured $\overline{\overline{PC}}_3$-graphs G_1 and G_2 with minimum degree at least δ in each colour, and such that G_1 has order $2^{K-2}(\delta + 1)$ if δ is odd or if δ is even and $K = 2$, order $2^{K-3}(2\delta + 1)$ if δ is even and $K > 2$, and such that G_2 has order $p-|V(G_1)|$. If necessary, change the colour sets of these graphs so that neither contains blue, and so that their colour sets are the same. Define G as the join of G_1 and G_2 in blue. Clearly G is K-edge-coloured, has order p, has minimum degree at least δ in each colour and by lemma 2.2 is a $\overline{\overline{PC}}_3$-graph. Hence if p satisfies equation (2E), there exists a graph of order p with the required properties.

It remains to show the necessity of equation (2E). Let G be a 2-edge-coloured $\overline{\overline{PC}}_3$-graph with minimum degree at least δ in each monochromatic subgraph. Then each vertex has total degree at least 2δ, so that G has order at least $2\delta + 1$. If G has order exactly $2\delta + 1$, then each vertex has degree δ in each colour, and each monochromatic subgraph is of odd order and regular of degree δ. As no graph can have an odd number of vertices of odd degree, δ is even.

Next, let G be a K-edge-coloured $\overline{\overline{PC}}_3$-graph with minimum degree at least δ in each monochromatic subgraph, where $K \geq 3$. If G is connected in two colours, red and blue say, then by theorem 2.7 V(G) can be partitioned into at least four non-empty sets $A_1,A_2,..,A_n$ such that each A_iA_j-edge is red or blue, $i \neq j$. For $i = 1,2,..,n$, let B_i be the graph induced in G by A_i, and let D_i be the graph obtained from B_i by recolouring each blue and red edge of B_i in a third colour present in G. This recolouring will not create any polychromatic triangles, so D_i is a $\overline{PC}_3$-graph. If v is any vertex in D_i, v is incident in G with at least δ edges in each of the K-2 colours with disconnected monochromatic subgraphs, and as all vA_j-edges are in the connected colours, $j \neq i$, these edges must be in D_i. D_i is therefore a (K-2)-edge-coloured $\overline{PC}_3$-graph with minimum degree at least δ in each colour, $i = 1,2,..,n$, so the induction assumption can be applied. For $i = 1,2,..,n$,

$$|A_i| = |V(D_i)|$$

$$\geq \begin{cases} 2^{K-3}(\delta + 1) & \delta \text{ odd} \\ 2^{K-4}(2\delta + 1) & \delta \text{ even} \end{cases}$$

$$p \geq |A_1| + |A_2| + |A_3| + |A_4| \qquad \text{as } n > 4$$

$$\geq \begin{cases} 2^{K-1}(\delta + 1) & \delta \text{ odd} \\ 2^{K-2}(2\delta + 1) & \delta \text{ even} \end{cases}$$

If G is not connected in two colours, then by theorem 2.7 it is connected in one colour only, blue say, and V(G) can be partitioned into at least two non-empty sets $A_1, A_2, .., A_n$ such that each A_iA_j-edge is blue, $i \neq j$. For $i = 1,2,..,n$, let B_i be the graph induced in G by A_i, and let D_i be the graph obtained from B_i by recolouring each blue edge in a second colour present in G. As above it can be shown that D_i is a (K-1)-edge-coloured $\overline{PC}_3$-graph with minimum degree at least δ in each monochromatic subgraph, and applying the induction assumption again gives the desired result.

Chapter 3

COMPLETE GRAPHS WITH NO BICHROMATIC TRIANGLES

1. Structure

A triangle containing exactly two distinct colours is a bichromatic triangle. A complete graph with no bichromatic triangles is a $\overline{BC}_3$-graph. Trivially, any 1-edge-coloured complete graph is a $\overline{BC}_3$-graph, and these graphs provide a set of examples of $\overline{BC}_3$-graphs of every possible order. A further set of examples of $\overline{BC}_3$-graphs is the complete graphs in which every triangle is polychromatic; these graphs are discussed in chapter 5.

Lemma 3.1

Every 2-edge-coloured complete graph contains a bichromatic triangle.

Proof

If a complete graph G contains two colours, and some vertex x of G is incident with edges of one colour only, blue say, then any vertex u incident with an edge of the other colour in G is also incident with a blue edge (x,u). Hence some vertex u of G is incident with both of the colours in G. This means that for some vertices v and w, (u,v) and (u,w) are differently coloured, and the triangle uvw contains at least two colours. Since G contains only two colours, uvw is bichromatic.

Theorem 3.2

There exists a k-edge-coloured $\overline{BC}_3$-graph if and only if $k \neq 2$, where k is a natural number.

Proof

Lemma 3.1 shows that if there exists a k-edge-coloured $\overline{BC}_3$-graph G, then $k \neq 2$. Trivially, any 1-edge-coloured complete graph is a

$\overline{BC}_3$-graph, so it is enough to construct a k-edge-coloured $\overline{BC}_3$-graph for each integer k, k > 2.

Let $c_1, c_2, .., c_k$ be k > 2 distinct colours, and let G_1 be a complete graph of order k - 1 in which every edge is c_1-coloured. Add a vertex z to G_1, and join z to the k - 1 vertices of G_1 by k - 1 differently coloured edges in the colours $c_2, c_3, .., c_k$; clearly G, the new graph, is a k-edge-coloured complete graph. Any bichromatic triangle in G must contain z, since G_1 is 1-edge-coloured. If zxy is any triangle of G containing z, (x,y) is c_1-coloured. The edges (z,x) and (z,y) are differently coloured, and neither is c_1-coloured, so zxy is polychromatic. G is a k-edge-coloured $\overline{BC}_3$-graph.

In the last chapter, it was found that monochromatic subgraphs are a significant factor in any characterisation of $\overline{PC}_3$-graphs. Monochromatic subgraphs are even more significant in $\overline{BC}_3$-graphs, as they can be used to completely characterise these graphs.

Theorem 3.3

A complete graph G is a $\overline{BC}_3$-graph if and only if each monochromatic subgraph of G consists of a set of disjoint complete graphs.

Proof

If G contains a bichromatic triangle uvw where (u,v) and (v,w) are blue say and (u,w) red say, then theconnected component of the blue subgraph of G containing u,v, and w does not contain the edge (u,w), and thus is not complete.

Now let the blue subgraph of G consist of connected components not all of which are complete, and let H be a connected component in this subgraph with edge (u,v) missing. Since H is connected, u must be connected to v by a path P containing blue edges only; let P' be the shortest such path. If P' is of length 2, say it is uwv, then uwv is a bichromatic triangle in G. Otherwise let $P' = uw_1w_2 \ldots w_nv$, $n \geq 2$; (u,w_2) cannot be blue since P' is the shortest path in blue between u and v, so uw_1w_2 is a bichromatic triangle.

Thus whereas the $\overline{PC}_3$-graphs were connected in either one or two colours, the $\overline{\overline{BC}}_3$-graphs are connected in no colours at all (apart from the trivial 1-edge-coloured case).

To completely describe the monochromatic subgraphs of $\overline{\overline{BC}}_3$-graphs, it is necessary to give the possible numbers and orders of the connected components in them. To describe the $\overline{\overline{BC}}_3$-graphs themselves, it is necessary to say how the monochromatic subgraphs mesh together. The following two lemmas go some way towards the latter objective.

Lemma 3.4

Let G be a $\overline{\overline{BC}}_3$-graph, and let H_1 and H_2 be connected components of two different monochromatic subgraphs of G. Then H_1 and H_2 have at most one vertex in common.

Proof

Let H_1 be a connected component in the c_1-coloured subgraph of G, and let H_2 be a connected component in the c_2-coloured subgraph of G, $c_1 \neq c_2$. By theorem 3.3, both H_1 and H_2 are complete graphs, so that if u and v are vertices in H_1, then (u,v) is c_1-coloured. If both u and v were also in H_2, (u,v) would have to be c_2-coloured, which is impossible, so H_1 and H_2 have at most one vertex in common.

Lemma 3.5

Let G be a $\overline{\overline{BC}}_3$-graph, and let A_1 and A_2 be the vertex sets of two connected components of a monochromatic subgraph of G. Then adjacent A_1A_2-edges are in different colours.

Proof

Let A_1 and A_2 be the vertex sets of connected components in the blue subgraph of G. The connected components are maximal, so no A_1A_2-edge is blue. Suppose that two adjacent A_1A_2-edges are the same colour, so that for instance the edges (u,v) and (u,w) are red, where u is in A_1 and v and w are in A_2. Then uvw is a bichromatic triangle in G, since (v,w) is blue by theorem 3.3.

Theorem 3.6

Let G be a k-edge-coloured $\overline{\overline{BC}}_3$-graph, $k > 2$, and let H be a connected component of some monochromatic subgraph of G. Then $|V(H)| \leq k - 1$, with equality possible.

Proof

Let H be a connected component in the blue subgraph of G. Since not all the edges in G are blue, by theorem 3.3 the blue subgraph of G is disconnected, and G contains a vertex v not in H. By lemma 3.5, every edge from v to H is differently coloured, and none of these edges can be blue as v is not in H. This leaves $k - 1$ possible colours, so there can be at most $k - 1$ vertices in H. The bound is attained in the graph constructed in the proof of theorem 3.2.

Theorem 3.6 gives a best possible upper bound on the order of the connected components in a monochromatic subgraph of a $\overline{\overline{BC}}_3$-graph. That the trivial lower bound of 1 is a best possible lower bound can also be seen from the graph constructed in the proof of theorem 3.2 - the vertex z is incident with no c_1-coloured edge.

It is desirable to obtain limits on the number of components in a monochromatic subgraph. Any 1-edge-coloured complete graph is a $\overline{\overline{BC}}_3$-graph, so the trivial lower bound of 1 cannot in general be improved.

Even specifying the number of colours present in the graph does not increase the lower bound above 2, as shown by the c_1-coloured subgraph in the $\overline{\overline{BC}}_3$-graphs constructed in the proof of theorem 3.2. A non-trivial lower bound can however be obtained by considering a different monochromatic subgraph.

Lemma 3.7

Let G be a $\overline{\overline{BC}}_3$-graph, and suppose that some monochromatic subgraph of G contains a connected component of order n. Then every other monochromatic subgraph of G contains at least n connected components.

Proof

Let the blue subgraph of G contain a connected component H of order n, and suppose that the red subgraph of G contains at most n - 1 connected components. Then some connected component of the red subgraph of G has at least two vertices in common with H, contradicting lemma 3.4.

An upper bound on the number of connected components in any monochromatic subgraph of a $\overline{\overline{BC}}_3$-graph G is provided by the fact that there must be fewer connected components than vertices in G for an edge in that colour to be present. That this trivial bound cannot be improved in general can be seen from the family of complete graphs with no more than one edge of each colour: these are $\overline{\overline{BC}}_3$-graphs since a bichromatic triangle requires two edges in the same colour.

Again a non-trivial bound is obtained by considering a different monochromatic subgraph, though only in a special case.

Lemma 3.8

Let G be a k-edge-coloured $\overline{\overline{BC}}_3$-graph, $k > 2$, and let some monochromatic subgraph of G contain a connected component H of order $k - 1$.

Then every other monochromatic subgraph of G contains exactly $k - 1$ connected components, and each of these has exactly one vertex in common with H.

Proof

Let H be a connected component in the blue subgraph of G. By lemma 3.4, it is enough to prove that every connected component in the red (say) subgraph of G has at least one vertex in common with H. If not, let u be a vertex in a connected component of the red subgraph which has no vertex in common with H, so that there is no blue or red edge from u to H. Then the $k - 1$ edges from u to H have at most $k - 2$ colours between them, so two of them, (u,v) and (u,w) say, must be the same colour (not blue). But since v and w are in H, by theorem 3.3 (v,w) is blue and uvw must be bichromatic.

It should be noted that H is a largest possible connected component of a monochromatic subgraph by theorem 3.6.

We finish the section by applying the results obtained so far to derive some results of Busolini [B17].

Lemma 3.9 (Busolini)

If G is a k-edge-coloured $\overline{BC}_3$-graph, $k > 2$, then no vertex of G is incident with more than $k - 2$ edges of any colour.

Proof

Each monochromatic subgraph consists of a set of disjoint complete graphs by theorem 3.3, and no complete graph has order more than $k - 1$ by theorem 3.6. Thus $k - 2$ is the maximum possible degree of any vertex in any monochromatic subgraph.

Theorem 3.10 (Busolini)

If G is a k-edge-coloured $\overline{\overline{BC}}_3$-graph of order p, k > 2, then $p \leq (k - 1)^2$.

Proof

Any vertex v of G is incident with at most k(k - 2) edges by lemma 3.9, so $p \leq k(k - 2) + 1 = (k - 1)^2$.

2. $\overline{\overline{BC}}_3$-Graphs and Combinatorial Structures

In the last section, the structure of the monochromatic subgraphs of $\overline{\overline{BC}}_3$-graphs was described. In this section, we investigate how these monochromatic subgraphs fit together, making use of affine planes and partial Latin rectangles.

These results can be applied to the problem of how large a k-edge-coloured $\overline{\overline{BC}}_3$-graph can be. An upper bound was given in theorem 3.10. and the first $\overline{\overline{BC}}_3$-graphs considered are those that attain this bound. It is convenient to assume these graphs contain k + 1 rather than k colours.

Theorem 3.11

Let G be a (k + 1)-edge-coloured $\overline{\overline{BC}}_3$-graph of order k^2, k > 1. Then each monochromatic subgraph of G consists of k disjoint complete graphs of order k, and if H_1 and H_2 are connected components in different monochromatic subgraphs of G, then H_1 and H_2 have exactly one vertex in common.

Proof

Every vertex of G is incident with $k^2 - 1$ edges in at most k + 1 colours. As at most k - 1 edges at a vertex can be the same colour by lemma 3.9, each vertex is incident with k - 1 edges in each of the colours present in G. Hence each monochromatic subgraph is regular of degree $k - 1$, and so by theorem 3.3 consists of k disjoint complete graphs of order k. The second part of the theorem follows from lemma 3.8.

Definition 3.12

An affine plane α is a set of points and disjoint set of lines together with an incidence relation between the points and lines such that:

i) any two distinct points lie on a unique line;

ii) given any line L and any point P not on L, there is a unique line M such that P is on M and L and M have no common point;

iii) there are three non-collinear points.

Note that in the above definition, every line is incident with more than one point.

Theorem 3.13

If G is a $(k + 1)$-edge-coloured $\bar{\bar{B}}C_3$-graph of order k^2, $k > 1$, then G is an affine plane.

Proof

Call the vertices of G points, and the connected components of the monochromatic subgraphs of G lines. A point u and a line are incident if u is in the subgraph of G corresponding to the line.

Since by theorem 3.3 every connected component of each monochromatic subgraph of G is complete, if there is a monochromatic path between two vertices u and v of G, then (u,v) is in the same colour. There is exactly one edge between them, so u and v have exactly one connected component of a monochromatic subgraph in common, and axiom (i) is satisfied .

Now let u be a point of G not on the line L, where L is a complete graph in the blue subgraph say of G. The blue subgraph of G consists of a set of non-trivial disjoint complete graphs by theorem 3.11, so if M is the blue complete subgraph of G containing u, then L and M have no common vertex. If N is any other connected component of a monochromatic subgraph of G containing u, then L and N have a common vertex by theorem 3.11 since N is not blue. Thus M is the unique line containing u which has no point in common with L, and axiom (ii) is satisfied.

The blue subgraph of G has more than one connected component by theorem 3.11, so let u and v be vertices in the same connected component L and w a vertex in a different one. L is the only line containing both of the points u and v by axiom (i), and since w is not contained in L, u, v, and w are non-collinear. G thus satisfies axiom (iii), and the proof is complete.

Among the facts known about affine planes (see for example Dembowski [D2] or Hughes and Piper [H12]) are the following: for an arbitrary affine plane α, there exists an integer $n > 1$ such that α has n^2 points, $n^2 + n$ lines, n points on every line, and $n + 1$ lines passing through every point; α is defined to be an affine plane of order n. It is easy to prove in view of theorem 3.13 that a $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph G of order k^2 exists if and only if an affine plane of order k exists.

A further result known concerning affine planes is that an affine plane of order k exists if and only if there exist $k - 1$ mutually orthogonal Latin squares of order k.

<u>Definition 3.14</u>

An $m \times n$ partial Latin rectangle L based upon the integers $1,2,..,s$, where $m,n \leq s$, is an array of m rows and n columns formed from the integers 1,2,..,s in such a way that the integers in each row and column are distinct. If all mn of the cells are occupied, L is an $m \times n$ Latin rectangle. If $m = n = s$, L is called a partial Latin square of order m, or a Latin square of order m if all m^2 cells are occupied.

Note that any set of s distinct symbols can be used instead of the integers 1,2,..,s in the above definition.

Definition 3.15

Let L_g and L_h be two $m \times n$ partial Latin rectangles on the integers 1,2,..,s, and let (a,b) be any ordered pair of integers satisfying $1 \leq a, b \leq s$. L_g and L_h are orthogonal if for each such pair of integers a and b there exists at most one pair of integers i and j such that a is the entry in the i'th row and j'th column of L_g, and b is the entry in the same position in L_h. In the case where L_g and L_h are orthogonal Latin squares, exactly one such pair of integers i and j exists.

The next result can be proved by simply using the known results on affine planes together with the relationship already derived between $\overline{\overline{BC}}_3$-graphs and affine planes. However, it is more instructive to prove it directly, as this gives a valuable insight into the interconnection of the monochromatic subgraphs of $\overline{\overline{BC}}_3$-graphs.

Theorem 3.16

For $k > 1$, there exists a $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph G of order k^2 if and only if there exist $k - 1$ mutually orthogonal Latin squares of order k.

Proof

Firstly, assume that such a $\overline{\overline{BC}}_3$-graph G exists, with colour set

$\{c_o, c_1, .., c_k\}$ say, $k > 1$. By theorem 3.11, each monochromatic subgraph of G consists of k disjoint complete graphs of order k. Label the vertex sets of the connected components of the c_o-coloured subgraph $A_1, A_2, .., A_k$, and those of the c_1-coloured subgraph $B_1, B_2, .., B_k$. Theorem 3.11 states that two sets A_i and B_j have exactly one vertex in common, so for $i,j = 1,2,..,k$ label this vertex $x_{i,j}$. For $h = 1,2,..,k$, the vertices in A_h are now labelled $x_{h,1}, x_{h,2}, .., x_{h,k}$.

Consider a vertex $x_{1,j}$ in A_1 and a set A_h, $h > 1$. There are k edges between $x_{1,j}$ and A_h, and these are differently coloured by lemma 3.5: they must be in the colours $c_1, c_2, .., c_k$. These edges can be represented by a k x 1 column C_j whose i'th entry $h_{i,j}$ signifies the vertex of A_h to which $x_{1,j}$ is joined by a c_i-coloured edge: $h_{i,j} = z$ if $(x_{1,j}, x_{h,z})$ is c_i-coloured. By lemma 3.5, all of the numbers in C_j are different.

The edges between A_1 and A_h can be represented by putting the columns C_j together in an array, $j = 1,2,..,k$. Let L_h be the k x k array whose entry $h_{i,j}$ in the i'th row and j'th column is defined by $h_{i,j} = z$ if $(x_{1,j}, x_{h,z})$ is c_i-coloured. The j'th column of L_h is C_j, $j = 1,2,..,k$, so the numbers in each column of L_h are different. Suppose two numbers in a row R_i of L_h were the same, so that $h_{i,m} = z = h_{i,n}$ for some i,m,n, and z, $m \neq n$. This means that the edges from $x_{h,z}$ in A_h to both $x_{1,m}$ and $x_{1,n}$ in A_1 are c_i-coloured, contradicting lemma 3.5. Hence no two numbers in a single row or column of L_h are the same, and L_h is a Latin square of order k on the integers $1,2,..,k$.

For $h = 2,3,..,k$, each set A_h has a Latin square L_h associated with it. Suppose that L_g and L_h were not orthogonal, $g \neq h$, where their entries are $g_{i,j}$ and $h_{i,j}$ respectively, $i,j = 1,2,..,k$. Then for some ordered pair (a,b) satisfying $1 \leq a, b \leq k$ there exist two distinct pairs of integers m,n and r,s such that $g_{m,n} = a = g_{r,s}$ and $h_{m,n} = b = h_{r,s}$.

The edges $(x_{1,n},x_{g,a})$ and $(x_{1,n},x_{h,b})$ must therefore both be c_m-coloured, and since the triangle $x_{1,n}x_{g,a}x_{h,b}$ cannot be bichromatic the edge $(x_{g,a},x_{h,b})$ must also be c_m-coloured. But since $g_{r,s} = a$ and $h_{r,s} = b$, the edges $(x_{1,s},x_{g,a})$ and $(x_{1,s},x_{h,b})$ are both c_r-coloured, giving a bichromatic triangle $x_{1,s}x_{g,a}x_{h,b}$. The Latin squares $L_2,L_3,..,L_k$ are therefore mutually orthogonal Latin squares of order k, and the first part of the theorem is proved.

Now assume that there exist $k - 1$ mutually orthogonal Latin squares $L_2,L_3, ..,L_k$ of order k on the integers $1,2,..,k$, $k > 1$. Let the k^2 vertices of a graph G be $x_{i,j}$, $i,j = 1,2,..,k$. Join $x_{g,a}$ to $x_{h,b}$ according to the following rules, where without loss of generality $1 \leqslant g \leqslant h \leqslant k$:-

1) If $g = h$, $(x_{g,a},x_{h,b})$ is c_o-coloured.

2) If $1 = g < h$, then $(x_{1,a},x_{h,b})$ is c_i-coloured, where b is the entry in the i'th row and a'th column of L_h.

3) If $1 < g < h$, then $(x_{g,a},x_{h,b})$ is c_i-coloured, where a and b appear in the same position in the i'th row of L_g and L_h respectively.

Consider any two vertices $x_{g,a}$ and $x_{h,b}$ in G. If $g = h$, then $x_{g,a}$ and $x_{h,b}$ are joined in a unique way according to step (1). If $1 = g < h$, then since L_h is a Latin square the number b appears exactly once in the a'th column of L_h; thus $x_{g,a}$ and $x_{h,b}$ are joined in a unique way according to step (2). If $1 < g < h$, then since L_g and L_h are orthogonal Latin squares, a and b appear in the same position in L_g and L_h respectively exactly once; thus $x_{g,a}$ and $x_{h,b}$ are joined in a unique way according to step (3), and G is a well-defined complete graph. It is clear that G contains the $k + 1$ colours $c_o,c_1,..,c_k$, and it only remains to show that G contains no bichromatic triangle.

By theorem 3.3, it is enough to prove that for $i = 0,1,..,k$ the c_i-coloured subgraph of G consists of a set of disjoint complete graphs.

By step (1), an edge $(x_{g,a},x_{h,b})$ is c_o-coloured when $g = h$. Put $A_i = \{x_{i,j} : j = 1,2,..,k\}$, so that the A_iA_j-edges of G are c_o-coloured if and only if $i = j$. Each set A_i therefore induces a complete c_o-coloured graph in G, and these graphs are disjoint.

Now let $0 < i \leq k$, and consider the c_i-coloured subgraph of G. Let $x_{1,j}$ be any vertex in A_1; by step (2) $x_{1,j}$ is joined by a c_i-coloured edge to the $k - 1$ vertices represented in the i'th row and j'th column of the Latin squares $L_2,L_3,..,L_k$. These vertices are in $A_2,A_3,..,A_k$ respectively. By step (3), these vertices are also joined to each other by c_i-coloured edges, so $x_{1,j}$ is in a complete c_i-coloured graph of order k which is a subgraph of G. This is true for all k vertices of A_1, and as each number appears only once in each row of a Latin square no vertex of G is joined by a c_i-coloured edge to more than one vertex in A_1. Hence the c_i-coloured subgraph of G contains k vertex-disjoint complete graphs of order k.

This means that there are at least $k.\frac{1}{2}k(k - 1)$ c_i-coloured edges in G, and this holds for $i = 0,1,..,k$. Since G contains $\frac{1}{2}k^2(k^2 - 1)$ edges, each monochromatic subgraph of G contains exactly $k.\frac{1}{2}k(k - 1)$ edges, and so consists of k disjoint complete graphs of order k. The proof is now complete by theorem 3.3.

It should be noted that $k - 1$ is the largest possible number of mutually orthogonal Latin squares of order k.

Some known results concerning orthogonal Latin squares (see for instance Denes and Keedwell [D3] or Hall [H3]) can now be applied to $\overline{\overline{BC}}_3$-graphs.

Corollary 3.17

If $k>1$ is a power of a prime number, there exists a $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph of order k^2.

Proof

If k is a power of a prime number, there exist $k - 1$ mutually orthogonal Latin squares of order k.

The case where k is a square of a prime was proved by Busolini [B17].

Corollary 3.18

For $k > 1$, if $k \equiv 1$ or $2 \pmod 4$, and the square-free part of k contains at least one prime factor $n \equiv 3 \pmod 4$, then there exists no $(k + 1)$- edge-coloured $\overline{BC}_3$-graph of order k^2.

Proof

If k satisfies the conditions above, by the Bruck-Ryser theorem there are not $k - 1$ mutually orthogonal Latin squares of order k.

Thus it can be seen that the bound given in theorem 3.10 on the order of a $(k + 1)$-edge-coloured $\overline{BC}_3$-graph can be achieved for some values of k but not for others. It is not known if this bound is attained by any values of k other than prime powers: further results on orthogonal Latin squares must be awaited.

Theorem 3.16 can be modified to give results when there are fewer than $k - 1$ mutually orthogonal Latin squares of order k.

Theorem 3.19

If there exist $r - 1$ mutually orthogonal Latin squares of order k, $1 < r \leq k$, then there exists a $(k + 1)$-edge-coloured $\overline{BC}_3$-graph of order rk.

Proof

The proof is similar to the second part of that of theorem 3.13, except that the rk vertices of G are $x_{i,j}$, $i = 1,2,..,r$ and $j = 1,2,..,k$.

Corollary 3.20

If $k = p_1^{\alpha(1)} p_2^{\alpha(2)} \ldots p_n^{\alpha(n)}$ where $1 < p_1 < p_2 < \ldots < p_n$, $n \geq 1$, are primes and $\alpha(1), \alpha(2), .., \alpha(n)$ are integers, then there exists a $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph of order $k.p_i^{\alpha(i)}$ for $i = 1,2,..,n$.

Proof

If k satisfies the conditions above, there exist $p_i^{\alpha(i)}-1$ mutually orthogonal Latin squares of order k, $i = 1,2,..,n$.

Corollary 3.20 makes use of the best general result known on orthogonal Latin squares. A perhaps better result can be obtained more directly by recolouring a known $\overline{\overline{BC}}_3$-graph.

Theorem 3.21

Let $k > 1$ be given, and suppose that k_1 is the largest prime power not greater than k. Then there exists a $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph of order k_1^2.

Proof

By corollary 3.17, there exists a $(k_1 + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph G of order k_1^2. The theorem is proved if the edges of G can be recoloured in such a way as to add $k - k_1$ colours to the graph without creating a bichromatic triangle. This can be done using a combination of the following two processes.

Process 1: Suppose that the blue subgraph say of G contains more than one non-trivial connected component. Recolour one of these blue complete graphs in red, where red is a colour not already present in G. The recoloured graph has one more colour than G, and since its blue and red subgraphs both consist of sets of disjoint complete graphs while the other monochromatic subgraphs of G remain unaltered, the recoloured graph is still a $\overline{\overline{BC}}_3$-graph by theorem 3.3.

Process 2: Suppose that the blue subgraph say of G contains a connected component H which has order $n > 2$. If u is any vertex of H, recolour the $n - 1$ edges joining u to the rest of H in $n - 1$ different colours which are not already present in G. The recoloured graph has $n - 1$ more colours than G, and as above is still a $\overline{\overline{BC}}_3$-graph.

It is easily checked that the required number of colours can be added to G using the above processes an appropriate number of times.

For the final result of this chapter, we give an analogue of the way the monochromatic subgraphs of $\overline{\overline{BC}}_3$-graphs fit together. This has already been done in the case of $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graphs of order k^2 in theorem 3.16, where orthogonal Latin squares were used. In the more general case, the analogue needs to be generalised from Latin squares to partial Latin rectangles (see definition 3.14).

<u>Theorem 3.22</u>

For $k > 1$, if there exists a $(k + 1)$-edge-coloured $\overline{\overline{BC}}_3$-graph G of order p then for some r, s, and t satisfying $1 < r < p$ and $1 < t \leq s < p$, there exist $r - 1$ mutually orthogonal $k \times t$ partial Latin rectangles $L_2, L_3, \ldots, L_r$ on the numbers $1, 2, \ldots, s$ such that for each L_h the numbers in every column are the same $|A_h|$ numbers from the set $\{1, 2, \ldots, s\}$, and such that $\sum_{h=2}^{r} |A_h| = p - t$.

<u>Proof</u>

Suppose that such a graph G exists, with colour set $\{c_o, c_1, \ldots, c_k\}$ say. Each monochromatic subgraph of G consists of a set of disjoint complete graphs of order at most k by theorems 3.3 and 3.6. Label the vertex sets of the connected components of the c_o-coloured subgraphs of G $A_1, A_2, \ldots, A_r$ and those of the c_1-coloured subgraph $B_1, B_2, \ldots, B_s$ for some $1 < r, s < p$. If $|A_1| = t$, then $s \geq t$ by lemma 3.7. Some of these sets may consist of only one vertex, so A_1 should be chosen to ensure that $t > 1$ (this is possible since G must contain a c_o-coloured edge).

Every vertex of G is in exactly one set A_i and one set B_j, and this pair is unique by lemma 3.4; label a vertex $x_{i,j}$ if it is in both A_i and B_j, $i = 1,2,..,r$ and $j = 1,2,..,s$. Since the labellings $B_1,B_2,..,B_s$ are arbitrary, choose them so that the vertices in A_1 are $x_{1,1},x_{1,2},..,x_{1,t}$. As in the proof of theorem 3.13, the edges between a vertex $x_{1,j}$ in A_1 and a set of vertices A_h can be represented by a $k \times 1$ column C_j, whose i'th entry signifies the vertex (if any) to which $x_{1,j}$ is joined by a c_i-coloured edge. No two numbers in the column are the same, although if A_h contains less than k vertices, some of these entries will be blank.

Putting these columns together gives a $k \times t$ array L_h representing the edges between A_1 and A_h: the entry $h_{i,j}$ in the i'th row and j'th column of L_h is z if $(x_{1,j},x_{h,z})$ is c_i-coloured, and is a blank if there is no c_i-coloured edge between $x_{1,j}$ and A_h. The j'th column of L_h is C_j, and so contains no number twice. If $h_{i,m} = z = h_{i,n}$ for some i, m, n and z, $m \neq n$, then both $(x_{1,m},x_{h,z})$ and $(x_{1,n},x_{h,z})$ are c_i-coloured edges, contradicting lemma 3.3. Hence no row contains any number twice, and L_h is a partial Latin rectangle on the integers $1,2,..,s$.

A number appears in a column of L_h if and only if $x_{h,z}$ is in A_h, and so L_h contains $|A_h|$ numbers in each column. Each number $1,2,..,s$ clearly appears in every column of L_h or in none at all.

There is a partial Latin rectangle L_h associated with each of the sets $A_2,A_3,..,A_s$. Since the sets $A_2,A_3,..,A_s$ include every vertex of G except for the t vertices in A_1, $\sum_{h=2}^{r}|A_h| = p - t$. The proof that these partial Latin rectangles are mutually orthogonal is exactly the same as in the proof of theorem 3.16.

Chapter 4

COMPLETE GRAPHS WITHOUT MONOCHROMATIC TRIANGLES

1. Ramsey Theory

Most of the interest in forbidden edge-coloured triangles in complete graphs has centred on monochromatic triangles. This is just a small part of the field of forbidden monochromatic subgraphs in arbitrary graphs, usually called Ramsey theory. The subject was initiated in 1930 by a paper of Ramsey [R1], which was concerned with logic and set theory rather than graph theory. The main theorems of the paper are on the r-subsets of a set; if $r = 2$, then these are just the edges of a graph. We give a special case of a theorem in the paper:

Theorem 4.1 (Ramsey)

For any positive integers k and n, there exists a least integer p_o such that if $p \geq p_o$, any k-edge-coloured complete graph on p vertices contains a monochromatic complete graph of order n as an induced subgraph.

In other words, if G is a k-edge-coloured complete graph containing no monochromatic complete graph of order n, then G has order less than p_o for some p_o dependent on k and n. The problem arising out of theorem 4.1 is to find the least integer p_o for each k and n.

Much of the earliest work on Ramsey theory (see for instance [E1, G3, K1]) concentrated on 2-edge-coloured graphs (coloured in blue and red say), but generalised the above theorem in that they tried to find the 2-edge-coloured complete graph of largest order which did not contain a blue complete graph of order m or a red complete graph of order n (the existence of such a graph is guaranteed by theorem 4.1).

Little progress was made, however, so the problem was generalised to other forbidden monochromatic subgraphs of complete graphs. One of the first generalisations was forbidden circuits, discussed in chapter 8; for surveys of the many other results of this work, see for instance [B13, H4]. At the same time, the number of colours in the complete graphs was increased (see for instance [E3, F2]), and the further generalisations include forbidden monochromatic subgraphs of arbitrary graphs (see [B14] for a survey).

In this chapter we deal with a problem arising directly from theorem 4.1; this was the only multicolour Ramsey problem studied in the early Ramsey theory papers.

Corollary 4.2

For each positive integer k, there exists a least integer $r_k(3)$ such that if $p \geqslant r_k(3)$, any k-edge-coloured complete graph of order p contains a monochromatic triangle as an induced subgraph.

The proof of corollary 4.2 follows from theorems 4.3 and 4.4 below. A complete graph with no monochromatic triangles is called an $\overline{MC}_3$-graph. Corollary 4.2 puts an upper limit of $r_k(3) - 1$ on the order of a k-edge-coloured $\overline{MC}_3$-graph. The rest of this chapter will be devoted to trying to find the integers $r_k(3)$.

2. The Integers $r_k(3)$

The problem of finding the integers $r_k(3)$ first seems to have been posed in the William Lowell Putman Mathematical Competition in 1953 [B15], where one of the questions was to find $r_2(3)$. (The same problem was posed later by Bostwick [B12].) The first paper on the subject was that of Greenwood and Gleason [G3], in which a number of important results was presented, including the following three.

Theorem 4.3 (Greenwood and Gleason)

For $k > 1$,

$$r_k(3) \leq kr_{k-1}(3) - k + 2$$

Proof

Let G be a k-edge-coloured complete graph of order $kr_{k-1}(3) - k + 2$; it is enough to prove that G must contain a monochromatic triangle. Let v be any vertex of G, so that v is incident with $kr_{k-1}(3) - k + 1$ edges. Since G contains k colours, v must be incident with at least $r_{k-1}(3)$ edges of some colour, say blue. Let B be the set of vertices in G adjacent in G to v by a blue edge, and let G_1 be the complete graph induced in G by B. If G_1 contains a blue edge (u,w), then vuw is a blue triangle and the theorem is proved. Otherwise, G_1 contains at most $k - 1$ colours, and as G_1 is a complete graph of order $r_{k-1}(3)$, G_1 contains a monochromatic triangle.

Corollary 4.4

$r_2(3) = 6$.

Proof

Trivially $r_1(3) = 3$, so by theorem 4.3 $r_2(3) \leq 6$. There exists a 2-edge-coloured complete graph of order 5 in which each monochromatic subgraph is a circuit (see graph (iv) of figure 6.1) so $r_k(3) > 5$.

Corollary 4.5 (Greenwood and Gleason)

$r_3(3) = 17$.

Proof

Theorem 4.3 and corollary 4.4 give $r_3(3) \leq 17$; Greenwood and Gleason [G3] constructed a 3-edge-coloured $\overline{MC}_3$-graph of order 16, giving the result.

Finding $r_4(3)$ has proved rather more difficult a problem than finding $r_3(3)$, and is still far from being solved. Greenwood and Gleason [G3] gave the bounds $41 < r_4(3) \leq 66$, the upper bound a consequence of theorem 4.3 and corollary 4.5, and the lower bound derived from constructing an appropriate $\overline{\overline{MC}}_3$-graph. Folkman [F5] and Whitehead [W3, W5] both improved the upper bound slightly, incidentally showing that the bound in theorem 4.3 need not be attained.

Lemma 4.6 (Folkman; Whitehead)

$r_4(3) \leq 65.$

Also in [W3], Whitehad greatly improved the lower bound on $r_4(3)$, showing that $r_4(3) > 49$. Chung [C11] constructed an $\overline{\overline{MC}}_3$-graph which further improved the bounds on $r_4(3)$.

Lemma 4.7 (Chung)

$r_4(3) > 50.$

Very little work has been done on $r_5(3)$. Theorem 4.3 and lemma 4.6 give an upper bound of 322 on $r_5(3)$, and Fredricksen [F6] has constructed a 5-edge-coloured $\overline{\overline{MC}}_3$-graph to give a lower bound of 159 on $r_5(3)$.

For values of k greater than 5, only general bounds on $r_k(3)$ are available. Greenwood and Gleason [G3] used theorem 4.3 to obtain the absolute upper bound $r_k(3) \leq \lfloor k!e \rfloor + 1$. In her thesis [C13], Chung pointed out that this could be slightly improved in view of lemma 4.6.

Theorem 4.8 (Chung)

For each $k > 3$,

$$r_k(3) \leq \lfloor k!(e-\tfrac{1}{24}) \rfloor + 1$$

Chung [C11] also gave a lower bound on $r_k(3)$ in the form of a recurrence relation, comparable to theorem 4.3.

Theorem 4.9 (Chung)

For each $k \geqslant 4$,

$$r_k(3) \geqslant 3r_{k-1}(3) + r_{k-3}(3) - 3$$

Theorem 4.9 was used by Chung [C12] to give an absolute lower bound on $r_k(3)$. This bound superseded that of Abbot and Hanson [A2], who had shown that $r_k(3) \geqslant c89^{\frac{1}{4}k}$ for $k \geqslant 4$ and some constant c.

Theorem 4.10 (Chung)

For each $k \geqslant 4$,

$$r_k(3) \geqslant (3 + t)^k c + 1$$

where $t = 0.103\ldots$ is the only positive root of $x^3 + 6x^2 + 9x - 1 = 0$, and $c = 50t^2 = 0.5454\ldots$

Finding $r_k(3)$ for a particular value of k requires two stages. If x is the putative value of $r_k(3)$, firstly x must be shown to be an upper bound for $r_k(3)$, i.e. that every k-edge-coloured complete graph of order at least x contains a monochromatic triangle. Theorem 4.3 provides the best available upper bound on $r_k(3)$, but lemma 4.6 shows that it is not always attained. Lemma 4.6 is the only improvement so far made on this bound in a specific case, and even there the bound is only improved from 66 to 65.

The second step is to show that x is also a lower bound for $r_k(3)$, which involves the construction of a k-edge-coloured $\overline{MC}_3$-graph of order $x - 1$.

Definition 4.11

A k-edge-coloured $\overline{\overline{MC}}_3$-graph of order $r_k(3) - 1$ is a k-extremal $\overline{\overline{MC}}_3$-graph. A graph is an extremal $\overline{\overline{MC}}_3$-graph if it is a k-extremal $\overline{\overline{MC}}_3$-graph for some k.

Only 1-, 2-, and 3-extremal $\overline{\overline{MC}}_3$-graphs are known. However, efforts to find larger and larger 4- and 5-edge-coloured $\overline{\overline{MC}}_3$-graphs have met with some success, and attempts to narrow the limits on $r_k(3)$ have centred on increasing the lower bounds. The next section describes techniques of construction of $\overline{\overline{MC}}_3$-graphs with this aim in mind.

3. Constructing $\overline{\overline{MC}}_3$-graphs

Firstly, it should be noted that the set of $\overline{\overline{MC}}_3$-graphs includes complete graphs with all triangles polychromatic, and complete graphs with all triangles bichromatic. Details of their construction are included in the full investigation of these sets of graphs in chapters 5 and 6 respectively. Here it need only be stated that they cannot raise the lower bound on $r_k(3)$ above $5^{\frac{1}{2}k} + 1$, and so cannot improve on theorem 4.10.

Perhaps the simplest method of constructing $\overline{\overline{MC}}_3$-graphs is to take the join in blue (say) of two $\overline{\overline{MC}}_3$-graphs which do not contain any blue edges. This is just a specific instance of the following method.

Lemma 4.12

Let G_1 and G_2 be $\overline{\overline{MC}}_3$-graphs such that G_1 contains a vertex v incident with no colour contained in G_2. Then the graph G obtained by substituting G_2 for v in G_1 is an $\overline{\overline{MC}}_3$-graph.

Proof

Note first that any vertex in G is also in either G_1 or G_2. Clearly no triangle in G containing vertices from G_1 only or from G_2 only can be monochromatic. If x is in $V(G_2)$ and y and z are in $V(G_1)$, then since the edges (x,y) and (x,z) are the same colour in G as (v,y) and (v,z) in G_1, the triangle xyz in G is the same colour as the triangle vyz in G_1 and cannot be monochromatic. If x and y are in $V(G_2)$ and z is in $V(G_1)$, then since (x,z) in G is the same colour as (v,z) in G_1, which must be differently coloured from (x,y) in G_2, again xyz cannot be monochromatic. Hence G cannot contain a monochromatic triangle, and since by lemma 2.11 it is complete, G must be an $\overline{\overline{MC}}_3$-graph.

Forming the join of two $\overline{\overline{MC}}_3$-graphs in blue is just the same as starting with two vertices v_1 and v_2 joined by a blue edge, and substituting one $\overline{\overline{MC}}_3$-graph for v_1 and the other $\overline{\overline{MC}}_3$-graph for v_2. In general, starting from an $\overline{\overline{MC}}_3$-graph G, a new $\overline{\overline{MC}}_3$-graph can be obtained by successively substituting other $\overline{\overline{MC}}_3$-graphs for the various vertices in G, subject only to colouring restrictions.

Theorem 4.13

For $k > 1$ and $i = 1,2,..,k - 1$

$$r_k(3) \geq (r_{k-i}(3) - 1)(r_i(3) - 1) + 1$$

Proof

Let G_1 be a (k-i)-edge-coloured $\overline{\overline{MC}}_3$-graph of order $r_{k-i}(3) - 1$. There exists an i-edge-coloured $\overline{\overline{MC}}_3$-graph G_2 of order $r_i(3) - 1$ containing none of the colours in G_1. Successively substitute copies of G_2 for each vertex of G_1; by lemma 4.12, the graph obtained at each stage is an $\overline{\overline{MC}}_3$-graph. The final graph obtained is k-edge-coloured, and has order $|V(G_1)||V(G_2)|$.

As the ratio $r_{j+1}(3):r_j(3)$ seems to increase as j gets larger, it is likely that $r_{k-i}(3)r_i(3) \geq r_{k-i-1}(3)r_{i+1}(3)$ for $k > 2i$. Putting

$i = 1$ in theorem 4.13 gives $r_k(3) \geq 2r_{k-1}(3) - 1$, inferior to the bound given in theorem 4.9. Thus it seems that theorem 4.13 is no improvement on the known bounds.

The method most used to construct $\overline{MC}_3$-graphs of comparatively large order is that of symmetric sum-free sets. (A general discussion of sum-free sets, including their applications to $\overline{MC}_3$-graphs, can be found in [W2].)

Definition 4.14

Let Γ be a group with operation +. A set S properly contained in Γ is said to be symmetric sum-free if for all x and y in S (x and y not necessarily distinct), x + y is not in S and x^{-1} and y^{-1} are both in S.

Theorem 4.15

Let Γ be a group whose non-zero elements can be partitioned into k disjoint symmetric sum-free sets $S_1, S_2, .., S_k$, $k > 1$. Then there exists a k-edge-coloured $\overline{MC}_3$-graph of order $|\Gamma|$.

Proof

Let the elements of Γ be the vertices of a graph G, and for $i = 1,2,..,k$ let two vertices x and y say be joined in G by a c_i-coloured edge if and only if $x - y$ (and $y - x$) belongs to the symmetric sum-free set S_i, where $c_1, c_2, .., c_k$ are distinct colours. Clearly G is a k-edge-coloured complete graph of order $|\Gamma|$. If xyz is any triangle in G, then since $(x - y) = (x - z) + (z - y)$, $(x - y)$, $(y - z)$, and $(x - z)$ cannot be in the same set S_i, so that xyz cannot be monochromatic and G is an $\overline{MC}_3$-graph.

For example, the non-zero elements of Z_5 (the integers modulo 5) can be partitioned into the symmetric sum-free sets $S_1 = \{1,4\}$ and

$S_2 = \{2,3\}$. From this can be constructed the graph G with vertex set $\{0,1,2,3,4\}$, c_1-coloured edges $\{(0,1),(1,2),(2,3),(3,4),(4,0)\}$ and c_2-coloured edges $\{(0,2),(1,3),(2,4),(3,0),(4,1)\}$. G is in fact the only 2-extremal $\overline{\overline{MC}}_3$-graph up to isomorphism (see theorem 4.14).

Whitehead [W4] showed that both of the 3-extremal $\overline{\overline{MC}}_3$-graphs can be directly obtained from symmetric sum-free sets. The largest known 5-edge-coloured $\overline{\overline{MC}}_3$-graph and a 4-edge-coloured $\overline{\overline{MC}}_3$-graph of order 49 were also obtained by this method [W5, F6]. Street [S6] proved that each $\overline{\overline{MC}}_3$-graph G which cannot itself be constructed using sum-free sets has a supergraph H which can be so constructed, although Heinrich [H5] pointed out that in some cases H must contain more colours than G. However, it is extremely difficult to find a partition of a large group into sum-free sets, even using a computer. A possible approach is that of Hill and Irving [H7], who noticed that in all cases except the 5-edge-coloured $\overline{\overline{MC}}_3$-graph of Fredricksen, the classes of the partition into sum-free sets are images of each other under group automorphisms. Applying this restriction, together with the divisibility condition it imposes on the group, greatly reduces the work involved in looking for a partition into sum-free sets. However, even using this method no concrete results were obtained in the case of $\overline{\overline{MC}}_3$-graphs because of the amount of work needed.

The largest known 4-edge-coloured $\overline{\overline{MC}}_3$-graph was constructed by Chung [C11, C12] as a special case of a method using adjacency matrices. (Briefly, a symmetric matrix A of order p is the adjacency matrix of a graph of order p if for each entry $a_{m,n}$ of A, $a_{m,n} = j$ if and only if the edge (m,n) of G is c_j-coloured, and $a_{m,n} = 0$ if and only if m = n or m is not adjacent to n in G.)

Let G be a k-edge-coloured complete graph with vertex set $\{1,2,..,p\}$, and adjacency matrix A. Clearly A is the sum of the adjacency matrices $M_1, M_2, .., M_k$ of the monochromatic subgraphs of G. In the product of M_i

with itself, the entry $(M_i^2)_{m,n}$ gives the number of c_i-coloured paths of length 2 joining points m and n in G (after multiplication by a suitable scalar). Since G has no c_i-coloured triangle, either $(M_i^2)_{m,n} = 0$ or $(M_i)_{m,n} = 0$ for each m and n, so the componentwise product $M_i{*}M_i^2 = 0$. Hence G is a $\overline{MC}_3$-graph if for each i, i = 1,2,..,k, $M_i{*}M_i^2 = 0$.

The difficulty lies in finding the matrices $M_1,M_2,..,M_k$. Chung showed that if $S_1,S_2,..,S_{k-1}$ and $T_1,T_2,..,T_{k-3}$ were the adjacency matrices of a (k-1)-edge-coloured and a (k-3)-edge-coloured $\overline{MC}_3$-graph respectively, then the matrices M_i could be constructed in a four-by-four array of blocks, each block consisting of one of the matrices S_j, T_j, or a simple matrix such as the identity matrix (for further details see [C12]). This produces the bound in theorem 4.9. Unfortunately, it is not obvious how to improve on Chungs construction, if indeed it can be improved upon.

Both of the methods outlined above which have been used to construct k-edge-coloured $\overline{MC}_3$-graphs of order at or near $r_k(3)$ transfer the graphical construction problem to an essentially combinatorial construction problem which is slightly easier to deal with. To progress by the method of sum-free sets in particular, either a great deal more work involving large computers or a lucky guess is needed. One of the problems may well be that there are very few k-edge-coloured $\overline{MC}_3$-graphs to be found with orders near $r_k(3)$. Indeed, Heinrich [H6] has shown that there are only two 2-edge-coloured $\overline{MC}_3$-graphs of order $r_2(3) - 2 = 4$ up to isomorphism, and only two 3-edge-coloured $\overline{MC}_3$-graphs of order $r_3(3) - 2 = 15$ up to isomorphism.

It is noticeable that no method aims to construct specifically extremal $\overline{MC}_3$-graphs. In the next section, we examine the extremal $\overline{MC}_3$-graphs in the hope that knowledge of some of their properties will aid their construction, or at least give a better estimate of their order.

4. Extremal $\overline{MC}_3$-Graphs

Very few general results are known about extremal $\overline{MC}_3$-graphs, a situation which stems in large part from the fact that only three examples are known (other than the trivial 1-extremal $\overline{MC}_3$-graph consisting of a single edge).

Theorem 4.16

There is a unique 2-extremal $\overline{MC}_3$-graph (up to isomorphism).

Proof

Suppose that G is a 2-extremal $\overline{MC}_3$-graph. By corollary 4.4 G has order 5, so each vertex is incident with 4 edges. Let the colours in G be blue and red. If a vertex v is incident with 3 edges of the same colour, *so that* (v,x), (v,y), *and* (v,z) *are* blue say, then either xyz is a red triangle or some edge in the triangle, (x,y) say, is blue, giving a blue triangle vxy. This cannot be so, hence each vertex of G is incident with two edges of each colour. It is easily checked that the only graph of order 5 which is regular of degree 2 is a circuit, so G must be isomorphic to the graph (iv) shown in figure 6.1(p. 87).

Greenwood and Gleason [G3] found the first 3-extremal $\overline{MC}_3$-graph, and another was found in a computer search by L. James and displayed in a paper by Kalbfleisch and Stanton [K2]. In the same paper, Kalbfleisch and Stanton proved that these were the only such graphs.

Theorem 4.17 (Kalbfleisch and Stanton)

There are exactly two 3-extremal $\overline{MC}_3$-graphs (up to isomorphism).

It has been previously noted that all of the known extremal $\overline{MC}_3$-graphs can be obtained by the method of symmetric sum-free sets, and that the classes of the sum-free partitions of the groups are images of each other under group automorphisms. This ensures that the monochromatic subgraphs of each extremal $\overline{\overline{MC}}_3$-graph are isomorphic. In fact, all of the monochromatic subgraphs of the 3-extremal $\overline{\overline{MC}}_3$-graphs are isomorphic. It would be remarkable, however, if for each k all the monochromatic subgraphs of the k-extremal $\overline{\overline{MC}}_3$-graphs were isomorphic.

The most useful general fact known about extremal $\overline{\overline{MC}}_3$-graphs is an upper bound on the maximum degree in their monochromatic subgraphs, which was derived in the proof of theorem 4.3.

Theorem 4.18 (Greenwood and Gleason)

Let G be a k-extremal $\overline{\overline{MC}}_3$-graph. Then the maximum degree in any of its monochromatic subgraphs is at most $r_{k-1}(3) - 1$.

For k = 2,3, the monochromatic subgraphs of the k-extremal $\overline{\overline{MC}}_3$-graphs are regular of degree $r_{k-1}(3) - 1$, so this result cannot be improved.

Theorem 4.19

Let G be an extremal $\overline{\overline{MC}}_3$-graph. Then each vertex is incident with an edge of each colour in C(G).

Proof

Suppose that the vertex v is incident with no blue edge in G, where blue is a colour in C(G). If a blue edge is substituted for v in G, the resultant graph is an $\overline{\overline{MC}}_3$-graph by lemma 4.12, contains as many colours as G, and has order larger than G; G cannot therefore be an extremal $\overline{\overline{MC}}_3$-graph.

Theorem 4.19 is probably very far from being a best possible result.

However, any improvement on it will improve Chungs lower bound on $r_4(3)$ (see [C11]), as the graph which provides the lower bound contains an isolated edge in one of its monochromatic subgraphs.

Connectivity has proved very important in complete graphs lacking other types of triangle. $\overline{PC}_3$-graphs have been shown to be connected in either one or two colours, and $\overline{BC}_3$-graphs in no colours at all (other than the trivial 1-edge-coloured case).

Theorem 4.20

For each $k \geqslant 1$, there exists a k-extremal $\overline{MC}_3$-graph connected in each colour.

Proof

Suppose that G is a k-extremal $\overline{MC}_3$-graph containing colours $c_1, c_2, .., c_k$. For $i = 1,2,..,k$, let the c_i-coloured subgraph of G contain $n_i \geqslant 1$ connected components, so that G contains a total of $N = \sum_{i=1}^{k} n_i$ connected components in its monochromatic subgraphs. If $N > k$, we shall construct a k-extremal $\overline{MC}_3$-graph with a total of $N - 1$ connected components in its monochromatic subgraphs. Repeated application of this technique eventually results in a k-extremal $\overline{MC}_3$-graph with a total of k connected components in its monochromatic subgraphs, i.e. which is connected in each colour.

If $N > k$, then some monochromatic subgraph of G, in colour c_i = blue say, is disconnected, and V(G) can be partitioned into two non-empty sets A_1 and A_2 such that no A_1A_2-edge is blue. Recolouring a (say) c_2-coloured (= red) A_1A_2-edge in blue reduces the number of connected components in the blue subgraph of G, butmay disconnect a connected component in the red subgraph. However, if it can be shown that an A_1A_2-edge is contained in a red circuit, then that edge can be recoloured in blue without disconnecting a connected component in the red subgraph, and the theorem is proved.

Choose x in A_1 and y in A_2, and construct a graph H as follows: add a vertex z to G, together with blue edges (x,z) and (y,z); if u is in A_1 and $u \neq x$, join u to z by an edge in the same colour as (u,y); if v is in A_2 and $v \neq y$, join v to z by an edge in the same colour as (v,x); finally, if u is in A_1, v is in A_2, $u \neq x$, $v \neq y$, and $xvuy$ is a monochromatic path, recolour (u,v) in blue. H is a k-edge-coloured complete graph of order $|V(G)| + 1 = r_k(3)$, and so must contain a monochromatic triangle.

Now (x,z) and (y,z) are blue, but (x,y) is not, so the triangle xyz is not monochromatic. These are the only blue edges incident with z, so (x,z) and (y,z) are not in a monochromatic triangle. If u and w are distinct vertices in A_1, $u \neq x \neq w$, then the triangle uwz in H is the same colour as uwy in G, and so cannot be monochromatic. Similarly, vwz cannot be monochromatic where v and w are distinct vertices in A_2, $v \neq y \neq w$. If uvz is a monochromatic triangle, in red say, where u is in A_1, v is in A_2 and $u \neq x$, $v \neq y$, then since (u,z) is the same colour as (y,u) and (v,z) is the same colour as (x,v), the path $xvuy$ is red. But then (u,v) would have been recoloured blue, so uvz cannot be red and the monochromatic triangle in H cannot contain z.

Thus the monochromatic triangle in H must be uvw say, where u, v, and w are also in G. As uvw is not monochromatic in G, some edge (u,v) must have been recoloured during the construction of H. The only edges recoloured were A_1A_2-edges which were recoloured in blue, so uvw is blue and without loss of generality u is in A_1 and v and w are in A_2. For (u,v) and (u,w) to have been recoloured, the paths $xvuy$ and $xwuy$ must be monochromatic in G. If (u,y) is red say, then $xvuw$ is a red circuit in G containing an A_1A_2-edge, as required.

Although theorem 4.20 in itself produces no useful lower bound on $r_k(3)$, if it could be extended to say that all extremal $\overline{MC}_3$-graphs were connected in each colour, this would at least improve the known lower bound on $r_4(3)$. We have only been able to obtain this result by placing other strict conditions on the extremal $\overline{MC}_3$-graphs. Theorem 4.20 gives an immediate corollary of this type.

Corollary 4.21

If there is a unique k-extremal $\overline{MC}_3$-graph up to isomorphism, it is connected in each colour.

Lemma 4.22

Let G be a k-extremal $\overline{MC}_3$-graph. Suppose that G contains a vertex v such that a monochromatic triangle is created if the colour of any edge incident with v is changed to one of the other $k - 1$ colours in C(G). Then G is connected in all k colours.

Proof

It is enough to show that for each vertex u in G other than v, u is connected to v in all k colours. If (u,v) is c_i-coloured, trivially u is connected to v in colour c_i. Let c_j be any other colour present in G. If (u,v) is recoloured in colour c_j, a monochromatic triangle uvw say is created. Then (v,w) and (u,w) form a c_j-coloured path from v to u in G, as required.

Theorem 4.23

If $r_k(3) = kr_{k-1}(3) - k + 2$, $k \geq 2$, then any k-extremal $\overline{MC}_3$-graph is connected in all k colours.

Proof

Let G be a k-extremal $\overline{MC}_3$-graph of order $k(r_{k-1}(3) - 1) + 1$. By theorem 4.18, each vertex of G has degree $r_{k-1}(3) - 1$ in each colour. If some edge (u,v) is changed from colour c_i to colour c_j, then u is incident with $r_{k-1}(3)$ c_j-coloured edges, impossible by theorem 4.19 unless a monochromatic triangle is created. Hence each vertex in G satisfies the conditions of lemma 4.22, and G is connected in all k colours.

Corollary 4.21 applies to the 2-extremal $\overline{\overline{MC}}_3$-graph, but not to the 3-extremal $\overline{\overline{MC}}_3$-graphs. It is not known whether any other k-extremal $\overline{\overline{MC}}_3$-graphs are unique up to isomorphism. The 2- and 3-extremal $\overline{\overline{MC}}_3$-graphs all achieve the upper bound on $r_k(3)$ in theorem 4.3 so that theorem 4.23 applies to them, but it is known (lemma 4.6) that this is not the case for a 4-extremal $\overline{\overline{MC}}_3$-graph.

The final results for this chapter give a lower bound on the number of monochromatic circuits in extremal $\overline{\overline{MC}}_3$-graphs. A graph obtained by deleting m vertices together with their incident edges from a graph G is an m-vertex-deleted subgraph of G.

Lemma 4.24

Let G be a k-extremal $\overline{\overline{MC}}_3$-graph, $k \geq 4$. If $0 \leq m < r_{k-1}(3) + r_{k-3}(3)$, then no m-vertex-deleted subgraph of G can contain a bipartite monochromatic subgraph.

Proof

Suppose S is a subset of V(G) such that if all the vertices in S together with their incident edges are removed from G, then the resultant graph G_1 contains a bipartite monochromatic subgraph, say in blue. The lemma is proved if it can be shown that $|S| \geq r_{k-1}(3) + r_{k-3}(3)$.

Since the blue subgraph of G_1 is bipartite, $V(G_1)$ can be partitioned into two non-empty sets A_1 and A_2 such that any blue edge in G_1 is an

A_1A_2-edge. Let H_1 and H_2 be the subgraphs of G_2 induced by A_1 and A_2 respectively, where G_2 is the graph obtained from G_1 by recolouring in blue all the A_1A_2-edges. As no A_1A_1-edge or A_2A_2-edge in G_1 (and therefore G_2) is blue, G_2 is an $\overline{\overline{MC}}_3$-graph containing at most k colours.

Consider the graph H_1: H_1 is an $\overline{\overline{MC}}_3$-graph containing at most k - 1 colours, and so $|A_1| = |V(H_1)| \leq r_{k-1}(3) - 1$. Similarly, $|A_2| \leq r_{k-1}(3) - 1$. Now

$$|V(G)| - |S| = |A_1| + |A_2|$$

$$\leq 2r_{k-1}(3) - 2$$

$$|S| \geq |V(G)| - 2r_{k-1}(3) + 2$$

and $\quad |V(G)| \geq 3r_{k-1}(3) + r_{k-3}(3) - 2$ by theorem 4.9

so that $\quad |S| \geq r_{k-1}(3) + r_{k-3}(3)$

Theorem 4.25

Let G be a k-extremal $\overline{\overline{MC}}_3$-graph, $k \geq 4$. Then each monochromatic subgraph of G contains at least $r_{k-1}(3) + r_{k-3}(3)$ circuits of odd length.

Proof

If the blue (say) subgraph of G has fewer circuits of odd length, then by deleting one vertex in each circuit from G, a graph is obtained whose blue subgraph has no odd circuits, i.e. which is bipartite. The theorem then follows from lemma 4.24.

Chapter 5

COMPLETE GRAPHS WITH ALL TRIANGLES POLYCHROMATIC

1. Structure and Construction

A complete graph in which every triangle is polychromatic is called a PC_3*-graph. The PC_3*-graphs have a particularly simple characterisation.

Theorem 5.1

A complete graph G is a PC_3*-graph if and only if adjacent edges of G are differently coloured.

Proof

If (x,y) and (y,z) are adjacent edges of G in the same colour, then the triangle xyz is not polychromatic and G is not a PC_3*-graph.

Conversely, assume that every pair of adjacent edges in G are differently coloured. If xyz is any triangle in G, all three edges (x,y), (y,z), and (x,z) are mutually adjacent and so differently coloured, which means that xyz is polychromatic. The choice of xyz was arbitrary, so G is a PC_3*-graph.

Note that a PC_3*-graph as defined here is what most authors define as a proper edge-colouring of the complete graph.

As with $\overline{\overline{BC}}_3$-graphs, the PC_3*-graphs can also be characterised in terms of their monochromatic subgraphs.

Theorem 5.2

A complete graph G is a PC_3*-graph if and only if each monochromatic subgraph of G consists of a non-empty set of non-adjacent edges together with a (possible empty) set of isolated vertices.

Proof

The theorem follows directly from theorem 5.1, since adjacent edges of G are differently coloured if and only if the edges in each monochromatic subgraph are non-adjacent.

Constructing a PC_3*-graph is somewhat more difficult than finding its structure. One approach is that of Hilton [H8], who utilises Latin rectangles (see definition 3.14). A p x p symmetric Latin rectangle is a p x p Latin rectangle in which the element in the i'th row and j'th column is the same as the element in the j'th row and i'th column.

Theorem 5.3 (Hilton)

There exists a PC_3*-graph of order p containing at most k colours if and only if there exists a p x p symmetric Latin rectangle with elements from the set $\{c_o,c_1,..,c_k\}$ in which for each i the element in the i'th row and i'th column is c_o.

Proof

Suppose that such a Latin rectangle L exists. Define G to be the complete graph with vertex set $\{v_1,v_2,..,v_p\}$ and for $i \neq j$ the edge (v_i,v_j) in colour c_r, where c_r is the entry in the i'th row and j'th column of L. The graph is well-defined since L is symmetric, and the colours in G are from the set $\{c_1,c_2,..,c_k\}$. Since all the entries in the i'th row of L are different, all of the edges incident with v_i in G are differently coloured. This is true for each vertex in G, so G is a PC_3*-graph of order p by theorem 5.1.

Now suppose that a PC_3*-graph G exists with vertex set $\{v_1,v_2,..,v_p\}$, and coloured from the set $\{c_1,c_2,..,c_k\}$. Let L be the p x p array whose entry in the i'th row and j'th column is c_o if $i = j$, and c_r if $i \neq j$ and (v_i,v_j) is c_r-coloured. Since all the edges incident at a vertex of G are differently coloured by theorem 5.1, the entries in any single row or column of L are different, and since (v_i,v_j) is the same as (v_j,v_i), L is a p x p symmetric Latin rectangle with elements from the set $\{c_o,c_1,\ldots,c_k\}$.

Thus results on the construction of Latin rectangles can be applied to the PC_3*-graphs. However, the standard line-by-line method of constructing Latin rectangles is by using transversals (see definition 5.6). Since this method can be applied directly to PC_3*-graphs, we continue by constructing PC_3*-graphs directly rather than by using Latin rectangles.

Given a particular PC_3*-graph G, it is trivial to construct G vertex-by-vertex. If no such specific graph is aimed at, a problem arises. When a new vertex v is added to the existing graph, v must be joined to the vertices in the rest of the graph in a way which will not create any monochromatic or bichromatic triangle.

<u>Lemma 5.4</u>

Let G be a PC_3*-graph, and let H be the complete graph obtained from G by adding a vertex v and a set of incident edges E_v, $|E_v| = |V(G)|$. Then H is a PC_3*-graph if and only if

i) each member of E is a different colour, and

ii) the colour c_u of the edge (u,v) in H is not incident with u in G, where u is an arbitrary vertex of G.

<u>Proof</u>

Suppose that H is a PC_3*-graph. Then (i) and (ii) follow straight from theorem 5.1, which states that adjacent edges in H must be differently coloured.

Now suppose that conditions (i) and (ii) hold. By theorem 5.1, to show that H is a PC_3*-graph it is sufficient to show that adjacent edges are differently coloured, so let e_1 and e_2 be adjacent edges in H. If both edges are incident with v, then both are in E_v and (i) applies. If only one of them is incident with v, but both are incident with u say in G, then (ii) applies. Either way, the edges are differently coloured. Otherwise, neither edge is incident with v, so that both edges are in G. But G is a PC_3*-graph, and adjacent edges in G must be differently coloured.

A suitable set of edges E_v can be chosen easily enough by using a set of completely new colours with each vertex added, so that any PC_3*-graph can be extended to a PC_3*-graph of larger order. However, when the number of colours is limited, this method may not be possible. Some other criterion for choosing the colours in E_v is needed.

Definition 5.5

G is a $(\leq k)$-edge-coloured graph if G is a k'-edge-coloured graph for some $k' \leq k$.

Definition 5.6

A collection $S_1,S_2,..,S_n$, $n \geq 1$, of finite non-empty sets has a transversal (alternatively, a system of distinct representatives) if there exists a set $\{s_1,s_2,..,s_n\}$ of distinct elements such that s_i is in S_i for $i = 1,2,..,n$.

Lemma 5.7

Let G be a $(\leq k)$-edge-coloured PC_3*-graph, with colours from the set $C = \{c_1,c_2,..,c_k\}$ and with vertex set $V(G) = \{v_1,v_2,..,v_p\}$. Define S_i to be the set of colours in C not incident in G with v_i, $i = 1,2,..,p$. Then G can be extended to a $(\leq k)$-edge-coloured PC_3*-graph H with order $p + 1$ if and only if the collection of sets $S_1,S_2,..,S_p$ has a transversal.

Proof

Suppose that such a transversal T exists, with members $\{s_1,s_2,..,s_p\}$ where s_i is a member of S_i for each i. To construct H, add a vertex v to G, and for i = 1,2,..,p add edges (v,v_i) in colour s_i; call the set of new edges E_v. H is clearly a complete graph of order p + 1, and is ($\leq$ k)-edge-coloured since all of its colours come from the set $\{c_1,c_2,..,c_k\}$. Since T is a transversal, the edges in E_v are differently coloured, and for each i the colour s_i is not adjacent to v_i in G since s_i is in S_i. Conditions (i) and (ii) in lemma 5.4 are therefore satisfied, and H is a PC_3*-graph.

Now suppose that G can be extended to the graph H. Let v be the vertex of H not present in G, and let E_v be the set of edges incident with v in H. Let the edges (v,v_i) be in colour s_i, i = 1,2,..,p, so that the colours in E_v are $\{s_1,s_2,..,s_p\}$. By condition (i) of lemma 5.4, the colours $s_1,s_2,..,s_p$ are distinct, and by condition (ii) for each i, the colour s_i is in S_i. The set $\{s_1,s_2,..s_p\}$ is therefore a transversal of $S_1,S_2,..,S_p$.

Theorem 5.8

G is a PC_3*-graph coloured from the finite set of colours C if and only if G can be constructed from a single vertex by performing the following operation a finite number of times: let H be the graph already obtained, with vertex set $\{v_1,v_2,..,v_{n-1}\}$, and let S_i be the set of colours from C not incident in H with the vertex v_i, i = 1,2,..,n-1; if a transversal $\{s_1,s_2,..,s_{n-1}\}$ exists of the collection of sets $S_1,S_2,..,S_{n-1}$, where s_i is in S_i, add a vertex v_n to H, and join v_n to v_i by an s_i-coloured edge, i = 1,2,..,n-1.

Proof

Suppose that G is constructed as above. Then by lemma 5.7, the graph obtained at each stage of the construction is a PC_3*-graph, and the graphs are clearly coloured from C.

Conversely, suppose that G is a PC_3*-graph coloured from C, where $V(G) = \{v_1,v_2,..,v_p\}$. For $j = 1,2,..,p$, let H_j be the subgraph of G induced by $\{v_1,v_2,..,v_j\}$, so that $H_p = G$. Each H_j is a PC_3*-graph with colours from the set C. For some such graph H_j, $1 \leq j < p$, let $S_1,S_2,..,S_j$ be the sets of colours not incident in H_j with the vertices $v_1,v_2,..,v_j$ respectively. Since H_j is extendable to H_{j+1}, the colours of the edges (v_i,v_{j+1}) in H_{j+1}, $i = 1,2,..,j$, are a transversal of $S_1,S_2,..,S_j$ by lemma 5.7, and H_{j+1} can be constructed from H_j as in the theorem. Since H_j is chosen arbitrarily, the theorem is proved.

2. Maximal PC_3*-Graphs

One obvious question arises from theorem 5.8 - when can a transversal be chosen from a given collection of sets? If one of the sets is empty, obviously no transversal can exist, so by lemma 5.7 the PC_3*-graph cannot be extended to a larger PC_3*-graph without first extending the colour set available. This section investigates those PC_3*-graphs which cannot be extended without increasing the colour set.

Definition 5.9

Let G be a ($\leq$ k)-edge-coloured PC_3*-graph. G is a k-maximal PC_3*-graph if there is no ($\leq$ k)-edge-coloured PC_3*-graph H containing G as a subgraph. G is a maximal PC_3*-graph if it is a k-maximal PC_3*-graph for some k.

It should be noted that a k-maximal PC_3*-graph G is not necessarily a K-maximal PC_3*-graph, K > k, since G can always be extended if enough colours are available.

A well-known result by P. Hall (see for instance [H3]) characterises those collections of sets with a transversal.

Theorem 5.10 (P. Hall)

A collection $S_1,S_2,..,S_n$ of finite non-empty sets, $n \geq 1$, has a transversal if and only if for $i = 1,2,..,n$ the union of any i of these sets contains at least i distinct elements.

Theorem 5.11

Let G be a PC_3*-graph coloured from the set $\{c_1,c_2,..,c_k\}$. G is a k-maximal PC_3*-graph if and only if for some non-zero integer m, G contains a set X of m distinct vertices and a set of at least $k - m + 1$ colours C(X) such that each vertex in X is incident with all of the colours in C(X).

Proof

Suppose first that such a set $X = \{v_1,v_2,..,v_m\}$ exists. For $i = 1,2,..,m$ let S_i be the set of colours from $\{c_1,c_2,..,c_k\}$ not incident with v_i. The union of the sets $S_1,S_2,..,S_m$ cannot contain any member of C(X), and so can contain at most $m - 1$ colours. The collection of sets $S_1,S_2,..,S_m$ cannot therefore have a transversal by theorem 5.10, and so by lemma 5.7 G is a k-maximal PC_3*-graph.

On the other hand, suppose that G is a k-maximal PC_3*-graph coloured from the set $\{c_1,c_2,..,c_k\}$, with vertex set $\{v_1,v_2,..,v_p\}$. For $i = 1,2,..,p$ let S_i be the set of colours from $\{c_1,c_2,..,c_k\}$ not incident with v_i. By lemma 5.7, the collection of sets $S_1,S_2,..,S_p$ can have no transversal. Thus by theorem 5.10, for some non-zero integer m there exists a collection of m of these sets whose union contains at most $m - 1$ elements. Without loss of generality, suppose that these sets are $S_1,S_2,..,S_m$. Then there exists a set of $k - m + 1$ colours C(X), none of whom appear in these sets. The vertices $v_1, v_2, \ldots, v_m$ must be incident with all of these colours, and the theorem is proved.

First we show that there exist PC_3*-graphs which are not maximal, and then that there exist PC_3*-graphs which are maximal.

Lemma 5.12

Let G be a ($\leq$ k)-edge-coloured PC_3*-graph which is not a k-maximal PC_3*-graph. Then G is not a K-maximal PC_3*-graph for any $K > k$.

Proof

Suppose to the contrary that G is a K-maximal PC_3*-graph for some $K > k$. Then by theorem 5.11, for some non-zero integer m G contains a set of m distinct vertices and a set of at least $K - m + 1$ colours such that each vertex in X is incident with all of the colours in C(X). But then each vertex in X would be incident with all of the colours in any subset of C(X) of order $k - m + 1$. Since G is ($\leq$ k)-edge-coloured, G would be a k-maximal PC_3*-graph by theorem 5.11.

Lemma 5.13

If G is a ($\leq$ k)-edge-coloured PC_3*-graph of order $p \leq \frac{1}{2}k$, then G is not a k-maximal PC_3*-graph.

Proof

Every vertex of G is incident with $p - 1$ edges, and so by theorem 5.1 is incident with $p - 1$ colours. If X is any set of vertices of G and C(X) the set of colours incident with all of the vertices of X, then $|X| \leq p$ and $|C(X)| \leq p - 1$. Hence $|C(X)| < \frac{1}{2}k < k - |X| + 1$, and by theorem 5.11 G is not a k-maximal PC_3*-graph.

Theorem 5.14

For each integer $p \geq 5$, there exists a PC_3*-graph of order p which is not a maximal PC_3*-graph.

Proof

For any $p \geq 5$, let G be a complete graph of order p in which every edge is differently coloured; G is a PC_3*-graph by theorem 5.1. G contains $k = \frac{1}{2}p(p - 1)$ colours, and since $p \leq \frac{1}{2}p(p - 1)$ G cannot be a k-maximal PC_3*-graph by lemma 5.13. Since G contains k colours, the theorem now follows from lemma 5.12.

A standard result (see for instance [F4]) concerns the k-edge-coloured PC_3*-graphs of largest order. These were studied in the context of finding the chromatic index of a complete graph (the chromatic index of a graph is the least number of colours it can contain without forcing two adjacent edges to be the same colour).

Definition 5.15

If p is the largest order of any k-edge-coloured PC_3*-graph, a k-edge-coloured PC_3*-graph of order p is a k-extremal PC_3*-graph.

A k-extremal PC_3*-graph is necessarily a k-maximal PC_3*-graph.

Theorem 5.16

Let G be a k-extremal PC_3*-graph, $k \geq 3$. If k is odd, G has order $k + 1$ and is a K-maximal PC_3*-graph for $k \leq K \leq 2k$. If k is even, G has order k and is a K-maximal PC_3*-graph for $k \leq K < 3.\frac{1}{2}k$.

Proof

By theorem 5.1, a vertex of G can be incident with no more than one edge of each colour. Hence the degree of a vertex in G can be at most k, and the order of G at most $k + 1$. For k odd, a construction of a k-edge-coloured PC_3*-graph of order $k + 1$ can be found in Fiorini and Wilson [F4].

Suppose that k is even, and that G is a k-edge-coloured PC_3*-graph of order $k + 1$. By theorem 5.2, each monochromatic subgraph contains a non-empty set of non-adjacent edges and a (possibly empty) set of isolated vertices; hence there can be no more than $\lfloor \frac{1}{2}(k + 1) \rfloor = \frac{1}{2}k$ edges of any one colour. But for k colours this gives no more than $\frac{1}{2}k^2$ edges altogether, whereas G contains $\frac{1}{2}k(k + 1)$ edges. Thus if k is even, a k-edge-coloured PC_3*-graph can have order at most k. A construction of such a graph can be made from a $(k - 1)$-extremal PC_3*-graph by recolouring one edge in a new colour.

Now let G be a k-extremal PC_3*-graph of order $k + 1$, k odd. Each vertex is incident with all k colours in G. Let K be an integer, $k \leq K \leq 2k$. Then $V(G)$ is a set of $k + 1$ vertices, all incident with a set of at least $K - (k + 1) + 1$ colours, so by theorem 5.11 G is a K-maximal PC_3*-graph.

If G is a k-extremal PC_3*-graph of order k, k even, each vertex is incident with all but one of the k colours in G. We claim that at least $\frac{1}{2}k$ colours are incident with all of the vertices of G. If not, then less than $\frac{1}{2}k$ colours are on $\frac{1}{2}k$ edges each, while the rest of the colours are on at most $\frac{1}{2}k - 1$ edges each. As G contains $\frac{1}{2}k(k - 1)$ edges, this is impossible. Now let K be an integer, $k \leq K < 3.\frac{1}{2}k$. Then $V(G)$ is a set of k vertices, all incident with a set of at least $K - k + 1 \leq \frac{1}{2}k$ colours, so G is a K-maximal PC_3*-graph by theorem 5.11.

Wallis [W1] has shown that for odd $k \geq 7$, there exist at least two non-isomorphic k-extremal PC_3*-graphs.

We have seen that if G is a k-extremal PC_3*-graph, G has even order. There is a weaker restriction on the order of k-maximal PC_3*-graphs.

<u>Theorem 5.17</u>

For each integer $p \geq 6$, $p \not\equiv 3 \pmod 4$, there exists a maximal PC_3*-graph of order p.

Proof

First, let p be even, so that $p - 1$ is odd. By theorem 5.16 there exists a $(p - 1)$-edge-coloured PC_3*-graph of order p which is a maximal PC_3*-graph.

Next, let $p \equiv 1 \pmod 4$ so that $p = 2m + 1$ for some even integer m, $m > 3$. By theorem 5.16, there exists a PC_3*-graph G_1 with vertex set $\{u_1,u_2,..,u_m\}$ and colour set $\{c_{m+1},c_{m+2},..,c_{2m-1}\}$. Also by theorem 5.16, by removing any vertex of an $(m + 1)$-edge-coloured PC_3*-graph of order $m + 2$, a PC_3*-graph G_2 can be constructed with vertex set $\{v_1,v_2,..,v_{m+1}\}$ and colour set $\{c_{m+1},c_{m+2},...,c_{2m+1}\}$. Construct the graph H from G_1 and G_2 by joining u_i to v_j by a c_k-coloured edge, where $j - i \equiv k \pmod{m + 1}$. H is a complete graph of order $2m + 1$, and is $(2m + 2)$-edge-coloured.

Suppose that e_1 and e_2 are adjacent edges in H which are the same colour, where $e_1 = (x,y)$ and $e_2 = (y,z)$ say. Take first the case where y is in G_1. Since G_1 is a PC_3*-graph, x and z cannot both be in G_1, so let x be in G_1 and z be in G_2. But then (x,y) is c_i-coloured for some i, $m + 1 \leqslant i \leqslant 2m - 1$, and (y,z) is c_j-coloured for some j, $0 \leqslant j \leqslant m$; e_1 and e_2 could not be the same colour. The only possibility remaining is that both x and z are in G_2, so that $y = u_r$ for some r, $1 \leqslant r \leqslant m$, and $x = v_s$, $z = v_t$ for some s,t, $1 \leqslant s \leqslant t \leqslant m + 1$. Let e_1 and e_2 be c_k-coloured for some k, $0 \leqslant k \leqslant m$. From the definition of k in the construction of H,

$$
\begin{aligned}
& t - r \equiv k \pmod{m + 1} \\
\text{and} \quad & s - r \equiv k \pmod{m + 1} \\
\text{so} \quad & t - s \equiv 0 \pmod{m + 1}
\end{aligned}
$$

As $0 \leqslant s, t \leqslant m + 1$, s and t must be equal. But then e_1 and e_2 are the same edges, so that H cannot contain two adjacent edges in the same colour. The second case, where y is in G_2, proceeds similarly and gives the same result. By theorem 5.1, therefore, H is a PC_3*-graph.

To show that H is a (2m + 2)-maximal PC_3*-graph, by theorem 5.11 it is enough to show that each vertex of G_1 is incident with the colours $c_o, c_1, \ldots, c_{m+2}$. G_1 is (m - 1)-edge-coloured, and has order m; since no two adjacent edges are the same colour, each vertex of G_1 is incident with each of the colours $c_{m+1}, c_{m+2}, \ldots, c_{2m-1}$, and in particular with the distinct colours c_{m+1} and c_{m+2} (since m > 3). Also, each vertex in G_1 is joined to the m + 1 vertices in G_2 by the colours $c_o, c_1, \ldots, c_m$, and as no two adjacent edges are the same colour in H, each vertex must be incident with each of the colours $c_o, c_1, \ldots, c_m$. H is therefore a maximal PC_3*-graph, and the theorem is proved.

It is not know if there exist maximal PC_3*-graphs of order p, $p \equiv 3 \pmod 4$. There is no maximal PC_3*-graph of order 3.

We finish the section by quoting a result of Hilton [H8] on what could be called "least maximal" PC_3*-graphs, the PC_3*-graphs which can be extended furthest without adding any new colours. The PC_3*-graphs of largest order given the number of colours in them are the k-extremal PC_3*-graphs, where k is odd; a k-edge-coloured PC_3*-graph could be called 'least maximal' if it could be extended to one of these.

<u>Theorem 5.18</u> (Hilton)

Let G be a k-edge-coloured PC_3*-graph of order p, where k is odd. Then G can be extended to a k-extremal PC_3*-graph if and only if each colour in C(G) is incidentwith at least 2p - k - 1 vertices.

3. Other Results

In this section, we investigate the interrelation between some of the characteristics of PC_3*-graphs, such as order and the number of colours in them.

Theorem 5.19

There exists a k-edge-coloured PC_3*-graph of order p, $k \geq 1$, if and only if

$$k \leq \tfrac{1}{2}p(p-1)$$

and

$$k \geq \begin{cases} p-1 & k \text{ odd} \\ p & k \text{ even} \end{cases}$$

Proof

There exists a k-edge-coloured PC_3*-graph of order k + 1 (k odd) or k (k even) by theorem 5.16. Suppose that G is a k-edge-coloured PC_3*-graph of order p, where $k < \tfrac{1}{2}p(p-1)$. It is enough to prove that there exists a (k + 1)-edge-coloured PC_3*-graph of order p. Since $k < \tfrac{1}{2}p(p-1)$, some colour of G must have more than one edge in its monochromatic subgraph. Recolour one of these edges in a new colour to create a (k + 1)-edge-coloured complete graph of order p. Since the recoloured edge is in a new colour, adjacent edges in the new graph must still be differently coloured, so that by theorem 5.1 it is a PC_3*-graph.

Recall that $Q'(k)$ is the largest number of edges in any monochromatic subgraph of a particular complete graph, and $q'(k)$ the least number of edges. Since a complete graph with all of its edges differently coloured is a PC_3*-graph by theorem 5.1, the best possible general lower bound on both $Q'(k)$ and $q'(k)$ for PC_3*-graphs is 1.

Theorem 5.20

The largest possible number of edges of any single colour in a k-edge-coloured PC_3*graph is $\lfloor \tfrac{1}{2}(k+1) \rfloor$.

Proof

By theorem 5.2, a monochromatic subgraph of a PC_3*-graph consists of a set of non-adjacent edges together with a (possibly empty) set of isolated vertices, so a PC_3*-graph of order p can have no more than $\frac{1}{2}p$ edges of any one colour. If k is odd, by theorem 5.19 a k-edge-coloured PC_3*-graph has order at most k + 1, and so has at most $\frac{1}{2}(k + 1)$ edges of any one colour. It is easily checked that it has $\frac{1}{2}(k + 1)$ edges in each of its monochromatic subgraphs.

If k is even, then by theorem 5.19 a k-edge-coloured PC_3*-graph has order at most k, and so has at most $\frac{1}{2}k$ edges in any one colour. A k-extremal PC_3*-graph contains $\frac{1}{2}k(k - 1)$ edges in k colours, and so some monochromatic subgraphs must contain at least $\frac{1}{2}(k - 1)$ edges. Since $\frac{1}{2}(k - 1)$ is not an integer, these monochromatic subgraphs must contain $\frac{1}{2}k$ edges.

Theorem 5.21

Let G be a k-edge-coloured PC_3*-graph, $k > 2$. Then the least number of edges in any monochromatic subgraph of G is q(k) edges, where

$$q(k) \leq \begin{cases} \frac{1}{2}(k + 1) & k \text{ odd} \\ \frac{1}{2}(k - 2) & k \text{ even} \end{cases}$$

with equality possible when k is odd.

Proof

Let k be odd. By theorem 5.20, no k-edge-coloured PC_3*-graph can contain more than $\frac{1}{2}(k + 1)$ edges of any one colour. It was seen in the proof of theorem 5.20 that k-extremal PC_3*-graphs contain $\frac{1}{2}(k + 1)$ edges of each colour, so equality is possible.

Now let k be even. A k-edge-coloured PC_3*-graph has order at most k by theorem 5.16, and so contains at most $\frac{1}{2}k(k - 1)$ edges.

Some monochromatic subgraph must therefore contain at most $\frac{1}{2}(k - 1)$ edges, which reduces to at most $\frac{1}{2}(k - 2)$ since $\frac{1}{2}(k - 1)$ is not an integer.

Thus for PC_3*-graphs, $1 \leq Q'(k) \leq \lfloor \frac{1}{2}(k + 1) \rfloor$ with equality possible, and $1 \leq q'(k) \leq \frac{1}{2}(k + 1)$ (k odd), $1 \leq q'(k) \leq \frac{1}{2}(k - 2)$ (k even).

The final results of this chapter concern the smallest and largest numbers $q(p)$ and $Q(p)$ respectively of edges in the monochromatic subgraphs of any particular PC_3*-graph G of order p. As with $Q'(k)$ and $q'(k)$, since a complete graph with all of its edges differently coloured is a PC_3*-graph by theorem 5.1, the best general lower bound on both $q(p)$ and $Q(p)$ is 1.

Theorem 5.22

The largest possible number of edges of any single colour in a PC_3*-graph of order p is $\lfloor \frac{1}{2}p \rfloor$.

Proof

By theorem 5.2, a monochromatic subgraph of a PC_3*-graph consists of a set of non-adjacent edges together with a (possibly empty) set of isolated vertices, so a PC_3*-graph of order p can have no more than $\lfloor \frac{1}{2}p \rfloor$ edges of any one colour. Equality can be obtained by taking $\lfloor \frac{1}{2}p \rfloor$ edges in one colour (together with an isolated vertex if p is odd), and joining each pair of non-adjacent vertices by an edge in a new colour. The resultant graph is complete and of order p, and is clearly a PC_3*-graph.

Theorem 5.23

Let G be a PC_3*-graph of order p. Then the least number of edges in any monochromatic subgraph of G is $q(p)$ edges, where $q(p) \leq \lfloor \frac{1}{2}p \rfloor$ with equality possible.

Proof

Since $q(p) \leqslant Q(p) \leqslant \lfloor \frac{1}{2}p \rfloor$ by theorem 5.22, it remains to show that equality is possible. If p is even, it was seen in theorem 5.20 that a (p - 1)-extremal PC_3*-graph of order p contains $\frac{1}{2}p$ edges of each colour. If p is odd, then there exists a p-extremal PC_3*-graph G of order p + 1, containing $\frac{1}{2}(p + 1)$ edges of each colour. Each vertex is incident with an edge of each colour, so if one of the vertices of G is removed together with its incident edges, the resultant PC_3*-graph of order p contains $\frac{1}{2}(p - 1)$ edges of each colour.

Thus for PC_3*-graphs $1 \leqslant Q(p) \leqslant \lfloor \frac{1}{2}p \rfloor$, and $1 \leqslant q(p) \leqslant \lfloor \frac{1}{2}p \rfloor$, with equality possible in both cases.

Chapter 6

COMPLETE GRAPHS WITH ALL TRIANGLES BICHROMATIC

1. Structure and Construction

A bichromatic triangle is a triangle containing exactly two colours. A BC_3*-graph is a complete graph in which every triangle is bichromatic. Trivially, if a graph contains only bichromatic triangles, it can contain no polychromatic triangles; every BC_3*-graph is also a $\overline{PC}_3$-graph, and the results of chapter 2 apply.

Theorem 2.7 states that if G is a $\overline{PC}_3$-graph, G is connected in either one or two colours, and when the edges in these colours are removed, the remaining graph is disconnected. In the special case where G is a BC_3*-graph, further information on this disconnected graph can be obtained.

Lemma 6.1

Let G be a BC_3*-graph. Then G is connected in either one or two colours, and when the edges in these colours are removed from G, the resultant graph consists of $n \geq 2$ disjoint BC_3*-graphs $H_1, H_2, .., H_n$. For $i = 1,..,n$, let $V(H_i) = A_i$; then the colour of an A_iA_j-edge in G depends only on the choice of i and j, $1 \leq i < j \leq n$.

Proof

Since G satisfies the conditions of theorem 2.7, it suffices to prove that $H_1, H_2, .., H_n$ are BC_3*-graphs. Since they are subgraphs of G, any triangle in them will be bichromatic. To prove they are complete, it is enough to show that any two vertices u and v in an arbitrary graph H_i are adjacent.

If G is connected in two colours, red and blue say, (the case where

G is connected in a single colour is proved similarly) there exist red and blue edges in G from H_i to the rest of the graph, so for some x and y in V(G) but not in A_i, let there be a blue xA_i-edge and a red yA_i-edge. Let x be in A_j, where $j \neq i$; then since all A_iA_j-edges are the same colour, (x,u) and (x,v) are both blue in G. Similarly, (y,u) and (y,v) are both red in G. The triangles uvx and uvy are bichromatic in G, so (u,v) can be neither blue nor red in G. Since the only edges removed from G to create H_i were blue and red, (u,v) must be an edge in H_i, and the lemma is proved.

So let G be a non-trivial BC_3*-graph, connected in blue and red say (or just in blue). G can be though of as n (possibly trivial) disjoint BC_3*-graphs $H_1,H_2,..,H_n$, none of which contains blue or red edges, and where each pair H_i and H_j is joined either by blue edges or by red edges (if G is connected in blue only, each pair is joined by blue edges).

As with $\overline{PC}_3$-graphs, the study of BC_3*-graphs is greatly facilitated by the use of related graphs (see definition 2.8). The related graphs of $\overline{PC}_3$-graphs were characterised in theorem 2.9; since the BC_3*-graphs form a subset of the $\overline{PC}_3$-graphs, this characterisation must be modified.

<u>Lemma 6.2</u>

If G is a BC_3*-graph, then its related graph R(G) is also a BC_3*-graph.

<u>Proof</u>

From theorem 2.9, R(G) is complete and contains at most two colours, so it is sufficient to prove that it contains no monochromatic triangle. Let $A_1,A_2,..,A_n$ be the vertex sets of the maximal connected components remaining after the removal from G of the edges contained in its connected monochromatic subgraphs, and let $v_r v_s v_t$ be a triangle in R(G).

If $v_r v_s v_t$ is monochromatic, in blue say, then since all of the $A_r A_s$-, $A_r A_t$-, and $A_s A_t$-edges in G must also be blue, G contains a monochromatic triangle. This is impossible as G is a BC_3*-graph, so $v_r v_s v_t$ is bichromatic.

Theorem 2.9 stated that related graphs contained at most two colours. Before a characterisation of the related graphs of BC_3*-graphs can be presented, the 1- and 2-edge-coloured BC_3*-graphs must be investigated.

<u>Lemma 6.3</u>

The only 1-edge-coloured BC_3*-graph up to isomorphism is the complete graph of order 2.

<u>Proof</u>

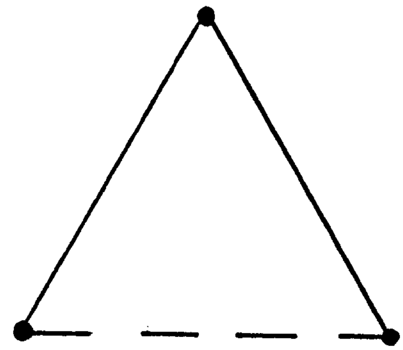
To contain a colour, the graph must contain an edge and therefore has order at least 2. Any BC_3*-graph of order greater than 2 contains a triangle, and so must contain more than one colour.

<u>Lemma 6.4</u>

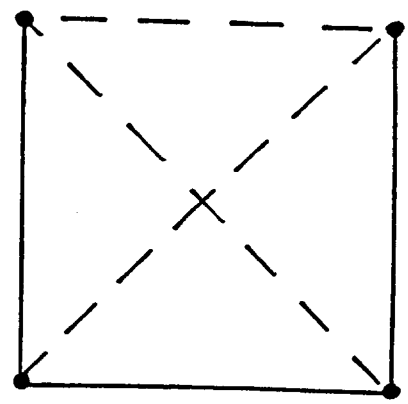
The graphs displayed in figure 6.1 are the only 2-edge-coloured BC_3*-graphs up to isomorphism.

<u>Proof</u>

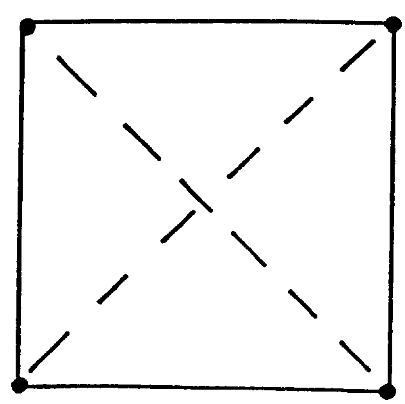
Let G be a 2-edge-coloured BC_3*-graph, coloured in blue and red say. Suppose some vertex of G has degree at least three in some colour, so that for instance (v,x), (v,y), and (v,z) are all blue edges in G. The triangle xyz cannot be red, and so contains a blue edge, (x,y) say. But then vxy is monochromatic; thus each vertex of G can have



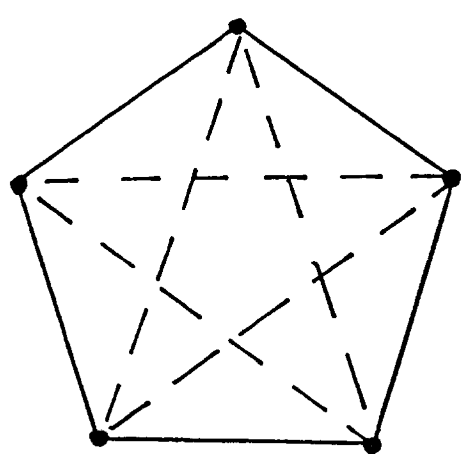
(i)



(ii)



(iii)



(iv)

Figure 6.1

degree no more than two in each colour, giving overall degree at most four and the order of G at most five.

If G has order five, then it is regular of degree two in each colour, and graph (iv) of figure 6.1 is the only possibility for G. If G has order four, each vertex of G has degree one in one colour, and degree two in the other colour. The monochromatic subgraphs of G can either be regular of degree one, regular of degree two, or consist of a path. Graphs (ii) and (iii) of figure 6.1 are the only complete graphs satisfying these conditions; it is easily checked that neither of these graphs contains a monochromatic triangle.

A BC$_3$*-graph of order three is a triangle, and so must be isomorphic to graph (i) of figure 6.1. A 2-edge-coloured complete graph must have order at least three, so the proof is completed.

Theorem 6.5

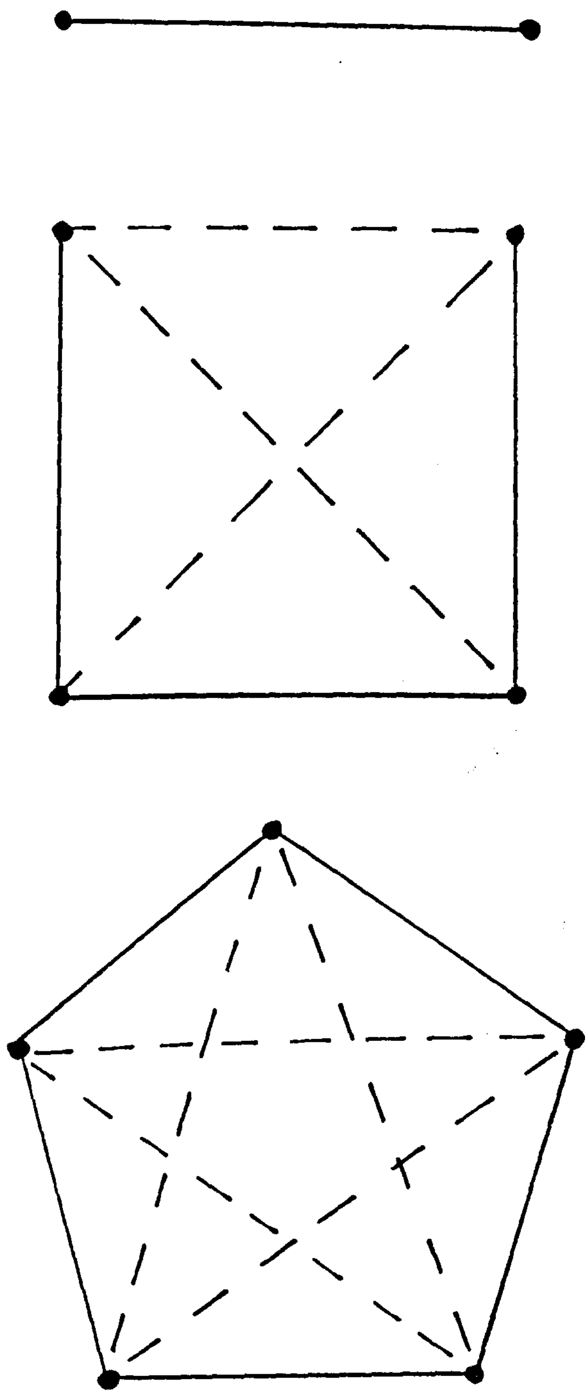
H is a related graph of some BC$_3$*-graph if and only if H is isomorphic to one of the graphs in figure 6.2.

Proof

If H is isomorphic to one of the graphs in figure 6.2, it is easily checked that H is a BC$_3$*-graph, and that it forms its own related graph.

If H is a related graph of some BC$_3$*-graph, by theorem 2.9 and lemma 6.2 H is a 1- or 2-edge-coloured BC *-graph. If H is a 1-edge-coloured BC$_3$*-graph, it is isomorphic to graph (i) of figure 6.2 by lemma 6.3. If H is a 2-edge-coloured BC$_3$*-graph, by lemma 6.4 it must be isomorphic to one of the graphs in figure 6.1. However, by theorem 2.9 H is connected in each of its colours, and so must be isomorphic to one of graphs (ii) and (iii) in figure 6.2.

It should be noted that in figure 6.2 graph (ii) is an induced



(i)

(ii)

(iii)

Figure 6.2

subgraph of graph (iii). From the definition of related graphs of BC_3*-graphs and theorem 6.5, if G is a BC_3*-graph connected in two colours either four or five connected components remain after the removal from G of the edges contained in its connected monochromatic subgraphs. It is sometimes more useful to say that five connected components remain, one of which may have an empty vertex set.

Theorem 6.6

Let G be a BC_3*-graph; then G is connected in either one or two colours.

i) If G is connected in one colour only, say blue, then V(G) can be partitioned into two non-empty sets A_1 and A_2 such that an edge is blue if and only if it is an A_1A_2-edge.

ii) If G is connected in two colours, say blue and red, then V(G) can be partitioned into five sets $A_1, A_2, .., A_5$, only one of which may be empty, such that an edge is blue if and only if it is an A_iA_j-edge, where $|j - i| \equiv \pm 1 \pmod 5$, and red if and only if it is an A_iA_j-edge, where $|j - i| \equiv \pm 2 \pmod 5$.

Proof

G is connected in one or two colours by lemma 6.1.

i) Let G be connected in one colour only, blue. The related graph R(G) is also connected in blue only by theorem 2.9, and so by theorem 6.5 it has order two. Hence when the blue edges are removed from G, by definition 2.8 and lemma 6.1 exactly two BC_3*-graphs remain, with vertex sets A_1 and A_2.

ii) Let G be connected in blue and red. The related graph R(G) is also connected in blue and red by theorem 2.9, and so by theorem 6.5 has order four or five. Hence when the blue and red edges are removed from G, by definition 2.8 and lemma 6.1 exactly five BC_3*-graphs remain, with subgraph of graph (iii). From the definition of related graphs of BC_3*-graphs and theorem 6.5, if G is a BC_3*-graph connected in two colours either four or five connected components remain after the removal from G of the edges contained in its connected monochromatic subgraphs. It is sometimes more useful to say that five connected components remain, one of which may have an empty vertex set.

Theorem 6.6

Let G be a BC_3*-graph; then G is connected in either one or two colours.

i) If G is connected in one colour only, say blue, then V(G) can be partitioned into two non-empty sets A_1 and A_2 such that an edge is blue if and only if it is an A_1A_2-edge.

ii) If G is connected in two colours, say blue and red, then V(G) can be partitioned into five sets $A_1, A_2, .., A_5$, only one of which may be empty, such that an edge is blue if and only if it is an A_iA_j-edge, where $|j - i| \equiv \pm 1 \pmod 5$, and red if and only if it is an A_iA_j-edge, where $|j - i| \equiv \pm 2 \pmod 5$.

Proof

G is connected in one or two colours by lemma 6.1.

i) Let G be connected in one colour only, blue. The related graph R(G) is also connected in blue only by theorem 2.9, and so by theorem 6.5 it has order two. Hence when the blue edges are removed from G, by definition 2.8 and lemma 6.1 exactly two BC_3*-graphs remain, with vertex sets A_1 and A_2.

ii) Let G be connected in blue and red. The related graph R(G) is also connected in blue and red by theorem 2.9, and so by theorem 6.5 has order four or five. Hence when the blue and red edges are removed from G, by definition 2.8 and lemma 6.1 exactly five BC_3*-graphs remain, with vertex sets $A_1, A_2, .., A_5$, where one of these sets may be empty. The theorem now follows from the structure of the related graphs in figure 6.2, since by definition 2.8 the A_iA_j-edges in G are the same colour as the edge (v_i, v_j) in R(G).

As with $\overline{\overline{PC}}_3$-graphs, the related graphs can be used to construct BC_3*-graphs as well as to describe their general structure. The $\overline{\overline{PC}}_3$-graphs were constructed from a single vertex by a series of substitutions of related graphs (see definition 2.10); lemma 2.11 ensured that the end product was a $\overline{\overline{PC}}_3$-graph. To ensure the production of a BC_3*-graph, somewhat more stringent conditions on the related graphs are necessary.

Lemma 6.7

Let H be the graph obtained by substituting G_1 for v in G_2, where G_1 and G_2 are complete graphs and v is a vertex in G_2. Then H is a BC_3*-graph if and only if G_1 and G_2 are BC_3*-graphs and G_1 contains no colour incident in G_2 with v.

Proof

Firstly, let G_1 and G_2 be BC_3*-graphs, and let G_1 contain no colour incident in G_2 with v. H is complete by lemma 2.11, so suppose that xyz is a triangle in H which is not bichromatic. All of the vertices of H are also either in G_1 or in G_2. Since every triangle in G_1 and G_2 is bichromatic, xyz cannot be wholly in G_1 or wholly in G_2. There are two cases to consider: G_1 can contain two vertices of xyz, or G_2 can contain them. If one vertex, say x, is in G_1 and the other two in G_2, then xyz is in the same colours as the triangle vyz in G_2. All triangles in G_2 are bichromatic, so this leads to a contradiction. Otherwise, both x and y say are in G_1 and z is in G_2. Since (x,z) and (y,z) in H are both the same colour as (v,z) in G_2, the triangle xyz must be monochromatic. But then (x,y) in G_1 is the same colour as (v,z) in G_2, again a contradiction since G_1 contains no colour incident in G_2 with v.

Secondly, let H be a BC_3*-graph. Since G_1 is a subgraph of H, G_1 must also be a BC_3*-graph. Let u be any vertex of G_1; then the subgraph of H induced by u together with all the vertices of G_2 except v is isomorphic to G_2. All of the triangles in this subgraph of H must be bichromatic so G_2 is a BC_3*-graph.

Now let z be any vertex in G_2. If G_1 contains an edge (x,y) in the same colour as the edge (v,z) in G_2, the triangle xyz in H would be monochromatic. This is impossible, so G_1 contains no colour incident in G_2 with v.

Theorem 6.8

Let G be a BC_3*-graph. Then G can be obtained from a single vertex by performing a finite series of substitutions of related graphs of BC_3*-graphs.

Proof

By theorem 2.12, G can be obtained from a single vertex by performing a finite series of substitutions of related graphs of $\overline{\overline{PC}}_3$-graphs. Let these related graphs by $R_1,R_2,..,R_n$, and the $\overline{\overline{PC}}_3$-graphs obtained at each stage $G_1,G_2,..,G_n$, where G_n is the graph G. It is enough to prove that for $i = 1,2,..,nR_i$ is the related graph of a BC_3*-graph.

If there exists an integer i, $1 \leq i \leq n$, such that G_i is not a BC_3*-graph, then there must be a largest such integer j. Since G_n is a BC_3*-graph $j < n$, so by substituting the related graph R_j for a vertex in G_j the BC_3*-graph G_{j+1} is obtained. But by lemma 6.7, G_{j+1} is a BC_3*-graph only if G_j is also a BC_3*-graph, a contradiction. Hence for

$i = 1,2,..,n$ G_i is a BC_3*-graph.

Now consider the related graph R_i, where $1 \leq i \leq n$. R_i is substituted for a vertex in G_{i-1} to create the graph G_i. As G_i is a BC_3*-graph, by lemma 6.7 so is R_i. By theorem 2.9, a related graph is 1- or 2-edge-coloured, so R_i must be one of the graphs given in lemmas 6.3 and 6.4. But a related graph is connected in each of its colours, also by theorem 2.9, so R_i must be one of the graphs in figure 6.2. Hence for $i = 1,2,..,n$ R_i is the related graph of a BC_3*-graph.

Lemma 6.7 is a much stronger condition than that required for $\overline{PC}_3$-graphs (see lemma 2.11): as well as the expected restriction that G_1 and G_2 should be BC_3*-graphs, there is also a colouring restriction involving G_1 and the edges incident with the vertex in G_2 for which G_1 is substituted. Thus although it is possible to say that the set of BC_3*-graphs is generated by the set of related graphs of BC_3*-graphs using the operation of substitution, it is not possible to say that the set of BC_3*-graphs is exactly the set generated in this way. It is nevertheless a remarkable result that all of the BC_3*-graphs are generated by a set containing just three non-isomorphic graphs.

2. Other Results

Theorem 6.9

There exists a k-edge-coloured BC_3*-graph or order p if and only if

$$k + 1 \leq p \leq \begin{cases} 5^{\frac{1}{2}k} & k \text{ even} \\ 2.5^{\frac{1}{2}(k-1)} & k \text{ odd} \end{cases} \qquad (6A)$$

Proof

By induction on k. The cases $k = 1$ and $k = 2$ are given by lemmas 6.3 and 6.4, so assume the theorem true for $k < K$, where $K > 2$. First we prove that if G is a K-edge-coloured BC_3*-graph of order p, then p satisfies equation (6A).

Since G is also a $\overline{\overline{PC}}_3$-graph, the lower bound on p is given by theorem 2.13. For the upper bound, remove from G the edges contained in its connected monochromatic subgraphs. If G is connected in two colours, then by theorem 6.6 either four or five $(K - 2)$-edge-coloured BC_3*-graphs remain, with vertex sets $A_1, A_2, .., A_5$, where A_5 may be empty. Then

$$p = |A_1| + |A_2| + \dots + |A_5|$$

so by the induction assumption,

$$p \leqslant 5.\begin{cases} 5^{\frac{1}{2}(K-2)} & K \text{ even} \\ 2.5^{\frac{1}{2}(K-3)} & K \text{ odd} \end{cases}$$

$$= \begin{cases} 5^{\frac{1}{2}K} & K \text{ even} \\ 2.5^{\frac{1}{2}(K-1)} & K \text{ odd} \end{cases}$$

Otherwise, G is connected in one colour only, and this case is proved similarly.

Next, we prove that if p is an integer satisfying equation (6A), where $k = K$, there exists a K-edge-coloured BC_3*-graph G of order p. We distinguish three cases (which may overlap). If $K + 1 \leqslant p \leqslant 2.5^{\frac{1}{2}(K-2)}$, G should be the join in blue of G_1 and a vertex z, where G_1 is a $(K - 1)$-edge-coloured BC_3*-graph of order $p - 1$ not containing the colour blue. The existence of G_1 is guaranteed by the induction assumption. As all $zV(G_1)$-edges are blue, and no edge in G_1 is blue, any triangle containing z is bichromatic; all triangles in G_1 are also bichromatic, so G is a BC_3*-graph. Clearly, G is a K-edge-coloured graph of order p, and so is the required graph.

Now let $2.5^{\frac{1}{2}(K-3)} + 2 \leq p \leq 2.5^{\frac{1}{2}(K-1)}$, and let G_5 be a $(K-2)$-edge-coloured BC_3*-graph of order $\lfloor 2.5^{\frac{1}{2}(K-3)} \rfloor$ containing no blue or red edges; the existence of G_5 is guaranteed by the induction assumption. Put $r = p - \lfloor 2.5^{\frac{1}{2}(K-3)} \rfloor$, and let r_1, r_2, r_3, and r_4 be integers satisfying $0 \leq r_i \leq \lceil 2.5^{\frac{1}{2}(K-3)} \rceil$, $i = 1,2,3,4$, summing to r, and such that r_1 and r_2 are non-zero. Then by the induction assumption, for $i = 1,2,3,4$, if $r_i \geq 1$ there exists a BC_3*-graph G_i of order r_i, coloured from the colour set of G_5. Now let H be a BC_3*-graph coloured in blue and red, isomorphic to graph (iii) of figure 6.2, with vertex set $v_1, v_2, .., v_5$. For $i = 1,2,..,5$ substitute G_i for v_i in H (if $r_i = 0$ this means removing v_i from H). By lemma 6.7 the resultant graph is a BC_3*-graph, and it has order p. Since G_1, G_2, and G_5 have non-empty vertex sets, G contains both blue and red edges, and so is K-edge-coloured.

For $p > 2.5^{\frac{1}{2}(K-1)}$, if p satisfies equation (6A) K must be even, and so $K - 2$ is also even. Let G_5 be a $(K-2)$-edge-coloured BC_3*-graph of order $5^{\frac{1}{2}(K-2)}$ containing no blue or red edges; again, the existence of G_5 is guaranteed by the induction assumption. Put $r = p - 5^{\frac{1}{2}(K-2)}$ and proceed as above, except that each r_i should satisfy $0 \leq r_i \leq 5^{\frac{1}{2}(K-2)}$. The proof is now complete.

Corollary 6.10

There are finitely many non-isomorphic k-edge-coloured BC_3*-graphs for each k.

Proof

A k-edge-coloured complete graph of order p has $\frac{1}{2}p(p-1)$ edges, and only k possible colours for each edge. Hence there can only be finitely many k-edge-coloured complete graphs of order p up to isomorphism. As there are only finitely many possible orders for k-edge-coloured complete graphs with all triangles bichromatic by theorem 6.9, there can only be finitely many k-edge-coloured BC_3*-graphs.

We now consider the limits on the number of edges of a single colour in a BC_3*-graph relative to the numberof colours contained in it. As before, denote by Q(k) the largest number of edges in any monochromatic subgraph of a particular k-edge-coloured complete graph, and denote by q(k) the least number of edges.

Theorem 6.11

Let G be a k-edge-coloured BC_3*-graph. Then the largest number of edges in a single colour which G may have is 5^{k-1}.

Proof

By induction on k. For k = 1,2 the theorem can be verified using lemmas 6.3 and 6.4. Suppose the theorem true for k < K, and let G be a K-edge-coloured BC_3*-graph, K > 2. If G is connected in one colour only, blue say, then when the blue edges are removed from G by theorem 6.6 two BC_3*-graphs remain, G_1 and G_2 say, each containing at most K - 1 colours. If red is any colour in G_1 or G_2 or both, then by the induction assumption there can be at most 5^{K-2} red edges in G_1 or G_2, giving at most 2.5^{K-2} red edges altogether. If $V(G_i) = A_i$ for i = 1,2 then by theorem 6.9 $|A_i| \leq 5^{\frac{1}{2}(K-1)}$ with equality possible when K is odd. Since by theorem 6.6 all the A_1A_2-edges are blue, and no other edges in G are blue, G contains $|A_1||A_2| \leq 5^{K-1}$ blue edges, with equality possible when K is odd.

If G is not connected in one colour only, then by theorem 6.6 it is connected in two colours, blue and red say, and when the blue and red edges are removed from G $n \leq 5$ BC_3*-graphs $G_1, G_2, .., G_n$ remain, each containing at most K - 2 colours. By the induction assumption, if green is any other colour in G, then each G_i can contain at most 5^{K-3} green edges, giving at most 5^{K-2} green edges altogether in G. If $V(G_i) = A_i$ for $i = 1,2,..,n$ then by theorem 6.9 $|A_i| \leqslant 5^{\frac{1}{2}(K-2)}$, with equality possible when K is even. Since by theorem 6.6 A_iA_j-edges can be blue (or red) for at most five classes of i and j, $1 \leqslant i < j \leqslant n$, and no other edge in G can be blue or red, G contains at most $5.5^{\frac{1}{2}(K-2)}.5^{\frac{1}{2}(K-2)} = 5^{K-1}$ blue (or red) edges, with equality possible when K is even.

<u>Theorem 6.12</u>

Let G be a k-edge-coloured BC_3*-graph. Then G contains k edges in some colour, but need not contain k + 1 edges in any colour.

<u>Proof</u>

It is easily checked that the extremal graph used in the proof of theorem 2.16 is a BC_3*-graph, so theorem 2.16 carries over.

Thus for BC_3*-graphs, $Q'(k)$ must satisfy $k \leqslant Q'(k) \leqslant 5^{k-1}$, with equality possible. The graph constructed in the proof of theorem 2.14 gives an attainable lower bound on $q'(k)$ of 1 for BC_3*-graphs. An upper bound on $q'(k)$ can be found from equation (6A) using the fact that some monochromatic subgraph of a graph must have a no more than average number of edges; for even k, this gives an upper bound of $\frac{5k - 5^{\frac{1}{2}k}}{2k}$. This bound can be slightly improved by using the same technique on the graphs remaining after the removal of the edges in the connected colours.

<u>Theorem 6.13</u>

Let G be a k-edge-coloured BC_3*-graph. Then some colour in G must have no more than $q'(k)$ edges, where

$$q'(k) \leq \begin{cases} 1 & k = 1 \\ 5 & k = 2 \\ \dfrac{4.5^{k-2} - 2.5^{\frac{1}{2}(k-2)}}{k-1} & k \text{ even}, k > 2 \\ \dfrac{5^{k-1} - 5^{\frac{1}{2}(k-1)}}{k-1} & k \text{ odd}, k > 2 \end{cases}$$

Equality is possible for $k = 1,2$.

Proof

The result for $k = 1,2$ is given by lemmas 6.3 and 6.4, so let G be a k-edge-coloured BC_3*-graph, $k > 3$, and suppose that G is connected in one colour only, blue say. By theorem 6.6, V(G) can be partitioned into two non-empty sets A_1 and A_2 such that all A_1A_2-edges are blue, but no other edge in G is blue. If G_1 and G_2 are the subgraphs of G induced by A_1 and A_2, then G_1 and G_2 are BC_3*-graphs with $k - 1$ colours between them and $\frac{1}{2}|A_1|(|A_1| - 1)$ and $\frac{1}{2}|A_2|(|A_2| - 1)$ edges respectively. Some colour in G_1 and G_2 must have a no more than average number of edges, so for BC_3*-graphs connected in one colour only,

$$q'(k) \leq \frac{|A_1|^2 - |A_1| + |A_2|^2 - |A_2|}{2(k-1)}$$

By theorem 6.9,

$$q'(k) \leq \begin{cases} \dfrac{4.5^{k-2} - 2.5^{\frac{1}{2}(k-2)}}{k-1} & k \text{ even}, k \geq 3 \\ \dfrac{5^{k-1} - 5^{\frac{1}{2}(k-1)}}{k-1} & k \text{ odd}, k \geq 3 \end{cases}$$

If G is not connected in one colour only, then by theorem 6.6 it is connected in two colours, blue and red say. V(G) can be partitioned into five sets $A_1, A_2, .., A_5$, one of which may be empty, such that an edge is an A_iA_j-edge if and only if it is blue or red, $1 \leq i < j \leq 5$. Let $G_1, .., G_5$ be the subgraphs of G induced by $A_1, .., A_5$; $G_1, .., G_5$ are

BC_3*-graphs with k - 2 colours between them. Again some colour in $G_1,\ldots,G_5$ must have a no more than average number of edges, so for BC_3*-graphs connected in two colours,

$$q'(k) \leqslant \frac{1}{k-2} \sum_{i=1}^{5} \frac{|A_i|^2 - |A_i|}{2}$$

So by theorem 6.9,

$$q'(k) \leqslant \begin{cases} \dfrac{5^{k-1} - 5^{\frac{1}{2}k}}{2(k-2)} & k \text{ even, } k \geqslant 3 \\[2ex] \dfrac{2.5^{k-2} - 5^{\frac{1}{2}(k-1)}}{k-2} & k \text{ odd, } k \geqslant 3 \end{cases}$$

$$q'(k) \leqslant \begin{cases} \dfrac{4.5^{k-2} - 2.5^{\frac{1}{2}(k-2)}}{k-1} & k \text{ even, } k \geqslant 3 \\[2ex] \dfrac{5^{k-1} - 5^{\frac{1}{2}(k-1)}}{k-1} & k \text{ odd, } k \geqslant 3 \end{cases}$$

Hence the bound is satisfied for all BC_3*-graphs.

Next, we consider the limits on the number of edges of a single colour in a BC_3*-graph relative to its order. Denote by Q(p) the largest number of edges in any monochromatic subgraph of a BC_3*-graph G of order p, and denote by q(p) the least number of edges.

<u>Theorem 6.14</u>

Let G be any BC_3*-graph of order p. Then the largest number of edges in a single colour which G may have is $\lfloor \frac{1}{4}p^2 \rfloor$.

<u>Proof</u>

If any monochromatic subgraph of G has more than $\lfloor \frac{1}{4}p^2 \rfloor$ edges, then by Turans theorem it contains a monochromatic triangle. As each triangle in G is bichromatic, this is impossible, so $Q(p) \leqslant \lfloor \frac{1}{4}p^2 \rfloor$ for all BC_3*-graphs.

Now let p be given. Suppose G is the graph formed by the join in blue of the BC_3*-graphs G_1 and G_2, where G_1 has order $\lfloor \frac{1}{2}p \rfloor$, G_2 has order $\lceil \frac{1}{2}p \rceil$, and neither contains a blue edge. It is easily checked that G is a BC_3*-graph and contains $\lfloor \frac{1}{4}p^2 \rfloor$ blue edges, so the bound $Q(p) \leq \lfloor \frac{1}{4}p^2 \rfloor$ is the best possible bound for BC_3*-graphs.

Theorem 6.15

Let G be a BC_3*-graph of order p. Then G contains p - 1 edges in some colour, but need not contain p edges in any colour.

Proof

The proof of theorem 2.14 carries through to BC_3*-graphs, since the extremal graph H_p is easily checked to be a BC_3*-graph.

Thus for BC_3*-graphs Q(p) satisfies $p - 1 \leq Q(p) \leq \lfloor \frac{1}{4}p^2 \rfloor$, with equality possible. The extremal graph constructed in the proof of theorem 2.14, H_p, also serves to show that the attainable lower bound on q(p) is 1 for BC_3*-graphs. An upper bound can be found by considering the average number of edges in the monochromatic subgraphs of the BC_3*-graphs. Theorem 6.9 gives a lower bound for the number of colours in the graph: if G is a BC_3*-graph of order p, where $5^{n-1} < p \leq 5^n$ for some n, then G contains at least 2n colours. Some colour in G has at most an average number of edges, so $\frac{p(p-1)}{4n}$ is an upper bound for q(p). However, this seems far from a best possible result, since a graph which is close to the upper bound given in theorem 6.9 need not be close to the upper bound on q(p) for BC_3*-graphs.

Next, we consider the relationship between the order of a BC_3*-graph and limits on degree in its monochromatic subgraphs. Firstly, we show that all BC_3*-graphs are subject to a minimum degree constraint, unlike

$\overline{PC}_3$-graphs where all constraints are related to order (see theorems 2.17, 2.18).

Lemma 6.16

Let G be a BC_3*-graph. Then there exists a vertex in G incident with no more than two edges of some colour.

Proof

By induction on the order p of G. If $p = 2$, then the result is clear. If $2 < p \leqslant 5$, G cannot be 1-edge-coloured by lemma 6.3, so some vertex of G is incident with at least two colours. As no vertex of G is incident with more than four edges, this vertex must be incident with no more than two edges in one of these colours.

Now assume the lemma true for $p < p_o$, where $p_o > 5$, and let G be a BC_3*-graph of order p_o. By lemmas 6.3 and 6.4, G must contain more than two colours. Remove from G the edges in the connected colours; by theorem 6.6 n BC_3*-graphs $G_1, G_2, .., G_n$ remain, where $n \geqslant 2$. Each of these graphs has order less than p_o, so in particular by the induction assumption there exists a vertex v in G_1 which is incident with no more than two edges of some colour in G_1. As no edge of this colour has been removed from G to obtain G_1, v is incident with no more than two edges of this colour in G.

Theorem 6.17

Let G be a k-edge-coloured BC_3*-graph of order p with at least δ edges of each colour incident with each vertex. Then $\delta = 1$ or 2, and

$$p \geqslant \begin{cases} 2^k & \delta = 1 \\ 5.2^k - 2 & \delta = 2,\ k > 1 \end{cases} \qquad (6B)$$

with equality possible.

Proof

Suppose that such a graph exists. If $k = 1$, then $\delta = 1$ and $p = 2$ by lemma 6.3. Otherwise, $\delta = 1$ or 2 by lemma 6.16, and p satisfies equation (6B) by theorem 2.18. The proof that equality is possible proceeds by induction on k. For $k = 1,2$ the theorem is given by lemmas 6.3 and 6.4, so let $K > 2$ and assume the theorem true for $k < K$. By the induction assumption there exists a $(K - 1)$-edge-coloured BC_3*-graph G_1 with minimum degree at least δ in each monochromatic subgraph, and of order 2^{K-1} if $\delta = 1$, and 5.2^{K-3} if $\delta = 2$. Let G_2 be a graph isomorphic to G_1, with the same colour set, and let G be the join of G_1 and G_2 in colour c, where c is not in G_1. G is clearly a K-edge-coloured BC_3*-graph of the required order, and each vertex of G is incident with at least δ c-coloured edges. If c_1 is any colour in G other than c, and v is any vertex in G, then v is in G_i for $i = 1$ or 2, and is incident with at least δ c_1-coloured edges in G_i. Since G_i is a subgraph of G, v is incident with at least δ c_1-coloured edges in G also.

Note that an upper bound on p is given by theorem 6.9.

Finally, we consider an upper limit on the number of edges of each colour at each vertex of a BC_3*-graph.

Theorem 6.18

There exists a BC_3*-graph G of order p with no vertex incident with more than Δ edges of any colour if and only if

$$p \leq \begin{cases} 2 & \Delta = 1 \\ 5.\tfrac{1}{2}\Delta & \Delta \text{ even} \\ \tfrac{1}{2}(5\Delta - 3) & \text{otherwise} \end{cases} \tag{6C}$$

Proof

Since G is also a $\overline{\overline{PC}}_3$-graph, p must satisfy equation (6C) by theorem 2.17.

Now let Δ be given, and let p satisfy equation (6C). If $p \leq 5$, there exists a BC_3*-graph of order p satisfying the requirements of the theorem by lemmas 6.3 and 6.4, so let $p > 5$ and assume the theorem true for orders less than p.

Let H be a graph coloured in blue and red, with vertex set $\{v_1,..,v_5\}$ and isomorphic to graph (iii) in figure 6.2. If BC_3*-graphs $G_1,..,G_5$, not containing blue or red, are substituted successively for the vertices $v_1,..,v_5$ in H, the resultant graph G is a BC_3*-graph by lemma 6.7. To ensure that G has order p, it is sufficient that $|V(G_1)| + |V(G_2)| +..+ |V(G_5)| = p$.

Consider a vertex v in G_1; the blue edges incident with v are the $vV(G_2)$- and $vV(G_5)$-edges (say), and the red edges incident with v are the $cV(G_3)$- and $vV(G_4)$-edges. Hence if $|V(G_2)| + |V(G_5)| \leq \Delta$, and if $|V(G_3)| + |V(G_4)| < \Delta$, v cannot be incident with more than Δ red or blue edges, and if $|V(G_1)| \leq \Delta$, v cannot be incident with more than Δ edges in any other colour either. In general, the conditions of the theorem are satisfied if there exist BC_3*-graphs $G_1,..,G_5$ with total order p, and $|V(G_i)| + |V(G_j)| \leq \Delta$ for any i and j, $1 \leq i < j \leq 5$.

Let $p = 5r + s$ for some integers r and s, $r > 0$ and $0 \leq s < 5$. Let G_i be a BC_3*-graph not containing red or blue, and with order $r + 1$ for $i = 1,..,s$ and order r for $i = 1 + s,..,5$. Clearly $|V(G_1)| +..+ |V(G_5)| = p$. To show that $|V(G_i)| + |V(G_j)| \leq \Delta$ for $1 \leq i < j \leq 5$, we take various cases separately.

Case 1: $s = 0$. Then $|V(G_i)| + |V(G_j)| = 2r$; but $p = 5r$ satisfies equation (6C), so $5r \leq 5.\frac{1}{2}\Delta$ giving $2r \leq \Delta$ as required.

Case 2: $s = 1$. Then $|V(G_i)| + |V(G_j)| \leq 2r + 1$; $p = 5r + 1$ satisfies equation (6C), so $5r + 1 \leqslant 5.\frac{1}{2}\Delta$, giving $\Delta \geqslant 2r + \frac{2}{5}$. Since Δ is an integer, this rounds up to $\Delta \geqslant 2r + 1$ as required.

Case 3: $s > 1$ and Δ is even. Then $|V(G_i)| + |V(G_j)| \leqslant 2r + 2$; $p = 5r + s$ satisfies equation (6C), so $5r + s \leqslant 5.\frac{1}{2}\Delta$, giving $2r + \frac{2}{5}s \leqslant \Delta$. Since Δ is an even integer and $0 < s < 5$, this rounds up to $2r + 2 \leqslant \Delta$, as required.

Case 4: $s > 1$ and Δ is odd. Then again $|V(G_i)| + |V(G_j)| \leqslant 2r + 2$; $p = 5r + s$ satisfies equation (6C), and $p > 5$, so $5r + s \leqslant 5.\frac{1}{2}\Delta - 3.\frac{1}{2}$, giving $2r + \frac{2s + 3}{5} \leqslant \Delta$. Since Δ is an odd integer and $s > 1$, this rounds up to $2r + 2 < \Delta$, as required.

The proof now follows by induction.

Chapter 7

COMPLETE GRAPHS WITHOUT POLYCHROMATIC CIRCUITS

A circuit C is polychromatic if no two edges of C are the same colour. In chapter 2, we investigated complete graphs without polychromatic circuits of length 3. In this chapter, we extend the scope of that investigation to include complete graphs without polychromatic circuits of other lengths.

1. Complete Graphs in Which No Circuit is Polychromatic

A complete graph in which no circuit is polychromatic is a $\overline{PC}$-graph. Note that a C_m is a circuit of length m.

Lemma 7.1

Let G be a complete graph containing a polychromatic C_m, $m \geq 4$. Then for each r, $1 \leq r < m - 2$, G contains a polychromatic C_{r+2} or a polychromatic C_{m-r}.

Proof

Let $v_1v_2...v_m$ be a polychromatic C_m in G, and for $r = 1,2,..,m - 3$ consider the edge (v_1,v_{r+2}). There cannot be two edges in $v_1v_2...v_m$ coloured the same as (v_1,v_{r+2}), so at least one of the circuits $v_1v_2...v_{r+1}v_{r+2}v_1$ and $v_1v_{r+2}v_{r+3}...v_mv_1$ must be polychromatic.

Theorem 7.2

Let G be a complete graph. Then G contains no polychromatic circuit if and only if G contains no polychromatic triangle.

Proof

If G contains a polychromatic triangle, this is also a polychromatic circuit. If G contains a polychromatic circuit but no polychromatic triangle, then G contains a polychromatic circuit of minimum length n, where $n \geq 4$. Putting $r = 1$ and applying lemma 7.1 shows that if G contains no polychromatic triangle it must contain a polychromatic circuit of length $n - 1$. This contradicts the minimality of n, so the theorem is proven.

Thus the $\overline{\overline{PC}}$-graphs arejust the $\overline{\overline{PC}}_3$-graphs, so that all the results of chapter 2 apply to $\overline{\overline{PC}}$-graphs. Here we adapt only two of those results, theorems 2.7 and 2.12.

Theorem 7.3

Let G be a $\overline{\overline{PC}}$-graph. It is connected in either one or two colours, and if the edges in these colours are removed from G then n connected components with vertex sets $A_1, A_2, .., A_n$ remain, $n \geq 2$. If G is connected in one colour only, then for $i \neq j$ every A_iA_j-edge is in that colour. If G is connected in two colours, then $n \geq 4$ and for $i \neq j$ every A_iA_j-edge is in one of the two connected colours, which colour being dependent only on i and j.

Theorem 7.4

Let G be a $\overline{\overline{PC}}$-graph. Then G can be obtained from a single vertex by performing a finite series of substitutions of related graphs.

2. Complete Graphs With No Polychromatic Circuit of Length n

A complete graph which does not contain a polychromatic circuit of length n is a $\overline{\overline{PC}}_n$-graph. The complete graphs with at most $n - 1$ vertices or $n - 1$ colours are $\overline{\overline{PC}}_n$-graphs, as are the $\overline{\overline{PC}}$-graphs. A further set of examples is the set of complete graphs in which no polychromatic circuit has length more than $n - 1$. Some of the graphs in this set (though not all of them, as is seen from the $(n-1)$-edge-coloured complete graphs) can be constructed by generalising a method of Erdos, Simonovits and Sos [E6].

Theorem 7.5

Let G be a $\overline{\overline{PC}}$-graph containing the vertices $v_1, v_2, .., v_m$, and let $G_1, G_2, .., G_m$ be complete graphs in which polychromatic circuits have length at most $n - 1$ for some $n > 3$. Then the graph H constructed by successively substituting G_i for v_i in G, $i = 1,2,..,m$, is a complete graph in which polychromatic circuits have length at most $n - 1$.

Proof

Define H_o to be G, and for $i = 1,2,..,m$ define H_i to be the graph obtained by substituting G_i for v_i in H_{i-1}, so that H_m is H; by theorem 2.11 each H_i is complete. Assume the theorem false, so that H_m contains a polychromatic circuit of length at least n. As H_o contains no such circuit, there exists a least integer j such that H_j contains a polychromatic circuit of length at least n, $0 < j \leq m$.

Suppose that $C = x_1 x_2 \ldots x_r$ is a polychromatic circuit in H_j, where $r \geq n$. H_j is constructed by substituting G_j for v_j in H_{j-1}, so every vertex in H_j is in either G_j or H_{j-1}. Since C cannot be wholly in H_{j-1} or wholly in G_j, two adjacent vertices of C, x_1 and x_r say, must be in H_{j-1} and G_j respectively. If x_2 were in G_j, then both (x_1, x_r) and (x_1, x_2) in H_j would be in the same colour as (x_1, v_j) in H_{j-1} by the definition of substitution, and C could not be polychromatic; x_2 must be in H_{j-1}.

Let s be the least integer for which x_s is in G_j, so that $2 < s \leq r$. Now (x_1, x_r) and (x_{s-1}, x_s) in H_j are the same colour as (x_1, v_j) and (x_{s-1}, v_j) respectively in H_{j-1}, so the circuit $C_a = v_j x_1 x_2 \ldots x_{s-1} v_j$ in H_{j-1} is in the same colours as the path $x_r x_1 x_2 \ldots x_{s-1} x_s$ in H_j. This path is part of C, and so must be polychromatic. This means that C_a is also polychromatic, so v_j must be contained in a polychromatic circuit in H_{j-1}.

H_o contains no polychromatic circuit, so let i be the least integer for which H_i contains a polychromatic circuit which includes v_j, so that $0 < i \leq j - 1$. Suppose $C_b = v_j z_1 z_2 \ldots z_r$ is a polychromatic circuit in H_i, $1 \leq r < n - 1$. H_i is constructed by substituting G_i for v_i in H_{i-1}, so every vertex in H_i is in either G_i or H_{i-1}. C_b cannot be wholly in H_{i-1}, so at least one vertex of C_b is in G_i. If both z_1 and z_r were in G_i, then (v_j,z_1) and (v_j,z_r) in H_i would be the same colour as (v_j,v_i) in H_{i-1}, and C_b would not be polychromatic; z_r say must be in H_{i-1}.

Now let s be the least integer such that z_s is in G_i, and t the largest integer such that z_t is in G_i, so that $1 \leq s \leq t < r$. As before, the circuit $v_i z_{t+1} z_{t+2} \ldots z_r v_j z_1 z_2 \ldots z_{s-1}$ in H_{i-1} is in the same colours as the path $z_t z_{t+1} \ldots z_r v_j z_1 z_2 \ldots z_s$ in H_i, and so is a polychromatic circuit in H_{i-1} containing v_j. This contradicts the minimality of i, so the theorem is proved.

Any graph constructed by the method just outlined is connected in either one or two colours by lemma 2.11(iii) and theorem 7.3. As the connectedness of the monochromatic subgraphs is a significant factor in the complete graphs discussed previously, especially the $\overline{\overline{PC}}$-graphs, it is natural to speculate on whether it is of any significance in $\overline{\overline{PC}}_n$-graphs, or even in complete graphs in which polychromatic circuits have length at most n - 1. In particular, is there a limit on the number of connected monochromatic subgraphs in such graphs? The (n - 1)-edge-coloured complete graphs provide examples connected in any number of colours up to n - 1, and figure 7.1 shows a 4-edge-coloured complete graph connected in all four colours in which any polychromatic circuit has length three. Thus if any general upper limit does exist, it is at least n.

We now consider $\overline{\overline{PC}}_n$-graphs which contain a polychromatic circuit of length greater than n, concentrating on the case n = 4. Lemma 7.1 showed that the existence of a large polychromatic circuit in a complete graph guaranteed the existence of some smaller polychromatic circuits, although their length was not precisely known. It is possible to specify the length of some of these circuits.

Lemma 7.6

Let G be a complete graph containing a polychromatic C_m, $m \geq 4$. If $m \equiv 2 \pmod r$ for any r satisfying $1 \leq r < m - 2$, then G contains a polychromatic C_{r+2}.

Proof

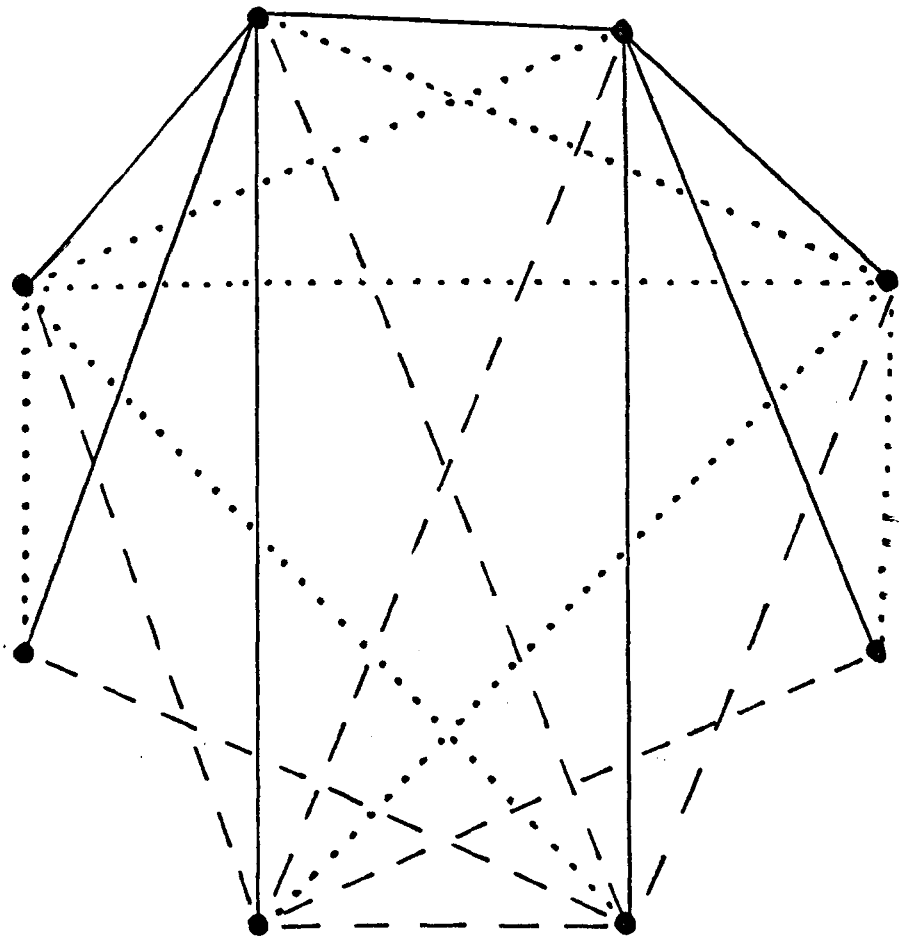
By induction on m. The lemma is true for m = 4 by theorem 7.2, so assume it is true for $m < M$ and let G be a complete graph with a polychromatic C_M, $M > 4$. Suppose that for some r, $M \equiv 2 \pmod r$, where $1 \leq r < M - 2$. By lemma 7.1, if G contains no polychromatic C_{r+2} it must contain a polychromatic C_{M-r}. But $M - r \equiv 2 \pmod r$ and $1 \leq r < m - r - 2$ (since $M - r \neq r + 2$), so the result follows by induction.

Theorem 7.7

Let G be a $\overline{\overline{PC}}_n$-graph, and suppose that G contains a polychromatic C_p, where $p \equiv m \pmod{n - 2}$, $0 \leq m < n - 2$. Then $m \neq 2$, and if $q < p$ and $q \equiv m \pmod{n - 2}$, then G contains a polychromatic C_q.

Proof

Straightforward from lemmas 7.1 and 7.6



(one colour omitted)

Figure 7.1

Corollary 7.8

Let G be a $\overline{\overline{PC}}_4$-graph. Then G contains no polychromatic circuit of even length, and if G contains a polychromatic circuit of length $2m + 1$ for some $m > 1$, then it contains a polychromatic circuit of length $2r + 1$ for $r = 1,2,..,m$.

Corollary 7.8 is just the particular case $n = 4$ of theorem 7.7. The conditions imposed on $\overline{\overline{PC}}_4$-graphs by corollary 7.8 are far more rigorous than the comparable conditions on $\overline{\overline{PC}}_n$-graphs for $n > 4$, and it is because of this that the next result has no parallel in $\overline{\overline{PC}}_n$-graphs, $n > 4$.

Theorem 7.9

Let G be a $\overline{\overline{PC}}_4$-graph containing a polychromatic circuit C. Then if P is any polychromatic path in G between two vertices of C, P contains at least one colour present in C. In particular, the subgraph induced in G by V(C) contains only the colours present in C.

Proof

Let $C = v_1v_2...v_m$, let P be a polychromatic path of length r in G between v_i and v_j, $1 \leq i < j \leq m$, and suppose that P contains no colour present in C. Call the path $v_iv_{i+1}...v_j$ R_1 and the path $v_jv_{j+1}...v_mv_1...v_i$ R_2. Neither R_1 nor R_2 has any colour in common with P. As C has odd length by corollary 7.8, one of R_1 and R_2, R_1 say, has odd length and the other has even length. If r is odd, then R_1 and P together form a polychromatic circuit in G of even length; if r is even, R_2 and P form the polychromatic circuit of even length (P can be assumed to have at most two vertices in common with C, otherwise segments of P can be considered separately). This contradicts corollary 7.8, so P and C have a colour in common. The proof is completed by noticing that every edge in the subgraph induced in G by V(C) is a polychromatic path between two vertices of C.

The $\overline{\overline{PC}}_4$-graphs are studied more fully in the next section. To finish this section, we give a result relating the number of colours in a $\overline{\overline{PC}}_n$-graph to its order. The result is due mainly to Erdos, Simonovits and Sos [E6], who conjectured that it was a best possible result.

Theorem 7.10

Let p and n be given integers, $p > 1$. If

$$1 \leq k \leq \tfrac{1}{2}p(n-2) + \left\lceil \frac{p}{n-1} \right\rceil - \tfrac{1}{2}s(n-s-1) - 1$$

where $s \equiv p \pmod{n-1}$, then there exists a k-edge-coloured $\overline{\overline{PC}}_n$-graph of order p.

Proof

First, consider the case where $p = r(n-1) + s$ for some r,s, $0 < s < n-1$. From theorems 2.13 and 7.2, there exists an r-edge-coloured $\overline{\overline{PC}}$-graph G of order $r + 1$, let $V(G) = \{v_0, v_1, .., v_r\}$. Let $G_1, G_2, .., G_r$ be complete graphs of order $n - 1$ and G_0 a complete graph of order s such that no two edges in these graphs are the same colour, and none of the colours is present in G. Any polychromatic circuit in $G_0, G_1, .., G_r$ has length at most $n - 1$, so by theorem 7.5 the graph H constructed by successively substituting G_i for v_i in G, $i = 0,1,..,r$, is a $\overline{\overline{PC}}_n$-graph. H has order p, and the colours in H are exactly the colours in G and $G_0, G_1, .., G_r$ by lemma 2.11. A complete graph of order m has $\tfrac{1}{2}m(m-1)$ edges, and since these are all differently coloured in $G_0, G_1, .., G_r$, H has $r + \tfrac{1}{2}s(s-1) + r\tfrac{1}{2}(n-1)(n-2) = \tfrac{1}{2}(n-2)p + \left\lceil \frac{p}{n-1} \right\rceil - \tfrac{1}{2}s(n-s-1) - 1$ colours.

For the case $p = r(n-1)$ for some r, the construction of H proceeds as above except that G_0 is ignored, and v_0 is removed from G; this reduces the number of colours in G by 1. It is easily checked that H again has order p, and again contains $\frac{1}{2}p(n - 2) + \left\lceil \frac{p}{n-1} \right\rceil - \frac{1}{2}s(n - s - 1) - 1$ colours (where $s = 0$).

The proof is now completed by if necessary identifying some of the colours in H until the required number of colours is achieved. This process creates no new polychromatic circuits, and H is as required.

3. $\overline{PC}_4$-Graphs With A Polychromatic Hamiltonian Circuit

Let G be a $\overline{PC}_4$-graph containing a polychromatic circuit C of length greater than 4. From theorem 7.9, the colours in the subgraph of G induced by V(C) are exactly the colours in C itself. In this section, we study more closely the subgraph of G induced by V(C). Since C is a Hamiltonian circuit in this subgraph, this is equivalent to studying a $\overline{PC}_4$-graph with a polychromatic Hamiltonian circuit.

From theorem 7.8, a $\overline{PC}_4$-graph with a polychromatic Hamiltonian circuit must have odd order p, and from theorem 7.9 is p-edge-coloured. The first result in this section proves that such graphs exist.

Theorem 7.11

For each odd integer p, $p \geq 5$, there exists a $\overline{PC}_4$-graph of order p with a polychromatic Hamiltonian circuit.

Proof

Put $p = 2n + 1$; the result is proved by the construction of a graph G_n of order $2n + 1$ with the required properties. G_2 is shown in figure 7.2; the colours $c_1, c_2, .., c_5$ are all different, and $x_1y_1x_2y_2x_3$ is a polychromatic Hamiltonian circuit.

Given G_{n-1}, construct G_n as follows. Add vertices y_n and x_{n+1} to G_{n-1}, and for $i = 1,2,..,n$ join y_n to x_i by c_{2n}-coloured edges, and join x_{n+1} to y_i by c_{2n+1}-coloured edges, where c_{2n} and c_{2n+1} are distinct colours not present in G_{n-1}. Also, for $j = 1,2,..,n - 1$ join y_n to y_j

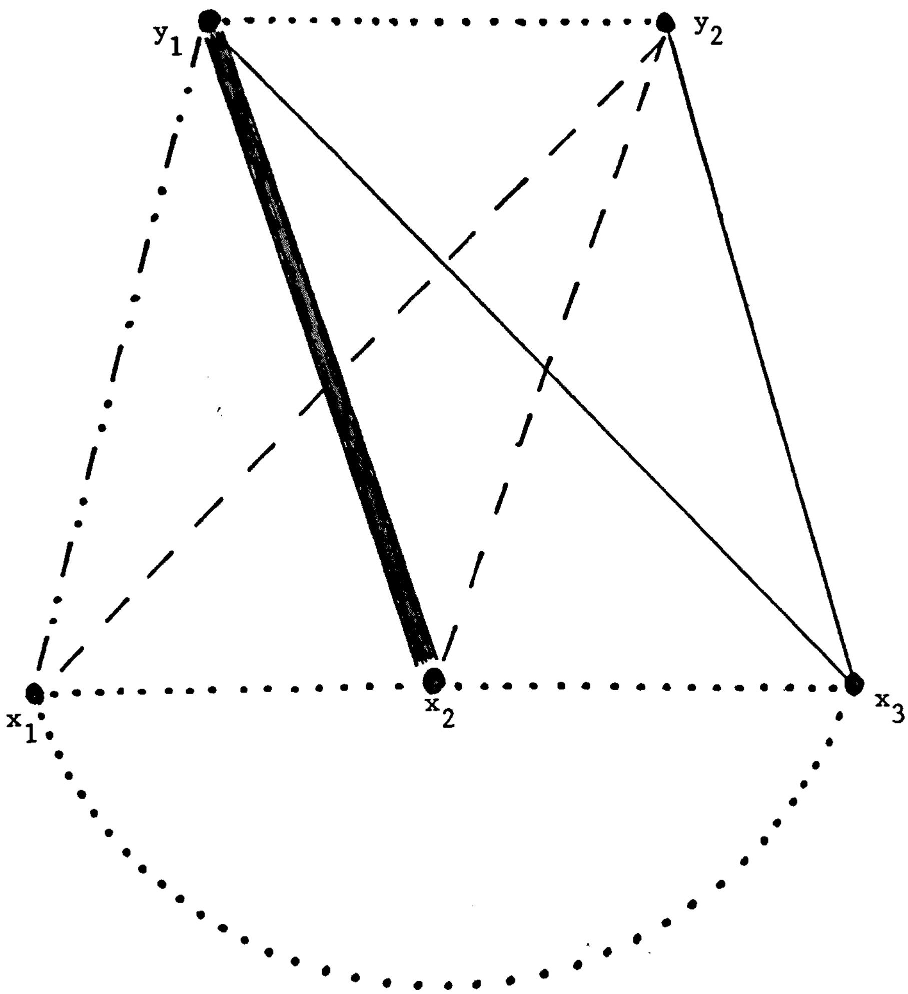


Figure 7.2

by a c_1-coloured edge, and for $j = 1,2,..,n$ join x_{n+1} to x_j by a c_1-coloured edge. The new graph G_n is complete, has order $2n + 1$, and has a polychromatic Hamiltonian circuit $x_1y_1x_2y_2...x_ny_nx_{n+1}$.

If G_n has any polychromatic C_4, it must include one of the new vertices, so suppose first that x_{n+1} is contained in a polychromatic C_4. As x_{n+1} is only incident with colours c_1 and c_{2n+1}, a c_1-coloured edge (x_i,x_{n+1}) must be in the polychromatic C_4, $1 \leqslant i \leqslant n$. If the c_1-coloured edges are removed from G_n, a complete bipartite graph results with x_i and x_{n+1} in the same section of the bipartition. There can be no paths of length 3 in this graph between x_i and x_{n+1}, so x_{n+1} cannot be in a polychromatic C_4.

Next suppose that y_n is contained in a polychromatic C_4. Since x_{n+1} cannot be in this C_4, the c_{2n+1}-coloured edge (x_{n+1},y_n) cannot be in the polychromatic C_4. The only other edges incident with y_n are c_1- or c_{2n}-coloured, so by a similar proof to that for x_{n+1} it can be shown that y_n is not contained in any polychromatic C_4. Thus G_n is a $\overline{\overline{PC}}_4$-graph, and the proof is complete.

For clarity of presentation, the next few results apply to an arbitrary $\overline{\overline{PC}}_4$-graph F of order p containing a polychromatic Hamiltonian circuit. Let one such circuit be $C = v_1v_2...v_p$, where p must be odd. For convenience, define $v_{p+i} = v_i$ for any i, so that for instance v_o can also be called v_p. Any edge of F which is not in C, i.e. the edges (v_i,v_{i+j}) for any i and for $j = 2,3,..,p - 2$, is called a chord of F. An edge (v_i,v_{i+j}), $j < p$, is called a type j edge of F; any type j edge is also a type $p - j$ edge. A type 1 edge is a member of C. Note that whether or not a particular edge is a type j edge, or a chord, depends on the choice of C.

Lemma 7.12

Let (v_i,v_j) be a chord of F, where $1 \leq i < j \leq p$. F contains exactly one type 1 edge (v_r,v_{r+1}) in the same colour as (v_i,v_j), and $i \leq r < j$ if and only if $j - i$ is odd. If $j - i$ is odd, the circuit $v_j v_{j+1} \cdots v_i v_j$ is polychromatic; if $j - i$ is even, the circuit $v_i v_{i+1} \cdots v_j v_i$ is polychromatic.

Proof

C must contain an edge (v_r,v_{r+1}) in the same colour as (v_i,v_j) by theorem 7.9, and since C is polychromatic this must be the only such edge. If $j - i$ is odd, the circuit $v_i v_{i+1} \cdots v_{j-1} v_j v_i$ has even order, and cannot be polychromatic by corollary 7.8; hence (v_r,v_{r+1}) is in this circuit, and $i \leq r < j$. The path $v_j v_{j+1} \cdots v_{i-1} v_i$ is polychromatic, consists of type 1 edges only, and does not contain (v_r,v_{r+1}). Thus no edge in this path is the same colour as (v_i,v_j), and the circuit $v_j v_{j+1} \cdots v_i v_j$ is polychromatic. The proof for $j - i$ even is similar.

The following two special cases of lemma 7.12 are useful enough to be given as separate results.

Lemma 7.13

For $i = 1,2,..,p$, the chord (v_i,v_{i+2}) of F is a different colour to (v_i,v_{i+1}) and (v_{i+1},v_{i+2}).

Lemma 7.14

For $i = 1,2,..,p$, the chord (v_i,v_{i+3}) of F is the same colour as exactly one of (v_i,v_{i+1}), (v_{i+1},v_{i+2}), and (v_{i+2},v_{i+3}).

Theorem 7.15

Let the type 1 edge (v_r,v_{r+1}) of F and the chord $e = (v_i,v_j)$ of F be the same colour, red say. Let S be the vertex set of the shortest path along C from v_r to the chord e which does not include v_{r+1}, and let T be the vertex set of the shortest path along C from v_{r+1} to the chord e which does not include v_r. Define

S_E = {s∈S: the distance from v_r to s along C is even},

S_O = {s∈S: the distance from v_r to s along C is odd},

T_E = {t∈T: the distance from v_{r+1} to t along C is even},

and T_O = {t∈T: the distance from v_{r+1} to t along C is odd},

Then every S_ET_E-edge and every S_OT_O-edge is red, but no S_ET_O-edge or S_OT_E-edge is red.

<u>Proof</u>

Let (v_m,v_n) be any S_ET_E-edge, where without loss of generality $i \leq m \leq r < n \leq j$. As $n - (r + 1)$ and $r - m$ are both even, $n - m$ is odd and by lemma 7.12 the circuit $v_mv_nv_{n+1}\cdots v_{m-1}v_m$ is polychromatic. Also from lemma 7.12, $j - i$ must be odd, giving $j - n + m - i$ as even. This means that the circuit $v_mv_nv_{n+1}\cdots v_jv_iv_{i+1}\cdots v_{m-1}v_m$ is of even order, and so by corollary 7.8 contains two edges of the same colour. As $v_mv_nv_{n+1}\cdots v_{m-1}v_m$ is polychromatic, (v_i,v_j) must be one of these edges. (v_r,v_{r+1}) is the only red type 1 edge, so the other red edge in $v_mv_nv_{n+1}\cdots v_jv_iv_{i+1}\cdots v_{m-1}v_m$ must be (v_m,v_n). Hence all of the S_ET_E-edges must be red.

Now let (v_m,v_n) be any S_OT_E-edge, where without loss of generality $i \leq m \leq r < n \leq j$. Then $n - (r + 1)$ is even and $r - m$ is odd, giving $n - m$ as even. Together with the inequality $m \leq r < n$, this means that from lemma 7.12 (v_r,v_{r+1}) must be a different colour from (v_m,v_n). Hence no S_OT_E-edges are red.

The proof for S_OT_O-edges and S_ET_O-edges is similar.

Corollary 7.16

If F contains a c-coloured type $2i + 1$ edge e, $1 \leq i < \frac{1}{2}p$, then F contains a c-coloured type $2j + 1$ edge for $j = 1,2,..,i$.

Proof

By induction in i. If $i = 1$ and e is a type 3 edge, by lemma 7.14 F must contain a type 1 edge in the same colour. Otherwise assume the corollary true for $i < I$, and let (v_m,v_n) be a c-coloured type $2I + 1$ edge, $I > 1$. It is enough to prove that there exists a c-coloured type $2I - 1$ edge in F. Since $n - m = 2I + 1$ is odd, F contains a c-coloured edge (v_r,v_{r+1}), where $m \leq r < n$. Without loss of generality, in the notation of theorem 7.15 let (v_m,v_n) be an S_ET_E-edge. If $m + 2 \leq r$, then (v_{m+2},v_n) is an S_ET_E-edge, otherwise $m + 2 > r$ and (v_m,v_{n-2}) is an S_ET_E-edge. Hence by lemma 7.15 one of these edges is c-coloured, and since both are type $2I - 1$ edges the result is proved.

If both of the paths induced by S and T in theorem 7.15 are long, it can be seen that one monochromatic subgraph of F contains a large number of edges. This apparent disparity in the number of edges in the various monochromatic subgraphs of F can be confirmed by proving that one monochromatic subgraph contains a single edge. Some preliminary results are needed.

Lemma 7.17

If F has a type 3 edge in each colour present, then the type 3 and the type 1 edges in each colour are adjacent.

Proof

If each colour in F has a type 3 edge, each colour has exactly one type 3 edge, as there are p type 3 edges and p colours in F. Assume that the c_1-coloured type 1 and type 3 edges are not adjacent. If

(v_r,v_{r+1}) is the c_1-coloured type 1 edge, by lemma 7.14 (v_{r-1},v_{r+2}) must be the c_1-coloured type 3 edge. For j = 0,1,2,3,4, take (v_{r+j-1},v_{r+j}) to be c_j-coloured. The type 3 edge (v_r,v_{r+3}) cannot be c_2-coloured otherwise $v_rv_{r+3}v_{r+2}v_{r-1}$ is polychromatic, so by lemma 7.14 (v_{r+1},v_{r+4}) is the c_2-coloured type 3 edge. Also by lemma 7.14, (v_r,v_{r+3}) must be c_3-coloured type 3 edge. But then $v_rv_{r+1}v_{r+4}v_{r+3}$ is polychromatic, a contradiction which gives the result.

Lemma 7.18

If F has a type 2 edge in each colour present, then for some colour c there is no c-coloured type 3 edge adjacent to the c-coloured type 1 edge.

Proof

As with type 3 edges, if there is a type 2 edge of each colour, there is exactly one type 2 edge of each colour.

Let (v_r,v_{r+1}) be a c_1-coloured edge, and suppose that any c_1-coloured type 3 edge is adjacent to it. Let the type 2 edges (v_r,v_{r+2}) and (v_{r-1},v_{r+1}) be c_o- and c_2-coloured edges respectively, whereby lemma 7.13 c_o, c_1, and c_2 are distinct colours. By the assumption, (v_{r-1},v_{r+2}) cannot be c_1-coloured, and since the circuit $v_rv_{r+1}v_{r-1}v_{r+2}$ cannot be polychromatic, (v_{r-1},v_{r+2}) must be either c_o- or c_2-coloured. Without loss of generality take it to be c_o-coloured. Then by lemmas 7.13 and 7.14, (v_{r-1},v_r) must also be c_o-coloured.

A type 2 edge can also be called a type p - 2 edge. From corollary 7.16, if there is a c_1-coloured type p - 2 edge, there is also a c_1-coloured type 3 edge. By lemma 7.14, the c_1-coloured type 3 edge must be either (v_{r-2},v_{r+1}) or (v_r,v_{r+3}). If (v_{r-2},v_{r+1}) is c_1-coloured, then the circuit $v_{r+1}v_{r+2}v_{r-1}v_{r-2}$ would be the same colour as the path

$v_{r-2}v_{r-1}v_rv_{r+1}v_{r+2}$, which is polychromatic; (v_r,v_{r+3}) must be the c_1-coloured type 3 edge.

The circuit $v_rv_{r+3}v_{r+1}v_{r-1}$ cannot be polychromatic, so the edge (v_{r+1},v_{r+3}) must be coloured in c_o, c_1, or c_2. As c_o- and c_2-coloured type 2 edges already exist, (v_{r+1},v_{r+3}) is c_1-coloured. Suppose that (v_{r+1},v_{r+2}) is c_3-coloured, where $c_o \neq c_3 \neq c_1$ since all type 1 edges are differently coloured. From lemma 7.12, the edge (v_{r-1},v_{r+3}) cannot be coloured in c_o, c_1, or c_3, so the circuit $v_{r-1}v_{r+2}v_{r+1}v_{r+3}$ is polychromatic. Since F can contain no polychromatic C_4, this is a contradiction, giving the result.

Lemma 7.19

There is no type 3 edge in one of the colours present in F.

Proof

Suppose that F contains a type 3 edge in each colour. From lemmas 7.17 and 7.18, F cannot also contain a type 2 edge in each colour, so that there must be two type 2 edges in the same colour, red say. If $p = 5$, a type 3 edge is also a type 2 edge, so there cannot be a type 3 edge in each colour. Now let $p > 5$, and let (v_i,v_{i+2}) be a red type 2 edge, and let (v_r,v_{r+1}) be the red type 1 edge. In the notation of theorem 7.15, $|S| + |T| = p - 1 > 4$, since S and T include all the vertices of F except v_{i+1}. Suppose that $|S| > 1$ and $|T| > 1$, so that v_{r-1} is included in S and v_{r+2} is included in T; by lemma 7.15 (v_{r-1},v_{r+2}) is a type 3 red edge. As $|S| + |T| > 4$, v_{r-2} is in S or v_{r+3} is in T, so one of (v_{r-2},v_{r+1}) and (v_r,v_{r+3}) must also be a red type 3 edge by theorem 7.15. Since there can only be one red type 3 edge, then $|S| = 1$ or $|T| = 1$, and the only possible red type 2 edges are (v_{r-2},v_r) and (v_{r+1},v_{r+3}). But then by theorem 7.15 both (v_{r-2},v_{r+1}) and (v_r,v_{r+3}) must be red type 3 edges, a contradiction proving the lemma.

Theorem 7.20

Let F be a $\overline{\overline{PC}}_4$-graph with a polychromatic Hamiltonian circuit. There exists a monochromatic subgraph of F containing a single edge.

Proof

Let F contain a type n edge in some colour, $1 < n < p - 1$. If n is odd, then by corollary 7.16 there is also a type 3 edge in that colour. If n is even, $p - n$ is odd and $p - n \geq 3$, so again there is a type 3 edge in that colour. From lemma 7.19, there is some monochromatic subgraph in F with no type 3 edge. This monochromatic subgraph can have no type n edge for $n = 2,3,..,p - 2$ either, and so has a single type 1 edge.

Theorem 7.21

Let F be a $\overline{\overline{PC}}_4$-graph. Then it cannot contain two edge-disjoint polychromatic Hamiltonian circuits.

Proof

If F contained two polychromatic Hamiltonian circuits, some monochromatic subgraph of F would contain a single edge by theorem 7.20. Then by theorem 7.9 both polychromatic Hamiltonian circuits must contain this edge, so they have an edge in common.

We finish the section by relating some of the characteristics associated with $\overline{\overline{PC}}_4$-graphs containing a polychromatic Hamiltonian circuit, namely the order, number of colours, number of edges in various colours, and the minimum and maximum degrees in monochromatic subgraphs.

It was proved in theorem 7.9 that the number of colours in a $\overline{\overline{PC}}_4$-graph with a polychromatic Hamiltonian circuit is equal to its order. Recall that Q(p) and q(p) are the largest and smallest numbers of edges respectively in the monochromatic subgraphs of a complete graph G of order p, and $Q'(k)$ and $q'(k)$ are the largest and smallest numbers of edges respectively in the monochromatic subgraphs of a k-edge-coloured complete graph G. Clearly for $\overline{\overline{PC}}_4$-graphs with a polychromatic Hamiltonian circuit, $Q(p)$ and $Q'(k)$ are equivalent, as are $q(p)$ and $q'(k)$. Theorem 7.20 shows that $q(p) = 1$ for all $\overline{\overline{PC}}_4$-graphs with a polychromatic Hamiltonian circuit.

Theorem 7.22

Let F be a $\overline{\overline{PC}}_4$-graph of order p containing a polychromatic Hamiltonian circuit. Then F can contain at most $\frac{1}{4}(p - 1)^2$ edges in any one colour, with equality possible.

Proof

By induction on n, where $p = 2n + 1$. The graphs G_n in the proof of theorem 7.11 have order $2n + 1$ and contain $n^2 = \frac{1}{4}(p - 1)^2$ blue edges, so it suffices to prove that the bound $\frac{1}{4}(p - 1)^2$ cannot be exceeded. If $n = 1$, F has order 3 and every edge is contained in a polychromatic Hamiltonian circuit, so there must be one edge in each colour. Assume the theorem true for $n < N$, and let F have order $p = 2N + 1$.

Suppose that the type 1 edge (v_p,v_1) is blue. If there is more than one blue edge in F, then by corollary 7.16 F must contain a blue type 3 edge. Assume first that this edge is (v_{p-1},v_2). Then by lemma 7.12, $v_2v_3...v_{p-1}v_2$ is polychromatic, and the graph F_1 obtained from F by removing v_1 and v_p together with their incident edges is a $\overline{\overline{PC}}_4$-graph with a polychromatic Hamiltonian circuit. Applying the induction assumption, F_1 can contain at most $\frac{1}{4}(p - 3)^2$ blue edges. Now consider the vertex v_p in F, and let j be any integer such that (v_j,v_p) is blue in F. Again by lemma 7.12, j must be odd, and since $j < p - 1$ v_p can be incident with at most $\frac{1}{2}(p - 1)$ blue edges in F. Similarly, v_1 is incident with at most $\frac{1}{2}(p - 1)$ blue edges, one of which is also incident with v_p. Hence there can be at most $\frac{1}{4}(p - 3)^2 + \frac{1}{2}(p - 1) + \frac{1}{2}(p - 1) - 1 = \frac{1}{4}(p - 1)^2$ blue edges in F.

If (v_{p-1}, v_2) is not a blue type 3 edge, then by lemma 7.14 either (v_{p-2}, v_1) or (v_p, v_3) must be, and the proof proceeds as before except that v_{p-1} and v_p or v_1 and v_2 respectively are removed to create F_1.

Theorem 7.23

Let F be a $\overline{\overline{PC}}_4$-graph of order p containing a polychromatic Hamiltonian circuit. Then F contains at least p - 2 edges of the same colour.

Proof

From theorem 7.20, some colour in F has a single edge, and therefore cannot have a type 2 edge. Thus some other colour, blue say, has at least two type 2 edges; let $e = (v_r, v_{r+2})$ be one such edge. In the notation of theorem 7.15, S and T are the vertex sets of the two paths along C from e to the blue type 1 edge; let $|S| = s$ and $|T| = t$. The only vertex not contained in these sets is v_{r+1}, so $s + t = p - 1$, which is even.

Now from theorem 7.15, all of the S_ET_E- and S_0T_0-edges are blue, so that there are at least $|S_E||T_E| + |S_0||T_0|$ blue edges in F. If s and t are both even, then $|S_E| = |S_0| = \frac{1}{2}s$ and $|T_0| = |T_E| = \frac{1}{2}t$, and there are $\frac{1}{2}st$ blue S_0T_0- and S_ET_E-edges in F. This value is minimised when s (or t) is as small as possible, i.e. when s = 2. This gives $t = p - 3$ blue S_ET_E-edges and S_0T_0-edges in F. But only one of these edges is a type 2 edge, and there are two type 2 blue edges in F, so F must contain at least p - 2 blue edges.

Otherwise s and t are both odd, and $|S_0| = \frac{1}{2}(s + 1)$, $|S_E| = \frac{1}{2}(s - 1)$, $|T_0| = \frac{1}{2}(t + 1)$, and $|T_E| = \frac{1}{2}(t - 1)$. This gives $\frac{1}{4}(s + 1)(t + 1) +$

$\frac{1}{4}(s - 1)(t - 1) = \frac{1}{2}(st + 1)$ S_0T_0- and S_ET_E-edges in F. This again is minimised when s is small and can only be less than p - 2 if s = 1. In this case there are no S_0T_0-edges, all of the $\frac{1}{2}(p - 1)$ S_ET_E-edges are adjacent, and the blue type 1 edge is (v_{r-1},v_r). Again there is another blue type 2 edge in F, and the same reasoning shows that it must be (v_{r-3},v_{r-1}), and that v_{r-3} is incident with another $\frac{1}{2}(p - 1)$ blue edges. Hence the two vertices v_{r+2} and v_{r+3} are each incident with $\frac{1}{2}(p - 1)$ blue edges, only one of which is counted twice, so F contains at least p - 2 blue edges.

So for $\overline{PC}_4$-graphs of order p with a polychromatic Hamiltonian circuit, $p - 2 \leq Q(p) \leq \frac{1}{4}(p - 1)^2$. The upper bound is a best possible bound, but it is likely that the lower bound can be improved.

We now turn to limits on degree in monochromatic subgraphs. Since some monochromatic subgraph contains a single edge, and therefore also contains some isolated vertices, there can be no meaningful lower limit on the number of edges of each colour incident with each vertex in F. The maximum degree can be determined, however.

Theorem 7.24

Let F be a $\overline{PC}_4$-graph of order p containing a polychromatic Hamiltonian circuit. The largest number of edges of a single colour incident with a vertex in F is $\frac{1}{2}(p - 1)$.

Proof

First suppose that the vertex v_1 is incident with more than $\frac{1}{2}(p - 1)$ blue edges. Then for some r, $1 < r < p$, both (v_1,v_r) and (v_1,v_{r+1}) must be blue. By lemma 7.12, (v_r,v_{r+1}) is the blue type 1 edge in F, and r is odd. Then for any odd i, neither (v_1,v_{r-i}) nor (v_1,v_{r+i+1}) can be blue by lemma 7.12, and it is easily checked that no more than $\frac{1}{2}(p - 1)$

blue edges can be incident with v_1.

To show that F does contain a vertex incident with $\frac{1}{2}(p - 1)$ edges of the same colour, by theorem 7.15 it is enough to show that there is a type 2 edge adjacent to a type 1 edge of the same colour. If this were not the case, then not all type 2 edges could be the same colour, so for some r (v_r, v_{r+2}) is a different colour to (v_{r+1}, v_{r+3}). The type 1 edges (v_r, v_{r+1}) and (v_{r+2}, v_{r+3}) are each adjacent to both of these type 2 edges, and so must be differently coloured from them. But then $v_r v_{r+2} v_{r+3} v_{r+1}$ is a polychromatic C_4, a contradiction, so the theorem is proved.

Chapter 8

ALTERNATING CIRCUITS IN COMPLETE GRAPHS

1. Alternating Circuits in 2-Edge-Coloured Complete Graphs

A circuit C is an alternating circuit if adjacent edges in C are differently coloured. An alternating circuit of length n is denoted an AC_n. A complete graph containing no alternating circuit is an $\overline{AC}$-graph, and a complete graph with no AC_n is an $\overline{AC}_n$-graph.

In this section, the graphs considered are assumed to be 2-edge-coloured, in blue and red. Some of the first results are analogous to results in chapter 7, on polychromatic circuits in complete graphs.

Lemma 8.1

A 2-edge-coloured alternating circuit has even length.

Proof

Straightforward.

One consequence of lemma 8.1 is that the smallest possible alternating circuit in a 2-edge-coloured graph is an AC_4.

As in the case of polychromatic circuits (lemma 7.1), the existence of small alternating circuits can be related to that of large alternating circuits.

Lemma 8.2

Let G be a 2-edge-coloured complete graph containing an AC_n, $n > 4$. Then for each even integer r, $2 \leq r \leq n - 2$, G contains an AC_{r+2} or an AC_{n-r}.

Proof

Let the AC_n be $v_1v_2...v_n$, and define $v_{n+1} = v_1$; n is even by lemma 8.1. Without loss of generality suppose that for $i = 1,2,..,\frac{1}{2}n$ (v_{2i},v_{2i+1}) is blue and (v_{2i-1},v_{2i}) is red. The lemma is trivial for $r = n - 2$, so let r be an even integer, $2 \leq r < n - 2$. If (v_1,v_{r+2}) is blue, then $v_1v_2...v_{r+2}v_1$ is an AC_{r+2}; otherwise, (v_1,v_{r+2}) is red and $v_1v_{r+2}v_{r+3}...v_nv_1$ is an AC_{n-r}.

Lemma 8.2 is of limited application: given any particular r and n, it is not known whether the graph in question contains an AC_{r+2} or an AC_{n-r} or both under the conditions of the lemma. However, in certain cases there is no such ambiguity.

Lemma 8.3

Let G be a 2-edge-coloured complete graph containing an AC_n, $n \geq 4$. If r is an even integer, $2 \leq r < n$, and $n \equiv 2 \pmod{r}$, then G contains an AC_{r+2}.

Proof

By induction on n. The lemma is true for $n = 4$, so assume it true for $n < m$ and consider an AC_m in G, where $m \equiv 2 \pmod{r}$ for some even r, $2 \leq r < m$. From lemma 8.2, G either contains an AC_{r+2}, in which case the lemma is proved, or $m \neq r + 2$ and G contains an AC_{m-r}. Now $m \equiv 2 \pmod{r}$, so $m - r \equiv 2 \pmod{r}$; also $m - r > 2$ so that $m - r \geq r + 2$ and hence $2 \leq r < m - r < m$. Then by the induction assumption, if G contains an AC_{m-r} then G contains an AC_{r+2}.

Theorem 8.4

Let G be a 2-edge-coloured complete graph. G contains an alternating circuit if and only if G contains an AC_4.

Proof

If G contains an alternating circuit, then by lemma 8.1 it has even length. Putting r = 2 in lemma 8.3 gives that G contains an AC_4. The proof is completed by the fact that an AC_4 is an alternating circuit.

The 2-edge-coloured $\overline{AC}$-graphs (or equivalently, the 2-edge-coloured $\overline{AC}_4$-graphs) can be characterised by restricting a result of Chen [C7] to the 2-edge-coloured case. A proof different to that of Chen is presented, requiring some preliminary results.

Theorem 8.5 (Chen)

Let G be a 2-edge-coloured $\overline{AC}$-graph. Then G is connected in exactly one colour.

Proof

A 2-edge-coloured complete graph can be considered as the union of a blue graph and its complement coloured in red, and so must be connected in at least one colour. It is enough to prove by induction on the order p of G that G cannot be connected in both colours.

It is easily checked that the graph in figure 8.1 is the only 2-edge-coloured complete graph with fewer than 5 vertices which is connected in both colours; clearly it contains an alternating circuit. Now assume that G is a 2-edge-coloured $\overline{AC}$-graph of order q connected in both colours, $q \geq 5$, and that the result is true for graphs of order less than q.

If v is any vertex in G, remove v and its incident edges and call the resultant graph H. As G is an $\overline{AC}$-graph, H must also be an $\overline{AC}$-graph, and by the induction assumption H is connected in one colour only, say blue. Then V(H) can be partitioned into two non-empty sets A_1 and A_2 such that every A_1A_2-edge is blue. G is connected in blue, so v must be incident with a blue edge; without loss of generality, suppose that

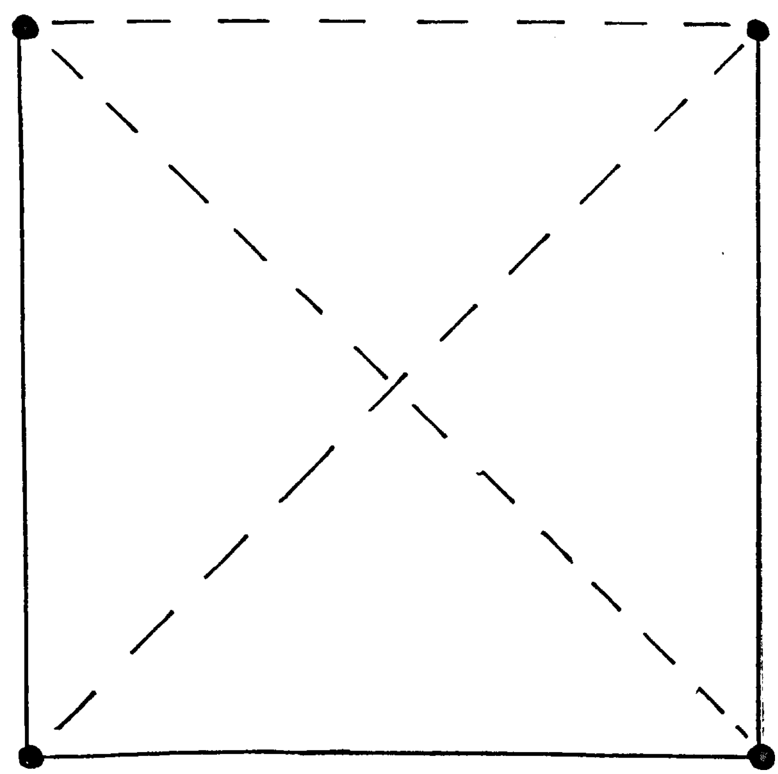


Figure 8.1

(v,u) is blue where u is in A_1. G is also connected in red, and since no A_1A_2-edge is red there are both vA_1- and vA_2-edges which are red. In particular there is a red edge (v,w) where w is in A_2. Also u must be incident with a red edge which is not (u,v); say that (u,x) is red where x must be in A_1. Now (x,w) is an A_1A_2-edge, and so must be blue; this means that uvwx is an AC_4, and G cannot be an $\overline{\overline{AC}}$-graph. This contradiction proves the theorem.

Corollary 8.6

Let G be a 2-edge-coloured $\overline{\overline{AC}}_4$-graph. Then G is connected in exactly one colour.

Proof

Theorems 8.4 and 8.5.

Lemma 8.7

If G is a 2-edge-coloured $\overline{\overline{AC}}$-graph, then G contains a vertex incident with edges of one colour only.

Proof

Let G be coloured in blue and red. By theorem 8.5 G is connected in exactly one colour, blue say. V(G) can be partitioned into two non-empty sets A_1 and A_2 such that all A_1A_2-edges are blue. G contains a red edge, so without loss of generality there is a red A_2A_2-edge (u,v). If there is a red A_1A_1-edge (w,x), then uvwx would be an AC_4, which is impossible. Hence any A_1A_1-edge is blue, and vertices in A_1 are incident with edges of one colour only.

Theorem 8.8 (Chen)

Let G be a 2-edge-coloured complete graph. Then G is an $\overline{\overline{AC}}$-graph if and only if V(G) can be partitioned into non-empty sets $A_1, A_2, .., A_n$,

$n \geq 2$, such that for $i \leq j$ the A_iA_j-edges of G are blue (say) if i is odd and red (say) if i is even.

Proof

Assume that V(G) can be partitioned as above, and suppose that G contains an alternating circuit C. Then there is a least integer i such that a vertex u of A_i is contained in C, $i \geq 1$. If the vertices adjacent in C to u are v and w, then v and w are in A_i or A_j, $j \geq i$. But both of these edges are the same colour, so C cannot be an alternating circuit and G is an $\overline{AC}$-graph.

Now assume that G is a 2-edge-coloured $\overline{AC}$-graph. By lemma 8.7 there is a vertex of G incident with edges of one colour only, say blue. Put A_1 as the set of vertices of G incident with blue edges only, and put B_1 as the set of vertices of G which are not in A_1. Since G contains two colours, both A_1 and B_1 are non-empty, and every vertex in B_1 is incident with a red B_1B_1-edge.

Now consider the complete graph H induced in G by B_1. Every vertex in H is incident with a red edge. If H contains only red edges, then put $A_2 = B_1$ and the theorem is proved. Otherwise, H is a 2-edge-coloured $\overline{AC}$-graph of order less than G, and the induction assumption can be applied. H contains a vertex incident with one colour only by lemma 8.7, which must be red. Thus V(H) can be partitioned into non-empty sets $A_2, A_3, \ldots, A_n$, $n \geq 2$, such that for $i \leq j$ the A_iA_j-edges of G are red if i is even, and blue if i is odd. $A_1, A_2, \ldots, A_n$ is the partition of V(G) required to prove the theorem.

Corollary 8.9

Let G be a 2-edge-coloured complete graph. Then G is an $\overline{AC}_4$-graph if and only if V(G) can be partitioned into non-empty sets $A_1, A_2, \ldots, A_n$, $n \geq 2$, such that for $i \leq j$ the A_iA_j-edges of G are blue (say) if i is odd and red (say) if i is even.

Proof

Theorems 8.4 and 8.9.

The 2-edge-coloured $\overline{\overline{AC}}$-graphs can be more intuitively described by a method of construction.

Theorem 8.10

The 1- and 2-edge-coloured $\overline{\overline{AC}}$-graphs (equivalently, the 1- and 2-edge-coloured $\overline{\overline{AC}}_4$-graphs) are exactly those graphs obtained from a single vertex by repeated application of the following procedure: add a vertex to the graph H already obtained, and join that vertex to all the vertices of H either by blue edges or by red edges.

Proof

By induction on the order p of G. The theorem is easily verified for $p \leq 3$, so assume the result for $p < q$, and let G be an $\overline{\overline{AC}}$-graph of order q, $q > 3$. By lemma 8.7 if G is 2-edge-coloured, or trivially if G is 1-edge-coloured, G contains a vertex v incident with edges of one colour only, blue say. Remove v together with its incident edges to obtain the $\overline{\overline{AC}}$-graph H of order $q - 1$. By the induction assumption, H can be constructed vertex by vertex, at each stage adding edges of one colour only. As G is obtained from H by adding the vertex v and joining v to the vertices of H by blue edges only, G can be obtained in the manner required.

Now suppose that G is a graph of order q obtained in the manner described in the theorem. G is complete, and either 1- or 2-edge-coloured. Suppose that G was derived by adding a vertex v to a graph H, and joining v to the vertices of H by edges of one colour only. By the induction assumption, H contains no alternating circuit, and so either G is an

$\overline{AC}$-graph as required, or v is contained in an alternating circuit. But v is incident with edges of one colour only, so this is impossible; the theorem is proved.

We now turn to alternating Hamiltonian circuits in 2-edge-coloured complete graphs.

Definition 8.11

An alternating circuit in a graph G which contains every vertex of G is an alternating Hamiltonian circuit or an AH of G. A complete graph which contains no AH is an $\overline{AH}$-graph.

As all 2-edge-coloured alternating circuits are of even order by lemma 8.1, any 2-edge-coloured complete graph of odd order is an $\overline{AH}$-graph. Not all 2-edge-coloured complete graphs of even order are $\overline{AH}$-graphs, since if the non-adjacent vertices of an AC coloured in blue and red are joined by blue and red edges in an arbitrary manner to create a complete graph G, then G is a 2-edge-coloured complete graph containing an AH. To complete this section, we characterise the 2-edge-coloured $\overline{AH}$-graphs.

Note that a 1-factor of a graph is a spanning subgraph regular of degree 1.

Lemma 8.12

Let G be a 2-edge-coloured complete graph. G is an $\overline{AH}$-graph if one of the monochromatic subgraphs of G contains no 1-factor.

Proof

Suppose to the contrary that G does contain an AH, and that G is coloured in red and blue say. Then both the red and blue subgraphs of the AH are 1-factors of G.

We quote a well-known result of Tutte (see for instance [B2]) concerning the existence of a 1-factor in a graph.

Lemma 8.13 (Tutte)

Let G be an arbitrary graph. G contains a 1-factor if and only if for each set S of vertices of G, when the vertices of S together with their incident edges are removed from G the number of connected components of odd order in the resultant graph is at most $|S|$.

Lemma 8.14 (Bankfalvi and Bankfalvi [B1])

Let G be a 2-edge-coloured complete graph of order at most 7 with a 1-factor in each monochromatic subgraph. Then G contains an AH.

Proof

Let F_1 and F_2 be 1-factors in the two monochromatic subgraphs of G, and let F be the union of F_1 and F_2. Then F is a spanning subgraph of G, regular of degree 2 and with adjacent edges differently coloured. A graph regular of degree 2 consists of a set of disjoint circuits, so F is a set of disjoint 2-edge-coloured alternating circuits. But the smallest possible 2-edge-coloured alternating circuit has length 4; F has order at most 7, and so must be a single circuit. Since F is a spanning subgraph of G, F is an AH.

Lemma 8.15 (Bankfalvi and Bankfalvi [B1])

Let F be a 2-edge-coloured complete graph of order 2n, $n > 3$, coloured in red and blue with 1-factors in both monochromatic subgraphs. Then G is an $\overline{\overline{AH}}$-graph if and only if V(G) can be partitioned into three sets X, Y, and Z, $2 \leq |X| = |Y|$ and $4 \leq |Z|$, such that every XX- and XZ-edge is blue and every YY- and YZ-edge is red.

Proof

We will not present a complete proof of the result - for that the reader is referred to [B1]. We shall, however, prove that G is an $\overline{AH}$-graph if the above condition holds.

Suppose that V(G) can be partitioned as above, but that G does contain an AH, called C. Since X, Y, and Z are all non-empty, and C contains vertices in each set, it must be possible to travel from Z to either X or Y along C. Without loss of generality, assume that C contains an XZ-edge. This edge is blue, so the next edge in C must be red. As all XX- and XZ-edges are blue, the next edge must be an XY-edge. All YY- and YZ-edges are red, so the next edge along C must also be an XY-edge. Clearly it is impossible to travel back to Z along C, as the above argument can be repeated ad infinitum. C cannot therefore be an AH, and G must therefore be an $\overline{AH}$-graph.

As in lemma 8.14, if G contains both red and blue 1-factors then the union F of a red 1-factor and a blue 1-factor is a set of disjoint alternating circuits which span G. If F is not a single circuit, then Bankfalvi and Bankfalvi show that if V(G) cannot be partitioned as in the theorem, two of the circuits can be 'linked' to form a single larger alternating circuit. Repetition of this process eventually produces a single spanning circuit, which is an AH.

Theorem 8.16

Let G be a 2-edge-coloured complete graph of order at least 8 and coloured in red and blue. Then G is an $\overline{AH}$-graph if and only if one of the following holds.

i) G contains a (possibly empty) set of vertices S such that when the vertices of S together with their incident edges are removed from G, a monochromatic subgraph of the resultant graph contains more than $|S|$ connected components of odd order;

ii) V(G) can be partitioned into three sets X, Y, and Z, $2 \leqslant |X| = |Y|$,

$4 \leq |Z|$, such that all XX- and XZ-edges are blue and all YY- and YZ-edges are red.

Proof

Suppose that G is an $\overline{\overline{AH}}$-graph and that condition (i) does not hold. Then by lemma 8.13 each monochromatic subgraph of G contains a 1-factor. Lemma 8.15 then gives that condition (ii) must hold.

Now suppose that condition (i) holds. Then by lemma 8.13 some monochromatic subgraph of G cannot contain a 1-factor, so G is an $\overline{\overline{AH}}$-graph by lemma 8.12. If condition (ii) holds but not condition (i), then by lemma 8.13 G contains a 1-factor in each monochromatic subgraph, and the theorem follows from lemma 8.15.

2. Small Alternating Circuits in Complete Graphs

We now relax the assumption that the graph G contains only two colours. If more colours are available, alternating circuits can have odd as well as even length. In particular, alternating circuits can have length 3, when they become polychromatic triangles. Clearly the $\overline{\overline{AC}}_3$-graphs are just the $\overline{\overline{PC}}_3$-graphs, and all the results of chapter 2 apply. Here, we just adapt theorem 2.7.

Theorem 8.17

Let G be an $\overline{\overline{AC}}_3$-graph. It is connected in either one or two colours, and if the edges in these colours are removed from G, n connected components with vertex sets $A_1, A_2, .., A_n$ remain, $n > 1$. If G is connected in one colour only, then for $i \neq j$ every A_iA_j-edge is in that colour. If G is connected in two colours, then $n \geq 4$ and for $i \neq j$ every A_iA_j-edge is in one of the connected colours, which colour being dependent only on i and j.

It was shown in chapter 7 that $\overline{\overline{PC}}_3$-graphs can contain no polychromatic circuits. This does not extend to alternating circuits, however.

Theorem 8.18

For each m and n, $m \geqslant n > 3$, there exists a complete graph of order m, containing an AC_n but no AC_3.

Proof

First, a complete graph of order n containing an AH but no AC_3 is constructed. If n is even, a 2-edge-coloured complete graph containing an AH was constructed in the previous section: if the AH has order n, this graph will suffice since no 2-edge-coloured graph can contain an AC_3. If n is odd, then there exists a 2-edge-coloured complete graph of order $n - 1$ with AH $v_1v_2\dots v_{n-1}$. Add a vertex v_n, and add edges (v_i,v_n) in the same colours as (v_i,v_{n-1}) for $i = 1,2,..,n - 2$. Add an edge (v_{n-1},v_n) in a colour not already present to give a complete graph with order n, and containing the AH $v_1v_2\dots v_n$. Any AC_3 must contain the edge (v_{n-1},v_n) as the rest of the graph is 2-edge-coloured. But for $i = 1,2,..,n - 2$ $v_iv_{n-1}v_n$ is a bichromatic triangle, so the graph is an $\overline{AC}_3$-graph.

The graph can now be brought up to the required order m by adding a set of $m - n$ vertices, and joining these vertices to each other and to the vertices already present by edges of one colour only. Clearly this can create no new alternating circuits.

The next smallest alternating circuit is the AC_4. The general $\overline{\overline{AC}}_4$-graph can be related to the 2-edge-coloured case discussed in the previous section.

Lemma 8.19

Let G be a k-edge-coloured complete graph, $k \geq 2$. G is an $\overline{AC}_4$-graph if and only if for $r = 1,2,..,k-1$ whenever the union H of any r monochromatic subgraphs of G is recoloured in one colour, the recoloured graph H_1 is the monochromatic subgraph of a 2-edge-coloured $\overline{AC}_4$-graph.

Proof

Let C be a set of colours contained in G, $1 \leq |C| < k$, and let H be the union of the monochromatic subgraphs of G which are coloured from the set C. H is not a complete graph since $|C| < k$. Suppose that when H is recoloured in blue to form the graph H_1, H_1 is not a monochromatic subgraph of a 2-edge-coloured $\overline{AC}_4$-graph. Recolour in red all of the edges of G which are not in H to form a graph H_2. The union of H_1 and H_2 is a 2-edge-coloured complete graph G_1. Since H_1 is a monochromatic subgraph of G_1, G_1 must contain an AC_4, say $v_1v_2v_3v_4$, where without loss of generality (v_1,v_2) and (v_3,v_4) are blue and the other edges red. Now consider the circuit $v_1v_2v_3v_4$ in the original graph G: the edges (v_1,v_2) and (v_3,v_4) are in colours from C, and the other edges are not. Thus $v_1v_2v_3v_4$ is an AC_4, and G cannot be an $\overline{AC}_4$-graph.

Now suppose that G is not an $\overline{AC}_4$-graph, so that G contains an AC_4 $v_1v_2v_3v_4$ say. If (v_1,v_2) and (v_3,v_4) are c_1- and c_2-coloured respectively (c_1 and c_2 not necessarily distinct), recolour the c_1- and c_2-edge-coloured edges of G in blue, where blue is not already present in G. Then the blue subgraph of G is not the monochromatic subgraph of a 2-edge-coloured $\overline{AC}_4$-graph, since if all the other edges of G were recoloured in red $v_1v_2v_3v_4$ would still be an AC_4.

Theorem 8.20

Let G be an $\overline{AC}_4$-graph of order p containing at least 2 colours, and let M be a monochromatic subgraph of G. Then M contains either an isolated vertex or a vertex of degree p − 1. Further, V(G) can be partitioned into non-empty sets $A_1, A_2, .., A_n$, $n \geq 2$, such that for $i \leq j$ any A_iA_j-edge is in M if and only if either M contains a vertex of degree p − 1 and i is odd or M contains an isolated vertex and i is even.

Proof

Let G be as above. By lemma 8.19 every monochromatic subgraph of G (including M) is also the monochromatic subgraph of a 2-edge-coloured $\overline{AC}_4$-graph. M must therefore contain either an isolated vertex or a vertex of degree p − 1 by lemma 8.7 taken together with theorem 8.4, and the rest of the theorem follows from corollary 8.9.

Theorem 8.21

If G is an $\overline{AC}_4$-graph, then G is connected in at most one colour.

Proof

If G contains a monochromatic subgraph M with a vertex of degree $|V(G)| - 1$, then M is connected since v is connected in M with every other vertex of G. All of the edges incident in G with v are in M, so v is an isolated vertex in any other monochromatic subgraph of G, and M is the only connected monochromatic subgraph of G.

Otherwise, if no monochromatic subgraph of G contains a vertex of degree $|V(G)| - 1$, then by theorem 8.20 every monochromatic subgraph of G contains an isolated vertex. In this case G has no connected monochromatic subgraphs.

Any 2-edge-coloured $\overline{AC}_4$-graph is connected in exactly one colour by corollary 8.6, and figure 8.2 shows an $\overline{AC}_4$-graph connected in no colours at all; theorem 8.21 cannot therefore be improved.

In the last section, it was shown in theorem 8.4 that a 2-edge-coloured

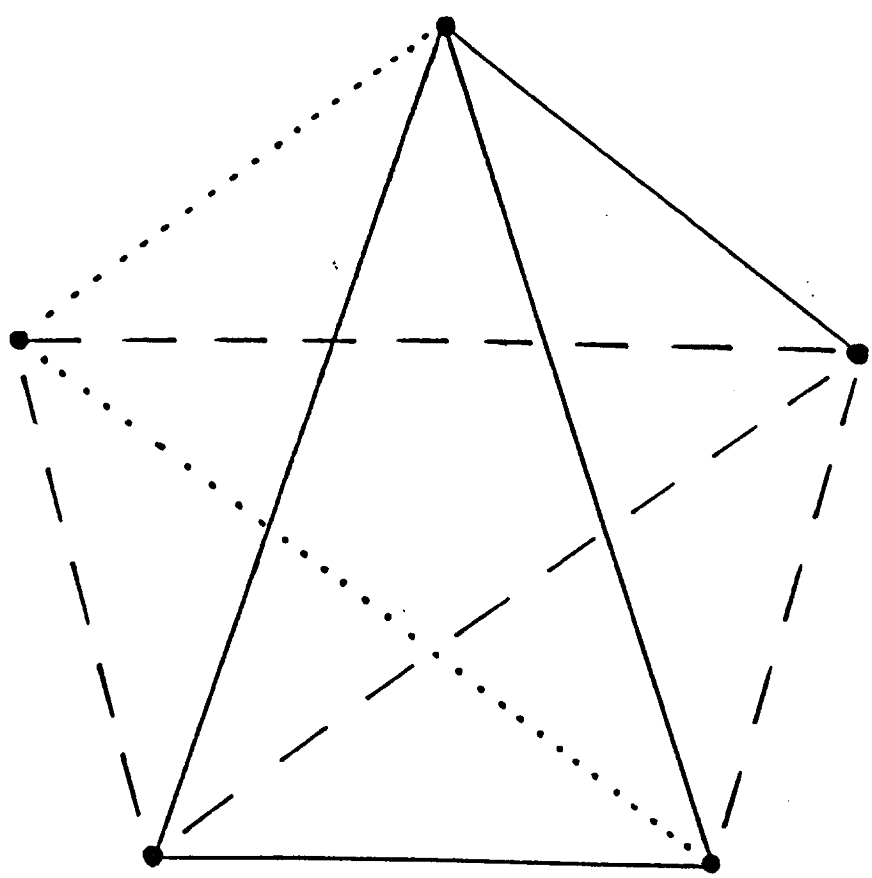


Figure 8.2

complete graph is an $\overline{\overline{AC}}$-graph if and only if it is an $\overline{\overline{AC}}_4$-graph. This does not hold for graphs with more than two colours (as is shown by a polychromatic triangle) but there is a close parallel.

Theorem 8.22

G is an $\overline{\overline{AC}}$-graph if and only if G is both an $\overline{\overline{AC}}_3$-graph and an $\overline{\overline{AC}}_4$-graph.

Proof

If G is an $\overline{\overline{AC}}$-graph, clearly it can contain no AC_3 or AC_4, so let G be both an $\overline{\overline{AC}}_3$-graph and an $\overline{\overline{AC}}_4$-graph. Suppose that G contains an alternating circuit, with $C = v_1v_2\ldots v_n$ a smallest such circuit in G, $n > 4$. Take first the case where (v_1,v_2) is the same colour as (v_3,v_4), blue say. The edge (v_1,v_4) must also be blue, otherwise $v_1v_2v_3v_4$ is an AC_4. Neither (v_n,v_1) nor (v_4,v_5) can be blue since C is an alternating circuit. But then $v_1v_4v_5\ldots v_n$ is an AC_{n-2}, contradicting the minimality of C.

The other case is where (v_1,v_2) is differently coloured from (v_3,v_4), say blue and red respectively. The edge (v_2,v_3) is differently coloured from both of these edges, in green say. Since $v_1v_2v_3v_4$ is not an AC_4, the edge (v_1,v_4) must be either red or blue; without loss of generality, let it be blue. Now consider the edge (v_1,v_3): since neither $v_1v_2v_3$ nor $v_1v_3v_4$ is an AC_3, (v_1,v_3) must be blue. But then $v_1v_3v_4\ldots v_n$ is an AC_{n-1}, again contradicting the minimality of C. G must therefore be an $\overline{\overline{AC}}$-graph.

Note that since a 2-edge-coloured graph can contain no AC_3, theorem 8.4 is just the restriction of the above result to the 2-edge-coloured case.

The earlier results in this section can now be adapted to give results on $\overline{\overline{AC}}$-graphs. The following two theorems are due to Chen [C7].

Theorem 8.23 (Chen)

If G is an $\overline{\overline{AC}}$-graph it is connected in exactly one colour.

Proof

By theorem 8.22, G is both an $\overline{\overline{AC}}_3$-graph and an $\overline{\overline{AC}}_4$-graph. Theorem 8.17 states that G is connected in either one or two colours, and theorem 8.21 states that it is connected in at most one colour.

Lemma 8.24

Let G be an $\overline{\overline{AC}}$-graph. Then some vertex of G is incident with edges of one colour only.

Proof

By theorem 8.22, G is an $\overline{\overline{AC}}_4$-graph. If every monochromatic subgraph of G contained an isolated vertex, G would be connected in no colours at all, contradicting theorem 8.23. Hence by theorem 8.20 some monochromatic subgraph of G contains a vertex of degree $|V(G)| - 1$, which must be incident with edges of that colour only.

Theorem 8.25 (Chen)

Let G be an $\overline{\overline{AC}}$-graph. Then V(G) can be partitioned into non-empty sets $A_1, A_2, \ldots, A_n$ such that for $i < j$ the colour of an A_iA_j-edge depends only on the choice of i.

Proof

By induction on the order p of G. The result is trivial for $p = 2$, so let $p > 2$ and assume the result for $\overline{\overline{AC}}$-graphs of smaller order.

By lemma 8.24, G contains a vertex v incident with edges of one colour only, say blue. Let H be the graph obtained from G by removing v together with its incident edges. H is an $\overline{\overline{AC}}$-graph of order $p - 1$, so the induction assumption can be applied. $V(H)$ can therefore be partitioned into non-empty sets $B_1, B_2, .., B_n$ such that for $i \leqslant j$ the colour of a B_iB_j-edge depends only on the choice of i. In particular, the B_1B_j-edges are all the same colour for $j \geqslant 1$.

If this colour is blue, set $A_i = B_i$ for $i = 2,3,..,n$ and $A_1 = B_1 \cup \{v\}$. $A_1, A_2, .., A_n$ partition $V(G)$, all of the A_1A_j-edges are the same colour for $j \geqslant 1$, and the theorem holds. If the B_1B_j-edges are not blue, $j \geqslant 1$, then set $A_1 = \{v\}$ and $A_{i+1} = B_i$ for $i = 1,2,..,n$. $A_1, A_2, .., A_{n+1}$ partition $V(G)$, all of the A_1A_j-edges are the same colour for $j \geqslant 1$, and again the theorem holds.

Theorem 8.26

The $\overline{AC}$-graphs are exactly those graphs obtained from a single vertex by repeated application of the following procedure: add a vertex to the graph H already obtained, and join that vertex to all the vertices of H by edges of the same colour.

Proof

By induction on the order p of a graph G. The theorem is trivial for $p = 2$, so assume G has order $p > 2$ and that the theorem holds for graphs of order less than p.

Suppose that G is derived in the manner described, by adding a vertex v to a graph H, and joining v to the vertices of H by blue edges say. Clearly G is complete. By the induction assumption, H contains no alternating circuit, and so either G is an $\overline{AC}$-graph as required, or v is contained in an alternating circuit. But v is incident with edges of one colour only, so this is impossible.

Now suppose that G is an $\overline{AC}$-graph. By lemma 8.24 G contains a vertex v incident with edges of one colour only, say blue. Let the graph obtained from G by removing v together with all of its incident edges be called H. H is an $\overline{AC}$-graph of order $p - 1$, so the induction assumption applies: H is constructed vertex by vertex, at each stage adding edges of one colour only. Since G is constructed from H by adding a vertex v and joining v to the vertices of H by blue edges, G is also constructed in the required way and the theorem is proved.

3. Restrictions on the Edges Incident with Each Vertex

Most of the papers written on alternating circuits in complete graphs have been concerned with the existence of an AC_n (especially an AH) when G is restricted in one of two ways: firstly, by allowing a maximum of Δ edges in any colour to be incident with any vertex of G; and secondly, by requiring a minimum of λ different colours to be incident with each vertex of G.

We shall deal first with the Δ problem, so for the next few results let G be a complete graph of order p with no more than Δ edges of each colour incident with each vertex. The problem, first posed by Daykin [D1], can be stated as follows: given integers Δ and p, for what values of n, $3 \leq n \leq p$, must G contain an AC_n? Daykin proved the following result:

Theorem 8.27 (Daykin)

Let $\Delta = 2$, and let G be a complete graph of order p, $p \geq 3\Delta$. If no vertex of G is incident with more than Δ edges of each colour, then G contains an AC_n for $n = 3,4,..,p$.

Theorem 8.27 is a best possible result in that the bound $p \geq 3\Delta$ cannot be improved: graph (iii) in figure 6.2 has order $3\Delta - 1$, but contains no AC_3 or AC_5.

Daykin's result applies to one particular value of Δ. In [B9], Bollobas and Erdos studied the case where Δ is an arbitrary integer.

They proved that if $p > 69\Delta$, G must contain an AC_n for $n = 3,4,..,p$. This bound was improved to $p \geq 17\Delta$ by Chen and Daykin [C8], and further improved by Shearer [S4].

Theorem 8.28 (Shearer)

Let G be a complete graph of order p. If $p > 7\Delta$, and no vertex of G is incident with more than Δ edges of each colour, then G contains an AC_n for $n = 3,4,..,p$.

The above result is the best known condition for the existence of alternating circuits of all possible lengths and for an arbitrary Δ. However, for alternating circuits of particular length it is possible to improve on this result.

Theorem 8.29

Let G be a complete graph of order p such that no vertex of G is incident with more than Δ edges of each colour. If $p \geq 2\Delta + 2$, G contains an AC_4.

Proof

Let d be the largest number of edges of a single colour incident with a vertex of G, and suppose that v is incident with d blue edges. It is sufficient to show that if $p \geq 2d + 2$, G contains an AC_4.

Suppose to the contrary that G is an $\overline{AC}_4$-graph. Since G contains no vertex incident with blue edges only, the blue subgraph of G contains an isolated vertex by theorem 8.20. Also by theorem 8.20, V(G) can be partitioned into non-empty sets $A_1,A_2,..,A_n$ such that for $i \leq j$ an A_iA_j-edge is blue if and only if i is even. In particular, no A_1A_i-edge is blue for any i, so that the vertices in A_1 are incident with no blue edges, and every A_2A_j-edge is blue, $j \geq 2$. Since v is the vertex of

G incident with most blue edges, v is in A_2, and if any other vertex w is incident with a blue edge then (v,w) is blue. Thus exactly d + 1 vertices of G are incident with blue edges. If the set of these vertices is called B, then V(G) is partitioned into A_1 and B, and A_1 contains at least d + 1 vertices.

Let u be in A_1, and let (u,v) be red say. Since u can be incident with no more than d red edges, some uA_1-edge (u,x) must be differently coloured, in green say. If w is any vertex in B other than v, then since xuvw cannot be an AC_4 and x is incident with no blue edges, then (x,w) is green. This is true for all d vertices of B other than v. But (u,x) is also green, giving that x is incident with at least d + 1 green edges. This contradicts the definition of d, so G must contain an AC_4.

Theorem 2.17 of chapter 2 can be adapted to yield a sharper result on $\overline{AC}_3$-graphs, since the $\overline{AC}_3$-graphs are just the $\overline{PC}_3$-graphs.

Theorem 8.30

There exists an $\overline{AC}_3$-graph G of order p with no vertex incident with more than Δ edges of any colour if and only if

$$p \leq \begin{cases} 2 \\ \frac{1}{2}.5\Delta \\ \frac{1}{2}(5\Delta - 3) \end{cases}$$

A much more general result is due to Chen [C7].

Theorem 8.31 (Chen)

There exists an $\overline{AC}$-graph G of order p such that no vertex of G is incident with more than Δ edges of each colour if and only if $p \leq \Delta + 1$.

Proof

If G is an $\overline{AC}$-graph of order p, $p \leq \Delta + 1$ by lemma 8.24.

Now suppose that $p \leq \Delta + 1$. Any complete graph of order p is regular of degree $p - 1$. Thus any graph of order p constructed in the manner described in theorem 8.26 is an $\overline{\overline{AC}}$-graph with no vertex incident with more than Δ edges.

Thus although the best available general result stated that if $p > 7\Delta$ G contained alternating circuits of every possible length (theorem 8.28), a bound as low as $p > \Delta + 1$ ensures that G contains some alternating circuit. It is likely that Shearers result can be improved - specific bounds such as theorems 8.30 and 8.31 are closer to the bound in theorem 8.32 than to that in theorem 8.28.

Few bounds which are definitely too low are known. If $p \leq \Delta + 1$, G need contain no alternating circuit at all, and G need contain no AC_3 if $p \leq \frac{1}{2}.5\Delta$. A general bound of a similar order can be found for alternating circuits of odd length.

Theorem 8.32

If n is odd and $n \leq p$, then there exists an $\overline{\overline{AC}}_n$-graph G of order p with no vertex incident with more than Δ edges of each colour if $p \leq 2\Delta + 1$ (Δ even) or $p \leq 2\Delta$ (Δ odd).

Proof

In view of lemma 8.1 and the fact that any complete subgraph of an $\overline{\overline{AC}}_n$-graph is an $\overline{\overline{AC}}_n$-graph, it is sufficient to show that there is a 2-edge-coloured complete graph G of order $2\Delta + 1$ (Δ even) or 2Δ (Δ odd) with no more than Δ edges of any colour incident with a vertex. For Δ even, this has already been done in the proof of theorem 2.18. For Δ odd, G is the join in blue of two complete graphs coloured in red, each of order Δ.

The only other result known is due to Bollobas and Erdos [B9].

Theorem 8.33 (Bollobas and Erdos)

For each $p \leq 2\Delta + 1$, there exists an $\overline{\overline{AH}}$-graph of order p such that no vertex is incident with more than Δ edges of any colour.

The second problem mentioned at the start of the section can be stated as follows: given integers p and n, $p \geq n$, what is the minimum integer λ such that whenever a complete graph G of order p has at least λ different colours incident with each vertex, G contains an AC_n? The problem was again introduced by Daykin [D1], who proved the following result:

Theorem 8.34 (Daykin)

Let n be an odd integer. Then there exists an $\overline{\overline{AH}}$-graph G of order 2n such that each vertex of G is incident with at least n colours.

This shows that $\lambda = \frac{1}{2}p$ will not necessarily give an AH, where p is the order of the complete graph. In [B9], Bollobas and Erdos showed that $\lambda = \frac{7}{8}p$ did force an AH, and this was improved by Shearer [S4].

Theorem 8.35 (Shearer)

Let G be a complete graph of order p. If each vertex of G is incident with at least $\frac{(5p + 8)}{7}$ edges of different colours, G contains an AH.

Chapter 9

COMPLETE GRAPHS WITHOUT MONOCHROMATIC CIRCUITS

1. Complete Graphs in which No Circuit is Monochromatic

A complete graph in which no circuit is monochromatic is called an $\overline{\overline{MC}}$-graph. A graph which does not contain a circuit is called a forest. Using this terminology, the $\overline{\overline{MC}}$-graphs can be trivially characterised in terms of their monochromatic subgraphs.

Theorem 9.1

Let G be a complete graph. G is an $\overline{\overline{MC}}$-graph if and only if each monochromatic subgraph of G is a forest.

A result of Beineke [B3] can be modified to limit the number of colours in an $\overline{\overline{MC}}$-graph of order p.

Theorem 9.2

There exists a k-edge-coloured $\overline{\overline{MC}}$-graph of order p if and only if

$$\tfrac{1}{2}p(p - 1) \geq k \geq \lfloor \tfrac{1}{2}(p + 1) \rfloor$$

Proof

If G is a k-edge-coloured $\overline{\overline{MC}}$-graph of order p, then each monochromatic subgraph of G is a forest by theorem 9.1. It is a well-known result that a forest of order p can contain at most p - 1 edges, so that $\tfrac{1}{2}p(p - 1) \leq k(p - 1)$. This gives $k \geq \tfrac{1}{2}p$, and since both k and p are integers, $k \geq \lfloor \tfrac{1}{2}(p + 1) \rfloor$. $\overline{\overline{MC}}$-graphs which attain this bound can be found in Beineke [B3].

The upper bound on k is given by the fact that a graph cannot contain more colours than edges. An intermediate value of k is achieved in the following way: suppose that G is a k-edge-coloured $\overline{\overline{MC}}$-graph of order p, $\frac{1}{2}p(p - 1) > k \geq \lfloor\frac{1}{2}(p + 1)\rfloor$; then some monochromatic subgraph of G contains more than one edge, and if one of these edges is recoloured in a new colour, an $\overline{\overline{MC}}$-graph of order p is created with one more colour than G.

For an arbitrary forest F to be a monochromatic subgraph of an $\overline{\overline{MC}}$-graph G, clearly F must be monochromatic and of the same order as G. In the absence of information on the other monochromatic subgraphs of G, these are the only restrictions on F.

Theorem 9.3

Let F be a monochromatic forest of order p. Then F is a monochromatic subgraph of some $\overline{\overline{MC}}$-graph of order p.

Proof

The required complete graph can be obtained from F by joining each pair of non-adjacent vertices by an edge such that no new edge is the same colour as the edges of F, and no two new edges are the same colour.

Thus if G is an $\overline{\overline{MC}}$-graph of order p, then each monochromatic subgraph of G is a forest, and any monochromatic forest of order p could be a monochromatic subgraph of G. To characterise the $\overline{\overline{MC}}$-graphs in a non-trivial way, the following question needs to be answered: if $S = \{F_1, F_2, .., F_k\}$ is a set of monochromatic forests of the same order but differently coloured, does there exist an $\overline{\overline{MC}}$-graph whose set of monochromatic subgraphs contains S? Equivalently, does there exist a graph, not necessarily complete, whose set of monochromatic subgraphs is exactly S? Theorem 9.3 answers the question in the affirmative for k = 1, but the following example shows that the question is not so easily solved for k = 2.

A connected forest is called a tree, and a tree in which one vertex is adjacent to all the other vertices is a star. If F_1 is a monochromatic star contained in a complete graph G of the same order, some vertex of G is incident with edges of one colour only, that of F_1. In a tree, each vertex is incident with an edge, so if F_1 is a monochromatic star and F_2 a monochromatic tree of the same order but differently coloured, F_1 and F_2 cannot be monochromatic subgraphs of the same complete graph.

For $k \geq 2$ the question of whether or not the forests in S are monochromatic subgraphs of an $\overline{MC}$-graph depends on the combination of forests in S, and so a simple solution is unlikely. Here we study just two cases: firstly, where each monochromatic subgraph of the complete graph is a tree; and secondly where each monochromatic subgraph of a complete graph is a forest, and where all of them are isomorphic.

So let $S_1 = \{T_1, T_2, .., T_k\}$ be a set of monochromatic trees of the same order p but differently coloured. We consider under what circumstances S_1 can be the set of monochromatic subgraphs of an $\overline{MC}$-graph.

Lemma 9.4

Let each monochromatic subgraph of the complete graph G be a tree. Then G is a k-edge-coloured graph of order 2k for some integer $k \geq 1$.

Proof

Suppose that G has order p. A tree of order p contains p - 1 edges, so each monochromatic subgraph of G contains p - 1 edges. G has $\frac{1}{2}p(p - 1)$ edges altogether, and so must contain $\frac{1}{2}p$ colours.

Theorem 9.3 shows that if G is a complete graph of order p, all of whose monochromatic subgraphs are forests, then there is no restriction on whether an arbitrary monochromatic forest of order p can form one of these monochromatic subgraphs. However, if the more stringent condition that every monochromatic subgraph is a tree is imposed on G, this no longer applies.

Theorem 9.5

Let T be a monochromatic tree of order 2k, where $k \geq 1$ is an integer. Then T is a monochromatic subgraph of a complete graph G of order 2k, each of whose monochromatic subgraphs is a tree, if and only if $\Delta(T) \leq k$, where $\Delta(T)$ is the maximum degree of T.

Proof

Suppose that the monochromatic subgraphs of the complete graph G of order 2k are all trees, where $k \geq 1$ is an integer. Each vertex of G is incident with $2k - 1$ edges, and by lemma 9.4 G is k-edge-coloured. Since each monochromatic subgraph is a tree, each vertex of G is incident with an edge of each colour, so that no vertex can be incident with more than k edges of any colour.

Now let T satisfy $\Delta(T) \leq k$, and let H be a graph with vertex set V(T) and such that if u and v are vertices of T (and H), the edge (u,v) is in E(H) if and only if (u,v) is not in E(T). To prove the theorem, it is enough to show that H can have $k - 1$ monochromatic subgraphs, each a tree.

The arboricity of a graph G is the minimum number $a_1(G)$ of forests with vertex sets V(G) whose union forms the graph. The arboricity of a graph is given by the formula

$$a_1(G) = \max_n \left\lceil \frac{q_n}{n - 1} \right\rceil$$

where q_n is the maximum number of edges in any subgraph of H of order n (see for instance [B2]). If it can be shown that $a_1(H) = k - 1$, so that H is the union of $k - 1$ trees, then by making each of these trees monochromatic and differently coloured H is as required.

Consider the graph H and an integer n satisfying $n \leq 2k - 2$.

Then

$$q_n \leq \tfrac{1}{2}n(n - 1)$$

$$\text{giving } \frac{q_n}{n - 1} \leq \tfrac{1}{2}n \leq k - 1$$

$$\text{so that } \left\lceil \frac{q_n}{n - 1} \right\rceil \leq k - 1 \qquad \text{for } n \leq 2k - 2 \tag{9A}$$

If n = 2k, then

$$\begin{aligned} q_{2k} &= |E(H)| \\ &= \tfrac{1}{2}.2k(2k - 1) - (2k - 1) \\ &= (k - 1)(2k - 1) \end{aligned}$$

$$\text{so that } \left\lceil \frac{q_{2k}}{2k - 1} \right\rceil = k - 1 \tag{9B}$$

Now let v be any vertex in H (and T), and let v be incident with $d \leq k$ edges in T. Removing v from H together with its incident edges removes $(2k - 1) - d$ edges from H, so removing a single vertex from H removes at least $k - 1$ edges. Thus

$$\begin{aligned} q_{2k-1} &\leq (k - 1)(2k - 1) - (k - 1) \\ &= (k - 1)(2k - 2) \end{aligned}$$

$$\text{Therefore} \qquad \frac{q_{2k-1}}{2k - 2} \leq k - 1$$

$$\text{so that } \left\lceil \frac{q_{2k-1}}{2k - 2} \right\rceil \leq k - 1 \tag{9C}$$

Hence equations (9A), (9B), and (9C) together give

$$a_1(H) = \max_n \left\lceil \frac{q_n}{n - 1} \right\rceil = k - 1$$

as required.

Theorem 9.6

Let G be a complete graph of order 2k whose monochromatic subgraphs $T_1, T_2, .., T_k$ are all trees. If the trees between them have n vertices of degree k, then each tree contains at least n - 2 endvertices, and if $k > n$, k - n of the trees contain at least n endvertices.

Proof

Suppose a tree T_i contains a vertex v of degeee k. Now v has degree 2k - 1 in G, and must be incident with an edge of each of the k colours in G, and so must have degree 1 in all of the trees except T_i. The second part of the theorem is given by considering the (at least) k - n trees which do not have a vertex of degree k.

The first part follows by considering a single tree T_i. T_i has 2k - 1 edges, giving the sum of the degrees of its vertices as 4k - 2. Since all 2k vertices have degree at least 1, no more than two vertices have degree k. Thus at the at least n - 2 vertices of G which have degree k in one of the trees other than T_i, T_i has degree 1.

2. $\overline{MC}$-Graphs with Isomorphic Monochromatic Subgraphs

We now turn to considering complete graphs, all of whose monochromatic subgraphs are isomorphic forests. As two graphs which differ only by the number of isolated vertices in them are essentially similar, the study of these graphs is greatly facilitated by the following definition:

Definition 9.7

Let H be a monochromatic graph. A graph G of order p isomorphically decomposes into H if each monochromatic subgraph of G is isomorphic to H', where the graph H' can be obtained from H by the addition of isolated vertices. If such a graph G exists which is complete, this is denoted by $p \in ID[H]$.

By taking H to have no isolated vertices at all, all monochromatic forests whose non-trivial connected components together are isomorphic to H can be dealt with at the same time. The isomorphic decomposition problem can now be stated as follows: for any monochromatic graph H without isolated vertices, for which integers p does $p \in ID[H]$? Here, we are only interested in the case where H is a forest.

Each of the $\frac{1}{2}p(p - 1)$ edges in a complete graph of order p is in exactly one of the monochromatic subgraphs. If each monochromatic subgraph has the same number q of edges, then q must divide $\frac{1}{2}p(p - 1)$, which proves:

Theorem 9.8

Let F be a monochromatic forest with q edges and no isolated vertices. Then $p \in ID[F]$ only if

$$p(p - 1) \equiv 0 \pmod{2q} \tag{9D}$$

Corollary 9.9 (Huang and Rosa, [H11])

Let F be a monochromatic forest with q edges and no isolated vertices, where q is a prime power. Then $p \in ID[F]$ only if

$$p \equiv 0 \text{ or } 1 \pmod{2q} \quad \text{if } q \text{ is even}$$
$$\text{and} \quad p \equiv 0 \text{ or } 1 \pmod{q} \quad \text{otherwise}$$

In the more general case, where F need not be a forest, a further necessary condition is that the greatest common divisor of the degrees of the vertices in F must divide p - 1, the degree of each vertex in the complete graph of order p. However, this condition becomes redundant when F is a forest, as any non-trivial forest contains a vertex of degree 1.

Erdos and Schonheim [E5] conjectured that condition (9D) was not only necessary but also 'asymptotically sufficient'. They proved this conjecture for connected graphs of order at most 4, and the general case was proved by Wilson [W6], who used a mainly combinatorial proof making use of block designs.

Theorem 9.10 (Wilson)

Let F be a monochromatic forest with q edges and no isolated vertices. Then for all sufficiently large p, p εID[F] if and only if $p(p - 1) \equiv 0 \pmod{2q}$.

Although theorem 9.10 deals with arbitrary forests, it is an asymptotic result and so of limited application. To obtain exact results it has been found necessary to restrict the choice of forest, and nearly all exact results pertain to simple types of tree. Proving that p εID[F] for a forest F usually involves direct construction, so when p is large the following lemma is often used:

Lemma 9.11

Let $p_1, p_2, .., p_m$ εID[F] (respectively let $p_1 + 1$, $p_2 + 2$, . ,$p_m + 1$ εID[F]) for some monochromatic forest F, and assume that there exists a complete m-partite graph H which isomorphically decomposes into F, and such that the i'th m-partition of H contains p_i vertices ($i = 1,2,..,m$). Then $\sum_{i=1}^{m} p_i$ εID[F] (resp. $(\sum_{i=1}^{m} p_i) + 1$ εID[F]).

The advantage of lemma 9.11 lies in the fact that it is usually easier to isomorphically decompose complete multipartite graphs than complete graphs. Indeed, in the case where $p_1, p_2, .., p_m$ are all equal, it is enough to find a suitable complete bipartite graph with each section of the bipartition containing p_1 vertices. However, lemma 9.11 still relies on the small values in ID[F] being known. These values are usually found by the method of differences (see for example Bermond and

and Sotteau [B6]), which is essentially a derivation of Bose's method of 'symmetrically repeated differences' (see Hall [H3]).

A labelling of a graph G is an assignment of a non-negative integer a_i to each vertex v_i of G such that no two vertices receive the same integer: a_i is the label of v_i. The weight of an edge (v_i, v_j) is the absolute difference $|a_i - a_j|$ of the labels of its incident vertices. Let T be a tree with q edges and q + 1 vertices; *a labelling of T is admissable (also called a ρ-valuation) if the set of labels of V(T) is a subset of {0,1,..,2q}* and if whenever b_i and b_j are two weights of edges in E(T), then $b_i \neq b_j$ and $b_i + b_j \neq 2q + 1$.

Now assume that T has an admissable labelling L. We want to construct a complete graph G which isomorphically decomposes into T. Label the 2q + 1 vertices of G 0,1,..,2q. Let e be an edge of G, with weight b say. There exists a unique edge of T with weight either b or 2q + 1 - b; let this edge be (i,j). Then e = (i + r, j + r) for some r, $0 \leq r \leq 2q$ (all integers taken modulo 2q + 1). Now colour e in colour c_r, and repeat the process for each edge in G. It is easily checked (see Rosa [R4]) that G isomorphically decomposes into T, so that $2q + 1 \in ID[T]$.

The preceding method was used by Rosa [R4] in response to a conjecture of Ringel [R2] that for every tree T with q edges, $2q + 1 \in ID[T]$. Rosa called an isomorphic decomposition obtained in such a way a cyclic decomposition, as each monochromatic subgraph of G can be obtained from the c_o-coloured subgraph by a rotation in G. He reported a conjecture of Kotzig that for any monochromatic tree T, some complete graph could be cyclically decomposed into T. We present in lemma 9.12 and theorem 9.13 some of the classes of trees found by Rosa to have admissable labellings.

Rosa also studied a labelling which involves stricter conditions, and which has been widely studied since (see Bloom [B8] and various articles in the American Mathematical Monthly [D4, G4-8]. A tree of order $q + 1$ is graceful if it can be labelled with the set $\{0,1,..,q\}$ such that the set of weights of $E(T)$ is $\{1,2,..,q\}$. (This terminology is due to Golomb [G2]; such a labelling is called a β-valuation by Rosa.) Clearly a graceful tree has an admissable labelling, so that if a tree with q edges is graceful, then $2q + 1 \in ID[T]$. A conjecture attributed to Kotzig is that all trees are graceful.

Before some results on graceful trees and admissable labellings can be presented, some nomenclature is needed. The base of a tree T is the tree obtained from T by removing its endvertices and their incident edges. A star is a tree whose base is a single vertex. A caterpillar is a tree whose base is a path or a single vertex. A lobster is a tree whose base is a caterpillar. A branch at a vertex v of a tree T is a maximal subtree of T containing v as an endvertex. A complete m-ary tree is one constructed from a star with m edges by repeatedly joining m new vertices to each endvertex of the existing tree. A generalisation of this are the trees of British Number Systems: a tree T of a British Number System $BNS(d_1,d_2,..,d_n)$ is constructed from a vertex v by at stage $i, i = 1,2,..,n$, joining d_i new vertices to v if $i = 1$, or to each endvertex of the existing tree if $i > 1$.

<u>Lemma 9.12</u>

If T is a tree with one of the following properties, then it is graceful:

a) T is a caterpillar;

b) T has less than 5 endvertices;

c) T has less than 16 edges;

d) there exists a vertex in T such that all the branches of T at v (except possibly one) are isomorphic caterpillars;

e) there exists a vertex v in T of degree 3 such that two branches of T

at v are paths, and the third is a caterpillar;

f) there exists a vertex v in T of degree 4 such that all four branches of T at v are paths;

g) T is a tree of a British Number System.

Sections a) - f) were proved by Rosa [R4], while a) and part of b) were rediscovered by Cahit and Cahit [C2]. Cahit [C1] conjectured that all complete binary trees were graceful; this was proved by Owens [O1], and independently by Stanton and Zarnke [S5], and by Koh, Rogers, and Tan [K4] as a special case of the result that all complete m-ary trees are graceful. This was extended by Beth and Sprague [B7] to a proof of section g). These last three papers used the technique of constructing a large graceful tree from a set of smaller graceful trees. These methods have since been greatly extended by Koh et al. [K3, K5, K6, K7, R3]. Haggard and McWha [H1] found a sufficient condition for a tree to be graceful in terms of its adjacency matrix, but were unable to find an algorithm to apply this condition. Other authors have since pursued this line of enquiry, so although Kotzig's conjecture is still unconfirmed, work on it is progressing.

Theorem 9.13

Let T be a monochromatic tree with q edges. If T has one of the following properties, then T has an admissable labelling and $2q + 1 \in ID[T]$:

a) T is graceful;

b) there exists a graceful connected subgraph of T containing the base of T (this subgraph may itself be the base of T).

Section a) and a weaker version of b) are due to Rosa [R4]; section b) is due to Kotzig [K8].

Huang and Rosa [H11] modified the method of differences outlined above to provide a proof that $2q \in ID[T]$ for certain trees with q edges.

This modification involved labelling as infinity a vertex of the complete graph and also an endvertex of the tree, making sure that no edge had weight q or q + 1, and then proceeding as before. Another variation enabled them to isomorphically decompose complete bipartite graphs into certain types of tree. Application of lemma 9.11 (and of corollary 9.9 in the case of the next result) produces the following results:

Lemma 9.14 (Huang and Rosa [H11])

Let T_q be a monochromatic lobster with $q = 2^n$ edges for some integer n. Then $p \in ID[T_q]$ if and only if $p \equiv 0$ or 1 (mod 2q).

Lemma 9.15 (Huang and Rosa [H11])

Let T_q be a monochromatic lobster with q edges. Then if $p \equiv 0$ or 1 (mod 2q), $p \in ID[T_q]$.

Two vertices of a tree are similar if there is an automorphism of T mapping one onto the other. An edge of a tree is symmetric if it joins two similar vertices. A tree is symmetric if it has a symmetric edge.

Lemma 9.16 (Huang and Rosa [H11])

Let T_q be a monochromatic caterpillar with q edges. If one of the following holds, then $p \in ID[T_q]$:

i) $q \equiv 1 \pmod 4$, $p \equiv 0$ or 1 (mod q) and $p \neq q, q + 1$, or $3q + 1$;

ii) $q \equiv 3 \pmod 4$, $p \equiv 0$ or 1 (mod q) and $p \neq q, q + 1$, or $3q$;

iii) T_q is symmetric, $p \equiv 0$ or 1 (mod q) and $p \neq q$.

One of the few classes of tree for which our knowledge is complete is the class of stars. Partial results were obtained by Hogarth [C3, H9] and Ae, Yamamoto, and Yoshida [A3]. The final result was obtained by Yamamoto et al. [Y1] using the adjacency matrices of complete graphs rather than the method of differences, and was independently obtained by Huang [H10].

<u>Lemma 9.17</u> (Yamamoto <u>et al</u>; Huang)

Let S_q be a monochromatic star with q edges. Then $p \in ID[S_q]$ if and only if $p(p - 1) \equiv 0 \pmod{2q}$ and $p \geq 2q$.

Huang and Rosa [H11] also looked at trees with at most 8 edges. Complete results were found in all cases, and can be summarised as follows:

<u>Lemma 9.18</u> (Huang and Rosa)

Let T_q be a monochromatic tree with q edges.

i) If $q = 2, 4,$ or 8 then $p \in ID[T_q]$ if and only if $p \equiv 0$ or $1 \pmod{2q}$.

ii) If $q = 3, 5,$ or 7, then $p \in ID[T_q]$ if and only if $p \equiv 0$ or $1 \pmod{q}$ and $p \geq 2q$ in some cases, $p > q$ in the other cases.

iii) If $q = 6$, then $p \in ID[T_q]$ if and only if $p(p - 1) \equiv 0 \pmod{2q}$ and $p \geq 2q$ in some cases, $p > q$ in the other cases.

Part (i) is a direct corollary of lemma 9.14, as every tree with at most 8 edges is a lobster.

Nearly all the papers concerned with the isomorphic decomposition problem have directed their attention towards just two types of graph - trees and circuits (for a survey of results on circuits, see for example Bermond and Sotteau [B6]). However, one result does exist on forests which are not trees, and concerns matchings (a matching is a set of independent edges).

<u>Lemma 9.19</u> (Schonheim and Bialostocki [S1])

Let M_q be a monochromatic matching with q edges. Then $p \in ID[M_q]$ if and only if $p(p - 1) \equiv 0 \pmod{2q}$ and $p \geq 2q$.

The case $q = 2$ was given by Bermond and Schonheim [B5].

In Wilsons Theorem (theorem 9.10) it was noted that $p(p-1) \equiv 0 \pmod{2q}$ is an asymptotically sufficient condition that $p \in ID[F_q]$, where F_q is a forest with q edges. This implies that for each forest F_q, there exists a least integer p_o (dependent on the choice of F) such that if $p \geq p_o$ and $p(p-1) \equiv 0 \pmod{2q}$, then $p \in ID[F_q]$; call this integer p_o the Wilson Threshold for F_q, and denote it by $WT(F_q)$.

As $p \in ID[F_q]$ implies that $p > q$, clearly for each forest F_q, $WT(F_q) \geq q + 1$. Lemmas 9.14, 9.17, and 9.19 give $WT(T_q) = q + 1$ when T_q is a lobster with $q = 2^n$ edges, and $WT(F_q) = 2q$ when F_q is a star or a matching with q edges. Lemma 9.18 splits the small trees into those with Wilson Threshold $q + 1$ and those with Wilson Threshold $2q$, where q is the number of edges in the tree. However, it should be remembered that all forests considered here are of essentially simple structure, and so a small Wilson Threshold might be expected.

3. Complete Graphs Without a Monochromatic C_n

In this section, we deal with the following problem: given an integer $n \geq 3$, which complete graphs have no monochromatic circuit of length n? The case $n = 3$ was considered in chapter 4, where it was noted that for each $k \geq 1$, there exists a smallest integer $r_k(3)$ such that every k-edge-coloured complete graph of order at least $r_k(3)$ contains a monochromatic triangle. These numbers are called the Ramsey numbers for triangles, as they are a generalisation of a concept introduced by Ramsey [R1].

A further generalisation of Ramsey's theorem involves circuits other than triangles. For integers $k \geq 1$ and $n \geq 3$, $r_k(C_n)$ is the least integer such that every k-edge-coloured complete graph of order at least $r_k(C_n)$ contains a monochromatic circuit of length n. The existence of such numbers is a consequence of Ramsey's theorem.

The integers $r_2(C_n)$ are usually considered in the context of a problem introduced by Chartrand and Schuster [C4, C5]: given integers m and n, $3 \leq m,n$, what is the least integer $r(C_m,C_n)$ such that every 2-edge-coloured complete graph of order at least $r(C_m,C_n)$ contains either a C_m in the first colour or a C_n in the second colour? Greenwood and Gleason [G3] had already shown that $r(C_3,C_3) = 6$ (see chapter 4); Chartrand and Schuster gave some other small values - $r(C_4,C_4) = 6$, $r(C_4,C_5) = 7$, $r(C_6,C_6) = 8$ - together with the values when m is small and n arbitrary:

$$r(C_m,C_n) = \begin{cases} 2n - 1 & m = 3 \text{ or } 5,\ n > 3 \\ \\ n + 1 & m = 4,\ n \geq 6 \end{cases}$$

Bondy and Erdos [B10, E2] extended Chartrand and Schusters lower bound construction for m odd to give

$$r(C_m,C_n) \geq 2n - 1 \quad m \text{ odd},\ n > 3$$

with equality for $n = m > 5$ or n sufficiently large. They also proved that for m even and n large enough, $r(C_m,C_n) = m + \frac{1}{2}n - 1$.

Chartrand and Schuster [C6] then gave the following lower bounds on $r(C_m,C_n)$ for m even:

$$r(C_m,C_n) \geq \begin{cases} 2m - 1 & n \text{ odd},\ n < \frac{3}{2}m \\ \\ n + \frac{1}{2}m - 1 & \text{otherwise} \end{cases}$$

and conjectured that these bounds together with those of Bondy and Erdos were sharp. Schuster [S2] strengthened the conjecture by finding $r(C_6,C_n)$ for n = 7, 8, and 9. The conjecture was confirmed independently by Rosta [R5, R6], and by Faudree and Schelp [F1].

<u>Theorem 9.20</u> (Rosta; Faudree and Schelp)

Let m and n be integers, $3 \leq m \leq n$. Then

$$r(C_m,C_n) = \begin{cases} 6 & m = n = 3,\ m = n = 4 \\ 2n - 1 & m \text{ odd},\ (m,n) \neq (3,3) \\ 2m - 1 & m \text{ even},\ n \text{ odd},\ n < \frac{3}{2}m \\ n + \frac{1}{2}m - 1 & \text{otherwise} \end{cases}$$

Two of the extremal graphs used in the proof of theorem 9.20 are the following: if n is odd and $n > 3$, the join in red of two complete graphs each of order $n - 1$ has order $r_2(C_n) - 1$ and is an $\overline{MC}_n$-graph; if n is even and $n > 4$, the join in red of two blue complete graphs, one of order $n - 1$ and the other of order $\frac{1}{2}n - 1$, has order $r_2(C_n) - 1$ and is an $\overline{MC}_n$-graph. These constructions can be generalised, though the cases where n is odd and n is even must be treated separately.

For n odd, the generalisation takes the form of what various authors (see [E3, F3]) call canonical colourings.

Theorem 9.21

For any odd integer $n \geq 3$, let G_o be a complete graph with vertex set $\{v_1,v_2,..,v_p\}$ containing no monochromatic circuits of odd length m, $m \leq n$, and let $G_1,G_2,..,G_p$ be $\overline{MC}_n$-graphs such that for $i = 1,2,..,p$ no colour in G_i is incident in G_o with v_i. The graph G obtained by successively substituting G_i for v_i in G_o, $i = 1,2,..,p$, is an $\overline{MC}_n$-graph.

Proof

G is complete by lemma 2.11, so it remains to prove that it contains no monochromatic C_n. Suppose to the contrary that G contains a monochromatic circuit M of length n, so that $M = u_1u_2...u_n$ say, in colour blue. Since $G_1,G_2,..,G_p$ are $\overline{MC}_n$-graphs, M must contain vertices from at least two of them. If two consecutive vertices of M were in G_i for some i, $1 \leq i \leq p$, then G_i would contain a blue edge; but G_i cannot itself contain M, and there are no blue edges incident in G_o with v_i by the conditions of the theorem, so this is impossible.

For $i = 1,2,..,n$ let $f(u_i) = v_j$ if u_i is a vertex of G_j, so that u_i is one of the vertices which replaces $f(u_i)$ to form G. Then by the method of construction of G, $f(u_1)f(u_2)\ldots f(u_n)$ is a closed walk in G_o in which each edge is blue; call this walk W. Now let H be the subgraph induced in the blue subgraph of G_o by $V(W)$; H must contain the closed walk W. Since H is a subgraph of G_o, it can contain no circuit of odd length m, $m \leq n$. H has order at most n, and so must be bipartite, with bipartition (X_1,X_2) say. Suppose that $f(u_1)$ is in X_1. Now $f(u_2)$ is different from $f(u_1)$ by the first part of the proof, and there is an edge in H joining them, so $f(u_2)$ must be in X_2. By similar reasoning, $f(u_{2i})$ is in X_2 and $f(u_{2i-1})$ is in X_1 for each i, and in particular $f(u_n)$ is in X_1 since n is odd. But W is a closed walk, so there is an edge in H between $f(u_1)$ and $f(u_n)$. This is a contradiction since H is bipartite, so the theorem is proved.

Before proceeding, some more notation is needed. For integers $k \geq 1$ and $n \geq 3$, n odd, $r_k{*}(C_n)$ is the least integer such that any k-edge-coloured complete graph of order at least $r_k{*}(C_n)$ contains a monochromatic C_m for some odd integer $m \leq n$. Clearly for each odd integer m, $3 \leq m \leq n$, $r_k{*}(C_n) \leq r_k(C_m)$.

The next result is a slight improvement of a result of Abbot [A1].

<u>Theorem 9.22</u>

Let $k \geq 1$ and $n \geq 3$ be integers, n odd. Then for $i = 1,2,..,k - 1$,

$$r_k(C_n) \geq (r_i{*}(C_n) - 1)(r_{k-i}(C_n) - 1) + 1$$

<u>Proof</u>

It is enough to prove that for $i = 1,2,..,k - 1$ there exists a k-edge-coloured $\overline{MC}_n$-graph of order $(r_i{*}(C_n) - 1)(r_{k-i}(C_n) - 1)$. There exists a complete i-edge-coloured graph G_o of order $p = r_i{*}(C_n) - 1$

containing no monochromatic circuits of odd length m, $m \leq n$, coloured from the set $\{c_1, c_2, .., c_i\}$. Also, there exist p (k-i)-edge-coloured $\overline{MC}_n$-graphs $G_1, G_2, .., G_p$ of order $r_{k-i}(C_n) - 1$ coloured from the set $\{c_{i+1}, c_{i+2}, .., c_k\}$. Then by theorem 9.21, there exists an $\overline{MC}_n$-graph of the required order coloured from the set $\{c_1, c_2, .., c_k\}$.

A precondition for applying theorem 9.22 is that some values of $r_k^*(C_n)$ are known. For k = 1, $r_1(C_3) \geq r_1^*(C_n)$ gives that $r_1^*(C_n) = 3$. This, together with the trivial observation that $r_1(C_n) = n$, gives a result due to Bondy and Erdos [B10].

Corollary 9.23 (Bondy and Erdos)

For n odd, $n \geq 3$,

$$r_k(C_n) \geq 2^{k-1}(n - 1) + 1$$

Proof

By repeated application of theorem 9.22.

For k = 2, $r_2^*(C_n) \leq r_2(C_3) = 6$. The only 2-extremal $\overline{MC}_3$-graph contains a monochromatic C_5 (see theorem 4.14), so for $n \geq 5$, $r_2^*(C_n) \leq 5$. Using this value in theorem 9.22 will not improve corollary 9.23. For k = 3 and n > 3, Erdos *et al*. [E3] showed that $r_k^*(C_n) = 9$, which again will not improve corollary 9.23. General values of $r_k^*(C_n)$ are not known for k > 3.

In the special case of n = 5, Abbot [A1] showed that $r_4^*(C_5) = 18$. Since it is known that $r_1(C_5) = 5$ and $r_2(C_5) = 9$, repeated application of theorem 9.22 yields a slight improvement on a result of Abbot [A1].

Corollary 9.24

For integers $k \geq 1$,

$$r_k(C_5) \geq \begin{cases} 4.17^{\frac{1}{4}(k-1)} + 1 & \text{if } k \equiv 1 \pmod 4 \\ 8.17^{\frac{1}{4}(k-2)} + 1 & \text{if } k \equiv 2 \pmod 4 \\ 16.17^{\frac{1}{4}(k-3)} + 1 & \text{if } k \equiv 3 \pmod 4 \\ 32.17^{\frac{1}{4}(k-4)} + 1 & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

Bondy and Erdos [B10] gave an upper bound on $r_k(C_n)$ of $(k + 2)!n$ for n odd. This was slightly improved to $r_k(C_n) \leq (k + 2)!(n - 1)$ by Erdos and Graham [E4], who also gave the bound $r_k(C_n) \leq ck^3(n - 1)\ [r_k(C_3)]^2$ for some constant c.

For even circuits, the construction used in the proof of theorem 9.20 can be generalised as follows:

Theorem 9.25

For any even integer $n > 2$, let G_1 and G_2 be $\overline{MC}_n$-graphs containing no red edges and such that G_2 has order at most $\frac{1}{2}n - 1$. The join in red of G_1 and G_2 is an $\overline{MC}_n$-graph.

Proof

G is clearly complete, so it remains to prove that it contains no monochromatic C_n. If such a circuit exists, it must be red since otherwise it would be contained in G_1 or G_2. But the red subgraph of G is bipartite, and any circuit of length n must contain $\frac{1}{2}n$ vertices from each half of the bipartition. Since the order of G_2 is at most $\frac{1}{2}n - 1$, this is impossible.

Theorem 9.26

For any integers $k \geq 1$, $n \geq 4$, n even,

$$r_k(C_n) \geq \tfrac{1}{2}(k + 1)(n - 2) + 2$$

Proof

By induction on k. The theorem is trivially true for $k = 1$, and is true for $k = 2$ by theorem 9.20, so suppose it true for $k < K$, $K \geq 3$. Then by the induction assumption there exists a $(K-1)$-edge-coloured $\overline{\overline{MC}}_n$-graph G_1 of order $\frac{1}{2}K(n - 2) + 1$ containing blue edges but no red edges. If G_2 is a blue complete graph of order $\frac{1}{2}n - 1$, then by theorem 9.25 there exists a K-edge-coloured $\overline{\overline{MC}}_n$-graph of order $\frac{1}{2}K(n - 2) + 1 + \frac{1}{2}n - 1 = \frac{1}{2}(K + 1)(n - 2) + 1$.

Theorem 9.27 slightly improves a result of Erdos and Graham [E4], which stated that $r_k(C_n) \geq \frac{1}{2}(k - 1)(n - 2) + 1$.

In the special case of $n = 4$, very good results have been obtained independently by Irving [I1] and Chung and Graham [C14].

Theorem 9.27 (Irving; Chung and Graham)

For $k > 1$,

$$r_k(C_4) \leq k^2 + k + 1$$

and for $k - 1$ a prime power

$$r_k(C_4) \geq k^2 - k + 2$$

REFERENCES

[A1] Abbot, H. L.: 'On Ramsey Theory for Circuits', J. Combinatorial Theory (B) 17 (1974), 65-68.

[A2] Abbot, H. L., Hanson, D.: 'A Problem of Schur and its Generalisations', Acta Arith. 20 (1972), 175-187.

[A3] Ae, T., Yamamoto, S., Yoshida, N.: 'Line-disjoint Decomposition of Complete Graphs into Stars', J. Combinatorial Theory, (to appear).

[B1] Bankfalvi, M., Bankfalvi, Z.: 'Alternating Hamiltonian Circuits in Complete Graphs', Theory of Graphs, Proc. Colloq. at Tihany (1966), pp. 11-18, Academic Press (New York-London), (1968).

[B2] Behzad, M., Chartrand, G., Lesniak-Foster, L.: 'Graphs and Digraphs', Prindle, Weber, and Schmidt (Boston, Massachusetts), (1979).

[B3] Beineke, L.: 'Decomposition of the Complete Graph into Forests', Magyar Tud. Akad. Mat. Kutato Int. Kozl. 9 (1964), 589-594.

[B4] Beineke, L., Wilson, R. J.: 'Selected Topics in Graph Theory', Academic Press (London), 1978.

[B5] Bermond, J. -C., Schonheim, J.: 'G-Decomposition of K_n where G has 4 Vertices or Less', Disc. Math. 19 (1977), 113-120.

[B6] Bermond, J. -C., Sotteau, D.: 'Graph Decompositions and G-Designs', Proc. 5th British Combinatorial Conf. (Univ. Aberdeen, 1975), pp. 53-72, Congressus Numerantium XV, Utilitas Math. (Winnipeg, Man.), 1976.

[B7] Beth, T., Sprague, A. P.: 'Trees of British Number Systems are Graceful', Arch. Math. (Basel) 33 (1979/80), 383-391.

[B8] Bloom, G.: 'A Chronology of the Ringel-Kotzig Conjecture and the Continuing Quest to Call All Trees Graceful', Topics in Graph Theory (New York, 1977), pp. 32-51, Ann. New York Acad. Sci. 328, New York Acad. Sci. (New York), 1978.

[B9] Bollobas, B., Erdos, P.: 'Alternating Hamiltonian Cycles', Israel J. Math. 23 (1976), 126-131.

[B10] Bondy, J. A., Erdos, P.: 'Ramsey Numbers for Cycles in Graphs', J. Combinatorial Theory (B) 14 (1973), 46-54.

[B11] Bondy, J. A., Murty, U. S. R.: 'Graph Theory with Applications', Macmillan (London), 1976.

[B12] Bostwick, C. W.: 'Problem E1321, Amer. Math. Monthly 65 (1958), 446, 66 (1959), 141-142.

[B13] Burr, S. A.: 'Generalised Ramsey Theory for Graphs - A Survey', Graphs and Combinatorics (Proc. Capital Conf. on Graph Theory and Combinatorics, George Washington Univ., 1973), pp. 52-75, Lecture Notes in Math. 406, Springer-Verlag (Berlin), 1974.

[B14] Burr, S. A.: 'A Survey of Non-Complete Ramsey Theory for Graphs', Topics in Graph Theory (New York), pp. 58-75, Ann. New York Acad. Sci. 328, New York Acad. Sci. (New York), 1979.

[B15] Bush, L. E.: 'The William Lowell Putnam Mathematical Competition', Amer. Math. Monthly 60 (1953), 539-542.

[B16] Busolini, D. T.: 'On a Problem of C. C. Chen and D. E. Daykin', Disc. Math. 24 (1978), 286-289.

[B17] Busolini, D. T.: '2-Coloured Triangles in Edge-Coloured Complete Graphs', Disc. Math. 24 (1978), 365-366.

[C1] Cahit, I.: 'Are All Complete Binary Trees Graceful?', Amer. Math. Monthly 83 (1976), 35-37.

[C2] Cahit, I., Cahit, R.: 'On the Graceful Numbering of Spanning Trees', Info. Proc. Lett. 3 (1974/75), 115-118.

[C3] Cain, P.: 'Decomposition of Complete Graphs in to Stars', Bull. Austral. Math. Soc. 10 (1974), 23-30.

[C4] Chartrand, G., Schuster, S.: 'On the Existence of Specified Cycles in Complementary Graphs', Bull. Amer. Math. Soc. 77 (1971), 995-998.

[C5] Chartrand, G., Schuster, S.: 'On a Variation of the Ramsey Number', Trans. Amer. Math. Soc. 173 (1972), 353-362.

[C6] Chartrand, G., Schuster, S.: 'A Note on Cycle Ramsey Numbers', Disc. Math. 5 (1973), 313-315.

[C7] Chen, C. C.: 'On the Structure of Complete Graphs Without Alternating Cycles', Disc. Math. 28 (1979), 1-5.

[C8] Chen, C. C., Daykin, D. E.: 'Graphs with Hamiltonian Cycles Having Adjacent Lines Different Colours', J. Combinatorial Theory (B) 21 (1976), 135-139.

[C9] Chen, C. C., Daykin, D. E.: Problem 6157, Amer. Math. Monthly 84 (1977), 491.

[C10] Chen, C. C., Daykin, D. E., Erdos, P.: 'Subgraphs with All Colours in a Line-Coloured Graph', Proc. 5th British Combinatorial Conf. (Univ. Aberdeen, Aberdeen 1975), pp. 101-112, Congressus Numerantium XV, Utilitas Math. (Winnipeg, Man.), 1976.

[C11] Chung, F. R. K.: 'On the Ramsey Numbers N(3,3,..,3;2)', Disc. Math. 5 (1973), 317-321.

[C12] Chung, F. R. K.: 'On Triangular and Cyclic Ramsey Numbers with k Colours', Graphs and Combinatorics (Proc. Capital Conf. on Graph Theory and Combinatorics, George Washington Univ., 1973), pp. 236-242, Lecture Notes in Math. 406, Springer-Verlag (Berlin), 1974.

[C13] Chung, F. R. K.: Ph.D. Thesis, Univ. Pennsylvania, 1974.

[C14] Chung, F. R. K., Graham, R. L.: 'On Multicolour Ramsey Theorems for Complete Bipartite Graphs', J. Combinatorial Theory (B) 18 (1975), 164-169.

[D1] Daykin, D. E.: 'Graphs with Cycles having Adjacent Lines Different Colours', J. Combinatorial Theory (B) 20 (1976), 149-152.

[D2] Dembowski, P.: 'Finite Geometries', Springer-Verlag (New York), 1968.

[D3] Denes, J., Keedwell, A. D.: 'Latin Squares and their Applications', Academic Press (New York-London), 1974.

[D4] Duke, R. A.: 'Can K_{2n+1} be Packed with Copies of an Arbitrary Tree with n Edges?', Amer. Math. Monthly 76 (1969), 1128-1130.

[E1] Erdos, P.: 'Some Remarks on the Theory of Graphs', Bull. Amer. Math. Soc. 53 (1947), 292-294.

[E2] Erdos, P.: 'A Solved and Unsolved Graph Colouring Problem', Problem 72-6*, SIAM Review 14 (1972), 364.

[E3] Erdos, P., Faudree, R. J., Rousseau, C. C., Schelp, R. H.: 'Generalised Ramsey Theory for Multiple Colours', J. Combinatorial Theory (B) 20 (1976), 250-264.

[E4] Erdos, P., Graham, R. L.: 'On Partition Theorems for Finite Graphs', Infinite and Finite Sets (Colloq. Keszthely, Hungary, 1973) vol. I, pp. 515-527, Colloq. Math. Soc. Janos Bolyai 10, North-Holland (Amsterdam), 1975.

[E5] Erdos, P., Schonheim, J.: 'Edge Decompositions of the Complete Graph into Copies of a Connected Graph', Proc. Conf. on Algebraic Aspects of Combinatorics (Univ. Toronto, Toronto, 1975), pp. 271-278, Congressus Numerantium XIII, Utilitas Math. (Winnipeg, Man.), 1975.

[E6] Erdos, P., Simonovits, M., Sos, V. T.: 'Anti-Ramsey Theorems', Infinite and Finite Sets (Colloq. Keszthely, Hungary, 1973) vol. II, pp. 633-643, Colloq. Math. Soc. Janos Bolyai 10, North-Holland (Amsterdam), 1975.

[F1] Faudree, R. J., Schelp, R. H.: 'All Ramsey Numbers for Cycles in Graphs', Disc. Math. 8 (1974), 313-329.

[F2] Faudree, R. J., Schelp, R. H.: 'Path Ramsey Numbers in Multicolourings', J. Combinatorial Theory (B) 19 (1975), 150-160.

[F3] Faudree, R. J., Schelp, R. H.: 'Some Problems in Ramsey Theory', Theory and Applications of Graphs (Proc. Internat. Conf., W. Michigan Univ., Kalamazoo, Mich. 1976), pp. 160-173, Lecture Notes in Math. 642, Springer (Berlin), 1978.

[F4] Fiorini, S., Wilson, R. J.: 'Edge-Colourings of Graphs', Pitman (London), 1977.

[F5] Folkman, J.: 'A Note on the Ramsey Number N(3,3,3,3)', J. Combinatorial Theory (A) 16 (1974), 371-379.

[F6] Fredricksen, H.: 'Schur Numbers and the Ramsey Numbers N(3,3,..,3;2)', J. Combinatorial Theory (A) 27 (1979), 376-377.

[G1] Galvin, F.: Problem 6034, Amer. Math. Monthly 82 (1975), 592.

[G2] Golomb, S.: 'How to Number a Graph', Graph Theory and Computing, pp. 23-27, Academic Press (New York), 1972.

[G3] Greenwood, R. E., Gleason, A. M.: 'Combinatorial Relations and Chromatic Graphs', Canad. J. Math. 7 (1955), 1-7.

[G4] Guy, R. K.: Research Problems, Amer. Math. Monthly 80 (1973), 1120-1128.

[G5] Guy, R. K.: Research Problems, Amer. Math. Monthly 82 (1975), 995-1004.

[G6] Guy, R. K.: Research Problems, Amer. Math. Monthly 84 (1977), 807-815.

[G7] Guy, R. K.: Research Problems, Amer. Math. Monthly 86 (1979), 847-852.

[G8] Guy, R. K., Klee, V.: Research Problems, Amer. Math. Monthly 78 (1971), 1113-1122.

[H1] Haggard, G., McWha, P.: 'Decomposition of Complete Graphs into Trees', Czech. Math. J. 25(104) (1975), 31-36.

[H2] Hahn, G.: 'Some Star Anti-Ramsey Numbers', Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory, and Computing (Louisiana State Univ., Baton Rouge, La. 1977), pp. 303-310, Congressus Numerantium XIX, Utilitas Math. (Winnipeg, Man.), 1977.

[H3] Hall, M.: 'Combinatorial Theory', Blaisdell (Waltham, Mass.), 1967.

[H4] Harary, F.: 'A Survey of Generalised Ramsey Theory', Graphs and Combinatorics (Proc. Capital Conf. on Graph Theory and Combinatorics, George Washington Univ. 1973), pp. 10-17, Lecture Notes in Math. 406, Springer-Verlag (Berlin), 1974.

[H5] Heinrich, K.: 'A Non-Imbeddable Proper Colouring', Combinatorial Mathematics IV (Proc. 4th Austral. Conf., Univ. Adelaide, Adelaide 1975), pp. 93-115, Lecture Notes in Math. 560, Springer (Berlin), 1976.

[H6] Heinrich, K.: 'Proper Colourings of K_{15}', J. Austral. Math. Soc. Ser. A 24 (1977), 465-495.

[H7] Hill, R., Irving, R. W.: 'On Group Partitions Associated with Lower Bound for Symmetric Ramsey Numbers', unpublished.

[H8] Hilton, A. J.: 'Embedding Incomplete Latin Rectangles and Extending the Edge-Colourings of Graphs', Nanta Math. 10 (1977), 201-206.

[H9] Hogarth, P.: 'Decomposition of Complete Graphs into 6-Stars and 10-Stars', Combinatorial Mathematics III (Proc. 3rd Austral. Conf. on Comb. Math., Univ. Queensland, 1974), pp. 136-142, Lecture Notes in Math. 452, Springer-Verlag (Berlin), 1975.

[H10] Huang, C.: 'Decomposition of Complete Graphs into Stars', unpublished.

[H11] Huang, C., Rosa, A.: 'Decomposition of Complete Graphs into Trees', Ars Comb. 5 (1978), 23-63.

[H12] Hughes, D. R., Piper, F. C.: 'Projective Planes', Springer-Verlag (New York), 1973.

[I1] Irving, R. W.: 'Generalised Ramsey Theory for Small Graphs', Disc. Math. 9 (1974), 251-264.

[K1] Kalbfleisch, J. G.: 'Chromatic Graphs and Ramseys Theorem', Ph.D. Thesis, Univ. Waterloo, 1966.

[K2] Kalbfleisch, J. G., Stanton, R. G.: 'Maximal Triangle-Free Edge-Chromatic Graphs in Three Colours', J. Combinatorial Theory 5 (1968), 9-20.

[K3] Koh, K. M., Lim, C. K., Rogers, D. G.: 'Graceful Graphs: I Sum of Graphs', S. E. Asian Bull. Math. 3 (1979), 58.

[K4] Koh, K. M., Rogers, D. G., Tan, T.: 'On Graceful Trees', Nanta Math. 10 (1977), 207-211.

[K5] Koh, K. M., Rogers, D. G., Tan. T.: 'Interlaced Trees: A Class of Graceful Trees', Combinatorial Mathematics VI (Proc. 6th Conf. on Combinatorial Math., Univ. New England 1978), pp. 65-78, Lecture Notes in Math. 748, Springer-Verlag (Berlin), 1979.

[K6] Koh, K. M., Rogers, D. G., Tan, T.: 'Two Theorems on Graceful Graphs', Disc. Math. 25 (1979), 141-148.

[K7] Koh, K. M., Rogers, D. G., Tan, T.: 'Products of Graceful Trees', Disc. Math. 31 (1980), 279-292.

[K8] Kotzig, A.: 'On Certain Vertex-Valuations of Finite Graphs', Utilitas Math. 4 (1973), 261-290.

[O1] Owens, F. W.: 'Complete d-nary Trees are Graceful', Notices Amer. Math. Soc. 23 (1976), A-354.

[R1] Ramsey, F. P.: 'On a Problem of Formal Logic', Proc. London Math. Soc. 30 (1930), 264-286.

[R2] Ringel, G.: Problem 25, Theory of Graphs and its Applications, (Proc. Symp., Smolenice 1963), Czech. Acad. Sci. (Prague), 1964.

[R3] Rogers, D. G.: 'A Graceful Algorithm', S. E. Asian Bull. Math. 2 (1978), 42-44.

[R4] Rosa, A.: 'On Certain Valuations of the Vertices of a Graph', Theory of Graphs (Proc. Internat. Symp. Rome 1966), pp. 349-355, Gordon and Brench (New York), 1967.

[R5] Rosta, V.: 'On a Ramsey Type Problem of J. A. Bondy and P. Erdos I', J. Combinatorial Theory (B) 15 (1973), 94-104.

[R6] Rosta, V.: 'On a Ramsey Type Problem of J. A. Bondy and P. Erdos II', J. Combinatorial Theory (B) 15 (1973), 105-120.

[S1] Schonheim, J., Bialostocki, A.: 'Decomposition of K_n into Subgraphs of Prescribed Type', Arch. Math. (Basel) 31 (1978), 105-112.

[S2] Schuster, S.: 'Progress on the Problem of Eccentric Hosts', Graph Theory and Applications (Proc. Conf. at W. Michigan Univ. 1972), pp. 283-290, Lecture Notes on Math. 303, Springer (Berlin), 1973.

[S3] Schwenk, A.: Solution to Problem 6034 (Amer. Math. Monthly 82 (1975), 592, Amer. Math. Monthly 84 (1977), 224.

[S4] Shearer, J.: 'A Property of the Coloured Complete Graph', Disc. Math. 25 (1979), 175-178.

[S5] Stanton, R. G., Zarnke, C. R.: 'Labelling of Balanced Trees', Proc. 4th Southeastern Conf. on Combinatorics, Graph Theory, and Computing (Boca Raton, Fa. 1973), pp. 479-495, Congressus Numerantium VIII, Utilitas Math. (Winnipeg, Man.), 1973.

[S6] Street, A. P.: 'Embedding Proper Colourings', Combinatorial Mathematics IV (Proc. 4th Austral. Conf., Univ. Adelaide, 1975), pp. 240-245, Lecture Notes in Math. 560, Springer (Berlin), 1976.

[W1] Wallis, W. D.: 'On 1-Factorisations of Complete Graphs', J. Austral. Math. Soc. 16 (1973), 167-171.

[W2] Wallis, W. D., Street, A. P., Wallis, J. S.: 'Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices', Lecture Notes in Math. 292, Springer-Verlag (Berlin-New York), 1972.

[W3] Whitehead, E. G.: 'Ramsey Numbers of the Form N(3,3,..,3;2)', Ph.D. Thesis, Univ. S. California, 1971.

[W4] Whitehead, E. G.: 'Algebraic Structure of Chromatic Graphs Associated with N(3,3,3;2)', Disc. Math. 1 (1971/72), 113-114.

[W5] Whitehead, E. G.: 'The Ramsey Number N(3,3,3,3;2)', Disc. Math. 4 (1973), 389-396.

[W6] Wilson, R. M.: 'Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph', Proc. 5th British Combinatorial Conf. (Univ. Aberdeen 1975), pp. 647-659, Congressus Numerantium XV, Utilitas Math. (Winnipeg, Man.), 1976.

[Y1] Yamamoto, S., Ikeda, H., Shige-Eda, S., Ushio, K., Hamada, N.: 'On Claw-Decomposition of Complete Graphs and Complete Bigraphs', Hiroshima Math. J. 5 (1975), 33-42.