CONTRIBUTIONS TO THE THEORY

OF CORRESPONDENCES AND HYPERSPACES

submitted by

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ABSTRACT

The object of this thesis is to study topological properties of correspondences, or set-valued mappings, and hyperspaces, i.e. spaces of subsets of given sets.

In chapter 1, we study correspondences under their purely set-theoretical aspect. We introduce canonical extensions and stress their usefulness in proving properties of correspondences.

Chapter 2 is devoted to topological structures on hyperspaces, including the finite topology and quasi-uniformities. Chapter 3 pursues that study. The unifying role of quasi-uniformities is discussed: not only do they often yield a common proof of different results, they also give way to new ones.

Finally, chapter 4 deals with so-called "covering topologies", such as the locally finite topology. We study their properties and compare them with the uniform and the finite topologies.

CONTENTS

I N T R O D U C T I O N

This thesis is a contribution to the theory of correspondences (set-valued mappings) and hyperspaces (spaces of subsets), with E. Michael's paper [8] and C. Berge's book [1] as general background.

We start with a study of the purely set-theoretical properties of correspondences. Our aim here is to present a detailed and unified treatment of a topic which has been mostly neglected or dealt with in a rather unsatisfactory manner in the literature available at present. Thus, we make a careful distinction between correspondences and what we call their canonical extensions. We try to define operations on correspondences in the most natural way possible. Our treatment of inverses, besides forming a common background for upper and lower inverses, enables us to give a sound presentation of results on the surjectivity and injectivity of correspondences.

We then deal with the finite topology and continuous correspondences, again underlining the role played by canonical extensions. This leads us to the concept of continuity at a set, rather than merely at a point, which in turn enables us to spread new light on topological results concerning correspondences, such as the continuity of composites. We also generalize one of the most useful properties of the finite topology, which states that "a compact union of compact sets is compact".

In the same vein, we study the hyperspace of a quasi-uniform space (i.e. a uniform space in which the symmetry axiom does not necessarily hold), and analyse the relations between topologies and quasi-uniformities in hyperspaces.

In chapter 3, we study topological properties of correspondences. One of the main thoughts here consists in trying to see how quasi-uniformities can be used to give a more or less unified picture of the properties considered. This investigation rests on the fact that every topological space is quasi-uniformizable; indeed, quasi-uniformities can be constructed which are compatible with the formation of quasi-uniformities in hyperspaces.

In the same spirit, we also study results on the continuity of the supremum of a real-valued function. After treating closed graphs, we prove two general supremum theorems, which then yield corresponding results on continuous selections.

In his review of E. Michael's paper [8] in Mathematical Reviews, vol. 13, 1952, p. 54, J.L. Kelley writes: "The finite topology (...) is used almost exclusively. (This may be a serious shortcoming; for example, a topology based on locally finite coverings seems to offer advantages.)" The object of our last chapter is to consider such "covering topologies". Besides comparing them with the uniform and the finite topologies, we study their separation properties, in particular from the point of view of bitopological spaces.

Each chapter is divided into sections. The theorems, corollaries and lemmas which compose the text are numbered serially in a single system that proceeds by chapters. For example, 3.4 refers to the fourth numbered item in the third chapter.

Finally, many thanks are due to our supervisors, Prof. A.P. Robertson and Mr. P.R. Baxandall, for their advice and their encouragement.

Chapter 1

SET THEORY OF CORRESPONDENCES

This chapter deals with the set-theoretical properties of correspondences. Its aim is to give an original presentation of general results, many of which will be used in later chapters.

In the literature, correspondences are also referred to as set-valued or multi-valued mappings. A correspondence between a set X and a set Y is usually defined to be a (single-valued) mapping of X into the set $\mathcal{P}(Y)$ of all subsets of Y.

Here, we prefer to adopt a slightly different point of view. Following N. Bourbaki in [2], we define a correspondence between X and Y to be a triple $R = (G,X,Y)$, where G is a subset of $X \times Y$, called the graph of R. This has the formal advantage that a function can then be viewed as a special sort of correspondence. It also enables us to make a definition for the composite of two correspondences which is consistent with that adopted for functions, whereas it does not make sense to compose a mapping $f: X \longrightarrow \mathcal{P}(Y)$ with a mapping $g: Y \longrightarrow \mathcal{P}(Z)$.

Of course, there is a very natural way to associate a mapping $X \longrightarrow \mathcal{P}(Y)$ to each correspondence R between X and Y. But we shall always distinguish between that mapping and R itself. From R, we can also deduce a mapping, which we denote by $\hat{R}$, of $\mathcal{P}(X)$ into $\mathcal{P}(Y)$: for each $A \in \mathcal{P}(X)$, we let

$\hat{R}(A)$ be the set of all elements of Y that correspond to elements of A under R. The relation between R and $\hat{R}$, on which the whole development of this chapter rests, turns out to be very useful in deducing properties of correspondences from similar properties of functions (cf. chapter 3). For instance, it enables us to prove very easily a result on the continuity of composites (2.6).

Finally, let us note here that the axiom of choice is assumed throughout.

§1. THE CONCEPT OF A CORRESPONDENCE

A correspondence between a set X and a set Y is defined to be a triple $R = (G,X,Y)$, where G is a subset of $X \times Y$, called the graph of R.

If $(x,y) \in G$, we say that y corresponds to x under R. We call X the source and Y the target of R. Instead of saying "let R be a correspondence between X and Y", we shall often say "let $R: X|Y$ be a correspondence", or simply "let $R: X|Y$".

We shall consider functions within this more general concept of a correspondence. Thus, a correspondence $R = (G,X,Y)$ is a function iff, for each $x \in X$, there exists a unique $y \in Y$ which corresponds to x under R.

If $G = \emptyset$, we say that $R = (\emptyset,X,Y)$ is the empty correspondence between X and Y.

If $A \subset X$, the set of elements of Y which correspond to elements of A under R is called the image of A under R and is denoted by R(A). If $x \in X$, the set $R(\{x\})$, which we also denote by R(x) or simply Rx, is called the section of R along x. Thus, $R(x) = \{y \in Y: (x,y) \in G\}$ for each $x \in X$. Moreover, $R(A) = \bigcup_{x \in A} R(x)$ for each $A \subset X$. Clearly, $R(\emptyset) = \emptyset$.

The above definition and notation for the image of a set under a correspondence are consistent with those usually adopted for functions. The notation $R(x)$ for the section of R along x, however, can lead to confusion if R happens to be a function. For then, $R(x)$ usually designates the unique element of Y which corresponds to x under R, i.e. the value of R at x. The section of R along x is then $\{R(x)\}$. Fortunately, this abuse of notation never leads to serious difficulties, as the sequel will show.

The graph of a correspondence $R: X|Y$ is a subset of $X \times Y$. The set of all triples $R = (G,X,Y)$ with $G \in \mathfrak{P}(X \times Y)$ is thus the set of all correspondences between X and Y; we denote it by $\Gamma(X,Y)$. It is not empty, since it contains the empty correspondence between X and Y.

Clearly, the mapping $G \mapsto (G,X,Y)$ is a bijection (said to be canonical) of $\mathfrak{P}(X \times Y)$ onto $\Gamma(X,Y)$. The existence of this bijection enables us, for instance, to translate immediately every proposition relating to the set $\mathfrak{P}(X \times Y)$ into a proposition relating to $\Gamma(X,Y)$, and vice-versa. The situation is the same as with functions, where we have a canonical bijection $G \mapsto (G,X,Y)$ of the set Y^X of all graphs of mappings of X into Y onto the set $\mathfrak{F}(X,Y)$ of all mappings of X into Y.

If $X = \emptyset$ or $Y = \emptyset$, then $X \times Y = \emptyset$ and so $\mathfrak{P}(X \times Y) = \{\emptyset\}$, so that $\Gamma(X,Y)$ has exactly one element, the empty correspondence between X and Y.

Before we go on to discuss canonical extensions, we find it desirable to introduce some terminology concerning functions between power sets. If X and Y are two sets and $f: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ is a mapping, the complement of f is defined to be the mapping $A \mapsto Y \setminus f(X \setminus A)$ of $\mathfrak{P}(X)$ into $\mathfrak{P}(Y)$, and is denoted by f^c. Clearly $(f^c)^c = f$. It follows that if $g: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ is another mapping, then the relations $g = f^c$ and $f = g^c$ are equivalent; if they hold, we say that f and g are complementary. Finally, if $f: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ and $g: \mathfrak{P}(Y) \to \mathfrak{P}(Z)$ are mappings, then $(g \circ f)^c = g^c \circ f^c$.

We say that $f: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ preserves unions if $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$ for each family $(A_i)_{i \in I}$ of subsets of X.

Adopting an analogous definition for intersections, it is clear that f preserves unions iff its complement f^c preserves intersections. If f preserves unions (resp. intersections), then f also preserves inclusions, in the sense that the relation $A \subset B$ implies $f(A) \subset f(B)$ for each A and B in $\mathcal{P}(X)$. If we want to prove that f preserves unions, it suffices to show that $f(A) = \bigcup_{x \in A} f(\{x\})$ for each subset A of X. For then, given any family $(A_i)_{i \in I}$ of subsets of X, we have, putting $A = \bigcup_{i \in I} A_i$,

$$f(A) = \bigcup_{x \in A} f(\{x\}) = \bigcup_{i \in I} \bigcup_{x \in A_i} f(\{x\}) = \bigcup_{i \in I} f(A_i).$$

Similarly, in order to prove that two union-preserving mappings $f: \mathcal{P}(X) \to \mathcal{P}(Y)$ and $g: \mathcal{P}(X) \to \mathcal{P}(Y)$ are equal, it is enough to show that $f(\{x\}) = g(\{x\})$ for each $x \in X$.

Canonical extensions

The canonical extension of a correspondence $R: X|Y$ to the power sets is defined to be the mapping $A \mapsto R(A)$ of $\mathcal{P}(X)$ into $\mathcal{P}(Y)$, and is denoted by $\hat{R}$. The restricted canonical extension of R is defined to be the mapping $x \mapsto R(x)$ of X into $\mathcal{P}(Y)$, and is denoted by $\dot{R}$.

If X is a set, then we shall always denote the canonical injection $x \mapsto \{x\}$ of X into $\mathcal{P}(X)$ by j_X. Using this notation, we see that $\dot{R} = \hat{R} \circ j_X$ for each $R: X|Y$.

The identity mapping of a set X onto itself will be denoted by Id_X. We have $\widehat{Id_X} = Id_{\mathcal{P}(X)}$, i.e. the canonical extension of the identity mapping of X is equal to the identity mapping of $\mathcal{P}(X)$.

If $R: X|Y$, then $\hat{R}: \mathcal{P}(X) \to \mathcal{P}(Y)$ preserves unions, since $R(A) = \bigcup_{x \in A} R(x)$ for each $A \in \mathcal{P}(X)$. Thus, not every mapping $f: \mathcal{P}(X) \to \mathcal{P}(Y)$ is the canonical extension of a correspondence between X and Y (cf. 1.1).

We now consider an important example of the canonical extension of a correspondence. Let X be a set, and define a correspondence $R = (G, X, \mathcal{P}(X))$ between X and $\mathcal{P}(X)$ by setting $G = \{(x,E) \in X \times \mathcal{P}(X) : x \in E\}$. The <u>star mapping</u> of X is defined to be the canonical extension $\hat{R}: \mathcal{P}(X) \longrightarrow \mathcal{P}(\mathcal{P}(X))$ of R, and is denoted by $\mathcal{S}_X$. Thus, $\mathcal{S}_X$ preserves unions. For each $A \in \mathcal{P}(X)$, the set $\mathcal{S}_X(A)$ (or simply $\mathcal{S}(A)$, when no confusion can arise) is called the <u>star</u> of A in X. Since $R(x) = \{E \in \mathcal{P}(X) : x \in E\}$ for each $x \in X$, we have $R(A) = \bigcup_{x \in A} R(x) = \{E \in \mathcal{P}(X) : E \cap A \neq \emptyset\}$ for each $A \in \mathcal{P}(X)$,

i.e. $$\mathcal{S}_X(A) = \{E \in \mathcal{P}(X) : E \cap A \neq \emptyset\}.$$

The <u>power set mapping</u> of X is defined to be the mapping $A \longmapsto \mathcal{P}(A)$ of $\mathcal{P}(X)$ into $\mathcal{P}(\mathcal{P}(X))$, and is denoted by $\mathcal{P}_X$. Since $\mathcal{P}(X) \setminus \mathcal{P}(A) = \{E \in \mathcal{P}(X) : E \cap (X \setminus A) \neq \emptyset\} = \mathcal{S}_X(X \setminus A)$ for each $A \in \mathcal{P}(X)$, the mappings $\mathcal{S}_X$ and $\mathcal{P}_X$ are complementary (in particular, it follows that $\mathcal{P}_X$ preserves intersections).

Finally, note that $\mathcal{S}_X(X) = \mathcal{P}_o(X)$, the set of all nonempty subsets of X.

The following concept of refinement will help us in clarifying the relation between a correspondence and its canonical extension. We shall also make use of it in the next section, where we characterize surjective and injective correspondences.

If $R = (G,X,Y)$ and $S = (H,X,Y)$ are two correspondences between X and Y, we say that R <u>refines</u> S, and we write $R \lhd S$, if $G \subset H$. Here, we make use of the canonical bijection of $\mathcal{P}(X \times Y)$ onto $\Gamma(X,Y)$ introduced above, and simply carry the order structure of the set $\mathcal{P}(X \times Y)$ (given by the relation "$G \subset H$") over to $\Gamma(X,Y)$, obtaining an order relation "$R \lhd S$". If the mapping $f: X \longrightarrow Y$ refines R, then we also say that f is a <u>selection</u> of R. This means that $f(x) \in R(x)$ for each $x \in X$. Thus, there exists a bijection of the set of all selections of R onto the product $\prod_{x\in X} R(x)$.

More generally, we have $R \lhd S$ iff $R(x)\subset S(x)$ for each $x\in X$. This is also equivalent to the statement: $R(A)\subset S(A)$ for each $A\subset X$. It follows that

$$R = S \text{ iff } \dot{R} = \dot{S} \text{ iff } \hat{R} = \hat{S}.$$

Thus, the mapping $R\mapsto\dot{R}$ of $\Gamma(X,Y)$ into $\mathfrak{F}(X,\mathcal{P}(Y))$ is injective; similarly for the mapping $R\mapsto\hat{R}$ of $\Gamma(X,Y)$ into $\mathfrak{F}(\mathcal{P}(X),\mathcal{P}(Y))$.

But the mapping $R\mapsto\dot{R}$ is also surjective. Indeed, if $F: X\longrightarrow\mathcal{P}(Y)$ is any member of $\mathfrak{F}(X,\mathcal{P}(Y))$, then F is clearly the restricted canonical extension of the correspondence between X and Y whose graph is the set $\{(x,y)\in X\times Y: y\in F(x)\}$. Thus, the mapping $R\mapsto\dot{R}$ is a bijection (said to be canonical) of $\Gamma(X,Y)$ onto $\mathfrak{F}(X,\mathcal{P}(Y))$. Passing on to graphs, we obtain a bijection of the set $\mathcal{P}(X\times Y)$ onto $(\mathcal{P}(Y))^X$.

In general, however, the mapping $R\mapsto\hat{R}$ of $\Gamma(X,Y)$ into $\mathfrak{F}(\mathcal{P}(X),\mathcal{P}(Y))$ is not surjective. The following result gives an easy, but useful characterization of those mappings $f: \mathcal{P}(X)\longrightarrow\mathcal{P}(Y)$ which are canonical extensions of correspondences between X and Y.

<u>1.1</u> THEOREM.- In order that a mapping $f: \mathcal{P}(X)\longrightarrow\mathcal{P}(Y)$ should be the canonical extension of a correspondence between X and Y, it is necessary and sufficient that f preserves unions.

If such a correspondence exists, then it is unique.

Proof.- It is only left to show "sufficiency". So suppose that f preserves unions, and consider the mapping $f\circ j_X$ of X into $\mathcal{P}(Y)$: we know that there exists a correspondence $R: X|Y$ such that $\dot{R} = f\circ j_X$. Then, we have $\hat{R}(\{x\}) = \dot{R}(x) = f(\{x\})$ for each $x\in X$. Since $\hat{R}$ and f both preserve unions, it follows that $\hat{R} = f$, as required.

The canonical bijection $R\mapsto\dot{R}$ is very useful in defining correspondences. It means that, if we want to define a correspondence $R\colon X|Y$, we only have to specify what $R(x)$ shall be for each $x\in X$. Let us give an example of this: the restriction of $R\colon X|Y$ to $A\subset X$, denoted by $R|A$, is defined by the relation $(R|A)(x) = R(x)$ for each $x\in A$ (in agreement with the definition and notation usually adopted for functions). Note that $\overset{\bullet}{\overline{R|A}} = \dot{R}|A$ and $\widehat{R|A} = \hat{R}|\mathcal{P}(A)$.

§2. COMPOSITES AND INVERSES

Composites

Let $R\colon X|Y$ and $S\colon Y|Z$ be two correspondences. We call composite of R and S, and we denote by $S\circ R$, the correspondence between X and Z which is uniquely determined by the relation $(S\circ R)(x) = S(R(x))$ for each $x\in X$.

This definition agrees with the usual definition for the composite of two functions. For the canonical extension of composites, we have the following result:

1.2 THEOREM.- Let $R\colon X|Y$ and $S\colon Y|Z$. Then we have

$$\widehat{S\circ R} = \hat{S}\circ\hat{R} \quad\text{and}\quad \overset{\bullet}{\overline{S\circ R}} = \hat{S}\circ\dot{R}.$$

Proof.- If $A\subset X$, then

$$(S\circ R)(A) = \bigcup_{x\in A} S(R(x)) = \bigcup_{x\in A}\ \bigcup_{y\in R(x)} S(y)$$
$$= \bigcup_{y\in R(A)} S(y) = S(R(A)),$$

since $R(A) = \bigcup_{x\in A} R(x)$. This shows the first part of the theorem. Using this, we obtain $\overset{\bullet}{\overline{S\circ R}} = \widehat{S\circ R}\circ j_X = \hat{S}\circ\hat{R}\circ j_X = \hat{S}\circ\dot{R}$, as desired.

Theorem 1.2 is useful in deducing other properties of composites. For instance, let $R\colon X|Y$, $S\colon Y|Z$ and $T\colon Z|W$; then $T\circ(S\circ R) = (T\circ S)\circ R$ (associativity). Indeed, putting $P = T\circ(S\circ R)$ and $Q = (T\circ S)\circ R$, we have
$\hat{P} = \hat{T}\circ\widehat{S\circ R} = \hat{T}\circ(\hat{S}\circ\hat{R}) = (\hat{T}\circ\hat{S})\circ\hat{R} = \widehat{T\circ S}\circ\hat{R} = \hat{Q}$, hence $P = Q$.

Similarly, we have $R\circ Id_X = Id_Y\circ R = R$ for each $R: X|Y$ (identity). For $\widehat{R\circ Id_X} = \widehat{R}\circ\widehat{Id_X} = \widehat{R}\circ Id_{\gamma(X)} = \widehat{R}$ and

$$\widehat{Id_Y\circ R} = \widehat{Id_Y}\circ\widehat{R} = Id_{\gamma(Y)}\circ\widehat{R} = \widehat{R},$$

hence the result.

The following theorem, which relates refinement and composites, will be found quite useful in characterizing surjective and injective correspondences, in the last part of this section.

1.3 THEOREM.- Let $R,R'\in\Gamma(X,Y)$ and $S,S'\in\Gamma(Y,Z)$. If $R\prec R'$ and $S\prec S'$, then $S\circ R\prec S'\circ R'$.

Proof.- We show that $S(R(x))\subset S'(R'(x))$ for each $x\in X$. Now, since $R\prec R'$, we have $R(x)\subset R'(x)$, hence $S(R(x))\subset S(R'(x))$. Using the fact that $S\prec S'$, we now obtain $S(R'(x))\subset S'(R'(x))$, hence the result.

Inverses

Let $R: X|Y$ be a correspondence. We call inverse correspondence (or simply inverse) of R, and we denote by R^{-1}, the correspondence between Y and X which is uniquely determined by the relation $R^{-1}(y) = \{x\in X:\ y\in R(x)\}$ for each $y\in Y$.

If $B\subset Y$, we call $R^{-1}(B)$ the inverse image of B under R.

This definition agrees with the usual definition for the inverse of a bijective function. Note however that the inverse correspondence is always defined, although it need not be a function. Also, the notation for the inverse image of a set under R agrees with the usual notation for the inverse image of a set under a function.

Clearly $(R^{-1})^{-1} = R$.

Closely related with this concept are the following two types of "inverses".

Let $R: X|Y$ be a correspondence. We call lower inverse of R, and we denote by R_*, the mapping

$$B \longmapsto \{x \in X:\ R(x) \cap B \neq \emptyset\} \quad \text{of } \mathcal{P}(Y) \text{ into } \mathcal{P}(X).$$

We call upper inverse of R, and we denote by R^*, the mapping

$$B \longmapsto \{x \in X:\ R(x) \subset B\} \quad \text{of } \mathcal{P}(Y) \text{ into } \mathcal{P}(X).$$

It is interesting to note that this definition yields a characterization of the concept of a function:

1.4 THEOREM.- In order that a correspondence $R: X|Y$ should be a function, it is necessary and sufficient that $R_* = R^*$.

Proof.- The condition is clearly necessary. Now suppose R is not a function. Thus, there exists an element $x \in X$ such that $R(x)$ does not have exactly one element.

If $R(x) = \emptyset$, then, considering the values of the functions R_* and R^* at the point $\emptyset \in \mathcal{P}(Y)$, we see that $R_*(\emptyset) = \emptyset$, whereas $x \in R^*(\emptyset)$; hence $R_* \neq R^*$.

If on the other hand $R(x) \neq \emptyset$, then we choose two distinct elements y, z in $R(x)$. Putting $B = Y \setminus \{z\}$, we have $x \in R_*(B) \setminus R^*(B)$. Thus, here again, $R_* \neq R^*$.

This shows that the condition stated is sufficient.

If $R: X|Y$ and $B \subset Y$, then

$$R^{-1}(B) = \bigcup_{y \in B} R^{-1}(y) = \{x \in X:\ R(x) \cap B \neq \emptyset\} = R_*(B),$$

so that $R_* = \widehat{R^{-1}}$; in particular, R_* preserves unions. Moreover, R_* and R^* are clearly complementary mappings, hence R^* preserves intersections.

Also note that $\hat{R} = \widehat{(R^{-1})^{-1}} = (R^{-1})_*$.

The following useful formulas relate the lower and upper inverses to the star and power set mappings introduced in §1:

1.5 THEOREM.- Let $R: X|Y$ be a correspondence. For each $B \subset Y$, we have:

a) $R_*(B) = \dot{R}^{-1}(\mathcal{J}(B))$ and $R^*(B) = \dot{R}^{-1}(\mathcal{P}(B))$.

b) $\mathcal{J}(R_*(B)) = \widehat{R}^{-1}(\mathcal{J}(B))$ and $\mathcal{P}(R^*(B)) = \widehat{R}^{-1}(\mathcal{P}(B))$.

Proof.- Indeed, we have

$$R_*(B) = \{x \in X\colon R(x) \cap B \neq \emptyset\} = \{x \in X\colon R(x) \in \mathcal{J}(B)\}$$

and $R^*(B) = \{x \in X\colon R(x) \subset B\} = \{x \in X\colon R(x) \in \mathcal{P}(B)\}$, which proves a).

To prove b), let A be an arbitrary element of $\mathcal{P}(X)$. Then we have

$$A \in \mathcal{J}(R_*(B)) \Longleftrightarrow A \cap R_*(B) \neq \emptyset \Longleftrightarrow R(A) \cap B \neq \emptyset \Longleftrightarrow R(A) \in \mathcal{J}(B)$$

and $A \in \mathcal{P}(R^*(B)) \Longleftrightarrow A \subset R^*(B) \Longleftrightarrow R(A) \subset B \Longleftrightarrow R(A) \in \mathcal{P}(B)$, hence the result.

We finally have the following result concerning the inverse of a composite:

1.6 THEOREM.- Let $R\colon X|Y$ and $S\colon Y|Z$ be correspondences. Then we have:

a) $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$.

b) $(S\circ R)_* = R_*\circ S_*$.

c) $(S\circ R)^* = R^*\circ S^*$.

Proof.- To prove b), let B be an arbitrary element of $\mathcal{P}(Y)$. Then we have, for each $x \in X$,

$$x \in (S\circ R)_*(B) \Longleftrightarrow S(R(x)) \cap B \neq \emptyset$$
$$\Longleftrightarrow R(x) \cap S_*(B) \neq \emptyset \Longleftrightarrow x \in R_*(S_*(B)),$$

hence the result. We then have

$\widehat{(S\circ R)^{-1}} = (S\circ R)_* = R_*\circ S_* = \widehat{R^{-1}}\circ\widehat{S^{-1}} = \widehat{R^{-1}\circ S^{-1}}$, which proves a). Part c) also follows from b), using the fact that R^* and R_* are complementary.

Surjectivity and injectivity

The following definition is a natural generalization of the notions of surjectivity, injectivity and bijectivity of functions.

Let $R\colon X|Y$ be a correspondence.
R is said to be surjective, if the family $(R(x))_{x\in X}$ is a

covering of Y.
R is said to be <u>injective</u>, if the sets of $(R(x))_{x \in X}$ are mutually disjoint.
R is said to be <u>bijective</u>, if it is both surjective and injective.

We now give characterizations for surjectivity and injectivity:

<u>1.7</u> THEOREM.- Let $R: X|Y$. Then the following statements are equivalent:

a) R is surjective.

b) The sections of R^{-1} are nonempty.

c) $Id_Y \lhd R \circ R^{-1}$.

Proof.- It is clear that a) and b) are equivalent. Now suppose that R is surjective. We claim that $y \in R(R^{-1}(y))$ for each $y \in Y$. Indeed, for each $y \in Y$ there exists an $x \in X$ such that $y \in R(x)$; but then $x \in R^{-1}(y)$ and so we have $y \in R(x) \subset R(R^{-1}(y))$, as desired. It follows that c) holds.

Finally, if $Id_Y \lhd R \circ R^{-1}$, then in particular $Y \subset R(R^{-1}(Y)) \subset R(X)$, hence R is surjective.

<u>1.8</u> THEOREM.- Let $R: X|Y$. Then the following statements are equivalent:

a) R is injective.

b) $R^{-1}|R(X)$ is a function.

c) $R^{-1} \circ R \lhd Id_X$.

Proof.- Again, a) and b) are clearly equivalent. Now suppose that R is injective. We claim that $R^{-1}(R(x)) \subset \{x\}$ for each $x \in X$. Indeed, for each $y \in R(x)$, the set $R^{-1}(y)$ has exactly one element; since $x \in R^{-1}(y)$, we must have $R^{-1}(y) = \{x\}$, so that

$$R^{-1}(R(x)) = \bigcup_{y \in R(x)} R^{-1}(y) \subset \{x\}, \text{ as desired.}$$

It follows that c) holds.

Finally, suppose that $R^{-1}\circ R \lhd Id_X$, and let $y \in R(X)$: we want to show that $R^{-1}(y)$ has exactly one element. Now there exists an $x \in X$ such that $y \in R(x)$; hence

$$x \in R^{-1}(y) \subset R^{-1}(R(x)) \subset \{x\},$$

showing that $R^{-1}(y) = \{x\}$. Hence R is injective.

Using the last two theorems, together with the equality $(R^{-1})^{-1} = R$ for a correspondence $R: X|Y$, we see that the relation $R\circ R^{-1} = Id_Y$ is true iff R is surjective and R^{-1} is injective; whenever this is the case, we shall say that R is <u>strongly surjective</u>.

Similarly, $R^{-1}\circ R = Id_X$ iff R is injective and R^{-1} is surjective; here, we shall say that R is <u>strongly injective</u>.

Note that, if R is bijective, it does not necessarily follow that R^{-1} is bijective. Indeed, R^{-1} need be neither surjective nor injective, as the following example shows: let X and Y each have at least two distinct elements. Choose an element $x_o \in X$, and let R be defined by $R(x_o) = Y$ and $R(x) = \emptyset$ for each $x \in X \setminus \{x_o\}$. Then R is clearly bijective, but R^{-1} is neither surjective nor injective.

If a correspondence R is a function, then R^{-1} is always bijective. Indeed, the sections of R are nonempty, hence R^{-1} is surjective. Also, $R|R^{-1}(Y)$ is a function, hence R^{-1} is injective. Thus, if R is a function, then R is (strongly) surjective iff $R\circ R^{-1} = Id_Y$, and R is (strongly) injective iff $R^{-1}\circ R = Id_X$.

As an application of the concept of strong surjectivity (resp. strong injectivity), we have the following result:

<u>1.9</u> THEOREM.- Let X_1, X_2, Y_1, Y_2 be four sets, and let

$S: X_1|X_2$, $T: Y_1|Y_2$ be two correspondences.

Finally, let F be the mapping $R \mapsto T\circ R\circ S^{-1}$

of $\Gamma(X_1,Y_1)$ into $\Gamma(X_2,Y_2)$.

If S and T are strongly surjective (resp. strongly injective), then F is surjective (resp. injective).

Proof.- Let $G: \Gamma(X_2,Y_2) \longrightarrow \Gamma(X_1,Y_1)$ be the mapping $R \mapsto T^{-1} \circ R \circ S$. If S and T are strongly surjective, then we have $F(G(R)) = F(T^{-1} \circ R \circ S) = (T \circ T^{-1}) \circ R \circ (S \circ S^{-1}) = R$, which shows that F is surjective. Similarly for injectivity.

If $R: X|Y$ and $S: Y|Z$ are surjective, then so is $S \circ R$. Indeed, we have $Id_Y \lhd R \circ R^{-1}$ and $Id_Z \lhd S \circ S^{-1}$, hence $Id_Z \lhd S \circ S^{-1} = S \circ Id_Y \circ S^{-1} \lhd S \circ (R \circ R^{-1}) \circ S^{-1} = (S \circ R) \circ (S \circ R)^{-1}$. Similarly for injectivity.

We finish this section with some considerations on the surjectivity and injectivity of canonical extensions. First of all, it is clear that the surjectivity of $\hat{R}$ implies that of R. But the corresponding result in the injective case is not true in general, even if it is assumed that the sections of R are nonempty. For example, let $X = \{1,2\}$, $Y = \{0,1,2\}$, and define $R: X|Y$ by putting $R(1) = \{0,1\}$ and $R(2) = \{0,2\}$. Then $\hat{R}$ is clearly injective, but R is not.

If $\dot{R}$ is surjective, then so is $\hat{R}$. If $\hat{R}$ is injective, then so is $\dot{R}$. But the converses of these two assertions are not true in general. Thus, if X has at least two distinct elements, and $R = Id_X$, then $\hat{R}$ is clearly surjective, since it is equal to $Id_{\mathcal{P}(X)}$. But $\dot{R}$ is not surjective, since it is equal to j_X. Again, let X have at least two distinct elements, choose an x_0 in X, and define $R: X|Y$ by putting $R(x_0) = X$ and $R(x) = \{x\}$ for each $x \in X \setminus \{x_0\}$. Then $\dot{R}$ is injective, but $\hat{R}$ is not, since $\hat{R}$ takes the same value, viz. X, at the two distinct points $\{x_0\}$ and X in $\mathcal{P}(X)$.

Finally, we have the following result:

1.10 THEOREM.- Let $R: X|Y$ be a correspondence. If R is strongly surjective (resp. strongly injective), then $\widehat{R}$ is surjective (resp. injective).

Proof.- If R is strongly surjective, then $R \circ R^{-1} = Id_Y$. Hence we have $\widehat{R} \circ \widehat{R^{-1}} = \widehat{R \circ R^{-1}} = Id_{\mathcal{P}(Y)}$, showing that $\widehat{R}$ is surjective. Similarly for injectivity.

§3. OPERATIONS ON FAMILIES OF CORRESPONDENCES

Unions and intersections

Let $(R_i)_{i \in I}$ be a family of correspondences between X and Y. We call union (resp. intersection) of this family, and we denote by $\bigcup_{i \in I} R_i$ (resp. $\bigcap_{i \in I} R_i$), the correspondence $R: X|Y$ which is uniquely determined by the relation $R(x) = \bigcup_{i \in I} R_i(x)$ (resp. $R(x) = \bigcap_{i \in I} R_i(x)$) for each $x \in X$.

If G_i is the graph of R_i for each $i \in I$, then the graph of $\bigcup_{i \in I} R_i$ (resp. $\bigcap_{i \in I} R_i$) is clearly equal to $\bigcup_{i \in I} G_i$ (resp. $\bigcap_{i \in I} G_i$). Because of this, we can transpose to correspondences conventions and results relating to unions and intersections of sets.

If $A \subset X$, then $(\bigcup_{i \in I} R_i)(A) = \bigcup_{i \in I} R_i(A)$ and $(\bigcap_{i \in I} R_i)(A) \subset \bigcap_{i \in I} R_i(A)$. The last inclusion relation cannot in general be replaced by an equality. For example, let X have at least two distinct elements x_1, x_2, let $Y \neq \emptyset$, and define correspondences R_1, R_2 between X and Y by setting $R_i(x_i) = Y$ and $R_i(x) = \emptyset$ for each $x \in X \setminus \{x_i\}$ and each $i = 1,2$. Then $R_1 \cap R_2$ is seen to be the empty correspondence between X and Y, so that $(R_1 \cap R_2)(X) = \emptyset$. But $R_1(X) = R_2(X) = Y$, so that $R_1(X) \cap R_2(X) = Y$ is not empty.

The following result, which follows directly from the relations above, relates unions (resp. intersections) and composites:

1.11 THEOREM.- Let $R_i\colon X|Y$ $(i \in I)$ and $S_j\colon Y|Z$ $(j \in J)$ be two families of correspondences. Then we have $\bigcup_{(i,j)\in I\times J} (S_j \circ R_i) = (\bigcup_{j\in J} S_j) \circ (\bigcup_{i\in I} R_i)$ and $\bigcap_{(i,j)\in I\times J} (S_j \circ R_i) \lhd (\bigcap_{j\in J} S_j) \circ (\bigcap_{i\in I} R_i)$.

We finally relate unions (resp. intersections) and inverses:

1.12 THEOREM.- Let $(R_i)_{i\in I}$ be a family of correspondences between X and Y. Then we have:

a) $(\bigcup_{i\in I} R_i)^{-1} = \bigcup_{i\in I} R_i^{-1}$ and $(\bigcap_{i\in I} R_i)^{-1} = \bigcap_{i\in I} R_i^{-1}$.

b) $(\bigcup_{i\in I} R_i)_*(B) = \bigcup_{i\in I} R_{i*}(B)$ and

$(\bigcap_{i\in I} R_i)_*(B) \subset \bigcap_{i\in I} R_{i*}(B)$ for each $B \in \mathcal{P}(Y)$.

c) $(\bigcup_{i\in I} R_i)^*(B) = \bigcup_{i\in I} R_i^*(B)$ and

$(\bigcap_{i\in I} R_i)^*(B) \supset \bigcap_{i\in I} R_i^*(B)$ for each $B \in \mathcal{P}(Y)$.

Proof.- Part a) follows directly from the definitions. Part b) then follows by using the relation between inverses and lower inverses. The fact that the upper and lower inverses of a correspondence are complementary mappings then yields a proof of c).

Products

Let $R_i\colon X_i|Y_i$ be a correspondence for each $i \in I$. We call product of the family $(R_i)_{i\in I}$, and we denote by $\prod_{i\in I} R_i$, the correspondence R between $\prod_{i\in I} X_i$ and $\prod_{i\in I} Y_i$ which is uniquely determined by the relation $R(x) = \prod_{i\in I} R_i(x_i)$ for each $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$.

This definition agrees with the usual definition for the product of a family of functions. Indeed, let $f_i\colon X_i \longrightarrow Y_i$ be a mapping for each $i \in I$, and let $f\colon \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} Y_i$ be the usual product of the family $(f_i)_{i\in I}$. Then we have $f(x) = (f_i(x_i))_{i\in I} = \prod_{i\in I}\{f_i(x_i)\}$ for each $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$, which agrees with our definition above.

If $R_i = (G_i, X_i, Y_i)$ for each $i \in I$, then the graph of the product $\prod_{i\in I} R_i$ is equal to the image of the set $\prod_{i\in I} G_i$ under the canonical bijection of $\prod_{i\in I}(X_i \times Y_i)$ onto $(\prod_{i\in I} X_i) \times (\prod_{i\in I} Y_i)$. A useful consequence of this is the following: if also $S_i = (H_i, X_i, Y_i)$ for each $i \in I$, and $R_i \prec S_i$ for each $i \in I$, then $\prod_{i\in I} R_i \prec \prod_{i\in I} S_i$. The converse is also true, provided $\prod_{i\in I} G_i \neq \emptyset$.

If X is the product of a family of sets $(X_i)_{i\in I}$, then $pr_i\colon X \to X_i$ denotes the projection of index $i \in I$. We sometimes use the notation pr_i^X, to avoid confusion. Concerning the image of a set under the product of a family of correspondences, we have:

<u>1.13</u> THEOREM.- Let $R_i\colon X_i | Y_i$ be a correspondence for each $i \in I$, and let $R = \prod_{i\in I} R_i$.

a) If $A_i \subset X_i$ for each $i \in I$, then

$$R(\prod_{i\in I} A_i) = \prod_{i\in I} R_i(A_i).$$

b) If $A \subset \prod_{i\in I} X_i$, then $R(A) \subset \prod_{i\in I} R_i(pr_i(A))$.

Proof.- Ad a): Putting $A = \prod_{i \in I} A_i$, we have

$$R(A) = \bigcup_{x \in A} R(x) = \bigcup_{(x_i)_{i \in I} \in A} \prod_{i \in I} R_i(x_i),$$

whereas $\prod_{i \in I} R_i(A) = \prod_{i \in I} \bigcup_{x_i \in A_i} R_i(x_i)$; hence the result.

Ad b): Since $A \subset \prod_{i \in I} pr_i(A)$, the result follows from part a).

Note also that, using the notation of 1.13 and putting $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$, the following diagram commutes for each $i \in I$, provided the sections of R are nonempty:

$$\begin{array}{ccc} X & \xrightarrow{R} & Y \\ {\scriptstyle pr_i^X}\downarrow & & \downarrow{\scriptstyle pr_i^Y} \\ X_i & \xrightarrow{R_i} & Y_i \end{array} \qquad \text{(i.e. } R_i \circ pr_i^X = pr_i^Y \circ R \text{ for each } i \in I).$$

Indeed, for each $x = (x_j)_{j \in I} \in X$ and each $i \in I$, we have $R_i(pr_i^X(x)) = R_i(x_i)$ and $pr_i^Y(R(x)) = pr_i^Y(\prod_{j \in I} R_j(x_j)) = R_i(x_i)$, since $R_j(x_j) \neq \emptyset$ for each $j \in I$.

If $R_1: X_1|Y_1$ and $R_2: X_2|Y_2$ are correspondences, we call <u>product</u> of R_1 and R_2, and we denote by $R_1 \times R_2$, the correspondence R between $X_1 \times X_2$ and $Y_1 \times Y_2$ which is uniquely determined by the relation $R(x) = R_1(x_1) \times R_2(x_2)$ for each $x = (x_1, x_2) \in X_1 \times X_2$.

Again, this definition agrees with the usual definition for the product of two functions. Indeed, let $f_1: X_1 \longrightarrow Y_1$ and $f_2: X_2 \longrightarrow Y_2$ be mappings, and let $f: X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ be the usual product of f_1 and f_2. Then we have $f(x) = (f_1(x_1), f_2(x_2)) = \{f_1(x_1)\} \times \{f_2(x_2)\}$ for each $x = (x_1, x_2) \in X_1 \times X_2$, which agrees with our definition above.

If $I = \{1,2\}$, the product of the family $(R_i)_{i \in I}$ is nothing but the correspondence $g \circ (R_1 \times R_2) \circ f^{-1}$, where $f: X_1 \times X_2 \longrightarrow \prod_{i \in I} X_i$ and $g: Y_1 \times Y_2 \longrightarrow \prod_{i \in I} Y_i$ are the canonical bijections. It follows from this that properties of the product of two correspondences can be deduced from properties of the product of a family of correspondences.

The concept of the product of two correspondences enables us to obtain a useful way of describing the graph of the composite of two correspondences:

1.14 THEOREM.- Let $R = (G,X,Y)$ and $S = (H,Y,Z)$ be two correspondences. Then the graph of the composite $S \circ R$ of R and S is equal to the set $(R \times Id_Z)^{-1}(H)$.

Proof.- For each $(x,z) \in X \times Z$, we have

$$z \in S(R(x)) \Longleftrightarrow \exists y \in R(x) \text{ with } z \in S(y), \text{ i.e. } (y,z) \in H,$$
$$\Longleftrightarrow (R \times Id_Z)(x,z) \cap H = (R(x) \times \{z\}) \cap H \neq \emptyset,$$

hence the result.

For the relation between products and composites, we have the following result, which can be deduced from 1.13:

1.15 THEOREM.- Let $R_i : X_i | Y_i$ and $S_i : Y_i | Z_i$ be correspondences for each $i \in I$. Then

$$\prod_{i \in I}(S_i \circ R_i) = (\prod_{i \in I} S_i) \circ (\prod_{i \in I} R_i).$$

As to inverses of products, we have:

1.16 THEOREM.- Let $R_i : X_i | Y_i$ be a correspondence for each $i \in I$. Then we have:

a) $(\prod_{i \in I} R_i)^{-1} = \prod_{i \in I} R_i^{-1}$.

b) $(\prod_{i \in I} R_i)_*(\prod_{i \in I} B_i) = \prod_{i \in I} R_{i*}(B_i)$ for each $(B_i)_{i \in I} \in \prod_{i \in I} \mathcal{P}(Y_i)$.

c) $(\prod_{i \in I} R_i)^*(\prod_{i \in I} B_i) = (\prod_{i \in I} R_i^*(B_i)) \cup (\prod_{i \in I} R_i)^*(\emptyset)$ for each $(B_i)_{i \in I} \in \prod_{i \in I} \mathcal{P}(Y_i)$.

Proof.- Part a) follows directly from the definitions;

part b) then follows, since
$(\prod_{i\in I} R_i^{-1})(\prod_{i\in I} B_i) = \prod_{i\in I} R_i^{-1}(B_i)$ by 1.13a. The easiest way to prove c) is to do it directly. We have, for each $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$,

$$\prod_{i\in I} R_i(x_i) \subset \prod_{i\in I} B_i \iff \exists\ i \in I \text{ with } R_i(x_i) = \emptyset \text{ or } R_i(x_i) \subset B_i \text{ for each } i \in I.$$

The results we have accumulated so far enable us to study surjectivity and injectivity of products.

1.17 THEOREM.- Let $R_i\colon X_i|Y_i$ for each $i \in I$.

a) In order that $\prod_{i\in I} R_i$ should be surjective, it is sufficient that R_i is surjective for each $i \in I$.

b) This condition is also necessary, provided $\prod_{i\in I} Y_i \neq \emptyset$.

Proof.- Put $Y = \prod_{i\in I} Y_i$. Since part a) is trivial if $Y = \emptyset$, we suppose from now on that $Y \neq \emptyset$. Then, the graph of Id_Y is nonempty. Now $Id_Y = \prod_{i\in I} Id_{Y_i}$, whereas $(\prod_{i\in I} R_i)\circ(\prod_{i\in I} R_i)^{-1} = \prod_{i\in I}(R_i\circ R_i^{-1})$, using 1.15 and 1.16. Using the characterization obtained in 1.7, we thus have

$$\prod_{i\in I} R_i \text{ is surjective} \iff Id_Y \prec (\prod_{i\in I} R_i)\circ(\prod_{i\in I} R_i)^{-1}$$
$$\iff Id_{Y_i} \prec R_i\circ R_i^{-1} \text{ for each } i \in I$$
$$\iff R_i \text{ is surjective for each } i \in I.$$

1.18 THEOREM.- Let $R_i\colon X_i|Y_i$ for each $i \in I$.

a) In order that $\prod_{i\in I} R_i$ should be injective, it is sufficient that R_i is injective for each $i \in I$.

b) This condition is also necessary, provided $\prod_{i\in I} R_i$ is not the empty correspondence between $\prod_{i\in I} X_i$ and $\prod_{i\in I} Y_i$.

Proof.- Put $X = \prod_{i\in I} X_i$ and $R = \prod_{i\in I} R_i$. Again, part a) is trivial if R is the empty correspondence, so suppose from now on that this is not the case. The proof can be carried out exactly analogous to that of 1.17, using this time the characterization obtained in 1.8, if we can only show that the graph of $R^{-1}\circ R$ is nonempty. But this graph is equal to the set $\{(x,x')\in X\times X:\ x'\in R^{-1}(R(x))\}$ $= \{(x,x')\in X\times X:\ R(x)\cap R(x')\neq\emptyset\}$, and therefore has the point (x,x) as element whenever the section of R along x is not empty.

Restricted products

If X and I are sets, then the diagonal mapping $d: X\longrightarrow X^I$ is defined to be that which is uniquely determined by the relation $pr_i\circ d = Id_X$ for each $i\in I$, where $pr_i: X^I\longrightarrow X$ is the projection of index i for each $i\in I$.

If $(A_i)_{i\in I}$ is a family of subsets of X, we then have $d^{-1}(\prod_{i\in I} A_i) = \{x\in X:\ d(x)\in\prod_{i\in I} A_i\} = \bigcap_{i\in I} A_i$.

Let $R_i: X|Y_i$ be a correspondence for each $i\in I$. We define the restricted product of the family $(R_i)_{i\in I}$ to be the correspondence $(\prod_{i\in I} R_i)\circ d$ between X and $\prod_{i\in I} Y_i$, where $d: X\longrightarrow X^I$ is the diagonal mapping.

We shall sometimes denote this restricted product by $\prod_{i\in I}{}' R_i$. Thus, we have $(\prod_{i\in I}{}' R_i)(x) = \prod_{i\in I} R_i(x)$ for each $x\in X$.

One can also define the restricted product of two correspondences R_1 and R_2. But we shall formulate our results only for the restricted product of a family of correspondences.

For each subset $A\subset X$, we have $(\prod_{i\in I}{}' R_i)(A)\subset\prod_{i\in I} R_i(A)$, since $(\prod_{i\in I} R_i)(d(A))\subset(\prod_{i\in I} R_i)(A^I) = \prod_{i\in I} R_i(A)$.

1.19 THEOREM.- Let X be a set, $(Y_i)_{i\in I}$ a family of sets, and R a correspondence between X and $Y = \prod_{i\in I} Y_i$.

We have:

a) $R \lhd \prod_{i\in I}{}'(pr_i^Y \circ R)$.

b) If R is itself the restricted product of a family $(R_i)_{i\in I}$, then, provided the sections of R are nonempty, $R_i = pr_i^Y \circ R$ for each $i \in I$, so that

$$R = \prod_{i\in I}{}'(pr_i^Y \circ R).$$

Proof.- Ad a): For each $x \in X$, we have

$R(x) \subset \prod_{i\in I} pr_i^Y(R(x))$, hence the result.

Ad b): For each $x \in X$ and each $i \in I$, we have

$pr_i^Y(R(x)) = pr_i^Y(\prod_{j\in I} R_j(x)) = R_i(x)$, since $R_j(x) \neq \emptyset$ for each $j \in I$.

Every product can be regarded as a restricted product, in the following sense:

1.20 THEOREM.- Let $R_i : X_i | Y_i$ for each $i \in I$, and put $X = \prod_{i\in I} X_i$. Then $\prod_{i\in I} R_i = \prod_{i\in I}{}'(R_i \circ pr_i^X)$.

Proof.- For each $x = (x_i)_{i\in I} \in X$, we have

$(\prod_{i\in I} R_i)(x) = \prod_{i\in I} R_i(x_i) = \prod_{i\in I} R_i(pr_i^X(x))$, hence the result.

As to inverses of restricted products, we have:

1.21 THEOREM.- Let $R_i : X | Y_i$ be a correspondence for each $i \in I$. Putting $Y = \prod_{i\in I} Y_i$, we have:

a) $(\prod_{i\in I}{}' R_i)^{-1} = \bigcap_{i\in I}(R_i^{-1} \circ pr_i^Y)$.

b) $(\prod_{i\in I}{}' R_i)_*(\prod_{i\in I} B_i) = \bigcap_{i\in I} R_{i*}(B_i)$

for each $(B_i)_{i\in I} \in \prod_{i\in I} \mathcal{P}(Y_i)$.

c) $(\prod_{i\in I}{}'R_i)^*(\prod_{i\in I} B_i) = (\bigcap_{i\in I} R_i{}^*(B_i)) \cup (\prod_{i\in I}{}'R_i)^*(\emptyset)$ for each $(B_i)_{i\in I} \in \prod_{i\in I} \mathcal{P}(Y_i)$.

Proof.- Letting $d\colon X \longrightarrow X^I$ be the diagonal mapping, so that $\prod_{i\in I}{}'R_i = (\prod_{i\in I} R_i)\circ d$, we can use 1.6 and 1.16 to write
$(\prod_{i\in I}{}'R_i)^{-1}(y) = d^{-1}(\prod_{i\in I} R_i{}^{-1}(y_i)) = \bigcap_{i\in I} R_i{}^{-1}(pr_i^Y(y))$ for each $y = (y_i)_{i\in I} \in Y$, hence the proof of a). Similarly, we have $(\prod_{i\in I}{}'R_i)_*(\prod_{i\in I} B_i) = d^{-1}(\prod_{i\in I} R_{i*}(B_i)) = \bigcap_{i\in I} R_{i*}(B_i)$, which proves b). Again, the easiest way to prove part c) is to do it directly.

We finally note the following: a necessary condition for $\prod_{i\in I}{}'R_i$ to be surjective is that R_i is surjective for each $i \in I$, provided $\prod_{i\in I} Y_i \neq \emptyset$; a sufficient condition for $\prod_{i\in I}{}'R_i$ to be injective is the existence of an index $i \in I$ such that R_i is injective.

Chapter 2

TOPOLOGICAL STRUCTURES ON HYPERSPACES

§1. THE FINITE TOPOLOGY

This section is devoted mainly to continuous correspondences as studied in [1]. Here again, as already noted at the beginning of chapter 1, we try to underline the importance of canonical extensions. In particular, this leads us to the new concept of continuity at a set, rather than merely at a point.

For convenience, let us first review the main ideas concerning continuous correspondences and the finite topology.

Continuous correspondences

Let R be a correspondence between a topological space X and a topological space Y, and let x_0 be a point of X.

R is said to be upper semi-continuous (abbr. usc) at x_0 if, for each open subset G of Y with $R(x_0) \subset G$, there exists a neighbourhood U of x_0 in X such that the relation $x \in U$ implies $R(x) \subset G$.

R is said to be lower semi-continuous (abbr. lsc) at x_0 if, for each open subset G of Y with $R(x_0) \cap G \neq \emptyset$, there exists a neighbourhood U of x_0 in X such that the relation

$x \in U$ implies $R(x) \cap G \neq \emptyset$.

R is said to be <u>continuous</u> at x_0 if it is both usc and lsc at x_0.

Expressed in terms of upper inverses, we see that R is usc at x_0 iff, for each open subset G of Y, the relation $x_0 \in R^*(G)$ implies $x_0 \in \overset{\circ}{\widehat{R^*(G)}}$. Passing over to the complement mapping, we obtain the following equivalent statement: for each closed subset C of Y, the relation $x_0 \in \overline{R_*(C)}$ implies $x_0 \in R_*(C)$.

Similarly, R is lsc at x_0 iff, for each open subset G of Y, the relation $x_0 \in R_*(G)$ implies $x_0 \in \overset{\circ}{\widehat{R_*(G)}}$; or, equivalently: for each closed subset C of Y, the relation $x_0 \in \overline{R^*(C)}$ implies $x_0 \in R^*(C)$.

If $A \subset X$, we say that R is continuous (resp. usc) (resp. lsc) on A if R is continuous (resp. usc) (resp. lsc) at every point of A. If R is continuous on X, we also say simply that R is continuous; similarly for usc and lsc. It is clear that R is usc (resp. lsc) iff $R^*(G)$ (resp. $R_*(G)$) is an open subset of X for each open $G \subset Y$, or, equivalently, iff $R_*(C)$ (resp. $R^*(C)$) is a closed subset of X for each closed $C \subset Y$.

For a mapping of X into Y, the above three notions of continuity are equivalent to each other, and agree with the usual definition. In order to avoid misunderstandings, we shall agree to refer to the usual concept of upper (resp. lower) semi-continuity of a real-valued function as that of <u>numerical upper</u> (resp. <u>lower</u>) <u>semi-continuity</u>.

Let now $(X, \mathcal{T})$ be a topological space (X is the underlying set and $\mathcal{T}$ is the set of open subsets).

The <u>upper semi-finite topology</u> induced by $\mathcal{T}$ on $\mathcal{P}(X)$ is defined to be that which is generated by the set of all sets of the form $\mathcal{P}(G)$, with $G \in \mathcal{T}$, and is denoted by $\mathcal{T}^*$.

The <u>lower semi-finite topology</u> induced by $\mathcal{T}$ on $\mathcal{P}(X)$ is defined to be that which is generated by the set of all sets of the form $\mathcal{J}(G)$, with $G \in \mathcal{T}$, and is denoted by $\mathcal{T}_*$.

The finite topology induced by $\mathcal{T}$ on $\mathcal{F}(X)$ is defined to be the join of $\mathcal{T}^*$ and $\mathcal{T}_*$, and is denoted by $\tilde{\tilde{\mathcal{T}}}$. In symbols, we have $\tilde{\tilde{\mathcal{T}}} = \mathcal{T}^* \vee \mathcal{T}_*$, using the "$\vee$" to denote the join of two topologies.

We shall use the abbreviation usf (resp. lsf) for "upper semi-finite" (resp. "lower semi-finite"). The fact that the mappings $\mathcal{F}_X$ and $\mathcal{J}_X$ are complementary shows that the usf (resp. lsf) topology of $\mathcal{F}(X)$ is the coarsest of all topologies on $\mathcal{F}(X)$ which have the property that, for each closed subset C of X, the set $\mathcal{J}(C)$ (resp. $\mathcal{F}(C)$) is closed in $\mathcal{F}(X)$.

If $\mathcal{Q} \subset \mathcal{F}(X)$, the finite topology of $\mathcal{Q}$ is defined to be that which is induced on $\mathcal{Q}$ by the finite topology of $\mathcal{F}(X)$, and will also be denoted by $\tilde{\tilde{\mathcal{T}}}$. Similarly for the usf and the lsf topologies of $\mathcal{Q}$.

The three topologies introduced above are all admissible, in the sense that the mapping $x \mapsto \{x\}$ is a homeomorphism of X onto a subspace of $\mathcal{F}(X)$. Indeed, putting $X' = \{\{x\} : x \in X\}$, we see that the image under the bijection $X \longrightarrow X'$ of any open subset G of X is equal to

$$\mathcal{F}(G) \cap X' = \mathcal{J}(G) \cap X'.$$

It is useful to note the following simple result:

2.1 THEOREM.-

a) The set of all sets of the form $\mathcal{F}(G)$, where G is an open subset of X, is a base for the usf topology of $\mathcal{F}(X)$.

b) If $\mathcal{B}$ is a base for the topology of X, then the set of all sets of the form $\mathcal{J}(G)$, where $G \in \mathcal{B}$, generates the lsf topology of $\mathcal{F}(X)$.

Proof.- Part a) follows from the fact that the mapping $\mathcal{F}_X$ preserves intersections, part b) from the fact that $\mathcal{J}_X$ preserves unions.

The continuity of canonical extensions

Let $R: X|Y$ be a correspondence between topological spaces. The relations $R_*(G) = \dot{R}^{-1}(\mathcal{J}(G))$ and $R^*(G) = \dot{R}^{-1}(\mathcal{P}(G))$ in 1.5a show that R is continuous (resp. usc) (resp. lsc) iff the restricted canonical extension $\dot{R}: X \to \mathcal{P}(Y)$ of R is continuous, when $\mathcal{P}(Y)$ is endowed with its finite (resp. usf) (resp. lsf) topology.

We now wish to investigate the continuity of $\hat{R}: \mathcal{P}(X) \to \mathcal{P}(Y)$. First of all, it is clear that R is continuous (resp. usc) (resp. lsc) at a point x_o in X iff $\hat{R}$ is continuous at $\{x_o\} \in \mathcal{P}(X)$, when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are endowed with their respective finite (resp. usf) (resp. lsf) topologies, since the mapping $x \mapsto \{x\}$ is a homeomorphism of X onto a subspace of $\mathcal{P}(X)$. We also have:

2.2 THEOREM.- Let $A_o \subset X$. If R is continuous (resp. usc) (resp. lsc) at every point of A_o, then $\hat{R}$ is continuous at the point $A_o \in \mathcal{P}(X)$, when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are endowed with their respective finite (resp. usf) (resp. lsf) topologies.

Proof.- It suffices to consider the "usc" and the "lsc" cases.

The "usc" case: Let G be any open subset of Y with $\hat{R}(A_o) \in \mathcal{P}(G)$, i.e. $R(A_o) \subset G$. Then, for each $x \in A_o$, we have $R(x) \subset G$, hence there exists an open neighbourhood U_x of x in X such that $R(U_x) \subset G$. Putting $U = \bigcup_{x \in A_o} U_x$, we have $R(U) \subset G$, so that $\hat{R}(\mathcal{P}(U)) \subset \mathcal{P}(G)$. Since $\mathcal{P}(U)$ is a neighbourhood of A_o in $\mathcal{P}(X)$, the result follows.

The "lsc" case: Let G be any open subset of Y with $\hat{R}(A_o) \in \mathcal{J}(G)$, i.e. $R(A_o) \cap G \neq \emptyset$, and choose a point $x_o \in A_o$ with $R(x_o) \cap G \neq \emptyset$. Now there exists an open neighbourhood U of x_o in X such that $R(x) \cap G \neq \emptyset$ for each $x \in U$. It follows that $R(A) \cap G \neq \emptyset$ for each $A \in \mathcal{P}(X)$ with $A \cap U \neq \emptyset$, so that $\hat{R}(\mathcal{J}(U)) \subset \mathcal{J}(G)$. Since $\mathcal{J}(U)$ is a neighbourhood of A_o in $\mathcal{P}(X)$, the result follows.

Summarizing our results, we obtain the following:

2.3 THEOREM.- The following statements are equivalent:

a) R is continuous (resp. usc) (resp. lsc).

b) $\dot{R}$ is continuous, when $\mathcal{P}(Y)$ is endowed with its finite (resp. usf) (resp. lsf) topology.

c) $\hat{R}$ is continuous, when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are endowed with their respective finite (resp. usf) (resp. lsf) topologies.

Note that the implication c)$\Longrightarrow$a) remains valid if, in c), we replace $\hat{R}$ by $\hat{R}|\mathcal{Q}$, where $\mathcal{Q} \subset \mathcal{P}(X)$ contains all singletons in X.

Considering the continuity of $\hat{R}$, we naturally come to the following concept of continuity at a set, rather than at a point: we shall say that a correspondence $R: X|Y$ between topological spaces is continuous (resp. usc) (resp. lsc) at a subset A_o of X if the canonical extension $\hat{R}: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ of R to the power sets is continuous at the point A_o of $\mathcal{P}(X)$, when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are endowed with their respective finite (resp. usf) (resp. lsf) topologies.

The statement of 2.2 can thus be rephrased as follows: if R is continuous (resp. usc) (resp. lsc) at every point of A_o, then R is continuous (resp. usc) (resp. lsc) at the set A_o.

In order to characterize this new concept of continuity, we need the following theorem, which is a special case of a more general result to be found in chapter 4 (cf. 4.10b).

2.4 THEOREM.- For each subset A of X, we have:

a) $\widehat{\mathcal{P}(A)}^{\circ} = \mathcal{P}(\mathring{A})$, the interior of the set $\mathcal{P}(A)$ being taken with respect to the finite (resp. usf) topology of $\mathcal{P}(X)$.

b) $\widehat{\mathcal{J}(A)}^{\circ} = \mathcal{J}(\mathring{A})$, the interior of the set $\mathcal{J}(A)$ being taken with respect to the finite (resp. lsf) topology of $\mathcal{P}(X)$.

We obtain:

<u>2.5</u> THEOREM.- Let $A_o \subset X$. Then we have:

a) R is usc at A_o iff, for each open subset G of Y, the relation $A_o \subset R^*(G)$ implies $A_o \subset \mathring{\widehat{R^*(G)}}$.

b) R is lsc at A_o iff, for each open subset G of Y, the relation $A_o \cap R_*(G) \neq \emptyset$ implies $A_o \cap \mathring{\widehat{R_*(G)}} \neq \emptyset$.

Proof.- We only prove a), the proof of b) being very similar. Now, by definition, R is usc at A_o iff, for each open subset G of Y with $\hat{R}(A_o) \in \mathcal{P}(G)$, the set $\hat{R}^{-1}(\mathcal{P}(G))$ is a neighbourhood of A_o in $\mathcal{P}(X)$, when $\mathcal{P}(X)$ is endowed with its usf topology. Now, the relation $\hat{R}(A_o) \in \mathcal{P}(G)$ is equivalent to $R(A_o) \subset G$, hence to $A_o \subset R^*(G)$. On the other hand, we have $\hat{R}^{-1}(\mathcal{P}(G)) = \mathcal{P}(R^*(G))$ by 1.5b, so that $\hat{R}^{-1}(\mathcal{P}(G))$ is a neighbourhood of A_o iff $A_o \in \mathring{\widehat{\mathcal{P}(R^*(G))}}$. By 2.4a, this is equivalent to $A_o \subset \mathring{\widehat{R^*(G)}}$, hence the result.

Clearly, we can obtain corresponding results for closed sets: for part a), the implication reads $A_o \cap \overline{R_*(C)} \neq \emptyset \Longrightarrow A_o \cap R_*(C) \neq \emptyset$, and for b), $A_o \subset \overline{R^*(C)} \Longrightarrow A_o \subset R^*(C)$. Also note that, because of 2.1b, in order to show that R is lsc at A_o, it suffices to show that the implication

$$A_o \cap R_*(G) \neq \emptyset \Longrightarrow A_o \cap \mathring{\widehat{R_*(G)}} \neq \emptyset$$

holds for each member G of a base $\mathcal{B}$ for the topology of Y.

If R is both usc and lsc at A_o, then it is also continuous at A_o. It is natural to ask whether the converse holds. This question is not as trivial as it may first sound: if

$\hat{R}: \mathcal{P}(X) \to \mathcal{P}(Y)$ is continuous at the point A_o when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ carry their respective finite topologies, there is, a priori, no reason why $\hat{R}$ should remain continuous at that point when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are endowed with coarser topologies. Nevertheless, the question can be answered in the affirmative. Indeed, suppose that R is continuous at A_o, and let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ carry their respective finite topologies. Let G be any open subset of Y with $A_o \subset R^*(G)$; we want to show that $A_o \subset \overset{\circ}{\overbrace{R^*(G)}}$. Now we have $R(A_o) \subset G$, and so $\hat{R}(A_o) \in \mathcal{P}(G)$. Since $\mathcal{P}(G)$ is open in $\mathcal{P}(Y)$, and $\hat{R}$ is continuous, it follows that $\hat{R}^{-1}(\mathcal{P}(G)) = \mathcal{P}(R^*(G))$ is a neighbourhood of A_o in $\mathcal{P}(X)$. Thus, $A_o \in \overset{\circ}{\overbrace{\mathcal{P}(R^*(G))}} = \mathcal{P}(\overset{\circ}{\overbrace{R^*(G)}})$, using 2.4a, and hence $A_o \subset \overset{\circ}{\overbrace{R^*(G)}}$, as desired. In the same way, we can show that R is lsc at A_o.

Our considerations on continuity at a set enable us to give a meaningful "local" result on the continuity of composites:

<u>2.6</u> THEOREM.- Let $R: X|Y$ and $S: Y|Z$ be correspondences between topological spaces. If R is continuous at a subset A_o of X, and S is continuous at $R(A_o)$, then $S \circ R$ is continuous at A_o. Similarly for "usc" and "lsc".

Proof: Endow $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ with their respective finite topologies. Then $\hat{R}$ is continuous at the point $A_o \in \mathcal{P}(X)$, and $\hat{S}$ is continuous at $\hat{R}(A_o) \in \mathcal{P}(Y)$, hence $\hat{S} \circ \hat{R}$ is continuous at A_o. Since $\widehat{S \circ R} = \hat{S} \circ \hat{R}$, the result follows. Similarly with the usf and lsf topologies.

Using 2.2, we then obtain the following result dealing with points:

<u>2.7</u> COROLLARY.- If R is continuous at a point x_o of X, and S is continuous at every point of $R(x_o)$, then

$S \circ R$ is continuous at x_0. Similarly for "usc" and "lsc".

Compact unions

One of the most useful properties of the finite topology is that a compact union of compact sets is compact. More precisely, if $\mathcal{Q} \subset \mathcal{P}(X)$ is compact (with respect to its finite topology) and if each set of $\mathcal{Q}$ is compact, then the union of the sets of $\mathcal{Q}$ is also compact. We now wish to extend this result, which also holds for the usf topology, to other compactness concepts.

Whenever we refer to a compactness concept, we shall not assume any separation axiom. Thus, a topological space X will be called compact (resp. Lindelöf) if every open covering of X has a finite (resp. countable) subcovering; we also say that X is $\aleph_0$- (resp. $\aleph_1$-) compact. More generally, if $\aleph$ is any cardinal number, we say that X is $\aleph$-compact if every open covering of X has a subcovering of cardinality less than $\aleph$. It is clear that $\aleph$-compactness is preserved under continuous mappings.

We say that X is paracompact (resp. para-Lindelöf) if every open covering of X has a locally finite (resp. locally countable) open refinement. The terms metacompact and meta-Lindelöf are defined in the same way, replacing "locally" by "point".

Finally, if we replace "every open covering of X" by "every countable open covering of X" in the above definitions of compactness, paracompactness and metacompactness, we obtain the corresponding concepts: X is countably compact, countably paracompact, or countably metacompact.

If, to every topological space X, there is associated a set $\alpha(X) \subset \mathcal{P}(\mathcal{P}(X))$ such that each member of $\alpha(X)$ is a set of open subsets of X, then we shall say that α is a carrier. We shall denote by $\alpha_0(X)$ the set of all sets of open subsets of X, by $\alpha_f(X)$ (resp. $\alpha_c(X)$) the set of all finite (resp. countable) sets of open subsets of X. Similarly for $\alpha_{lf}(X)$

(locally finite), $\alpha_{lc}(X)$ (locally countable), $\alpha_{pf}(X)$ (point finite) and $\alpha_{pc}(X)$ (point countable).

If α, β are carriers, then a subset $A \subset X$ will be said to be an <u>(α,β)-subset</u> of X if every covering $\mathcal{U} \in \alpha(X)$ of A has a refinement $\mathcal{H} \in \beta(X)$. If X is an (α,β)-subset of itself, we also say that X is an <u>(α,β)-space</u>.

Clearly, A is an (α_o,α_f)- (resp. (α_o,α_c)-) (resp. (α_c,α_f)-) subset of X iff it is compact (resp. Lindelöf) (resp. countably compact).

Also, if A is an (α_o,α_{lf})- (resp. (α_o,α_{pf})-) subset of X, then it is paracompact (resp. metacompact). Conversely, if X is paracompact (resp. metacompact) and $A \subset X$ is closed, then A is an (α_o,α_{lf})- (resp. (α_o,α_{pf})-) subset of X. Similar results hold for (α_o,α_{lc})- (para-Lindelöf), (α_o,α_{pc})- (meta-Lindelöf), (α_c,α_{lf})- (countably paracompact) and (α_c,α_{pf})- (countably metacompact) subsets of X.

In the following results, we shall suppose that we are given a cardinal number $\aleph$ and two carriers α and β. Further, we shall assume that, for each topological space X, the set $\beta(X)$ is closed under the formation of unions of families of cardinality less than $\aleph$, i.e. that $\bigcup_{i \in I} \mathcal{U}_i \in \beta(X)$ whenever the index set I has cardinality less than $\aleph$ and $\mathcal{U}_i \in \beta(X)$ for each $i \in I$. This condition is satisfied in the following special cases:

a) $\aleph = \aleph_0$; $\beta = \alpha_f, \alpha_{lf}, \alpha_{pf}, \alpha_{lc}$.

b) $\aleph = \aleph_1$; $\beta = \alpha_c, \alpha_{pc}$.

<u>2.8</u> THEOREM.- Let X be a topological space, and suppose that $\mathcal{Q} \subset \mathcal{S}(X)$ is $\aleph$-compact with respect to its upper semi-finite topology. If each $A \in \mathcal{Q}$ is an (α,β)-subset of X, then $\bigcup_{A \in \mathcal{Q}} A$ is also an (α,β)-subset of X.

Proof.- Let $\mathcal{U} \in \alpha(X)$ be a covering of $\bigcup_{A \in \mathcal{Q}} A$. Then $\mathcal{U}$ is a covering of each set $A \in \mathcal{Q}$; since A is an

(α,β)-subset of X, there exists a refinement $\mathcal{H}_A \in \beta(X)$ of $\mathcal{G}$, for each $A \in \mathcal{Q}$. Putting $H_A = \bigcup_{H \in \mathcal{H}_A} H$, then H_A is an open subset of X with $A \subset H_A$, and hence $\mathcal{V}(H_A)$ is an open subset of $\mathcal{V}(X)$ with $A \in \mathcal{V}(H_A)$.

Thus, $(\mathcal{V}(H_A))_{A \in \mathcal{Q}}$ is an open covering of $\mathcal{Q}$ in $\mathcal{V}(X)$. Since $\mathcal{Q}$ is $\aleph$-compact, there exists a subset $\mathcal{B}$ of $\mathcal{Q}$, of cardinality less than $\aleph$, such that $(\mathcal{V}(H_A))_{A \in \mathcal{B}}$ still covers $\mathcal{Q}$. Then we have $\bigcup_{A \in \mathcal{Q}} A \subset \bigcup_{A \in \mathcal{B}} H_A = \bigcup_{A \in \mathcal{B}} \bigcup_{H \in \mathcal{H}_A} H$, so that

$$\mathcal{H} = \bigcup_{A \in \mathcal{B}} \mathcal{H}_A \in \beta(X)$$

is a covering of $\bigcup_{A \in \mathcal{Q}} A$ which clearly refines $\mathcal{G}$.

We now give a table which enumerates special cases of this general result:

X	$\aleph$	α	β						
arbitrary	$\aleph_0$	α_o	α_f	comp.	unions of	comp.	sets	are	comp.
para-comp.	$\aleph_0$	α_o	α_{lf}	comp.	"	" closed	"	"	para-comp.
meta-comp.	$\aleph_0$	α_o	α_{pf}	comp.	"	" closed	"	"	meta-comp.
arbitrary	$\aleph_0$	α_c	α_f	comp.	"	" count. comp.	"	"	count. comp.
count. para-comp.	$\aleph_0$	α_c	α_{lf}	comp.	"	" closed	"	"	count. para-comp.
count. meta-comp.	$\aleph_0$	α_c	α_{pf}	comp.	"	" closed	"	"	count. meta-comp.
arbitrary	$\aleph_1$	α_o	α_c	Lind.	"	" Lind.	"	"	Lind.
para-Lind.	$\aleph_0$	α_o	α_{lc}	comp.	"	" closed	"	"	para-Lind.
meta-Lind.	$\aleph_1$	α_o	α_{pc}	Lind.	"	" closed	"	"	meta-Lind.

If we now consider correspondences, we obtain the following result:

<u>2.9</u> THEOREM.- Let X and Y be two topological spaces, with X $\mathcal{R}$-compact, and let $R: X|Y$ be an upper semi-continuous correspondence.

If the sections of R are (α,β)-subsets of Y, then R(X) is also an (α,β)-subset of Y.

Proof.- Since $R(X) = \bigcup_{x \in X} R(x)$, it suffices, by 2.8, to show that the set $\mathcal{Q} = \{R(x):\ x \in X\}$ is $\mathcal{R}$-compact with respect to its upper semi-finite topology. But this is the case, since $\mathcal{Q}$ is the image of the $\mathcal{R}$-compact space X under the continuous mapping $\dot{R}$.

Finally, if we further assume that every closed subset of an (α,β)-space is an (α,β)-subset of that space (which is the case in each of the examples considered above), we have:

<u>2.10</u> COROLLARY.- Let X and Y be two topological spaces, with Y an (α,β)-space. If $R: X|Y$ is upper semi-continuous and with closed sections, then the image under R of every $\mathcal{R}$-compact subset of X is an (α,β)-subset of Y.

We finish this section with some results involving relatively compact sets. A subset A of a topological space X is said to be <u>relatively compact</u> if A is contained in a compact subset of X. If X is Hausdorff, this is equivalent to requiring that $\overline{A}$ be compact. The easiest way to obtain a result in this direction is to make direct use of 2.9:

<u>2.11</u> COROLLARY.- Let R be an upper semi-continuous correspondence between a topological space X and a topological space Y. If the sections of R are compact, then the image under R of every relatively compact subset A of X is a relatively compact subset of Y.

Proof.- Indeed, A is contained in some compact set K, and R(K) is compact.

In order to obtain a result where the sections of R are only assumed to be relatively compact, we use theorem 5.3 of [8], which asserts that the mapping $A \mapsto \overline{A}$ of $\mathcal{B}(X)$ into itself is continuous with respect to the finite topology, provided X is normal.

2.12 THEOREM.- Let X be a normal T_1-space, and suppose that $\mathcal{Q} \subset \mathcal{C}(X)$ is compact with respect to its finite topology. If each $A \in \mathcal{Q}$ is relatively compact, then $\bigcup_{A \in \mathcal{Q}} A$ is also relatively compact.

Proof.- Since X is normal, it follows, by the above mentioned theorem, that the set $\{\bar{A}: A \in \mathcal{Q}\}$ is also compact. Now $\bar{A}$ is compact for each $A \in \mathcal{Q}$ and so, by 2.8, it follows that $\bigcup_{A \in \mathcal{Q}} \bar{A}$ is a compact set. Since it contains $\bigcup_{A \in \mathcal{Q}} A$, the result follows.

Again, we can deduce a result on correspondences, which can be proved exactly like 2.9:

2.13 THEOREM.- Let R be a continuous correspondence between a compact space X and a normal T_1-space Y. If the sections of R are relatively compact subsets of Y, then R(X) is relatively compact in Y.

2.14 COROLLARY.- Let R be a continuous correspondence between a topological space X and a normal T_1-space Y. If the sections of R are relatively compact in Y, then the image under R of every relatively compact subset A of X is a relatively compact subset of Y.

Proof.- This can now be proved like 2.11: A is contained in some compact set K, and R(K) is relatively compact by 2.13, hence R(A) is also relatively compact.

§2. QUASI-UNIFORMITIES ON HYPERSPACES

In the appendix to his basic paper [8], E. Michael apologizes to the reader for originally not having been aware that the finite (Vietoris) topology on $\mathcal{C}(X)$ is the join of the usf and the lsf topologies. He also mentions that, if $(X,\mathcal{U})$ is a uniform space, then the (Bourbaki) uniformity induced on $\mathcal{P}(X)$ by $\mathcal{U}$ can also be viewed as the join of two other uniformities on $\mathcal{P}(X)$. In fact, as noted by N. Levine and W.J. Stager in [7], these two structures are only quasi-uniformities, since they do not satisfy the symmetry axiom.

In this section, we intend to study the hyperspace of a quasi-uniform space. Besides having an interest of their own, quasi-uniformities will be found quite useful, in chapter 3, in proving results about continuous correspondences.

As mentioned above, a quasi-uniformity is defined with the same axioms as for a uniformity, except that it is no more required that U^{-1} be an entourage for each entourage U.

Most of the concepts and elementary results concerning base, uniformity generated by a set, relative uniformity, uniform continuity (which is now referred to as quasi-uniform continuity), and topology induced by a uniformity, can be treated in the same way, as they do not involve the symmetry axiom. Perhaps the first notable exception is the formula $\bar{A} = \bigcap_U U^{-1}(A)$ for the closure of a set in a quasi-uniform space $(X,\mathcal{U})$, where U runs through the set $\mathcal{U}$ of entourages of X. Indeed, $x \in \bar{A}$ iff $U(x) \cap A \neq \emptyset$ for each $U \in \mathcal{U}$, i.e. $x \in U^{-1}(A)$ for each U.

If $\mathcal{U}$ is a quasi-uniformity on a set X, the conjugate quasi-uniformity $\mathcal{U}^{-1}$ is defined to be that which has the sets U^{-1} as entourages, where U runs through $\mathcal{U}$.

If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on X, we denote by $\mathcal{U} \vee \mathcal{V}$ the join of $\mathcal{U}$ and $\mathcal{V}$. If $\mathcal{T}(\mathcal{U})$ denotes the topology induced by $\mathcal{U}$, we have $\mathcal{T}(\mathcal{U} \vee \mathcal{V}) = \mathcal{T}(\mathcal{U}) \vee \mathcal{T}(\mathcal{V})$, i.e. $\mathcal{T}(\mathcal{U} \vee \mathcal{V})$ is the join of $\mathcal{T}(\mathcal{U})$ and $\mathcal{T}(\mathcal{V})$.

If $\mathcal{U}$ is a uniformity, we also say that $\mathcal{U}$ is symmetric. Thus, the quasi-uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ is symmetric for each quasi-uniformity $\mathcal{U}$; it is the smallest uniformity containing $\mathcal{U}$. If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are quasi-uniform spaces, and $f: X \to Y$ is $(\mathcal{U},\mathcal{V})$-quasi-uniformly continuous, it is also $(\mathcal{U}^{-1},\mathcal{V}^{-1})$-quasi-uniformly continuous, hence $(\mathcal{U} \vee \mathcal{U}^{-1},\mathcal{V} \vee \mathcal{V}^{-1})$-uniformly continuous.

For a more detailed discussion of quasi-uniform spaces, we refer to [9].

Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$, we shall denote by U^*, U_* and $\widetilde{U}$ the following subsets of $\mathcal{P}(X) \times \mathcal{P}(X)$:

$$U^* = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X): \ B \subset U(A)\},$$

$$U_* = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X): \ A \subset U^{-1}(B)\},$$

$$\widetilde{U} = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X): \ B \subset U(A) \text{ and } A \subset U^{-1}(B)\}.$$

Clearly, $(U^*)^{-1} = (U^{-1})_*$, $(U_*)^{-1} = (U^{-1})^*$ and $\widetilde{U} = U^* \cap U_*$ for each U. Also, the relation $U \subset V$ implies $U^* \subset V^*$, $U_* \subset V_*$ and $\widetilde{U} \subset \widetilde{V}$.

2.15 THEOREM.- The sets U^* (resp. U_*) (resp. $\widetilde{U}$) form a base for a quasi-uniformity $\mathcal{U}^*$ (resp. $\mathcal{U}_*$) (resp. $\widetilde{\mathcal{U}}$) on $\mathcal{P}(X)$, as U runs through $\mathcal{U}$. Furthermore, we have $(\mathcal{U}^*)^{-1} = (\mathcal{U}^{-1})_*$, $(\mathcal{U}_*)^{-1} = (\mathcal{U}^{-1})_*$ and $\widetilde{\mathcal{U}} = \mathcal{U}^* \vee \mathcal{U}_*$.

Proof.- Let $\mathcal{B}$ be the set of all sets U^*. We have:

a) The intersection of two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$:
If U,V are entourages, then $(U \cap V)^* \subset U^* \cap V^*$.

b) Every set of $\mathcal{B}$ contains the diagonal in $\mathcal{P}(X)$:
Indeed, $A \subset U(A)$ for each entourage U and each $A \in \mathcal{P}(X)$.

c) Every set of $\mathcal{B}$ contains the square of a set of $\mathcal{B}$:

If U is an entourage, then $(U^*)^2 \subset (U^2)^*$. Indeed, if $(A,B) \in (U^*)^2$, there exists a set $C \in \mathcal{P}(X)$ with $(A,C) \in U^*$ and $(C,B) \in U^*$. Hence $C \subset U(A)$ and $B \subset U(C)$, and so $B \subset U(C) \subset U^2(A)$, showing that $(A,B) \in (U^2)^*$. Since every entourage of $\mathcal{U}$ contains some square U^2, the result follows.

This proves our claim concerning $\mathcal{U}^*$. The result concerning $\mathcal{U}_*$ then follows immediately. Finally, note that the sets $U^* \cap V_*$ form a base for the quasi-uniformity $\mathcal{U}^* \vee \mathcal{U}_*$ on $\mathcal{P}(X)$, as U and V independently run through $\mathcal{U}$. Since $\widetilde{U \cap V} \subset \widetilde{U} \cap \widetilde{V} \subset U^* \cap V_*$, the result concerning $\widetilde{\mathcal{U}}$ follows.

Note that, if $\mathcal{U}$ is symmetric, then $\mathcal{U}^*$ and $\mathcal{U}_*$ are conjugate quasi-uniformities, and so $\widetilde{\mathcal{U}}$ is also symmetric.

With the notation just introduced, we call $\mathcal{U}^*$ (resp. $\mathcal{U}_*$) the <u>upper</u> (resp. <u>lower</u>) <u>quasi-uniformity</u> induced by $\mathcal{U}$ on $\mathcal{P}(X)$. We call $\widetilde{\mathcal{U}}$ the <u>Bourbaki</u> <u>quasi-uniformity</u> induced by $\mathcal{U}$ on $\mathcal{P}(X)$.

Since $\widetilde{\mathcal{U}} = \mathcal{U}^* \vee \mathcal{U}_*$, we have $\mathcal{T}(\widetilde{\mathcal{U}}) = \mathcal{T}(\mathcal{U}^*) \vee \mathcal{T}(\mathcal{U}_*)$.

If $\mathcal{Q} \subset \mathcal{P}(X)$, the Bourbaki quasi-uniformity of $\mathcal{Q}$ is defined to be that which is induced on $\mathcal{Q}$ by the Bourbaki quasi-uniformity of $\mathcal{P}(X)$, and will also be denoted by $\widetilde{\mathcal{U}}$. This will not lead to confusion, for $\widetilde{\mathcal{U}_A} = \widetilde{\mathcal{U}}_{\mathcal{P}(A)}$ for each $A \subset X$; $\mathcal{U}_A$ denotes here the quasi-uniformity induced by $\mathcal{U}$ on A.

Similar remarks hold concerning $\mathcal{U}^*$ and $\mathcal{U}_*$.

The three quasi-uniformities introduced above are all <u>admissible</u>, in the sense that the mapping $x \mapsto \{x\}$ is an isomorphism of X onto a subspace of $\mathcal{P}(X)$. Indeed, putting $X' = \{\{x\} : x \in X\}$ and denoting by f the bijection $X \to X'$, we see that the image under the bijection $f \times f: X \times X \to X' \times X'$ of any entourage U of X is equal to

$$U^* \cap (X' \times X') = U_* \cap (X' \times X') = \widetilde{U} \cap (X' \times X').$$

<u>Quasi-uniformly continuous correspondences</u>

Let $R: X|Y$ be a correspondence between quasi-uniform spaces.

R is said to be <u>quasi-uniformly upper semi-continuous</u> if, for each entourage V of Y, there exists an entourage U of X such that the relation $(x,y) \in U$ implies $R(y) \subset V(R(x))$.

R is said to be <u>quasi-uniformly lower semi-continuous</u> if, for each entourage V of Y, there exists an entourage U of X such that the relation $(x,y) \in U$ implies $R(x) \subset V^{-1}(R(y))$.

R is said to be <u>quasi-uniformly continuous</u> if it is both quasi-uniformly usc and quasi-uniformly lsc.

<u>2.16</u> THEOREM.- The following statements are equivalent:

a) R is quasi-uniformly continuous (resp. quasi-uniformly usc) (resp. quasi-uniformly lsc).

b) $\dot{R}$ is quasi-uniformly continuous, when $\mathcal{P}(Y)$ is endowed with its Bourbaki (resp. upper) (resp. lower) quasi-uniformity.

c) $\hat{R}$ is quasi-uniformly continuous, when $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are endowed with their respective Bourbaki (resp. upper) (resp. lower) quasi-uniformities.

Proof.- The equivalence of a) and b) is obvious. Also, because of the relation $\dot{R} = \hat{R} \cdot j_X$, we see that b) follows from c). We now show that a) implies c). First consider the "usc" case. So let V be any entourage of Y. Then there is an entourage U of X such that $R(y) \subset V(R(x))$ for each $(x,y) \in U$. Hence, if $A,B \in \mathcal{P}(X)$ are such that $B \subset U(A)$, we easily see that $R(B) \subset V(R(A))$, so that

$$(R(A),R(B)) \in V^* \quad \text{for each} \quad (A,B) \in U^*.$$

The "lsc" case can be treated in the same manner. Finally, the "Bourbaki" case follows from the other two.

The following result on the quasi-uniform continuity of composites is a direct consequence of the last theorem; the proof goes exactly like that of 2.6.

<u>2.17</u> THEOREM.- Let $R: X|Y$ and $S: Y|Z$ be correspondences between quasi-uniform spaces. If R and S are both quasi-uniformly continuous, then so is $S \circ R$.

Similarly for "quasi-uniformly usc" and "quasi-uniformly lsc".

At this stage, it might be interesting to consider two other natural concepts of quasi-uniform continuity for correspondences, and to study their relationship to those we have introduced above.

So let $R: X|Y$ be a correspondence between quasi-uniform spaces. We shall say that R is <u>strongly</u> (resp. <u>weakly</u>) <u>quasi-uniformly continuous</u> if, for each entourage V of Y, there exists an entourage U of X such that the relation $(x,y) \in U$ implies $R(x) \times R(y) \subset V$ (resp. $(R(x) \times R(y)) \cap V \neq \emptyset$).

Putting $S = R \times R$, we see that R is strongly (resp.

weakly) quasi-uniformly continuous iff $S^*(V)$ (resp. $S_*(V)$) is an entourage of X for each entourage V of Y.

2.18 THEOREM.- Suppose that the sections of $\dot{R}$ are nonempty.

a) If R is strongly quasi-uniformly continuous, then it is also quasi-uniformly continuous.

b) If R is quasi-uniformly upper (resp. lower) semi-continuous, then it is also weakly quasi-uniformly continuous.

Proof.- Putting $S = R \times R$, it suffices to show that

$$S^*(V) \subset (\dot{R} \times \dot{R})^{-1}(\tilde{V})$$

and that $(\dot{R} \times \dot{R})^{-1}(V^*) \subset S_*(V)$ (resp. $(\dot{R} \times \dot{R})^{-1}(V_*) \subset S_*(V)$) for each entourage V of Y.

Suppose first that $R(x) \times R(y) \subset V$. Since $R(x) \neq \emptyset$, we can choose a point $z \in R(x)$. Then, for each $w \in R(y)$, we have $(z,w) \in V$, hence $w \in V(z) \subset V(R(x))$, so that $R(y) \subset V(R(x))$.

Similarly, $R(x) \subset V^{-1}(R(y))$.

Next, suppose that $R(y) \subset V(R(x))$. Since $R(y) \neq \emptyset$, we can choose a point $w \in R(y)$. Hence $w \in V(R(x))$, so there exists a point $z \in R(x)$ such that $w \in V(z)$. It follows that $(z,w) \in (R(x) \times R(y)) \cap V$.

Similarly, the relation $R(x) \subset V^{-1}(R(y))$ implies $(R(x) \times R(y)) \cap V \neq \emptyset$.

Finally, let us note the following result on composites:

2.19 THEOREM.- Let $R: X|Y$ and $S: Y|Z$ be correspondences between quasi-uniform spaces. If R and S are both strongly (resp. weakly) quasi-uniformly continuous, then so is $S \circ R$.

Proof.- It suffices to use the relation

$$(S \circ R) \times (S \circ R) = (S \times S) \circ (R \times R)$$

of 1.15. Thus, in the case of strong quasi-uniform continuity, we have $((S \circ R) \times (S \circ R))^*(W) = (R \times R)^*((S \times S)^*(W))$ for each entourage W of Z.

Compact, precompact and bounded unions

We now deal with results that go along the same lines as theorem 2.8 on compact unions. We start with a generalization to quasi-uniform spaces of the statement that compact unions of closed sets are closed in a uniform space.

2.20 THEOREM.- Let $(X,\mathcal{U})$ be a quasi-uniform space, and suppose that $\mathcal{Q}\subset\mathcal{P}(X)$ is compact with respect to its quasi-uniformity $(\mathcal{U}^{-1})^*$. If each $A\in\mathcal{Q}$ is closed, then $A_o = \bigcup_{A\in\mathcal{Q}} A$ is also closed.

Proof.- Suppose that $x_o\in X\setminus A_o$; we show that $x_o\notin \overline{A}_o$. Now, for each $A\in\mathcal{Q}$, $x_o\notin A$ and so there exists an entourage U_A of X such that $x_o\notin U_A^{-1}(A)$. Then, there exists an entourage V_A of X with $V_A^2\subset U_A$. Now, the set $(V_A^{-1})^*(A)$ is a neighbourhood of the point A in $\mathcal{P}(X)$, for each $A\in\mathcal{Q}$. Since $\mathcal{Q}$ is compact, there exists a finite subset $\mathcal{B}$ of $\mathcal{Q}$ such that $\mathcal{Q}\subset\bigcup_{A\in\mathcal{B}}(V_A^{-1})^*(A)$. It follows that $A_o\subset\bigcup_{A\in\mathcal{B}}V_A^{-1}(A)$. Putting $V=\bigcap_{A\in\mathcal{B}}V_A$, we see that $V\in\mathcal{U}$ and that $V^{-1}(A_o)\subset V^{-1}(\bigcup_{A\in\mathcal{B}}V_A^{-1}(A))=\bigcup_{A\in\mathcal{B}}V^{-1}(V_A^{-1}(A))\subset\bigcup_{A\in\mathcal{B}}U_A^{-1}(A)$. Hence, $x_o\notin V^{-1}(A_o)$, and the result follows.

The following theorem, which constitutes a new result even for uniform spaces, deals with precompact (resp. bounded) sets. A quasi-uniform space $(X,\mathcal{U})$ is said to be precompact if, for each $U\in\mathcal{U}$, there exists a finite set $F\subset X$ with $X = U(F)$. It is said to be bounded if, for each $U\in\mathcal{U}$, there exist a finite set $F\subset X$ and an integer $n>0$ with $X = U^n(F)$. Precompactness and boundedness are preserved under quasi-uniformly continuous mappings.

2.21 THEOREM.- Let X be a quasi-uniform space, and suppose that $\mathcal{Q}\subset\mathcal{P}(X)$ is precompact (resp. bounded) with respect to its upper quasi-uniformity. If each $A\in\mathcal{Q}$ is precompact (resp. bounded) then so is $A_o = \bigcup_{A\in\mathcal{Q}} A$.

Proof.-

The precompact case: Let U be any entourage of X. There exists an entourage V such that $V^2 \subset U$. Since $\mathcal{A}$ is precompact, there exists a finite subset $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A} \subset V^*(\mathcal{B})$. Now each set $B \in \mathcal{B}$ is precompact, and so there exists a finite subset F_B of B such that $B \subset V(F_B)$. Then, the set $F = \bigcup_{B \in \mathcal{B}} F_B$ is a finite subset of A_o. Moreover, we claim that $A_o \subset U(F)$. Indeed, for each $A \in \mathcal{A}$ there exists a set $B \in \mathcal{B}$ with $A \in V^*(B)$. But then

$$A \subset V(B) \subset V(V(F_B)) \subset V^2(F) \subset U(F).$$

The bounded case: Let U be any entourage of A_o. Since $\mathcal{A}$ is bounded, there exist a finite subset $\mathcal{B}$ of $\mathcal{A}$ and an integer $N > 0$ such that $\mathcal{A} \subset (U^*)^N(\mathcal{B})$. Now each set $B \in \mathcal{B}$ is bounded and so there exist a finite subset F_B of B and an integer $n_B > 0$ such that $B \subset U^{n_B}(F_B)$. Then, the set $F = \bigcup_{B \in \mathcal{B}} F_B$ is a finite subset of A_o. Put

$$n = N + \max\{n_B : B \in \mathcal{B}\};$$

we claim that $A_o \subset U^n(F)$. Indeed, for each $A \in \mathcal{A}$ there exists a set $B \in \mathcal{B}$ with $A \in (U^*)^N(B)$. But then, since $(U^*)^N \subset (U^N)^*$, we have

$$A \subset U^N(B) \subset U^N(U^{n_B}(F_B)) \subset U^{N+n_B}(F) \subset U^n(F).$$

Using restricted canonical extensions, as in the proof of 2.9, we can deduce:

2.22 THEOREM.- Let $R: X|Y$ be a quasi-uniformly upper semi-continuous correspondence between quasi-uniform spaces. If $A \subset X$ is precompact (resp. bounded) and $R(x)$ is precompact (resp. bounded) for each $x \in A$, then $R(A)$ is precompact (resp. bounded).

Relations between topologies and quasi-uniformities

Let $(X, \mathcal{U})$ be a quasi-uniform space. Our aim will now consist in finding out when the finite topology $\widetilde{\mathcal{T}(\mathcal{U})}$ induced by the topology $\mathcal{T}(\mathcal{U})$ of X coincides with the topology $\mathcal{T}(\widetilde{\mathcal{U}})$ induced by the Bourbaki quasi-uniformity.

We first have the following two theorems:

2.23 THEOREM.-

a) $\mathcal{T}(\mathcal{U}_*)$ is finer than $(\mathcal{T}(\mathcal{U}))_*$.

b) $\mathcal{T}(\mathcal{U}^*)$ is coarser than $(\mathcal{T}(\mathcal{U}))^*$.

Proof.-

a) We show that the set $\mathcal{S}(G)$ is open in $\mathcal{P}(X)$ with respect to the topology $\mathcal{T}(\mathcal{U}_*)$ for each $\mathcal{T}(\mathcal{U})$-open subset G of X. For this, let $A_0 \in \mathcal{S}(G)$, and choose a point $x_0 \in A_0 \cap G$. Since G is $\mathcal{T}(\mathcal{U})$-open, there exists an entourage $U \in \mathcal{U}$ with $U(x_0) \subset G$. Now $U_*(A_0)$ is a $\mathcal{T}(\mathcal{U}_*)$-neighbourhood of A_0 in $\mathcal{P}(X)$. We claim that $U_*(A_0) \subset \mathcal{S}(G)$. Indeed, if $A \in U_*(A_0)$, then $A_0 \subset U^{-1}(A)$ and so, in particular, there exists a point $x \in A$ with $x_0 \in U^{-1}(x)$. Thus, $x \in A \cap U(x_0)$, showing that $A \in \mathcal{S}(U(x_0)) \subset \mathcal{S}(G)$.

b) We show that, for each $A_0 \in \mathcal{P}(X)$, and each entourage $U \in \mathcal{U}$, the set $U^*(A_0)$ is a $(\mathcal{T}(\mathcal{U}))^*$-neighbourhood of A_0 in $\mathcal{P}(X)$. Indeed, the set $U(A_0)$ is a $\mathcal{T}(\mathcal{U})$-neighbourhood of A_0 in X, and so $\mathcal{P}(U(A_0))$ is a $(\mathcal{T}(\mathcal{U}))^*$-neighbourhood of A_0 in $\mathcal{P}(X)$. Since $U^*(A_0) = \mathcal{P}(U(A_0))$, the result follows.

2.24 THEOREM.- Let $A_0 \in \mathcal{P}(X)$.

a) In order that every $(\mathcal{T}(\mathcal{U}))^*$-neighbourhood of A_0 in $\mathcal{P}(X)$ contains a $\mathcal{T}(\mathcal{U}^*)$-neighbourhood of A_0, it suffices that the sets $U(A_0)$ form a base of neighbourhoods of A_0 in X, as U runs through $\mathcal{U}$.

b) In order that every $\mathcal{T}(\mathcal{U}_*)$-neighbourhood of A_0 in $\mathcal{P}(X)$ contains a $(\mathcal{T}(\mathcal{U}))_*$-neighbourhood of A_0, it suffices that A_0 be $\mathcal{U}^{-1}$-precompact.

Proof.-

a) We show that the set $\mathcal{P}(G)$ contains a $\mathfrak{T}(\mathcal{U}^*)$-neighbourhood of A_0 in $\mathcal{P}(X)$ for each $\mathfrak{T}(\mathcal{U})$-open subset G of X with $A_0 \in \mathcal{P}(G)$. Now, since $A_0 \subset G$, by hypothesis there exists an entourage $U \in \mathcal{U}$ such that $U(A_0) \subset G$. But then, $U^*(A_0)$ is a $\mathfrak{T}(\mathcal{U}^*)$-neighbourhood of A_0 in $\mathcal{P}(X)$. Also, $U^*(A_0) \subset \mathcal{P}(G)$, since the relation $A \in U^*(A_0)$ implies

$$A \subset U(A_0) \subset G.$$

b) We show that, for each entourage $U \in \mathcal{U}$, the set $U_*(A_0)$ contains a $(\mathfrak{T}(\mathcal{U}))_*$-neighbourhood of A_0 in $\mathcal{P}(X)$. Now, there exists an entourage $V \in \mathcal{U}$ with $V^2 \subset U$. Since A_0 is $\mathcal{U}^{-1}$-precompact, there exists a finite family $(a_i)_{i \in I}$ of elements of A_0 such that $A_0 \subset \bigcup_{i \in I} V^{-1}(a_i)$. Now $V(a_i)$ is a $\mathfrak{T}(\mathcal{U})$-neighbourhood of a_i for each $i \in I$, and so the set $\mathcal{O} = \bigcap_{i \in I} \mathcal{J}(V(a_i))$ is a $(\mathfrak{T}(\mathcal{U}))_*$-neighbourhood of A_0 in $\mathcal{P}(X)$. We now show that $\mathcal{O} \subset U_*(A_0)$. So let $A \in \mathcal{O}$. For each $x_0 \in A_0$, there exists an index $i \in I$ with $x_0 \in V^{-1}(a_i)$, and then $(x_0, a_i) \in V$. Now $A \in \mathcal{J}(V(a_i))$, and so we can choose a point $x \in A \cap V(a_i)$. It follows that $(a_i, x) \in V$, and hence $(x_0, x) \in V^2 \subset U$, showing that $x_0 \in U^{-1}(x) \subset U^{-1}(A)$. We have thus shown that $A_0 \subset U^{-1}(A)$, i.e. $A \in U_*(A_0)$, as required.

<u>2.25</u> COROLLARY.-

a) Let $\mathcal{A}$ be the set of all subsets A of X which have the property that the sets $U(A)$ form a base of neighbourhoods of A in X, as U runs through $\mathcal{U}$. Then the topologies $(\mathfrak{T}(\mathcal{U}))^*$ and $\mathfrak{T}(\mathcal{U}^*)$ coincide on $\mathcal{A}$.

b) Let $\mathcal{A}$ be the set of all $\mathcal{U}^{-1}$-precompact subsets of X. Then the topologies $(\mathfrak{T}(\mathcal{U}))_*$ and $\mathfrak{T}(\mathcal{U}_*)$ coincide on $\mathcal{A}$.

It follows that, on the set $\mathcal{A}$ of part a) above (which includes the $\mathfrak{T}(\mathcal{U})$-compact subsets of X), the topology $\widehat{\mathfrak{T}(\mathcal{U})}$ is coarser than $\mathfrak{T}(\widetilde{\mathcal{U}})$. Indeed, $(\mathfrak{T}(\mathcal{U}))_*$ is coarser than $\mathfrak{T}(\mathcal{U}_*)$

by 2.23a, and so $\widetilde{\mathfrak{T}(\mathcal{U})} = (\mathfrak{T}(\mathcal{U}))^* \vee (\mathfrak{T}(\mathcal{U}))_*$ is coarser than $\mathfrak{T}(\mathcal{U}^*) \vee \mathfrak{T}(\mathcal{U}_*) = \mathfrak{T}(\tilde{\mathcal{U}})$ on $\mathcal{Q}$. Similarly, it follows that $\widetilde{\mathfrak{T}(\mathcal{U})}$ is finer than $\mathfrak{T}(\tilde{\mathcal{U}})$ on the set of $\mathcal{U}^{-1}$-precompact subsets of X.

In the next two theorems, we obtain converses of 2.24.

2.26 THEOREM.- Let $A_o \in \mathcal{P}(X)$. In order that every $(\mathfrak{T}(\mathcal{U}))^*$-neighbourhood of A_o in $\mathcal{P}(X)$ contains a $\mathfrak{T}(\tilde{\mathcal{U}})$-neighbourhood of A_o, it is necessary that the sets $U(A_o)$ form a base of neighbourhoods of A_o in X, as U runs through $\mathcal{U}$.

Proof.- Let G be any $\mathfrak{T}(\mathcal{U})$-open subset of X containing A_o. Then $\mathcal{P}(G)$ is a $(\mathfrak{T}(\mathcal{U}))^*$-neighbourhood of A_o in $\mathcal{P}(X)$. By hypothesis, there exists an entourage $U \in \mathcal{U}$ such that $\tilde{U}(A_o) \subset \mathcal{P}(G)$. We now claim that $U(A_o) \subset G$. Indeed, suppose, to get a contradiction, that $U(A_o) \not\subset G$, and choose a point $x_o \in U(A_o) \setminus G$. Letting $A = A_o \cup \{x_o\}$, we see that $A \subset U(A_o)$ and $A_o \subset U^{-1}(A_o) \subset U^{-1}(A)$, so that $A \in \tilde{U}(A_o)$. It follows that $A \in \mathcal{P}(G)$, i.e. $A \subset G$, contradicting the fact that $x_o \notin G$.

2.27 COROLLARY.- In order that every $(\mathfrak{T}(\mathcal{U}))^*$- (resp. $\widetilde{\mathfrak{T}(\mathcal{U})}$-) neighbourhood of A_o in $\mathcal{P}(X)$ contains a $\mathfrak{T}(\mathcal{U}^*)$- (resp. $\mathfrak{T}(\tilde{\mathcal{U}})$-) neighbourhood of A_o, it is necessary and sufficient that the sets $U(A_o)$ form a base of neighbourhoods of A_o in X, as U runs through $\mathcal{U}$.

2.28 THEOREM.- Let $\mathcal{D}$ be the discrete topology on X, and let $A_o \in \mathcal{P}(X)$. In order that every $\mathfrak{T}(\mathcal{U}_*)$-neighbourhood of A_o in $\mathcal{P}(X)$ contains a $\tilde{\mathcal{D}}$-neighbourhood of A_o, it is necessary that A_o be $\mathcal{U}^{-1}$-precompact.

Proof.- Let U be any entourage in $\mathcal{U}$. Then $U_*(A_o)$ is a $\mathfrak{T}(\mathcal{U}_*)$-neighbourhood of A_o in $\mathcal{P}(X)$. By hypothesis, there exist a subset G_o of X and a finite family $(G_i)_{i\in I}$ of subsets of X such that, putting $\mathcal{O} = \mathcal{P}(G_o) \cap \bigcap_{i\in I} \mathcal{J}(G_i)$, we have $A_o \in \mathcal{O} \subset U_*(A_o)$. For each $i \in I$, choose a point x_i in $A_o \cap G_i$, and let $C = \{x_i : i \in I\}$. Then clearly $C \in \mathcal{O}$, and hence $C \in U_*(A_o)$. Thus, $A_o \subset U^{-1}(C)$, showing that A_o is $\mathcal{U}^{-1}$-precompact.

<u>2.29</u> COROLLARY.- In order that every $\mathcal{T}(\mathcal{U}_*)$- (resp. $\mathcal{T}(\tilde{\mathcal{U}})$-) neighbourhood of A_o in $\mathcal{P}(X)$ contains a $(\mathcal{T}(\mathcal{U}))_*$- (resp. $\widetilde{\mathcal{T}(\mathcal{U})}$-) neighbourhood of A_o, it is necessary and sufficient that A_o is $\mathcal{U}^{-1}$-precompact.

The results 2.26 - 2.29 could also have been formulated and proved for the <u>space</u> $\mathcal{F}(X)$ <u>of all closed subsets of</u> X instead of the whole space $\mathcal{P}(X)$, provided it is further assumed that X is a T_1-space. Indeed, if A_o is assumed to be closed in 2.26, then the set A constructed in the proof will also be closed. Similarly, the set C constructed in the proof of 2.28 will be closed.

We conclude with a result which compares the topologies $\mathcal{T}(\mathcal{U}_*)$ and $(\mathcal{T}(\mathcal{U}))^*$.

<u>2.30</u> THEOREM.- Let $(X,\mathcal{U})$ be a quasi-uniform T_1-space with at least two distinct points. Then the topology $\mathcal{T}(\mathcal{U}_*)$ of $\mathcal{P}(X)$ is <u>not</u> coarser than the topology $(\mathcal{T}(\mathcal{U}))^*$.

Proof.- It suffices to show that there is a point A_o in $\mathcal{P}(X)$ and an entourage U in $\mathcal{U}$ such that, for each open subset G of X with $A_o \in \mathcal{P}(G)$, the set $U_*(A_o)$ does not contain the set $\mathcal{P}(G)$. We claim that we can take $A_o = X$. Indeed, since every open subset of X containing X must be equal to X itself, and since

$$U_*(X) = \{A \in \mathcal{P}(X) : X \subset U^{-1}(A)\} = \{A \in \mathcal{P}(X) : U^{-1}(A) = X\}$$

for each entourage $U \in \mathcal{U}$, what we have to show reduces to the following: there exists an entourage $U \in \mathcal{U}$ such that the set $\{A \in \mathcal{P}(X) : U^{-1}(A) = X\}$ is different from $\mathcal{P}(X)$.

For this, let x_o, x_1 be two distinct points of X. Since X is a T_1-space, there exists an entourage $U \in \mathcal{U}$ such that $x_o \notin U(x_1)$. Then, $x_1 \notin U^{-1}(x_o)$, so that $U^{-1}(x_o) \neq X$. Hence the result.

Again, this result is valid for the space $\mathcal{F}(X)$ instead of $\mathcal{P}(X)$, since X is clearly a closed subset of itself, and since the one-point set $\{x_o\}$ constructed in the proof above is also closed, X being assumed a T_1-space.

Pseudometrizability

Having split up the Bourbaki uniformity into two other structures, it seems natural to attempt something similar with the Hausdorff pseudometric.

A quasi-pseudometric on a set X is defined with the same axioms as for a pseudometric, except that we no longer require symmetry. Thus, if d is a quasi-pseudometric on X, the relation $d(x,y) = d(y,x)$ does not necessarily hold for each $x,y \in X$.

Just as for a pseudometric, the sets

$$V_\varepsilon = V_\varepsilon^d = \{(x,y) \in X \times X\colon d(x,y) < \varepsilon\}$$ form a base for a quasi-uniformity $\mathcal{U}(d)$ on X. We also use the notation

$$W_\varepsilon = W_\varepsilon^d = \{(x,y) \in X \times X\colon d(x,y) \leq \varepsilon\}.$$

The conjugate d^{-1} of d is defined by the relation

$$d^{-1}(x,y) = d(y,x) \quad \text{for each} \quad (x,y) \in X \times X.$$

We have $(\mathcal{U}(d))^{-1} = \mathcal{U}(d^{-1})$, i.e. quasi-uniformities induced by conjugate quasi-pseudometrics are themselves conjugate.

If d_1, d_2 are two quasi-pseudometrics on a set X, then the join $d_1 \vee d_2$ of d_1 and d_2 is defined by the relation

$$(d_1 \vee d_2)(x,y) = \sup\{d_1(x,y),\ d_2(x,y)\}$$

for each $(x,y) \in X \times X$. We have $\mathcal{U}(d_1 \vee d_2) = \mathcal{U}(d_1) \vee \mathcal{U}(d_2)$.

If a quasi-pseudometric d is a pseudometric, we say that it is symmetric. Thus, the quasi-pseudometric $d \vee d^{-1}$ is symmetric for each quasi-pseudometric d.

In what follows, (X,d) will denote an arbitrary quasi-pseudometric space.

<u>2.31</u> THEOREM.- The function $d^*: \mathcal{P}(X)\times\mathcal{P}(X)\longrightarrow[0,\infty]$, defined by
$$d^*(A,B) = \begin{cases} 0, \text{ if } B=\emptyset \\ \sup\limits_{y\in B} d(A,y), \text{ if } B\neq\emptyset \end{cases},$$
is a quasi-pseudometric on $\mathcal{P}(X)$.

N.B.: We use the convention $\inf\emptyset=\infty$, so that
$$d(\emptyset,y) = \inf_{x\in\emptyset} d(x,y) = \infty \quad \text{for each } y\in B.$$

Proof.- It is clear that $d^*(A,A) = 0$ for each $A\in\mathcal{P}(X)$. We now show that the triangle inequality holds. For this, let $A,B,C\in\mathcal{P}(X)$; we want to show that
$$d^*(A,C)\leq d^*(A,B) + d^*(B,C).$$
Suppose first that $C=\emptyset$. Then we have $d^*(A,C) = 0$, so that inequality trivially holds. Suppose now that $C\neq\emptyset$. If also $B=\emptyset$, then $d^*(B,C) = \infty$, so that inequality trivially holds. If on the other hand $B\neq\emptyset$, then we can show that $d(A,c)\leq d^*(A,B) + d^*(B,C)$ for each $c\in C$. Indeed, let $\varepsilon>0$. Then there exists a point $b\in B$ such that
$$d(b,c)\leq d(B,c) + \varepsilon.$$
Since $d(x,c)\leq d(x,b) + d(b,c)$ for each $x\in A$, it follows that
$$d(A,c)\leq d(A,b) + d(b,c)\leq d(A,b) + d(B,c) + \varepsilon$$
$$\leq d^*(A,B) + d^*(B,C) + \varepsilon,$$
hence the result.

It follows immediately from this theorem that the function $d_*: \mathcal{P}(X)\times\mathcal{P}(X)\longrightarrow[0,\infty]$, defined by
$$d_*(A,B) = \begin{cases} 0, \text{ if } A=\emptyset \\ \sup\limits_{x\in A} d(x,B), \text{ if } A\neq\emptyset \end{cases},$$
is a quasi-pseudometric on $\mathcal{P}(X)$, and that the relations $(d^*)^{-1} = (d^{-1})_*$ and $(d_*)^{-1} = (d^{-1})^*$ hold.

Finally, the function $\tilde{d}: \mathcal{P}(X)\times\mathcal{P}(X)\longrightarrow[0,\infty]$, defined by
$$\tilde{d}(A,B) = \begin{cases} 0, \text{ if } A=B=\emptyset \\ \sup\left\{\sup\limits_{y\in B} d(A,y),\ \sup\limits_{x\in A} d(x,B)\right\}, \text{ otherwise} \end{cases},$$

is a quasi-pseudometric on $\mathcal{P}(X)$. Furthermore, $\tilde{d} = d^* \vee d_*$.

Note that, if d is symmetric, then d^* and d_* are conjugate, hence $\tilde{d}$ is also symmetric.

With the notation introduced above, we call d^* (resp. d_*) the <u>upper</u> (resp. <u>lower</u>) <u>quasi-pseudometric</u> induced by d on $\mathcal{P}(X)$. We call $\tilde{d}$ the <u>Hausdorff</u> <u>quasi-pseudometric</u> induced by d on $\mathcal{P}(X)$.

The three quasi-pseudometrics introduced above are <u>admissible</u>, in the sense that the mapping $x \mapsto \{x\}$ is an isomorphism of X onto a subspace of $\mathcal{P}(X)$. Indeed, it is clear that $d^*(\{x\},\{y\}) = d_*(\{x\},\{y\}) = \tilde{d}(\{x\},\{y\}) = d(x,y)$ for each $(x,y) \in X \times X$.

We now show that the quasi-uniformity induced by $\tilde{d}$ is precisely $\widetilde{\mathcal{U}(d)}$. But first:

<u>2.32</u> LEMMA.- For each $\varepsilon > 0$, we have $V_\varepsilon^{d^*} \subset (V_\varepsilon^d)^* \subset W_\varepsilon^{d^*}$.

Proof.-

Ad $V_\varepsilon^{d^*} \subset (V_\varepsilon^d)^*$: Suppose $(A,B) \in V_\varepsilon^{d^*}$, i.e. $d^*(A,B) < \varepsilon$. We have to show that $(A,B) \in (V_\varepsilon^d)^*$, i.e. $B \subset V_\varepsilon^d(A)$. This is obvious if $B = \emptyset$. If $B \neq \emptyset$, then $d^*(A,B) = \sup_{y \in B} d(A,y) < \varepsilon$, and so $d(A,y) < \varepsilon$ for each $y \in B$. Thus, for each $y \in B$, there exists a point $x \in A$ such that $d(x,y) < \varepsilon$, i.e. $y \in V_\varepsilon^d(x) \subset V_\varepsilon^d(A)$, as required.

Ad $(V_\varepsilon^d)^* \subset W_\varepsilon^{d^*}$: Suppose $(A,B) \in (V_\varepsilon^d)^*$, i.e. $B \subset V_\varepsilon^d(A)$. We have to show that $(A,B) \in W_\varepsilon^{d^*}$, i.e. $d^*(A,B) \leq \varepsilon$. This is obvious if $B = \emptyset$. If $B \neq \emptyset$, then we have to show that $d^*(A,B) = \sup_{y \in B} d(A,y) \leq \varepsilon$. Now, since $B \subset V_\varepsilon^d(A)$, for each $y \in B$ there exists a point $x \in A$ with $y \in V_\varepsilon^d(x)$, i.e. $d(x,y) < \varepsilon$; hence $d(A,y) < \varepsilon$ for each $y \in B$, and the result follows.

<u>2.33</u> THEOREM.- We have $(\mathcal{U}(d))^* = \mathcal{U}(d^*)$, $(\mathcal{U}(d))_* = \mathcal{U}(d_*)$ and $\widetilde{\mathcal{U}(d)} = \mathcal{U}(\tilde{d})$.

Proof.- The relation $(\mathcal{U}(d))^* = \mathcal{U}(d^*)$ follows directly from the lemma. Indeed, the sets $(V_\varepsilon^d)^*$ form a base for $(\mathcal{U}(d))^*$, whereas the sets $V_\varepsilon^{d^*}$ (resp. $W_\varepsilon^{d^*}$) form a base for $\mathcal{U}(d^*)$.

Then, we have $((\mathcal{U}(d))^{-1})^* = (\mathcal{U}(d^{-1}))^* = \mathcal{U}((d^{-1})^*)$
$= \mathcal{U}((d_*)^{-1}) = (\mathcal{U}(d_*))^{-1}$,

hence $(\mathcal{U}(d))_* = \mathcal{U}(d_*)$. Finally,

$$\widetilde{\mathcal{U}(d)} = (\mathcal{U}(d))^* \vee (\mathcal{U}(d))_* = \mathcal{U}(d^*) \vee \mathcal{U}(d_*) = \mathcal{U}(d^* \vee d_*) = \mathcal{U}(\tilde{d}).$$

2.34 THEOREM.-

a) If d is finite, then d^*, d_* and $\tilde{d}$ are all finite on the set of nonempty bounded subsets of X.

b) For each $A, B \in \mathcal{P}(X)$, we have

$d^*(A,B) = 0$ iff $B \subset \bar{A}$, where closure is taken with respect to d^{-1},

and $d_*(A,B) = 0$ iff $A \subset \bar{B}$, where closure is taken with respect to d.

c) If d is finite and symmetric, then $\tilde{d}$ is a metric on the set of nonempty closed bounded subsets of X.

Proof.- Part a) is clear. To prove part b), it suffices to show the first equality only. Now

$d^*(A,B) = 0$ iff $d(A,y) = 0$ for each $y \in B$
iff for each $y \in B$ and each $\varepsilon > 0$, there exists a point $x \in A$ with $d(x,y) < \varepsilon$, i.e. $y \in V_\varepsilon^d(x)$
iff $y \in V_\varepsilon^d(A)$ for each $y \in B$ and each $\varepsilon > 0$
iff $B \subset \bigcap_{\varepsilon>0} V_\varepsilon^d(A) = \bigcap_{\varepsilon>0} (V_\varepsilon^{d^{-1}})^{-1}(A) = \bar{A}$.

Finally, part c) is clear, since

$\tilde{d}(A,B) = 0 \Longrightarrow d^*(A,B) = d_*(A,B) = 0$
$\Longrightarrow B \subset \bar{A}$ and $A \subset \bar{B}$ by part b).

Chapter 3

TOPOLOGICAL PROPERTIES OF CORRESPONDENCES

§1. PROPERTIES OF UNIONS AND PRODUCTS

Pervin's quasi-uniformity

As mentioned at the beginning of chapter 2, we intend to show how quasi-uniformities on hyperspaces can be used to prove results about topologies on hyperspaces and about continuous correspondences.

The first step in this direction is to note that every topological space is quasi-uniformizable. In [10], W.J. Pervin shows how a topology $\mathcal{T}$ on a set X induces a quasi-uniformity $\mathcal{U}(\mathcal{T})$ on X with the property that the topology induced by $\mathcal{U}(\mathcal{T})$ is equal to $\mathcal{T}$ itself. We shall refer to $\mathcal{U}(\mathcal{T})$ as Pervin's quasi-uniformity induced by $\mathcal{T}$ on X.

For convenience, we recall how $\mathcal{U}(\mathcal{T})$ is constructed. If X is a set, we write $U_X(A) = (A \times A) \cup ((X \setminus A) \times X)$ for each subset $A \subset X$. If $\mathcal{T}$ is a topology on X, then the sets $U_X(G)$, with G running through $\mathcal{T}$, form a subbase for $\mathcal{U}(\mathcal{T})$. If we only let G run through a subbase of $\mathcal{T}$, the sets $U_X(G)$ still generate $\mathcal{U}(\mathcal{T})$.

The following is a useful property of Pervin's quasi-uniformity: if $\mathcal{T}, \mathcal{T}'$ are two topologies on X, then

$$\mathcal{U}(\mathcal{T} \vee \mathcal{T}') = \mathcal{U}(\mathcal{T}) \vee \mathcal{U}(\mathcal{T}').$$

Indeed, the set $\mathcal{T}\cup\mathcal{T}'$ is a subbase for $\mathcal{T}\vee\mathcal{T}'$. Hence, the sets $U_X(G)$, with G running through $\mathcal{T}\cup\mathcal{T}'$, generate $\mathcal{U}(\mathcal{T}\vee\mathcal{T}')$. But the sets $U_X(G)$ also generate $\mathcal{U}(\mathcal{T})\vee\mathcal{U}(\mathcal{T}')$.

It also follows from the relation just proved, in particular, that $\mathcal{T}$ is finer than $\mathcal{T}'$ iff $\mathcal{U}(\mathcal{T})$ is finer than $\mathcal{U}(\mathcal{T}')$. Indeed, if $\mathcal{T}$ is finer than $\mathcal{T}'$, then $\mathcal{T}\vee\mathcal{T}' = \mathcal{T}$ and so $\mathcal{U}(\mathcal{T})\vee\mathcal{U}(\mathcal{T}') = \mathcal{U}(\mathcal{T}\vee\mathcal{T}') = \mathcal{U}(\mathcal{T})$, showing that $\mathcal{U}(\mathcal{T})$ is finer than $\mathcal{U}(\mathcal{T}')$.

As far as hyperspaces are concerned, the most useful result is that Pervin's quasi-uniformity is compatible with the formation of quasi-uniformities in hyperspaces. Indeed, N. Levine and W.J. Stager have shown in [7], theorems 2.1.2 and 2.1.3, that the sets $(U_X(G))^*$ (resp. $(U_X(G))_*$) generate $(\mathcal{U}(\mathcal{T}))^*$ (resp. $(\mathcal{U}(\mathcal{T}))_*$), as G runs through $\mathcal{T}$, and that $(U_X(G))^* = U_{\mathcal{P}(X)}(\mathcal{G}(G))$ (resp. $(U_X(G))_* = U_{\mathcal{P}(X)}(\mathcal{J}(G))$). This proves parts a) and b) of the following theorem. Using this, we then have

$$\widetilde{\mathcal{U}(\mathcal{T})} = (\mathcal{U}(\mathcal{T}))^*\vee(\mathcal{U}(\mathcal{T}))_* = \mathcal{U}(\mathcal{T}^*)\vee\mathcal{U}(\mathcal{T}_*) = \mathcal{U}(\mathcal{T}^*\vee\mathcal{T}_*) = \mathcal{U}(\widetilde{\mathcal{T}}),$$

proving part c).

<u>3.1</u> THEOREM.- Let $(X,\mathcal{T})$ be a topological space. Then we have:

a) $(\mathcal{U}(\mathcal{T}))^* = \mathcal{U}(\mathcal{T}^*)$.
b) $(\mathcal{U}(\mathcal{T}))_* = \mathcal{U}(\mathcal{T}_*)$.
c) $\widetilde{\mathcal{U}(\mathcal{T})} = \mathcal{U}(\widetilde{\mathcal{T}})$.

<u>3.2</u> COROLLARY.- The topology induced by $(\mathcal{U}(\mathcal{T}))^*$ (resp. $(\mathcal{U}(\mathcal{T}))_*$) (resp. $\widetilde{\mathcal{U}(\mathcal{T})}$) is equal to $\mathcal{T}^*$ (resp. $\mathcal{T}_*$) (resp. $\widetilde{\mathcal{T}}$).

We now give an example of the use of 3.2, and show how the following two known results can be deduced from a single result about quasi-uniform spaces:

<u>3.3</u> THEOREM.- Let $(X,\mathcal{T})$ be a topological space. Then the topological space $(\mathcal{F}(X),\mathcal{T}_*)$ is a T_0-space.

3.4 THEOREM.- Let $(X,\mathcal{U})$ be a uniform space. Then the uniform space $(\mathfrak{F}(X),\tilde{\mathcal{U}})$ is Hausdorff.

In order to do this, we use the fact that, in quasi-uniform spaces, the T_0 separation axiom can be characterized in terms of entourages only: a quasi-uniform space $(X,\mathcal{U})$ with base $\mathcal{B}$ for its set of entourages $\mathcal{U}$ is a T_0-space iff the set $\bigcap_{U\in\mathcal{B}} U$ is anti-symmetric (cf. [9], theorem 3.1). We now prove:

3.5 THEOREM.- Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the quasi-uniform space $(\mathfrak{F}(X),\mathcal{U}_*)$ is a T_0-space.

Proof.- We show that the set $\mathcal{Q} = \bigcap_{U\in\mathcal{U}} U_* \cap (\mathfrak{F}(X)\times\mathfrak{F}(X))$ is anti-symmetric. Now, using the relation $(U_*)^{-1} = (U^{-1})^*$, we obtain $\mathcal{Q}^{-1} = \bigcap_{U\in\mathcal{U}} (U^{-1})^* \cap (\mathfrak{F}(X)\times\mathfrak{F}(X))$. Thus, if $(A,B)\in\mathcal{Q}\cap\mathcal{Q}^{-1}$, we have $A\subset\bigcap_{U\in\mathcal{U}} U^{-1}(B) = \bar{B}$ and $B\subset\bigcap_{U\in\mathcal{U}} U^{-1}(A) = \bar{A}$. Since A and B are closed, it follows that $A = B$.

Proof of 3.3.- By 3.5, the quasi-uniform space $(\mathfrak{F}(X),(\mathcal{U}(\mathfrak{T}))_*)$ is a T_0-space. Since the topology induced by $(\mathcal{U}(\mathfrak{T}))_*$ is equal to $\mathfrak{T}_*$, the result follows.

Proof of 3.4.- Since $\mathcal{U}_*$ is coarser than $\tilde{\mathcal{U}}$, the conclusion in 3.5 holds with $\tilde{\mathcal{U}}$ in place of $\mathcal{U}_*$. Thus, the uniform space $(\mathfrak{F}(X),\tilde{\mathcal{U}})$ is a T_0-space, hence it is Hausdorff.

Pervin's quasi-uniformity enables us to consider topological spaces as a special sort of quasi-uniform space. As the following result shows, it does much more than that: it transforms the morphisms of topological spaces into those of the associated quasi-uniform spaces. More precisely:

3.6 THEOREM.- Let $(X,\mathcal{T})$ and $(X',\mathcal{T}')$ be two topological spaces. Then a mapping $f: X \to X'$ is $(\mathcal{T},\mathcal{T}')$-continuous iff it is $(\mathcal{U}(\mathcal{T}),\mathcal{U}(\mathcal{T}'))$-quasi-uniformly continuous.

Proof.- If f is quasi-uniformly continuous, then it is continuous, since the topology induced by $\mathcal{U}(\mathcal{T})$ (resp. $\mathcal{U}(\mathcal{T}')$) is equal to $\mathcal{T}$ (resp. $\mathcal{T}'$). Now suppose f is continuous. To show that it is quasi-uniformly continuous, it suffices to show that $(f\times f)^{-1}(U_{X'}(G))$ is an entourage in X for each open subset G of X'. Now, since f is continuous, $f^{-1}(G)$ is an open subset of X. Thus, the result will follow if we can show that $(f\times f)^{-1}(U_{X'}(G)) = U_X(f^{-1}(G))$. Now, since $U_{X'}(G) = (G\times G)\cup((X'\setminus G)\times X')$, we have

$$(f\times f)^{-1}(U_{X'}(G)) = (f^{-1}(G)\times f^{-1}(G))\cup((X\setminus f^{-1}(G))\times X)$$
$$= U_X(f^{-1}(G)), \text{ as desired.}$$

3.7 COROLLARY.- If the quasi-uniformity $\mathcal{U}(\mathcal{T})$ is the inverse image of $\mathcal{U}(\mathcal{T}')$ under the mapping f, then the topology $\mathcal{T}$ is the inverse image of $\mathcal{T}'$.

As an example of the usefulness of that result, we consider the mapping $A\mapsto\overline{A}$ of $\mathcal{P}(X)$ into itself. In what follows (3.8 - 3.10), we let $f: \mathcal{P}(X)\to\mathcal{P}(X)$ be any mapping such that

$$A\subset f(A)\subset\overline{A} \quad \text{for each} \quad A\in\mathcal{P}(X).$$

3.8 THEOREM.- Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the quasi-uniformity $\mathcal{U}_*$ of $\mathcal{P}(X)$ is the inverse image of itself under the mapping f.

Proof.- It suffices to show that $U_*\subset(f\times f)^{-1}((U\circ U)_*)$ and $(f\times f)^{-1}(U_*)\subset(U\circ U)_*$ for each entourage U, i.e. that the relation $A\subset U^{-1}(B)$ implies

$$f(A)\subset(U\circ U)^{-1}(f(B))$$

and that the relation $f(A))\subset U^{-1}(f(B))$ implies

$$A\subset(U\circ U)^{-1}(B).$$

Now, if $A\subset U^{-1}(B)$, then

$$f(A) \subset \bar{A} \subset U^{-1}(A) \subset U^{-1}(U^{-1}(B)) \subset U^{-1}(U^{-1}(f(B))),$$

while if $f(A) \subset U^{-1}(f(B))$, then

$$A \subset f(A) \subset U^{-1}(f(B)) \subset U^{-1}(\bar{B}) \subset U^{-1}(U^{-1}(B)).$$

Hence the result.

3.9 COROLLARY 1.- Let $(X,\mathcal{T})$ be a topological space. Then the topology $\mathcal{T}_*$ of $\mathcal{P}(X)$ is the inverse image of itself under the mapping f.

Proof.- By 3.8, the quasi-uniformity $(\mathcal{U}(\mathcal{T}))_*$ is the inverse image of itself under f. Since $(\mathcal{U}(\mathcal{T}))_* = \mathcal{U}(\mathcal{T}_*)$, the result follows by using 3.7.

3.10 COROLLARY 2.- Let $(X,\mathcal{U})$ be a uniform space. Then each of the quasi-uniformities $\mathcal{U}^*$, $\mathcal{U}_*$ and $\tilde{\mathcal{U}}$ of $\mathcal{P}(X)$ is the inverse image of itself under the mapping f.

Proof.- Since the result holds for $\mathcal{U}_*$, and since $\mathcal{U}^*$ and $\mathcal{U}_*$ are conjugate, the result also holds for $\mathcal{U}^*$. It is clear from the proof of 3.8 that the result holds for $\tilde{\mathcal{U}}$.

Let us finally note that 3.6 also extends to correspondences, in the following sense:

3.11 THEOREM.- Let $(X,\mathcal{T})$ and $(X',\mathcal{T}')$ be two topological spaces, and let R be a correspondence between X and X'. Then R is $(\mathcal{T},\mathcal{T}')$-continuous (resp. usc) (resp. lsc) iff it is $(\mathcal{U}(\mathcal{T}),\mathcal{U}(\mathcal{T}'))$-quasi-uniformly continuous (resp. quasi-uniformly usc) (resp. quasi-uniformly lsc).

Proof.- We have

R is $(\mathcal{T},\mathcal{T}')$-continuous

iff $\hat{R}$ is $(\tilde{\mathcal{T}},\tilde{\mathcal{T}'})$-continuous

iff $\hat{R}$ is $(\mathcal{U}(\tilde{\mathcal{T}}),\mathcal{U}(\tilde{\mathcal{T}'}))$-quasi-uniformly cont.,

while R is $(\mathcal{U}(\mathcal{T}),\mathcal{U}(\mathcal{T}'))$-quasi-uniformly continuous

iff $\hat{R}$ is $(\widetilde{\mathcal{U}(\mathcal{T})},\widetilde{\mathcal{U}(\mathcal{T}')})$-quasi-uniformly conts.

Since $\widetilde{\mathcal{U}(\mathcal{T})} = \mathcal{U}(\tilde{\mathcal{T}})$ and $\widetilde{\mathcal{U}(\mathcal{T}')} = \mathcal{U}(\tilde{\mathcal{T}'})$, the result follows. Similarly for the "usc" and the "lsc" cases.

In the remaining parts of this section, we study the continuity of unions and products, trying to use the ideas developed so far.

Continuity of unions

3.12 THEOREM.- Let $(X,\mathcal{U})$ be a quasi-uniform space. Let $\mathcal{P}(X)$ be endowed with its quasi-uniformity $\tilde{\mathcal{U}}$ (resp. $\mathcal{U}^*$) (resp. $\mathcal{U}_*$), and let $\mathcal{P}(\mathcal{P}(X))$ be endowed with its quasi-uniformity $\tilde{\tilde{\mathcal{U}}}$ (resp. $\mathcal{U}^{**}$) (resp. $\mathcal{U}_{**}$).

Then, the mapping $f: \mathcal{A} \longmapsto \bigcup_{A\in\mathcal{A}} A$ of $\mathcal{P}(\mathcal{P}(X))$ into $\mathcal{P}(X)$ is quasi-uniformly continuous.

Proof.- For each $A \in \mathcal{P}(X)$, we have $f(\{A\}) = A$, so that $f\circ j_{\mathcal{P}(X)} = Id_{\mathcal{P}(X)}$. Hence, $f(\mathcal{A}) = \bigcup_{A\in\mathcal{A}} A = \bigcup_{A\in\mathcal{A}} f(\{A\})$ for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$, so that f preserves unions. Hence, there exists a (unique) correspondence R between $\mathcal{P}(X)$ and X with $\hat{R} = f$. Also, $\dot{R} = \hat{R}\circ j_{\mathcal{P}(X)} = f\circ j_{\mathcal{P}(X)} = Id_{\mathcal{P}(X)}$ is quasi-uniformly continuous. It follows that $f = \hat{R}$ is also quasi-uniformly continuous.

We can deduce a corresponding result for topological spaces:

3.13 COROLLARY.- Let $(X,\mathcal{T})$ be a topological space. Let $\mathcal{P}(X)$ be endowed with its topology $\tilde{\mathcal{T}}$ (resp. $\mathcal{T}^*$) (resp. $\mathcal{T}_*$), and let $\mathcal{P}(\mathcal{P}(X))$ be endowed with its topology $\tilde{\tilde{\mathcal{T}}}$ (resp. $\mathcal{T}^{**}$) (resp. $\mathcal{T}_{**}$). Then the mapping $f: \mathcal{A} \longmapsto \bigcup_{A\in\mathcal{A}} A$ of $\mathcal{P}(\mathcal{P}(X))$ into $\mathcal{P}(X)$ is continuous.

Proof.- By 3.12, f is $(\widetilde{\widetilde{\mathcal{U}(\mathcal{T})}},\widetilde{\mathcal{U}(\mathcal{T})})$-quasi-uniformly continuous, hence continuous. Now the topology induced by $\widetilde{\mathcal{U}(\mathcal{T})}$ is equal to $\tilde{\mathcal{T}}$, and the topology induced by $\widetilde{\widetilde{\mathcal{U}(\mathcal{T})}} = \widetilde{\mathcal{U}(\tilde{\mathcal{T}})}$ is equal to $\tilde{\tilde{\mathcal{T}}}$. Hence the result.

Similarly for $\mathcal{T}^*$ and $\mathcal{T}_*$.

We now consider unions of families of correspondences.

3.14 THEOREM.- Let R be the union of a finite family $(R_i)_{i\in I}$ of correspondences between a quasi-uniform space X and a quasi-uniform space Y. If R_i is quasi-uniformly upper (resp. lower) semi-continuous for each $i \in I$, then so is R.

Proof.- We consider the "upper" case; the "lower" case can be dealt with in the same way.
So let V be any entourage of Y. Then, for each $i \in I$ there exists an entourage U_i of X such that the relation $(x,y) \in U_i$ implies $R_i(y) \subset V(R_i(x))$. Then, putting $U = \bigcap_{i\in I} U_i$, U is an entourage of X, and the relation $(x,y) \in U$ implies $R(y) \subset V(R(x))$. Hence the result.

3.15 COROLLARY.- Let $(X,\mathcal{U})$ be a quasi-uniform space, I a finite index set, and let F be the mapping $(A_i)_{i\in I} \mapsto \bigcup_{i\in I} A_i$ of $(\mathcal{P}(X))^I$ into $\mathcal{P}(X)$. If $\mathcal{P}(X)$ is endowed with its quasi-uniformity $\tilde{\mathcal{U}}$ (resp. $\mathcal{U}^*$) (resp. $\mathcal{U}_*$), and $(\mathcal{P}(X))^I$ with the induced product quasi-uniformity, then F is quasi-uniformly continuous.

Proof.- For each $j \in I$, define a correspondence R_j between $(\mathcal{P}(X))^I$ and X by setting $R_j(A) = A_j$ for each $A = (A_i)_{i\in I} \in (\mathcal{P}(X))^I$. Denoting by pr_j the projection $(\mathcal{P}(X))^I \longrightarrow \mathcal{P}(X)$ of index j, we have $\dot{R}_j = pr_j$, which is quasi-uniformly continuous, showing that R_j is quasi-uniformly continuous (resp. quasi-uniformly usc) (resp. quasi-uniformly lsc). Putting $R = \bigcup_{j\in I} R_j$, we thus have that R is quasi-uniformly continuous (resp. quasi-uniformly usc) (resp. quasi-uniformly lsc).
Now $R(A) = \bigcup_{i\in I} A_i$ for each $A = (A_i)_{i\in I} \in (\mathcal{P}(X))^I$, so that $\dot{R} = F$. Hence the result.

Of course, Pervin's quasi-uniformity can be used, in conjunction with 3.11, to deduce from 3.14 and 3.15 corresponding results for topological spaces. But this would yield only global results. Also, one might expect stronger results when dealing with topological spaces, since the latter can be considered as included among the quasi-uniform spaces. It happens to be so in this case, where the finiteness of the index set I is no longer necessary in the case of lower semi-continuity. Finally, we are able to formulate the following result in terms of continuity at a set, rather than merely at a point.

<u>3.16</u> THEOREM.- Let R be the union of a family $(R_i)_{i\in I}$ of correspondences between a topological space X and a topological space Y, and let A_o be a subset of X.

a) If R_i is usc at A_o for each $i \in I$, then R is usc at A_o, provided I is <u>finite</u>.

b) If R_i is lsc at A_o for each $i \in I$, then R is lsc at A_o, for <u>arbitrary</u> I.

Proof.- Ad a): Let G be any open subset of Y with $A_o \subset R^*(G)$. Since $R^*(G) = \bigcap_{i\in I} R_i^*(G)$, we have $A_o \subset R_i^*(G)$, hence $A_o \subset \overset{\circ}{\widehat{R_i^*(G)}}$, for each $i \in I$. Thus, $A_o \subset \bigcap_{i\in I} \overset{\circ}{\widehat{R_i^*(G)}} = \overset{\circ}{\widehat{\bigcap_{i\in I} R_i^*(G)}} = \overset{\circ}{\widehat{R^*(G)}}$, since I is finite.

Ad b): Let G be any open subset of Y with $A_o \cap R_*(G) \neq \emptyset$. Since $R_*(G) = \bigcup_{i\in I} R_{i*}(G)$, there exists an index $i_o \in I$ such that $A_o \cap R_{i_o*}(G) \neq \emptyset$. Hence $A_o \cap \overset{\circ}{\widehat{R_{i_o*}(G)}} \neq \emptyset$, and so

$$\emptyset \neq A_o \cap \Big(\bigcup_{i\in I} \overset{\circ}{\widehat{R_{i*}(G)}}\Big) \subset A_o \cap \overset{\circ}{\widehat{\bigcup_{i\in I} R_{i*}(G)}} = A_o \cap \overset{\circ}{\widehat{R_*(G)}},$$

hence the result.

Note that 3.16a is not true in general, if I is not assumed to be finite. For example, let $X = Y = [0,1]$ and, for each $n \geq 1$, define a continuous function $f_n: X \to Y$ by setting $f_n(x) = \begin{cases} nx, & \text{if } x \in [0,1/n] \\ 1, & \text{if } x \in [1/n,1] \end{cases}$ for each $x \in X$. Then the union R of $(f_n)_{n\geqslant 1}$ is not upper semi-continuous at the point $0 \in X$, since $R(0) = \{0\}$ and $1 \in R(x)$ for each $x \in]0,1]$.

3.17 COROLLARY.- Let $(X,\mathcal{T})$ be a topological space, I an index set, and let F be the mapping $(A_i)_{i\in I} \longmapsto \bigcup_{i\in I} A_i$ of $(\mathcal{P}(X))^I$ into $\mathcal{P}(X)$.

a) If $\mathcal{P}(X)$ is endowed with its topology $\mathcal{T}^*$ (resp. $\widetilde{\mathcal{T}}$), and $(\mathcal{P}(X))^I$ with the induced product topology, then F is continuous, provided I is finite.

b) If $\mathcal{P}(X)$ is endowed with its topology $\mathcal{T}_*$, and $(\mathcal{P}(X))^I$ with the induced product topology, then F is continuous, for arbitrary I.

Proof.- This follows from 3.16, in the same way that 3.15 followed from 3.14.

Continuity of products

We start with the main result, from which all the remaining results of this section will be deduced.

3.18 THEOREM.- Let $(X_i)_{i\in I}$ be a family of quasi-uniform spaces, and let F be the mapping $(A_i)_{i\in I} \longmapsto \prod_{i\in I} A_i$ of $\prod_{i\in I} \mathcal{P}(X_i)$ into $\mathcal{P}(\prod_{i\in I} X_i)$.

Endow hyperspaces with their respective Bourbaki (resp. upper) (resp. lower) quasi-uniformities, and endow product spaces with their respective product quasi-uniformities.

Then the mapping F induces an isomorphism of $\prod_{i\in I} \mathcal{P}_0(X_i)$ onto a subspace of $\mathcal{P}_0(\prod_{i\in I} X_i)$.

Proof.- First of all, it is clear that $F\big|\prod_{i\in I} \mathcal{P}_0(X_i)$ is injective and that $F(\prod_{i\in I} \mathcal{P}_0(X_i)) \subset \mathcal{P}_0(\prod_{i\in I} X_i)$.

We only consider the case of upper quasi-uniformities; the other two cases can be dealt with in a similar way.

Put $G = F \times F$. For each $j \in I$, let $pr_j: \prod_{i \in I} X_i \longrightarrow X_j$ and $Pr_j: \prod_{i \in I} \mathcal{S}(X_i) \longrightarrow \mathcal{S}(X_j)$ be the projections, and put $g_j = pr_j \times pr_j$ and $G_j = Pr_j \times Pr_j$.

For each $i \in I$, the sets U_i^* form a base for the upper quasi-uniformity of $\mathcal{S}(X_i)$, as U_i runs through the set of entourages of X_i. Let $\mathcal{Q}$ be the set of all sets of the form $G_i^{-1}(U_i^*)$, where $i \in I$ and U_i is an entourage of X_i. Then the set of all finite intersections

$$W^*(U_{i_1}, \ldots, U_{i_n}) = G_{i_1}^{-1}(U_{i_1}^*) \cap \ldots \cap G_{i_n}^{-1}(U_{i_n}^*)$$

of sets of $\mathcal{Q}$ is a base for the product quasi-uniformity of $\prod_{i \in I} \mathcal{S}(X_i)$.

On the other hand, let $\mathcal{B}$ be the set of all sets of the form $g_i^{-1}(U_i)$, where $i \in I$ and U_i is an entourage of X_i. Then the set of all finite intersections

$$W(U_{i_1}, \ldots, U_{i_n}) = g_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap g_{i_n}^{-1}(U_{i_n})$$

of sets of $\mathcal{B}$ is a base for the product quasi-uniformity of $\prod_{i \in I} X_i$. It follows that the sets $(W(U_{i_1}, \ldots, U_{i_n}))^*$ form a base for the upper quasi-uniformity of $\mathcal{S}(\prod_{i \in I} X_i)$.

Putting $Y = \prod_{i \in I} \mathcal{S}_o(X_i)$, the result will follow if we can show that

$$G^{-1}((W(U_{i_1}, \ldots, U_{i_n}))^*) \cap (Y \times Y) = W^*(U_{i_1}, \ldots, U_{i_n}) \cap (Y \times Y).$$

Now, for each $((A_i)_{i \in I}, (B_i)_{i \in I}) \in Y \times Y$, we have

$((A_i)_{i \in I}, (B_i)_{i \in I}) \in G^{-1}((W(U_{i_1}, \ldots, U_{i_n}))^*)$

iff $(\prod_{i \in I} A_i, \prod_{i \in I} B_i) \in (W(U_{i_1}, \ldots, U_{i_n}))^*$

iff $\prod_{i \in I} B_i \subset W(U_{i_1}, \ldots, U_{i_n})(\prod_{i \in I} A_i)$, whereas

$((A_i)_{i \in I}, (B_i)_{i \in I}) \in W^*(U_{i_1}, \ldots, U_{i_n})$

iff $(A_{i_k}, B_{i_k}) \in U_{i_k}^*$ for each $k = 1, \ldots, n$

iff $B_{i_k} \subset U_{i_k}(A_{i_k})$ for each $k = 1,\dots,n$.

Now, using the fact that the sets A_i and B_i are nonempty, we have $\prod_{i\in I} B_i \subset W(U_{i_1},\dots,U_{i_n})(\prod_{i\in I} A_i)$

iff $\forall (y_i)_{i\in I} \in \prod_{i\in I} B_i \ \exists (x_i)_{i\in I} \in \prod_{i\in I} A_i$ such that

$(x_{i_k}, y_{i_k}) \in U_{i_k}$ for each $k = 1,\dots,n$

iff $B_{i_k} \subset U_{i_k}(A_{i_k})$ for each $k = 1,\dots,n$,

and we are done.

3.19 COROLLARY 1.- Let X be a quasi-uniform space, $(Y_i)_{i\in I}$ a family of quasi-uniform spaces and, for each $i \in I$, R_i a correspondence between X and Y_i with non-empty sections. Let R be the restricted product of the family $(R_i)_{i\in I}$. Then R is quasi-uniformly upper (resp. lower) semi-continuous iff each R_i is.

Proof.- With $Y = \prod_{i\in I} Y_i$, we have $R_i = \tilde{pr}_i^Y \circ R$ for each $i \in I$, and so the quasi-uniform upper (resp. lower) semi-continuity of R implies that of each R_i.

Conversely, suppose that each R_i is quasi-uniformly upper (resp. lower) semi-continuous, and let h be the mapping $(A_i)_{i\in I} \longmapsto \prod_{i\in I} A_i$ of $\prod_{i\in I} \mathcal{P}_o(Y_i)$ into $\mathcal{P}(\prod_{i\in I} Y_i)$. Endowing hyperspaces with their respective upper (resp. lower) quasi-uniformities, and product spaces with their respective product quasi-uniformities, we see by 3.18 that h is quasi-uniformly continuous. On the other hand, the function $g: X \longrightarrow \prod_{i\in I} \mathcal{P}_o(Y_i)$, defined by $g(x) = (R_i(x))_{i\in I}$ for each $x \in X$, is quasi-uniformly continuous. Since

$$R(x) = \prod_{i\in I} R_i(x) = h((R_i(x))_{i\in I}) = h(g(x))$$

for each $x \in X$, it follows that $\dot{R} = h \circ g$ is quasi-uniformly continuous. Hence the result.

3.20 COROLLARY 2.- Let $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$ be two families of quasi-uniform spaces indexed by the same set I. For each $i \in I$, let R_i be a correspondence between X_i and Y_i with nonempty sections, and let R be the product of the family $(R_i)_{i \in I}$.

If each R_i is quasi-uniformly upper (resp. lower) semi-continuous, then so is R. The converse also holds, provided the X_i are nonempty.

Proof.- Put $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. Since R is the restricted product of the family $(R_i \cdot pr_i^X)_{i \in I}$, it follows by 3.19 that the quasi-uniform upper (resp. lower) semi-continuity of each R_i implies that of R.

Conversely, assume the X_i are nonempty, and suppose that R is quasi-uniformly upper (resp. lower) semi-continuous. Choose a point $(a_i)_{i \in I} \in \prod_{i \in I} X_i$. For each $j \in I$, let g_j be the mapping of X_j into X such that, for each $x_j \in X_j$, $pr_j^X(g_j(x_j)) = x_j$ and $pr_i^X(g_j(x_j)) = a_i$ whenever $i \neq j$; then g_j is quasi-uniformly continuous. Also, the relation $R_j(x_j) = pr_j^Y(R(g_j(x_j)))$ holds for each $x_j \in X_j$, so that $R_j = pr_j^Y \cdot R \cdot g_j$ is quasi-uniformly upper (resp. lower) semi-continuous.

We now consider topological spaces. Let us first make some notational remarks. If X is a topological space, then $\mathcal{K}(X)$ will denote the set of all compact subsets of X, and $\mathcal{K}_0(X) = \mathcal{K}(X) \setminus \{\emptyset\}$.

Also, if $(X_i, \mathcal{T}_i)_{i \in I}$ is a family of topological spaces, we denote the product topology on $\prod_{i \in I} X_i$ by $\prod_{i \in I} \mathcal{T}_i$. Similarly for product quasi-uniformities. If $(X_i, \mathcal{U}_i)_{i \in I}$ is a family of quasi-uniform spaces, then, with the notation just introduced, we have $\mathcal{T}(\prod_{i \in I} \mathcal{U}_i) = \prod_{i \in I} \mathcal{T}(\mathcal{U}_i)$.

3.21 THEOREM.- Let $(X_i, \mathcal{T}_i)_{i \in I}$ be a family of topological spaces, and let F be the mapping

$$(A_i)_{i\in I} \longmapsto \prod_{i\in I} A_i$$

of $\prod_{i\in I} \mathcal{B}_o(X_i)$ into $\mathcal{B}_o(\prod_{i\in I} X_i)$. Endowing products with their respective product topologies, we have:

a) The mapping F induces a homeomorphism of $\prod_{i\in I} \mathcal{K}_o(X_i)$ onto a subspace of $\mathcal{K}_o(\prod_{i\in I} X_i)$, when hyperspaces are endowed with their respective usf topologies.

b) F is continuous at every $(A_i)_{i\in I}$ with A_i compact (resp. arbitrary) for each $i \in I$, when hyperspaces are endowed with their respective usf (resp. lsf) topologies.

Proof.- By 3.18, the mapping F induces an isomorphism, and hence a homeomorphism, of the space $(\prod_{i\in I} \mathcal{B}_o(X_i), \prod_{i\in I} (\mathcal{U}(\mathcal{T}_i))^*)$ onto a subspace of $(\mathcal{B}_o(\prod_{i\in I} X_i), (\prod_{i\in I} \mathcal{U}(\mathcal{T}_i))^*)$. Now the topology induced by $\prod_{i\in I} (\mathcal{U}(\mathcal{T}_i))^*$ is equal to the product of the topologies induced by the $(\mathcal{U}(\mathcal{T}_i))^*$, i.e. it is equal to $\prod_{i\in I} \tilde{\mathcal{T}}_i^*$. On the other hand, by 2.25a the topology $\mathcal{T}((\prod_{i\in I} \mathcal{U}(\mathcal{T}_i))^*)$ coincides with $(\mathcal{T}(\prod_{i\in I} \mathcal{U}(\mathcal{T}_i)))^* = (\prod_{i\in I} \mathcal{T}_i)^*$ on the set $\mathcal{K}_o(\prod_{i\in I} X_i)$, so that part a) is proved.

To prove b), note that the topology $\mathcal{T}((\prod_{i\in I} \mathcal{U}(\mathcal{T}_i))_*)$ is finer than the topology $(\mathcal{T}(\prod_{i\in I} \mathcal{U}(\mathcal{T}_i)))_* = (\prod_{i\in I} \mathcal{T}_i)_*$ on the set $\mathcal{B}_o(\prod_{i\in I} X_i)$, by 2.23a. Similarly, if $\prod_{i\in I} A_i$ is compact, then every $(\prod_{i\in I} \mathcal{T}_i)^*$-neighbourhood of $\prod_{i\in I} A_i$ in $\mathcal{B}_o(\prod_{i\in I} X_i)$ contains a $\mathcal{T}((\prod_{i\in I} \mathcal{U}(\mathcal{T}_i))^*)$-neighbourhood of $\prod_{i\in I} A_i$, by 2.24a.

The following local results, which are similar to the global results of 3.19 and 3.20, can now be deduced from 3.21:

3.22 COROLLARY 1.- Let X be a topological space, $(Y_i)_{i\in I}$ a family of topological spaces and, for each $i \in I$, R_i, a correspondence between X and Y_i with nonempty sections. Let R be the restricted product of the family $(R_i)_{i\in I}$, and let $x_o \in X$.

a) If each R_i is usc at x_o, then so is R, provided $R_i(x_o)$ is compact for each $i \in I$.

b) If each R_i is lsc at x_o, then so is R.

c) If R is usc (resp. lsc) at x_o, then so is each R_i.

3.23 COROLLARY 2.- Let $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$ be two families of topological spaces indexed by the same set I. For each $i \in I$, let R_i be a correspondence between X_i and Y_i with nonempty sections. Let R be the product of the family $(R_i)_{i\in I}$, and let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$.

a) If R_i is usc at x_i for each $i \in I$, then R is usc at $(x_i)_{i\in I}$, provided $R_i(x_i)$ is compact for each $i \in I$.

b) If R_i is lsc at x_i for each $i \in I$, then R is lsc at $(x_i)_{i\in I}$.

c) If R is usc (resp. lsc) at $(x_i)_{i\in I}$, then R_i is usc (resp. lsc) at x_i for each $i \in I$.

Note that the last two results represent an advance over previous results, where the proof rests heavily on the finiteness of the index set I. Also note that, for arbitrary (resp. finite) I, 3.22a (resp. 3.22b) remains true without the assumption that the sections of the R_i are nonempty.

§2. SUPREMUM THEOREMS

This section deals mainly with results concerning real-valued functions. We first show that the numerical semi-continuity of a real-valued function is equivalent to the semi-continuity of an associated correspondence. This is dealt with in the quasi-uniform (3.24) as well as in the topological case (3.26).

We then study the continuity of the supremum of a real-valued function as well as closed graphs, obtaining results which are then used to prove two supremum theorems (3.41 and 3.42). These, in turn, yield a general result on continuous selections (3.43).

Numerically semi-continuous functions

The additive uniformity $\mathcal{U}_{\mathbb{R}}$ of the real line $\mathbb{R}$ has as a base of entourages the sets $V_a = \{(x,y) \in \mathbb{R} \times \mathbb{R}: |x - y| < a\}$, as a runs through the set of real numbers > 0.

The sets $V_a^+ = \{(x,y) \in \mathbb{R} \times \mathbb{R}: y < x + a\}$ form, as a runs through the set of real numbers > 0, a base for a quasi-uniformity $\mathcal{U}_{\mathbb{R}}^+$ on $\mathbb{R}$, which we shall call the upper additive quasi-uniformity of the real line. Similarly for the lower additive quasi-uniformity $\mathcal{U}_{\mathbb{R}}^-$ of the real line, defined by the sets $V_a^- = \{(x,y) \in \mathbb{R} \times \mathbb{R}: y > x - a\}$.

Clearly $V_a^- = (V_a^+)^{-1}$ for each $a > 0$, hence $\mathcal{U}_{\mathbb{R}}^+$ and $\mathcal{U}_{\mathbb{R}}^-$ are conjugate, with $\mathcal{U}_{\mathbb{R}} = \mathcal{U}_{\mathbb{R}}^+ \vee \mathcal{U}_{\mathbb{R}}^-$, since $V_a = V_a^+ \cap V_a^-$ for each $a > 0$.

Now let $(X,\mathcal{U})$ be any quasi-uniform space. A function $f: X \to \mathbb{R}$ will be said to be (numerically) quasi-uniformly upper (resp. lower) semi-continuous if it is quasi-uniformly continuous when $\mathbb{R}$ is endowed with its upper (resp. lower) additive quasi-uniformity.

Clearly, f is quasi-uniformly continuous (when $\mathbb{R}$ is endowed with its additive uniformity) iff it is both numerically quasi-uniformly usc and numerically quasi-uniformly lsc. Also, f is numerically quasi-uniformly usc, when X is endowed with $\mathcal{U}$, iff it is numerically quasi-uniformly lsc, when X is endowed with $\mathcal{U}^{-1}$.

3.24 THEOREM.- Let $(X,\mathcal{U})$ be a quasi-uniform space, and let $f: X \to \overline{\mathbb{R}}$ be a numerical function on X. Define a correspondence R between X and $\mathbb{R}$ by setting

$$R(x) =]-\infty, f(x)]$$

for each $x \in X$. Then the following are equivalent:

a) f is numerically quasi-uniformly usc (resp. lsc).

b) R is quasi-uniformly usc (resp. lsc), when $\mathbb{R}$ is endowed with its additive uniformity $\mathcal{U}_{\mathbb{R}}$.

c) R is quasi-uniformly usc (resp. lsc), when $\mathbb{R}$ is endowed with its upper (resp. lower) additive quasi-uniformity $\mathcal{U}_{\mathbb{R}}^{+}$ (resp. $\mathcal{U}_{\mathbb{R}}^{-}$).

Proof.- We first consider the "upper" case. For this, it suffices to show that the following three statements are equivalent, for each $a > 0$ and each $x,y \in X$:

(i) $f(y) < f(x) + a$.

(ii) $R(y) \subset V_a(R(x))$.

(iii) $R(y) \subset V_a^{+}(R(x))$.

(i)$\Longrightarrow$(ii): Suppose $w \in R(y)$. If $w \in R(x)$, then clearly $w \in V_a(R(x))$. If on the other hand $w \notin R(x)$, then we have $f(x) < w \leq f(y) < f(x) + a$, hence $|f(x) - w| < a$, i.e. $w \in V_a(f(x)) \subset V_a(R(x))$. Thus, $R(y) \subset V_a(R(x))$.

(ii)$\Longrightarrow$(iii): This is clear, since $V_a \subset V_a^{+}$.

(iii)$\Longrightarrow$(i): Since $f(y) \in R(y) \subset V_a^{+}(R(x))$, there exists a point $z \in R(x)$ such that $(z, f(y)) \in V_a^{+}$, i.e. $f(y) < z + a$. But $z \leq f(x)$, and so $f(y) < f(x) + a$, as desired.

By going over to $\mathcal{U}^{-1}$ and by using the relations $\mathcal{U}_{\mathbb{R}}^{-} = (\mathcal{U}_{\mathbb{R}}^{+})^{-1}$ and $\mathcal{U}_{\mathbb{R}}^{-1} = \mathcal{U}_{\mathbb{R}}$, we can deduce the result for the "lower" case.

We now consider topologies. The intervals of the form $[-\infty, a[$ (resp. $]a, \infty]$), with $a \in \overline{\mathbb{R}}$, together with the set $\overline{\mathbb{R}}$ (the extended real line), can be taken as open sets for a topology on $\overline{\mathbb{R}}$, which we shall denote by $\mathcal{T}_{\overline{\mathbb{R}}}^{+}$ (resp. $\mathcal{T}_{\overline{\mathbb{R}}}^{-}$), and which we shall call the <u>upper</u> (resp. <u>lower</u>) <u>topology of the</u>

extended real line $\overline{\mathbb{R}}$. Denoting the natural topology of $\overline{\mathbb{R}}$ by $\mathfrak{T}_{\overline{\mathbb{R}}}$, we have $\mathfrak{T}_{\overline{\mathbb{R}}} = \mathfrak{T}^+_{\overline{\mathbb{R}}} \vee \mathfrak{T}^-_{\overline{\mathbb{R}}}$.

If $(X,\mathfrak{T})$ is any topological space, then a function $f\colon X \to \overline{\mathbb{R}}$ is (numerically) upper (resp. lower) semi-continuous at a point $x_0 \in X$ iff it is continuous at x_0, when $\overline{\mathbb{R}}$ is endowed with its upper (resp. lower) topology.

We shall denote the restriction of $\mathfrak{T}_{\overline{\mathbb{R}}}$ to $\mathbb{R}$ by $\mathfrak{T}_{\mathbb{R}}$; similarly for $\mathfrak{T}^+_{\mathbb{R}}$ and $\mathfrak{T}^-_{\mathbb{R}}$. We have $\mathfrak{T}^+_{\mathbb{R}} = \mathfrak{T}(\mathcal{K}^+_{\mathbb{R}})$, $\mathfrak{T}^-_{\mathbb{R}} = \mathfrak{T}(\mathcal{K}^-_{\mathbb{R}})$ and $\mathfrak{T}_{\mathbb{R}} = \mathfrak{T}(\mathcal{K}_{\mathbb{R}})$.

3.25 LEMMA.- Let X be a set and $f\colon X \to \overline{\mathbb{R}}$ a real-valued function defined on X. Define a correspondence R between X and $\overline{\mathbb{R}}$ by setting $R(x) = [-\infty, f(x)]$ for each $x \in X$. Let G be any subset of $\overline{\mathbb{R}}$, and let $a = \inf(\overline{\mathbb{R}} \setminus G)$, $b = \inf G$. Then we have:

a) $f^{-1}([-\infty, a[) \subset R^*(G)$.

b) $f^{-1}(]b, \infty]) \subset R_*(G)$.

c) If G is open and $G \neq \overline{\mathbb{R}}$, then equality holds in a).

d) If G is open and $-\infty \notin G$, then equality holds in b).

Proof.-

Ad a): Suppose $f(x) < a$. Then, for each $y \in R(x)$, we have $y \leq f(x) < a$, hence $y \in G$, showing that $R(x) \subset G$.

Ad b): Suppose $f(x) > b$. Then there exists an element $y \in G$ with $y < f(x)$. Hence $y \in R(x) \cap G$, showing that $R(x) \cap G \neq \emptyset$.

Ad c): By a), we have $f^{-1}([-\infty, a[) \subset R^*(G)$. On the other hand, using b) we have $f^{-1}(]a, \infty]) \subset R_*(\overline{\mathbb{R}} \setminus G)$; taking complements, we obtain $R^*(G) \subset f^{-1}([-\infty, a])$. So it suffices to show that $R^*(G) \cap f^{-1}(a) = \emptyset$. Now suppose $f(x) = a$. Then clearly $a \in R(x)$. On the other hand, $\overline{\mathbb{R}} \setminus G$ is a nonempty closed set, and so it contains its infimum a. Thus, $a \in R(x) \setminus G$, so that $R(x) \not\subset G$, as desired.

Ad d): By b), we have $f^{-1}(]b,\infty])\subset R_*(G)$. On the other hand, using a) we have $f^{-1}([-\infty,b[)\subset R^*(\overline{\mathbb{R}}\setminus G)$; taking complements, we obtain $R_*(G)\subset f^{-1}([b,\infty])$. So it suffices to show that $R_*(G)\cap f^{-1}(b)=\emptyset$. Now suppose $f(x)=b$. Since G is open and $-\infty\notin G$, we must have $b\notin G$, and so $y>b$ for each $y\in G$. On the other hand, clearly $y\leq f(x)=b$ for each $y\in R(x)$, and so $R(x)\cap G=\emptyset$.

3.26 THEOREM.- Let X be a topological space, and let $f\colon X\to\overline{\mathbb{R}}$ be a real-valued function defined on X. Define a correspondence R between X and $\overline{\mathbb{R}}$ by setting $R(x)=[-\infty,f(x)]$ for each $x\in X$, and let $x_0\in X$. Then the following are equivalent:

a) f is numerically usc (resp. lsc) at x_0.

b) R is usc (resp. lsc) at x_0, when $\overline{\mathbb{R}}$ is endowed with its topology $\mathfrak{T}_{\overline{\mathbb{R}}}$.

c) R is usc (resp. lsc) at x_0, when $\overline{\mathbb{R}}$ is endowed with its topology $\mathfrak{T}^+_{\overline{\mathbb{R}}}$ (resp. $\mathfrak{T}^-_{\overline{\mathbb{R}}}$).

Proof.- We first consider upper semi-continuity.

a) $\Longrightarrow$ b): Let G be any open subset of $\overline{\mathbb{R}}$ with $x_0\in R^*(G)$; we have to show that $x_0\in\overset{\circ}{\widehat{R^*(G)}}$. Now this is trivial if $G=\overline{\mathbb{R}}$. If $G\neq\overline{\mathbb{R}}$, then we can put $a=\inf(\overline{\mathbb{R}}\setminus G)$ and use 3.25c to obtain $R^*(G)=f^{-1}([-\infty,a[)$. Hence,

$$x_0\in\overset{\circ}{\widehat{f^{-1}([-\infty,a[)}}=\overset{\circ}{\widehat{R^*(G)}}.$$

b) $\Longrightarrow$ c): This is clear, since $\mathfrak{T}_{\overline{\mathbb{R}}}$ is finer than $\mathfrak{T}^+_{\overline{\mathbb{R}}}$.

c) $\Longrightarrow$ a): Let $a\in\overline{\mathbb{R}}$ be such that $x_0\in f^{-1}([-\infty,a[)$. Since $[-\infty,a[\neq\overline{\mathbb{R}}$ is open, and since $\inf(\overline{\mathbb{R}}\setminus[-\infty,a[)=\inf[a,\infty]=a$, we see by 3.25c that $f^{-1}([-\infty,a[)=R^*([-\infty,a[)$, hence

$$x_0\in\overset{\circ}{\widehat{R^*([-\infty,a[)}}=\overset{\circ}{\widehat{f^{-1}([-\infty,a[)}}.$$

We now turn to lower semi-continuity.

a) $\Longrightarrow$ b): Let G be any open subset of $\overline{\mathbb{R}}$ with $x_0\in R_*(G)$;

we have to show that $x_o \in \overset{\circ}{\widehat{R_*(G)}}$. Now, this is trivial if $-\infty \in G$. If $-\infty \notin G$, then we can put $a = \inf G$ and use 3.25d to obtain $R_*(G) = f^{-1}(]a,\infty])$. Hence, $x_o \in \overset{\circ}{\widehat{f^{-1}(]a,\infty])}} = \overset{\circ}{\widehat{R_*(G)}}$, as desired.

b) $\Longrightarrow$ c): This is clear, since $\hat{\mathcal{T}}_{\overline{\mathbb{R}}}$ is finer than $\tilde{\mathcal{T}}_{\overline{\mathbb{R}}}$.

c) $\Longrightarrow$ a): Let $a \in \overline{\mathbb{R}}$ be such that $x_o \in f^{-1}(]a,\infty])$. Since $]a,\infty]$ is open, and since $-\infty \notin]a,\infty]$ and $\inf]a,\infty] = a$, we see by 3.25d that

$$f^{-1}(]a,\infty]) = R_*(]a,\infty]),$$

hence $x_o \in \overset{\circ}{\widehat{R_*(]a,\infty])}} = \overset{\circ}{\widehat{f^{-1}(]a,\infty])}}$.

<u>Continuity of the supremum of a real-valued function</u>

In the following, we denote by $\mathcal{B}_o^+$ (resp. $\mathcal{B}_o^-$) the <u>set of all nonempty subsets of $\mathbb{R}$ which are bounded above</u> (resp. <u>below</u>).

<u>3.27</u> THEOREM.- Let $f: \mathcal{B}_o^+ \longrightarrow \mathbb{R}$ be the mapping $A \longmapsto \sup A$, and $g: \mathcal{B}_o^- \longrightarrow \mathbb{R}$ the mapping $A \longmapsto \inf A$.

a) f is quasi-uniformly usc (resp. lsc), when $\mathcal{B}_o^+$ is endowed with the upper (resp. lower) quasi-uniformity induced by the upper (resp. lower) additive quasi-uniformity of $\mathbb{R}$.

b) g is quasi-uniformly lsc (resp. usc), when $\mathcal{B}_o^-$ is endowed with the upper (resp. lower) quasi-uniformity induced by the lower (resp. upper) additive quasi-uniformity of $\mathbb{R}$.

Proof.- a) We first consider the "upper" case. Let $\varepsilon > 0$; we shall show that $f(B) \leq f(A) + \varepsilon$ for each $A, B \in \mathcal{B}_o^+$ with $B \subset V_\varepsilon^+(A)$. Indeed, suppose $B \subset V_\varepsilon^+(A)$. Then, for each $y \in B$ there exists an element $x \in A$ with $y < x + \varepsilon$; it follows that $y < f(A) + \varepsilon$ for each $y \in B$, hence

$f(B) \leq f(A) + \varepsilon$.

By considering conjugates, and using the fact that $\mathcal{H}_{\mathbb{R}}^{+}$ and $\mathcal{H}_{\mathbb{R}}^{-}$ are conjugate, we obtain the corresponding result for the "lower" case.

b) This can be proved like a). Alternatively, this can be deduced from a), since

$$g(A) = \inf A = - \sup(-A) = -f(-A) \quad \text{for each} \quad A \in \mathcal{B}_{0}^{-}.$$

We now consider topologies. Note that Pervin's quasi-uniformity only yields the first half of the following theorem. But first, we state a simple lemma which will be found useful in the sequel.

3.28 LEMMA.- Let X be a topological space and $f: X \to \overline{\mathbb{R}}$ a real-valued function on X. Suppose that f attains its minimum at a point $a \in X$, and let f_0 be the restriction of f to $X \setminus \{a\}$.

If $\{a\}$ is open (resp. closed), then the upper (resp. lower) semi-continuity of f_0 implies that of f.

Proof.- Put $A = X \setminus \{a\}$.

$\{a\}$ open: It is clear that f is usc at a, since a is isolated. The upper semi-continuity of f at a point $x \in A$ follows from that of f_0 at x and from the minimality of f at a.

$\{a\}$ closed: The lower semi-continuity of f at a follows from the minimality of f at a. Also, f is lsc at every $x \in A$, since f_0 is lsc at x and A is open.

In conjunction with the above lemma, note that, if $(X, \mathcal{T})$ is a topological space, then $\{\emptyset\}$ is open in $(\mathcal{P}(X), \mathcal{T}^*)$, since $\{\emptyset\} = \mathcal{P}(\emptyset)$, while it is closed in $(\mathcal{P}(X), \mathcal{T}_*)$, since

$$\{\emptyset\} = \mathcal{P}(X) \setminus \mathcal{J}(X).$$

3.29 THEOREM.- Let $f: \mathcal{P}(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ be the mapping $A \mapsto \sup A$, and $g: \mathcal{P}(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ the mapping $A \mapsto \inf A$.

a) f is usc (resp. lsc), when $\mathcal{P}(\overline{\mathbb{R}})$ is endowed with the usf (resp. lsf) topology induced by the upper (resp.

lower) topology of $\overline{\mathbb{R}}$.

b) g is lsc (resp. usc), when $\mathcal{B}(\overline{\mathbb{R}})$ is endowed with the usf (resp. lsf) topology induced by the lower (resp. upper) topology of $\overline{\mathbb{R}}$.

Proof.-

a) We first consider the mapping $F: \mathcal{B}([0,1]) \longrightarrow [0,1]$, defined by $F(\emptyset) = 0$ and $F(A) = \sup A$ for each $A \in \mathcal{B}([0,1])$, and prove a similar result for F. Denote by F_o the restriction of F to $\mathcal{B}_o([0,1])$.

The "upper" case: We endow $\mathcal{B}([0,1])$ with the topology $(\mathcal{T}^+_{\mathbb{R}})^*$ and show that F is usc (strictly speaking, $\mathcal{T}^+_{\mathbb{R}}$ refers to the topology induced on [0,1] by the upper topology of the real line). By 3.28, it suffices to show that F_o is usc. Now, by 3.27a, F_o is quasi-uniformly usc, when $\mathcal{B}_o([0,1])$ is endowed with $(\mathcal{U}^+_{\mathbb{R}})^*$, hence it is usc. Since $\mathcal{T}((\mathcal{U}^+_{\mathbb{R}})^*)$ is coarser than $(\mathcal{T}(\mathcal{U}^+_{\mathbb{R}}))^* = (\mathcal{T}^+_{\mathbb{R}})^*$, the result follows.

The "lower" case: We endow $\mathcal{B}([0,1])$ with the topology $(\mathcal{T}^-_{\mathbb{R}})_*$ and show that F is lsc. By 3.28, it suffices to show that F_o is lsc. So let $A_o \in \mathcal{B}_o([0,1])$ and $\varepsilon > 0$. Then there exists an element $x_o \in A_o$ such that $x_o > F_o(A_o) - (\varepsilon/2)$. We shall show that $F_o(A) > F_o(A_o) - \varepsilon$ for each $A \in \mathcal{S}_{[0,1]}(G)$, where $G = \{x \in [0,1]: x > x_o - (\varepsilon/2)\}$. Since $\mathcal{S}_{[0,1]}(G)$ is an open neighbourhood of A_o in $\mathcal{B}_o([0,1])$, the result will follow. So suppose $A \in \mathcal{S}_{[0,1]}(G)$. Then there exists an element $x \in A$ with $x > x_o - (\varepsilon/2)$; hence, we have
$F_o(A) \geqslant x > x_o - (\varepsilon/2) > (F_o(A_o) - (\varepsilon/2)) - (\varepsilon/2) = F_o(A_o) - \varepsilon$,
as desired.

We now consider the mapping f itself. There exists an increasing mapping φ which is a homeomorphism of $\overline{\mathbb{R}}$ onto [0,1] with respect to the natural topologies of those sets. In particular, we have $\varphi(\sup A) = \sup \varphi(A)$ for each $A \in \mathcal{B}(\overline{\mathbb{R}})$. Also, φ is a homeomorphism with respect to the upper (resp. lower) topologies of $\overline{\mathbb{R}}$ and [0,1]. Thus, in order to show that f is usc (resp. lsc), it suffices to show that the mapping $\varphi \circ f$ is usc (resp. lsc). On the other hand, it also follows that $\hat{\varphi}$ is continuous, when $\mathcal{P}([0,1])$ is endowed with the usf (resp. lsf) topology induced by the upper (resp. lower) topology of $[0,1]$. Since $\varphi(f(A)) = F(\varphi(A))$ for each $A \in \mathcal{P}(\overline{\mathbb{R}})$, it follows that $\varphi \circ f = F \circ \hat{\varphi}$ is usc (resp. lsc). Hence the result.

b) This can be deduced from a), as in the proof of 3.27.

The following theorem constitutes the main result on the continuity of the supremum of a real-valued function.

3.30 THEOREM.- Let X be a topological space, and $h: X \to \overline{\mathbb{R}}$ a real-valued function defined on X.

Let $F, G: \mathcal{P}(X) \to \overline{\mathbb{R}}$ be defined by $F(A) = \sup h(A)$, $G(A) = \inf h(A)$ for each $A \in \mathcal{P}(X)$, and let $A_0 \in \mathcal{P}(X)$.

a) If h is usc (resp. lsc) at every point of A_0, then F is usc (resp. lsc) at the point A_0 of $\mathcal{P}(X)$, when $\mathcal{P}(X)$ is endowed with its usf (resp. lsf) topology.

b) If h is lsc (resp. usc) at every point of A_0, then G is lsc (resp. usc) at the point A_0 of $\mathcal{P}(X)$, when $\mathcal{P}(X)$ is endowed with its usf (resp. lsf) topology.

Proof.- We only prove part a), the proof of part b) being very similar.

Let $\mathcal{P}(\overline{\mathbb{R}})$ be endowed with the usf (resp. lsf) topology induced by the upper (resp. lower) topology of $\overline{\mathbb{R}}$. Then $\hat{h}$ is continuous at the point A_0. But the mapping $f: \mathcal{P}(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$, defined by $f(B) = \sup B$ for each $B \in \mathcal{P}(\overline{\mathbb{R}})$, is usc (resp. lsc) by 3.29a. Since $F(A) = \sup h(A) = f(h(A))$ for each $A \in \mathcal{P}(X)$, it follows that $F = f \circ \hat{h}$ is usc (resp. lsc) at the point $A_0 \in \mathcal{P}(X)$.

3.31 COROLLARY 1.- Let $R: X|Y$ be a correspondence between topological spaces, and $h: Y \to \overline{\mathbb{R}}$ a real-valued function defined on Y. Let $f, g: X \to \overline{\mathbb{R}}$ be defined by $f(x) = \sup h(R(x))$, $g(x) = \inf h(R(x))$ for each $x \in X$, and let $x_0 \in X$.

a) If R is usc (resp. lsc) at x_o, and h is usc (resp. lsc) at every point of $R(x_o)$, then f is usc (resp. lsc) at x_o.

b) If R is usc (resp. lsc) at x_o, and h is lsc (resp. usc) at every point of $R(x_o)$, then g is lsc (resp. usc) at x_o.

Proof.- To prove a), let $F:\mathcal{P}(Y)\to\overline{\mathbb{R}}$ be defined by $F(A) = \sup h(A)$ for each $A\in\mathcal{P}(X)$.
Then $f(x) = \sup h(R(x)) = F(R(x))$ for each $x\in X$, i.e. $f = F\cdot\dot{R}$. The result thus follows from 3.30a.

Similarly for part b).

3.32 COROLLARY 2.- Let $R:X|Y$ be a correspondence between topological spaces, and $h:X\times Y\to\overline{\mathbb{R}}$ a real-valued function defined on $X\times Y$. Let $f,g:X\to\overline{\mathbb{R}}$ be defined by

$$f(x) = \sup\{h(x,y):\ y\in R(x)\},$$
$$g(x) = \inf\{h(x,y):\ y\in R(x)\}$$

for each $x\in X$, and let $x_o\in X$.

a) If R is usc (resp. lsc) at x_o, with $R(x_o)$ compact (resp. arbitrary), and h is usc (resp. lsc) at every point (x_o,y) with $y\in R(x_o)$, then f is usc (resp. lsc) at x_o.

b) If R is usc (resp. lsc) at x_o, with $R(x_o)$ compact (resp. arbitrary), and h is lsc (resp. usc) at every point (x_o,y) with $y\in R(x_o)$, then g is lsc (resp. usc) at x_o.

Proof.- Again, we only prove part a). Let S be the restricted product of Id_X and R; thus, we have
$S(x) = \{x\}\times R(x) = \{(x,y)\in X\times Y:\ y\in R(x)\}$ for each $x\in X$. By 3.22a (resp. 3.22b), S is usc (resp. lsc) at x_o. Moreover, h is usc (resp. lsc) at each point of the set $S(x_o)$. By 3.31a, it follows that the mapping $f_1:X\to\overline{\mathbb{R}}$, defined by $f_1(x) = \sup h(S(x))$ for each $x\in X$, is usc (resp. lsc) at

x_0. Now $f_1(x) = \sup\{h(x,y): y \in R(x)\} = f(x)$ for each $x \in X$, i.e. $f = f_1$. Hence the result.

Closed graphs

If f, g are two continuous mappings of a topological space X into a Hausdorff space Y, then the set

$$\{x \in X: f(x) = g(x)\}$$

is known to be closed in X.

We now generalize this result to correspondences.

3.33 THEOREM.- Let R, S be two correspondences between a topological space X and a Hausdorff (resp. regular) space Y, and let $\mathcal{P}(X)$ be endowed with its finite topology. Let $\mathcal{Y} = \{A \in \mathcal{P}(X): R(A) \subset S(A)\}$.

Suppose that R is lsc and S usc at a subset A_0 of X, and that $S(A_0)$ is compact (resp. closed).

Then the relation $A_0 \in \overline{\mathcal{Y}}$ implies $A_0 \in \mathcal{Y}$.

Proof.- Suppose that $A_0 \notin \mathcal{Y}$. Thus, $R(A_0) \not\subset S(A_0)$ and hence there exists an element $y \in R(A_0) \setminus S(A_0)$. Then, since $y \notin S(A_0)$, there exist open subsets V, W of Y with $y \in V$, $S(A_0) \subset W$ and $V \cap W = \emptyset$. Now R is lsc at A_0; since $y \in R(A_0) \cap V$, we have $A_0 \cap R_*(V) \neq \emptyset$, and hence $A_0 \cap \overset{\circ}{\widehat{R_*(V)}} \neq \emptyset$. Since S is usc at A_0 and $A_0 \subset S^*(W)$, we have $A_0 \subset \overset{\circ}{\widehat{S^*(W)}}$. Hence,

$$A_0 \in \mathcal{P}(\overset{\circ}{\widehat{S^*(W)}}) \cap \mathcal{J}(\overset{\circ}{\widehat{R_*(V)}}) = \overset{\circ}{\widehat{\mathcal{P}(S^*(W))}} \cap \overset{\circ}{\widehat{\mathcal{J}(R_*(V))}},$$

so that $\mathcal{O} = \mathcal{P}(S^*(W)) \cap \mathcal{J}(R_*(V))$ is a neighbourhood of the point A_0 in $\mathcal{P}(X)$. Moreover, for each $A \in \mathcal{O}$ we have $R(A) \cap V \neq \emptyset$ and $S(A) \subset W$, so that, since $V \cap W = \emptyset$, $R(A) \not\subset S(A)$, i.e. $A \notin \mathcal{Y}$. Thus, $\mathcal{O} \cap \mathcal{Y} = \emptyset$, showing that $A_0 \notin \overline{\mathcal{Y}}$.

In the above theorem, as well as in what follows, a regular space is not assumed to satisfy the T_1 separation axiom. An analogous remark holds for normal spaces.

A regular (resp. normal) T_1-space will also be referred to as a T_3- (resp. T_4-) space.

3.34 COROLLARY.- Let $G = \{x \in X: R(x) \subset S(x)\}$. Suppose that R is lsc and S usc at a point $x_0 \in X$, and that $S(x_0)$ is compact (resp. closed).

Then the relation $x_0 \in \overline{G}$ implies $x_0 \in G$.

Proof.- Let $\mathcal{P}(X)$ be endowed with its finite topology; then j_X is continuous.

Let $\mathcal{H} = \{A \in \mathcal{P}(X): R(A) \subset S(A)\}$; then $G = j_X^{-1}(\mathcal{H})$.

Now suppose $x_0 \in \overline{G}$. Since $j_X(G) \subset \mathcal{H}$, we then have $\{x_0\} = j_X(x_0) \in \overline{j_X(G)} \subset \overline{\mathcal{H}}$. By 3.33, it follows that $j_X(x_0) = \{x_0\} \in \mathcal{H}$, hence $x_0 \in j_X^{-1}(\mathcal{H}) = G$, as desired.

Let $R = (G,X,Y)$ be a correspondence between topological spaces. We call closure of R, and we denote by $\overline{R}$, the correspondence $(\overline{G},X,Y)$ between X and Y.

If $x_0 \in X$, we say that the graph of R is closed at x_0 if $\overline{R}(x_0) = R(x_0)$.

Here, $\overline{G}$ denotes the closure of G in the product space $X \times Y$. Clearly, G is closed at x_0 iff the relation $(x_0,y) \in \overline{G}$ implies $(x_0,y) \in G$ for each $y \in Y$.

As an example of the use of 3.34, we deduce the following result.

3.35 THEOREM.- Let $R = (G,X,Y)$ be a correspondence between a topological space X and a Hausdorff (resp. regular) space Y. Suppose that R is usc at a point x_0 of X and that $R(x_0)$ is compact (resp. closed).

Then G is closed at x_0.

Proof.- Define a mapping $f: X \times Y \to Y$ and a correspondence S between $X \times Y$ and Y by setting $f(x,y) = y$ and $S(x,y) = R(x)$ for each $(x,y) \in X \times Y$. Then, for each

$y \in Y$, f is continuous at (x_0,y) and S is usc at (x_0,y); indeed, if $pr_1: X \times Y \to X$ and $pr_2: X \times Y \to Y$ are the projections, then $f = pr_2$ and $S = R \circ pr_1$. Moreover, $S(x_0,y) = R(x_0)$ is compact (resp. closed) for each $y \in Y$.

$$\text{Now } G = \{(x,y) \in X \times Y: y \in R(x)\}$$
$$= \{(x,y) \in X \times Y: f(x,y) \in S(x,y)\}.$$

By 3.34, it follows that the relation $(x_0,y) \in \overline{G}$ implies $(x_0,y) \in G$ for each $y \in Y$. Hence G is closed at x_0.

The following two results on closed graphs will be used later (for proofs, cf. [1], theorems 7 and 2' of chapter 6):

<u>3.36</u> THEOREM.- Let R, S be two correspondences between a topological space X and a Hausdorff space Y. Suppose that the graph of R is closed at a point x_0 of X, that S is usc at x_0, and that $S(x_0)$ is compact.

Then $T = R \cap S$ is usc at x_0 and $T(x_0)$ is compact.

<u>3.37</u> THEOREM.- Let R be the intersection of a family of correspondences $(R_i)_{i \in I}$ between a topological space X and a T_2- (resp. T_3-) space Y. Let $x_0 \in X$ and suppose that, for each $i \in I$, R_i is usc at x_0, with $R_i(x_0)$ compact (resp. closed, and that at least one set of the family $(R_i(x_0))_{i \in I}$ is compact).

Then R is usc at x_0 and, unless $I = \emptyset$, $R(x_0)$ is compact.

We conclude with a result on composites:

<u>3.38</u> THEOREM.- Let $R = (G,X,Y)$ and $S = (H,Y,Z)$ be correspondences between topological spaces. If R is usc at a point $x_0 \in X$ and $R(x_0)$ is compact, and if H is closed at every point of $R(x_0)$, then the graph of $S \circ R$ is closed at x_0.

Proof.- By 1.14, the graph F of $S \circ R$ is given by

$$F = (R \times Id_Z)_*(H).$$

So suppose $(x_o,z) \in \overline{F} = \overline{(R \times Id_Z)_*(H)}$. Now $R \times Id_Z$ is usc at (x_o,z), hence $(x_o,z) \in (R \times Id_Z)_*(\overline{H})$, and so there exists a point $y \in R(x_o)$ with $(y,z) \in \overline{H}$. But H is closed at y, and so $(y,z) \in H$, showing that $(x_o,z) \in (R \times Id_Z)_*(H) = F$, as desired.

Continuous selections

We prove two supremum theorems, 3.41 and 3.42. These will be derived from more general results, to which we now proceed.

3.39 THEOREM.- Let X, Y be topological spaces, Z a Hausdorff (resp. regular) space, and let R_1, R_2 be two correspondences between $X \times Y$ and Z.

Let $x_o \in X$, and suppose that R_1 is lsc and R_2 usc at each point $(x_o,y) \in X \times Y$, and that $R_2(x_o,y)$ is compact (resp. closed).

a) We have: the graph of the correspondence $S_o:X|Y$, defined by $S_o(x) = \{y \in Y: R_1(x,y) \subset R_2(x,y)\}$ for each $x \in X$, is closed at x_o.

b) Let further $R:X|Y$ be usc at x_o, with $R(x_o)$ compact.

Then, provided Y is Hausdorff, the correspondence $S:X|Y$, defined by $S(x) = \{y \in R(x): R_1(x,y) \subset R_2(x,y)\}$ for each $x \in X$, is usc at x_o, and $S(x_o)$ is compact.

Proof.- To prove a), let G be the graph of S_o; thus, $G = \{(x,y) \in X \times Y: R_1(x,y) \subset R_2(x,y)\}$. By 3.34, the relation $(x_o,y) \in \overline{G}$ implies $(x_o,y) \in G$, for each $y \in Y$. Hence G is closed at x_o.

Since $S = R \cap S_o$, part b) then follows by using 3.36.

3.40 THEOREM.- Let X, Y, Z be topological spaces, let

$R_1:X|Z$, $R_2:Y|Z$, and let $x_o \in X$.

Suppose that (i) R_1 is usc at x_o, and $R_1(x_o)$ is compact.

(ii) For each $z \in R_1(x_o)$, R_2^{-1} is usc at z and $R_2^{-1}(z)$ is compact.

a) We have: the correspondence $S_o:X|Y$, defined by $S_o(x) = \{y \in Y: R_1(x) \cap R_2(y) \neq \emptyset\}$ for each $x \in X$, is usc at x_o, and $S_o(x_o)$ is compact.

b) Let further $R:X|Y$ be usc at x_o, with $R(x_o)$ compact (resp. closed). Then, provided Y is a T_2- (resp. T_3-) space, the correspondence $S:X|Y$, defined by

$$S(x) = \{y \in R(x): R_1(x) \cap R_2(y) \neq \emptyset\}$$

for each $x \in X$, is usc at x_o, and $S(x_o)$ is compact.

Proof.- We have $S_o(x) = R_2^{-1}(R_1(x))$ for each $x \in X$, so that $S_o = R_2^{-1} \circ R_1$ is usc at x_o. Further, $S_o(x_o) = R_2^{-1}(R_1(x_o))$ is compact by 2.9.

Since $S = R \cap S_o$, part b) follows by using 3.37.

<u>3.41</u> THEOREM.- Let X, Y be two topological spaces, with Y Hausdorff, and let $R:X|Y$ and $h:Y \to \overline{\mathbb{R}}$.

Let $f:X \to \overline{\mathbb{R}}$ be defined by $f(x) = \sup h(R(x))$ for each $x \in X$, and define $S:X|Y$ by setting

$$S(x) = \{y \in R(x): h(y) = f(x)\}$$

for each $x \in X$.

Suppose that R is continuous at a point x_o of X, with $R(x_o)$ compact, and that h is continuous. Then f is continuous at x_o, and S is usc at x_o, with $S(x_o)$ compact. If further $R(x_o) \neq \emptyset$, then also $S(x_o) \neq \emptyset$.

The same result is true if "sup" is replaced by "inf" in the definition of f.

Proof.- This follows from 3.31 and 3.39.

<u>3.42</u> THEOREM.- Let X, Y be topological spaces, with Y a T_2- (resp. T_3-) space, let $R:X|Y$ and $h:Y \to \overline{\mathbb{R}}$.

Let $f:X \to \overline{\mathbb{R}}$ be defined by $f(x) = \sup h(R(x))$ for each $x \in X$, and define $S:X|Y$ by setting

$$S(x) = \{y \in R(x): h(y) = f(x)\}$$

for each $x \in X$.

Suppose that R is continuous at a point x_o of X, with $R(x_o)$ compact (resp. closed), that h is continuous at every point of $R(x_o)$, and that h^{-1} is usc at the point $f(x_o)$ of $\overline{\mathbb{R}}$, with $h^{-1}(f(x_o))$ compact. Then f is continuous at x_o, and S is usc at x_o, with $S(x_o)$ compact.

The same result is true if "sup" is replaced by "inf" in the definition of f.

Proof.- This follows from 3.31 and 3.40.

Note that 3.41 is a local version of the corresponding theorem on p. 116 of [1]. 3.42, however, yields a supremum theorem concerning closed sets, rather than merely compact sets. This will now enable us to give a new proof of lemma 6 of [3] on continuous selections. Filippov's proof depends heavily on geometric considerations, whereas ours follows from a purely topological result.

Let (X,d) be a metric space, and $\mathcal{Q}$ a set of subsets of X. We say that X has the <u>nearest point property</u> with respect to $\mathcal{Q}$ if, for each $(x,A) \in X \times \mathcal{Q}$, there exists exactly one point $a \in A$ such that $d(x,a) = d(x,A)$. For example, the Euclidean space $\mathbb{R}^n$ has the nearest point property with respect to the set of all closed, convex, nonempty subsets of $\mathbb{R}^n$.

A finite dimensional normed vector space is called a <u>Minkowski</u> space.

<u>3.43</u> THEOREM.- Let $\mathcal{Q}$ be a set of compact (resp. closed) subsets of a metric (resp. Minkowski) space X, endowed with its finite topology. Suppose that X has the nearest point property with respect to $\mathcal{Q}$.

Then, for each $A_o \in \mathcal{Q}$ and $x_o \in A_o$, there exists a continuous mapping $F:\mathcal{Q} \to X$ such that $F(A_o) = x_o$ and $F(A) \in A$ for each $A \in \mathcal{Q}$.

Proof.- Let "d" denote the metric of X (resp. the metric induced by the norm of X). Then, for each $A \in \mathcal{Q}$ there exists a unique point $x \in A$ with

$$d(x_o, x) = d(x_o, A);$$

we set $F(A) = x$. We now show that F is continuous.

Let h_o be the mapping $x \mapsto d(x_o, x)$ of X into $\mathbb{R}$. Define a correspondence R between $\mathcal{Q}$ and X by setting $R(A) = A$ for each $A \in \mathcal{Q}$. Then R is continuous, since $\dot{R} = Id_{\mathcal{B}(X)}|\mathcal{Q}$, which is clearly continuous, when $\mathcal{B}(X)$ is endowed with its finite topology.

Finally, let $f: \mathcal{Q} \to \mathbb{R}$ be defined by

$$f(A) = d(x_o, A) = \inf \{d(x_o, x): x \in A\}$$

for each $A \in \mathcal{Q}$, and define a correspondence S between $\mathcal{Q}$ and X by setting

$$S(A) = \{x \in R(A): h_o(x) = f(A)\} = \{x \in A: d(x_o, x) = d(x_o, A)\}$$

for each $A \in \mathcal{Q}$. Since $S(A) = \{F(A)\}$ for each $A \in \mathcal{Q}$, the result will follow if we can show that S is usc.

Case 1: X metric, $\mathcal{Q}$ a set of compact subsets of X. Since h_o is continuous, the upper semi-continuity of S follows from 3.41.

Case 2: X Minkowski, $\mathcal{Q}$ a set of closed subsets of X. Here, we claim that h_o is a proper mapping, i.e. that h_o is a continuous, closed mapping and that h_o^{-1} has compact sections. Now h_o is continuous and the space $\mathbb{R}$ is locally compact, hence it suffices to show that $h_o^{-1}(B)$ is compact for each compact subset B of $\mathbb{R}$. Now, if $B \subset \mathbb{R}$ is compact, there exists a finite real number M such that $|y| \leq M$ for each $y \in B$. Then, $\|x - x_o\| \leq M$ for each element $x \in h_o^{-1}(B)$, so that $h_o^{-1}(B)$ is bounded. Moreover, $h_o^{-1}(B)$ is closed, since B is closed and h_o is continuous. It follows that $h_o^{-1}(B)$ is compact.

We now show that the requirements set on h in 3.42 are satisfied by the mapping $h: x \mapsto h_o(x)$ of X into $\overline{\mathbb{R}}$. Indeed, since h_o is continuous, so is h. Moreover, h_o^{-1} has compact sections, in particular $h^{-1}(f(A_o)) = h_o^{-1}(f(A_o))$ is compact. Finally, h_o is a closed mapping and so

$$(h_o^{-1})^{-1}(A) = h_o(A)$$

is closed for each closed subset A of X, showing that h_o^{-1} is usc; since $\mathbb{R}$ is an open subset of $\overline{\mathbb{R}}$, it follows that h^{-1} is usc at $f(A_o)$. Hence we can apply 3.42 and the result follows.

<u>3.44</u> COROLLARY.- Let X be a topological space, Y a metric (resp. Minkowski) space, and $\mathcal{Q}$ a set of compact (resp. closed) subsets of Y. Let R be a correspondence between X and Y with sections in $\mathcal{Q}$.

Suppose that R is continuous, and that Y has the nearest point property with respect to $\mathcal{Q}$. Then, for each $x_o \in X$ and $y_o \in R(x_o)$, there exists a continuous selection f of R with $f(x_o) = y_o$.

Proof.- Let $\mathcal{Q}$ be endowed with its finite topology. By 3.43, there exists a continuous mapping $F: \mathcal{Q} \to Y$ such that $F(R(x_o)) = y_o$ and $F(A) \in A$ for each $A \in \mathcal{Q}$. Let f be the mapping $x \mapsto F(R(x))$ of X into Y. Then f is continuous, $f(x_o) = F(R(x_o)) = y_o$ and $f(x) = F(R(x)) \in R(x)$ for each $x \in X$.

Chapter 4

COVERING TOPOLOGIES

§1. DEFINITIONS AND FUNDAMENTAL PROPERTIES

Defining topologies with carriers

If X is a topological space, then the set of all finite intersections of sets of the form $\mathcal{S}(G)$, with G open in X, is a base for the lsf topology of $\mathcal{G}(X)$. It is natural to ask what happens if one considers, for instance, locally finite families of open subsets of X, rather than merely finite ones. In this way, one can define "covering topologies " on $\mathcal{G}(X)$.

More precisely, we shall consider carriers α (in the sense of chapter 2, §1) which satisfy the following axioms, for each topological space X:

(C1) The union of a finite family of elements of $\alpha(X)$ belongs to $\alpha(X)$.

(C2) If $G \subset X$ is open, then $\{G\} \in \alpha(X)$.

(C3) If $\mathcal{H} \in \alpha(X)$ and H_G is an open subset of G for each $G \in \mathcal{H}$, then $\{H_G: G \in \mathcal{H}\} \in \alpha(X)$.

(C4) If $\mathcal{H} \in \alpha(X)$ and $\mathcal{V} \subset \mathcal{H}$, then $\mathcal{V} \in \alpha(X)$.

A carrier α satisfying those four axioms for each topological space X will be called a <u>covering carrier</u>. The choice of these axioms will be made clear in the sequel, as we develop the properties of topologies defined by carriers. We first give some immediate consequences of these axioms:

<u>4.1</u> THEOREM.- For each topological space X, we have:

a) If $\mathcal{G} \in \alpha(X)$ and $G_0 \subset X$ is open, then
$$\mathcal{G} \cup \{G_0\} \in \alpha(X).$$

b) If $\mathcal{G}$ is a finite set of open subsets of X, then
$$\mathcal{G} \in \alpha(X).$$

c) If $\mathcal{G} \in \alpha(X)$ and $G_0 \subset X$ is open, then
$$\{G \cap G_0 : G \in \mathcal{G}\} \in \alpha(X).$$

d) If $\mathcal{G} \in \alpha(X)$, then $\mathcal{G} \setminus \{\emptyset\} \in \alpha(X)$.

Proof.- Parts a) and b) follow from (C2) and (C1), part c) follows from (C3) and part d) from (C4).

Note that the statement of 4.1d implies (C4), if (C3) is assumed. Indeed, let $\mathcal{G} \in \alpha(X)$ and $\mathcal{H} \subset \mathcal{G}$, and define H_G for each $G \in \mathcal{G}$ by $H_G = \begin{cases} G, & \text{if } G \in \mathcal{H} \\ \emptyset, & \text{if } G \notin \mathcal{H} \end{cases}$. Then $\{H_G : G \in \mathcal{G}\} \in \alpha(X)$.
But $\mathcal{H} = \begin{cases} \{H_G : G \in \mathcal{G}\}, & \text{if } \emptyset \in \mathcal{H} \neq \mathcal{G} \text{ or } \mathcal{H} = \mathcal{G} \\ \{H_G : G \in \mathcal{G}\} \setminus \{\emptyset\}, & \text{if } \emptyset \notin \mathcal{H} \neq \mathcal{G} \end{cases}$.

The two most important examples of covering carriers which we shall consider are the <u>finite carrier</u> α_f and the <u>locally finite carrier</u> α_{lf}. In order to simplify the notation of this chapter, we shall use the symbol γ in place of α_f and λ in place of α_{lf}. Thus, for each topological space X, $\gamma(X)$ is the set of all finite sets of open subsets of X, and $\lambda(X)$ is the set of all locally finite subsets of open subsets of X.

We define an order relation between carriers by setting $\alpha \leq \beta$ iff $\alpha(X) \subset \beta(X)$ for each topological space X. It is clear from 4.1b that $\gamma \leq \alpha$ for each covering carrier α, i.e. γ is the smallest covering carrier.

In the following, we denote by $\langle \mathcal{G} \rangle$ the set
$$\langle \mathcal{G} \rangle = \bigcap_{G \in \mathcal{G}} \mathcal{S}(G) = \{A \in \mathcal{P}(X) : A \cap G \neq \emptyset \text{ for each } G \in \mathcal{G}\}$$
for each $\mathcal{G} \subset \mathcal{P}(X)$.

Let $(X,\mathcal{T})$ be a topological space. The <u>topology</u> $\mathcal{T}_*(\alpha)$ <u>of</u> $\mathcal{P}(X)$ is defined to be that which is generated by the set of all sets of the form $\langle\mathcal{B}\rangle$, where $\mathcal{B}\in\alpha(X)$.

If $\mathcal{Q}\subset\mathcal{P}(X)$, the topology $\mathcal{T}_*(\alpha)$ of $\mathcal{Q}$ is defined to be that which is induced on $\mathcal{Q}$ by the topology $\mathcal{T}_*(\alpha)$ of $\mathcal{P}(X)$.

The <u>topology</u> $\widetilde{\mathcal{T}}(\alpha)$ <u>of</u> $\mathcal{Q}$ is defined to be the join of the topologies $\mathcal{T}^*$ and $\mathcal{T}_*(\alpha)$ of $\mathcal{Q}$.

Generally speaking, we shall refer to the topologies $\mathcal{T}_*(\alpha)$ and $\widetilde{\mathcal{T}}(\alpha)$ as <u>covering topologies</u>.

The topology $\mathcal{T}_*(\gamma)$ (resp. $\widetilde{\mathcal{T}}(\gamma)$) is nothing but the lower semi-finite (resp. finite) topology $\mathcal{T}_*$ (resp. $\widetilde{\mathcal{T}}$). We shall refer to the topology $\mathcal{T}_*(\lambda)$ (resp. $\widetilde{\mathcal{T}}(\lambda)$) as the <u>lower semi-locally finite</u> (resp. <u>locally finite</u>) topology.

It is clear that the topology $\mathcal{T}_*$ (resp. $\widetilde{\mathcal{T}}$) is coarser than $\mathcal{T}_*(\alpha)$ (resp. $\widetilde{\mathcal{T}}(\alpha)$) for each covering carrier α.

In the rest of this section, we study some of the basic properties of covering topologies. In §2, we study covering topologies with respect to separation, and in §3 we compare them with other topologies considered in chapter 2.

If $G_o\subset X$ and $\mathcal{B}\subset\mathcal{P}(X)$, we denote by $\langle G_o,\mathcal{B}\rangle$ the set $\mathcal{P}(G_o)\cap\langle\mathcal{B}\rangle=\{A\in\mathcal{P}(X): A\subset G_o \text{ and } A\cap G\neq\emptyset \text{ for each } G\in\mathcal{B}\}$. We also set $\langle\mathcal{B}\rangle_{\mathcal{Q}}=\langle\mathcal{B}\rangle\cap\mathcal{Q}$ and $\langle G_o,\mathcal{B}\rangle_{\mathcal{Q}}=\langle G_o,\mathcal{B}\rangle\cap\mathcal{Q}$ for each $\mathcal{Q}\subset\mathcal{P}(X)$.

<u>4.2</u> THEOREM.- Let $(X,\mathcal{T})$ be a topological space.

a) The sets $\langle\mathcal{B}\rangle$, with $\mathcal{B}$ running through $\alpha(X)$, form a base for the topology $\mathcal{T}_*(\alpha)$ of $\mathcal{P}(X)$.

b) The sets $\langle G_o,\mathcal{B}\rangle$, where $G_o\subset X$ is open, $\mathcal{B}\in\alpha(X)$ and $\bigcup\mathcal{B}\subset G_o$, form a base for the topology $\widetilde{\mathcal{T}}(\alpha)$ of $\mathcal{P}(X)$.

Proof.-

a) Let $(\mathcal{B}_i)_{i\in I}$ be a finite family of elements of $\alpha(X)$.

Then $\bigcap_{i\in I}\langle \mathcal{Y}_i\rangle = \langle \bigcup_{i\in I} \mathcal{Y}_i\rangle$. Since $\bigcup_{i\in I} \mathcal{Y}_i \in \alpha(X)$ by (C1), the result follows.

b) A base for the topology $\tilde{\mathfrak{T}}(\alpha)$ of $\mathcal{P}(X)$ is formed by the sets $\mathcal{P}(G_o) \cap \langle \mathcal{Y}\rangle = \langle G_o, \mathcal{Y}\rangle$, where G_o is open and $\mathcal{Y} \in \alpha(X)$. Putting $\mathcal{Y}' = \{G \cap G_o : G \in \mathcal{Y}\}$, we have $\mathcal{Y}' \in \alpha(X)$ by 4.1c and $\bigcup \mathcal{Y}' \subset G_o$. Moreover, $\langle G_o, \mathcal{Y}\rangle = \langle G_o, \mathcal{Y}'\rangle$; for if A is any member of $\langle G_o, \mathcal{Y}\rangle$, then $A \cap (G \cap G_o) = A \cap G \neq \emptyset$ for each $G \in \mathcal{Y}$.

From now on, when we refer to a <u>basic open set</u> $\langle G_o, \mathcal{Y}\rangle$, we shall always assume that $\bigcup \mathcal{Y} \subset G_o$.

<u>4.3</u> THEOREM.- For each basic open set $\langle G_o, \mathcal{Y}\rangle$ for the topology $\tilde{\mathfrak{T}}(\alpha)$ of $\mathcal{P}(X)$, there exists an element $\mathcal{H} \in \alpha(X)$ such that $\langle G_o, \mathcal{Y}\rangle \cap \mathcal{P}_o(X) = \langle \bigcup \mathcal{H}, \mathcal{H}\rangle$.

Proof.- Put $\mathcal{H} = \mathcal{Y} \cup \{G_o\}$. Then $\mathcal{H} \in \alpha(X)$ by 4.1a, and $\langle G_o, \mathcal{Y}\rangle \cap \mathcal{P}_o(X) = \langle \bigcup \mathcal{H}, \mathcal{H}\rangle$; for if A is any member of $\langle G_o, \mathcal{Y}\rangle \cap \mathcal{P}_o(X)$, then $A \cap G_o = A \neq \emptyset$.

Thus, if $\emptyset \neq A_o \subset X$, then for each basic open set $\langle G_o, \mathcal{Y}\rangle$ with $A_o \in \langle G_o, \mathcal{Y}\rangle$, there exists an element $\mathcal{H} \in \alpha(X)$ with $A_o \in \langle \bigcup \mathcal{H}, \mathcal{H}\rangle \subset \langle G_o, \mathcal{Y}\rangle$. We also have:

<u>4.4</u> COROLLARY.- The sets $\langle \bigcup \mathcal{Y}, \mathcal{Y}\rangle_{\mathcal{P}_o(X)}$, where $\mathcal{Y}$ runs through $\alpha(X)$, form a base for the topology $\tilde{\mathfrak{T}}(\alpha)$ of $\mathcal{P}_o(X)$.

We now try to determine under what conditions covering topologies are admissible.

<u>4.5</u> THEOREM.- Let X be a topological space and let h be the bijection $x \mapsto \{x\}$ of X onto $\mathcal{Q} = \{\{x\} : x \in X\}$. If $\mathcal{Q}$ is endowed with its topology $\tilde{\mathfrak{T}}_*(\alpha)$ (resp. $\tilde{\mathfrak{T}}(\alpha)$), then we have:

a) h is open.

b) A necessary and sufficient condition for h to be continuous is that $\bigcap \mathcal{H}$ be open for each $\mathcal{H} \in \alpha(X)$.

Proof.- Since the topology $\mathcal{T}^*$ of $\mathcal{P}(X)$ is admissible, it is sufficient to consider only the topology $\mathcal{T}_*(\alpha)$.

a) For each open subset G of X, we have

$$h(G) = \{\{x\} : x \in G\} = \langle\{G\}\rangle_{\mathcal{Q}},$$

which is an open subset of $\mathcal{Q}$, since $\{G\} \in \alpha(X)$ by (C2).

b) h is continuous iff $h^{-1}(\langle\mathcal{H}\rangle_{\mathcal{Q}})$ is an open subset of X for each $\mathcal{H} \in \alpha(X)$. Since

$$h^{-1}(\langle\mathcal{H}\rangle_{\mathcal{Q}}) = \{x \in X : \{x\} \in \langle\mathcal{H}\rangle_{\mathcal{Q}}\} = \bigcap \mathcal{H},$$

the result follows.

<u>4.6</u> COROLLARY.- The following statements are equivalent:

a) The topology $\mathcal{T}_*(\alpha)$ of $\mathcal{P}(X)$ is admissible.

b) The topology $\tilde{\mathcal{T}}(\alpha)$ of $\mathcal{P}(X)$ is admissible.

c) $\bigcap \mathcal{H}$ is an open subset of X for each $\mathcal{H} \in \alpha(X)$.

We shall say that a covering carrier α is <u>admissible</u> if, for each topological space X, the topology $\tilde{\mathcal{T}}(\alpha)$ of $\mathcal{P}(X)$ is admissible. Thus, the finite carrier γ and the locally finite carrier λ are both admissible (if $\mathcal{H} \in \lambda(X)$, then $\bigcap \mathcal{H} \doteq \emptyset$ if $\mathcal{H}$ is infinite).

Interiors and closures

<u>4.7</u> THEOREM.- Let X be a set, let $\mathcal{Q}, \mathcal{B} \subset \mathcal{P}(X)$ and $A_o, B_o \subset X$. Then we have:

a) A sufficient condition for $\langle A_o, \mathcal{Q}\rangle \subset \langle B_o, \mathcal{B}\rangle$ is that $A_o \subset B_o$ and that every set of $\mathcal{B}$ contains a set of $\mathcal{Q}$. If $\bigcup \mathcal{Q} \subset A_o$ and $\emptyset \notin \mathcal{Q}$, then this condition is also necessary.

b) A necessary condition for $\langle A_o, \mathcal{Q}\rangle \cap \langle B_o, \mathcal{B}\rangle \neq \emptyset$ is that $A_o \cap B \neq \emptyset \ \forall B \in \mathcal{B}$ and $A \cap B_o \neq \emptyset \ \forall A \in \mathcal{Q}$.

If $\bigcup\mathcal{A} \subset A_o$ and $\bigcup\mathcal{B} \subset B_o$, then this condition is also sufficient.

Proof.-

a) The condition is obviously sufficient. We now show necessity. Suppose first that $A_o \not\subset B_o$, and choose a point $x_o \in A_o \setminus B_o$. Since $\emptyset \notin \mathcal{A}$, we can also choose a point $x_A \in A$ for each $A \in \mathcal{A}$. Let $C = \{x_o\} \cup \{x_A : A \in \mathcal{A}\}$. Since $x_o \notin B_o$, we have $C \in \langle A_o, \mathcal{A}\rangle \setminus \langle B_o, \mathcal{B}\rangle$, contrary to the hypothesis.

Suppose next that there exists a set $B \in \mathcal{B}$ which does not contain any set of $\mathcal{A}$. Thus, we can choose a point $x_A \in A \setminus B$ for each $A \in \mathcal{A}$. Again, let $C = \{x_A : A \in \mathcal{A}\}$. Since $C \cap B = \emptyset$, we have $C \in \langle A_o, \mathcal{A}\rangle \setminus \langle B_o, \mathcal{B}\rangle$, contrary to the hypothesis.

b) The condition is obviously necessary. We now show sufficiency. By hypothesis, we can choose a point $x_A \in A \cap B_o$ for each $A \in \mathcal{A}$ and a point $y_B \in A_o \cap B$ for each $B \in \mathcal{B}$. Putting $C = \{x_A : A \in \mathcal{A}\} \cup \{y_B : B \in \mathcal{B}\}$, we see that $C \in \langle A_o, \mathcal{A}\rangle \cap \langle B_o, \mathcal{B}\rangle$.

<u>4.8</u> COROLLARY.- Let X be a T_1-space, let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ and $A_o, B_o \subset X$. Then we have:

a) If $\bigcup\mathcal{A} \subset A_o$, $\emptyset \notin \mathcal{A}$ and $\mathcal{A}$ is locally finite, then $\langle A_o, \mathcal{A}\rangle_{\mathfrak{F}(X)} \subset \langle B_o, \mathcal{B}\rangle_{\mathfrak{F}(X)}$ iff $A_o \subset B_o$ and every set of $\mathcal{B}$ contains a set of $\mathcal{A}$.

b) If $\bigcup\mathcal{A} \subset A_o$, $\bigcup\mathcal{B} \subset B_o$, and $\mathcal{A}, \mathcal{B}$ are locally finite, then $\langle A_o, \mathcal{A}\rangle_{\mathfrak{F}(X)} \cap \langle B_o, \mathcal{B}\rangle_{\mathfrak{F}(X)} \neq \emptyset$ iff $A_o \cap B \neq \emptyset$ $\forall B \in \mathcal{B}$ and $A \cap B_o \neq \emptyset$ $\forall A \in \mathcal{A}$.

Proof.- Indeed, the two sets C constructed in the proof of 4.7 are then closed.

Let X be a topological space. It follows immediately from the definitions that $\overline{\mathcal{J}(A)} \subset \mathcal{J}(\overline{A})$ for each subset A of X, where the closure of $\mathcal{J}(A)$ is taken with respect to the topology $\mathfrak{T}^*$ of $\mathcal{P}(X)$. Passing over to complements, we obtain the relation $\mathcal{P}(\mathring{A}) \subset \overset{\circ}{\widehat{\mathcal{P}(A)}}$. Similarly, we have the relations $\overline{\mathcal{P}(A)} \subset \mathcal{P}(\bar{A})$ and $\mathcal{J}(\mathring{A}) \subset \overset{\bullet}{\widehat{\mathcal{J}(A)}}$ in the space $(\mathcal{P}(X), \mathfrak{T}_*)$.

<u>4.9</u> THEOREM.- For each $\mathcal{Q} \subset \mathcal{P}(X)$ and each subset A_o of X, we have:

a) $\overline{\langle \mathcal{Q} \rangle} \subset \langle \overline{\mathcal{Q}} \rangle$ in $(\mathcal{P}(X), \mathfrak{T}^*)$.

b) $\langle \mathring{\mathcal{Q}} \rangle \subset \overset{\bullet}{\widehat{\langle \mathcal{Q} \rangle}}$ in $(\mathcal{P}(X), \mathfrak{T}_*)$, provided $\mathcal{Q}$ is finite.

c) $\overline{\langle A_o, \mathcal{Q} \rangle} \subset \langle \bar{A}_o, \overline{\mathcal{Q}} \rangle$ in $(\mathcal{P}(X), \tilde{\mathfrak{T}})$.

d) $\langle \mathring{A}_o, \mathring{\mathcal{Q}} \rangle \subset \overset{\circ}{\widehat{\langle A_o, \mathcal{Q} \rangle}}$ in $(\mathcal{P}(X), \tilde{\mathfrak{T}})$, provided $\mathcal{Q}$ is finite.

Here $\overline{\mathcal{Q}} = \{\bar{A} : A \in \mathcal{Q}\}$ and $\mathring{\mathcal{Q}} = \{\mathring{A} : A \in \mathcal{Q}\}$.

Proof.- We only consider closures; the results concerning interiors can be proved in the same way.

a) $\overline{\langle \mathcal{Q} \rangle} = \overline{\bigcap_{A \in \mathcal{Q}} \mathcal{J}(A)} \subset \bigcap_{A \in \mathcal{Q}} \overline{\mathcal{J}(A)} \subset \bigcap_{A \in \mathcal{Q}} \mathcal{J}(\bar{A}) = \langle \overline{\mathcal{Q}} \rangle$.

c) $\overline{\langle A_o, \mathcal{Q} \rangle} = \overline{\mathcal{P}(A_o) \cap \langle \mathcal{Q} \rangle} \subset \overline{\mathcal{P}(A_o)} \cap \overline{\langle \mathcal{Q} \rangle} \subset \mathcal{P}(\bar{A}_o) \cap \langle \overline{\mathcal{Q}} \rangle = \langle \bar{A}_o, \overline{\mathcal{Q}} \rangle$.

<u>4.10</u> THEOREM.- Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{Q} \subset \mathcal{P}(X)$ and A_o a subset of X with $\bigcup \mathcal{Q} \subset A_o$.

Then we have:

a) $\langle \bar{A}_o, \overline{\mathcal{Q}} \rangle \subset \overline{\langle A_o, \mathcal{Q} \rangle}$ with respect to the topology $\tilde{\mathfrak{T}}(\alpha)$.

b) $\overset{\circ}{\widehat{\langle A_o, \mathcal{Q} \rangle}} \subset \langle \mathring{A}_o, \mathring{\mathcal{Q}} \rangle$ with respect to the topology $\tilde{\mathfrak{T}}(\alpha)$.

Proof.-

a) Let $B_o \in \langle \bar{A}_o, \overline{\mathcal{Q}} \rangle$. To show that $B_o \in \overline{\langle A_o, \mathcal{Q} \rangle}$, we have to show that $\langle G_o, \mathcal{G} \rangle \cap \langle A_o, \mathcal{Q} \rangle \neq \emptyset$ for each basic open set $\langle G_o, \mathcal{G} \rangle$ with $B_o \in \langle G_o, \mathcal{G} \rangle$. But for this, it is sufficient, by 4.7b, to show that $G_o \cap A \neq \emptyset$ for each $A \in \mathcal{Q}$ and $G \cap A_o \neq \emptyset$ for each $G \in \mathcal{G}$. Now $B_o \cap \bar{A} \neq \emptyset$ for each $A \in \mathcal{Q}$; since $B_o \subset G_o$, we also have $G_o \cap \bar{A} \neq \emptyset$, hence $G_o \cap A \neq \emptyset$. On the other hand, $B_o \subset \bar{A}_o$; since $B_o \cap G \neq \emptyset$ for each $G \in \mathcal{G}$, we also have $G \cap \bar{A}_o \neq \emptyset$, hence $G \cap A_o \neq \emptyset$, as desired.

b) Let $B_o \in \overset{\circ}{\overline{\langle A_o, \mathcal{Q}\rangle}}$. Then there exists a basic open set $\langle G_o, \mathcal{G}\rangle$ with $B_o \in \langle G_o, \mathcal{G}\rangle \subset \langle A_o, \mathcal{Q}\rangle$. Clearly $\emptyset \notin \mathcal{G}$, and so, by 4.7a, $G_o \subset A_o$ and every set of $\mathcal{Q}$ contains a set of $\mathcal{G}$. But then $B_o \subset G_o \subset A_o$, i.e. $B_o \subset \mathring{A}_o$. On the other hand, if $A \in \mathcal{Q}$, there exists a set $G \in \mathcal{G}$ with $G \subset A$, so that $G \subset \mathring{A}$; since $B_o \cap G \neq \emptyset$, it follows that $B_o \cap \mathring{A} \neq \emptyset$. Thus, we have shown that $B_o \in \langle \mathring{A}_o, \mathring{\mathcal{Q}}\rangle$.

In the same way, we obtain, using 4.8:

<u>4.11</u> THEOREM.- Let $(X,\mathcal{T})$ be a T_1-space, $\mathcal{Q} \subset \mathcal{P}(X)$ and A_o a subset of X with $\bigcup \mathcal{Q} \subset A_o$. Then we have:

a) $\langle \overline{A}_o, \overline{\mathcal{Q}}\rangle_{\mathfrak{F}(X)} \subset \overline{\langle A_o, \mathcal{Q}\rangle_{\mathfrak{F}(X)}}$ with respect to the topology $\tilde{\mathfrak{F}}(\lambda)$ of $\mathfrak{F}(X)$, provided $\mathcal{Q}$ is locally finite.

b) $\overset{\circ}{\overline{\langle A_o, \mathcal{Q}\rangle_{\mathfrak{F}(X)}}} \subset \langle \mathring{A}_o, \mathring{\mathcal{Q}}\rangle_{\mathfrak{F}(X)}$ with respect to the topology $\tilde{\mathfrak{F}}(\lambda)$ of $\mathfrak{F}(X)$.

§2. SEPARATION

In this section, we study separation properties of covering topologies. Besides being generalizations of the results in [8], some of our results will be new even for the case of the finite topology, for we shall consider hyperspaces as bitopological spaces as well as topological spaces.

A <u>bitopological space</u> $(X,\mathcal{P},\mathcal{Q})$ is a set X together with two topologies $\mathcal{P}$ and $\mathcal{Q}$ on X. Thus, for example, if $(X,\mathcal{T})$ is a topological space, then $(\mathcal{P}(X),\mathcal{T}^*,\mathcal{T}_*)$ is a bitopological space.

Bitopological spaces were introduced by J.C. Kelly in [5]. We refer to that paper for definitions concerning separation properties of bitopological spaces. Nevertheless, we shall repeat here, for convenience, definitions of terms we need.

Separation by open sets

A bitopological space $(X,\mathcal{P},\mathcal{Q})$ is called semi-pairwise Hausdorff if, given any two distinct elements of X, there exist a $\mathcal{P}$-open set U and a $\mathcal{Q}$-open set V such that U contains one of the two points, V contains the other and $U \cap V = \emptyset$.

4.12 THEOREM.- If $(X,\mathfrak{T})$ is regular, then $(\mathfrak{F}(X),\mathfrak{T}^*,\mathfrak{T}_*)$ is semi-pairwise Hausdorff.

Proof.- Let A, B be two distinct elements of $\mathfrak{F}(X)$. We may assume that $A \setminus B \neq \emptyset$ say. Choose a point $x_0 \in A \setminus B$. Since X is regular, there exist open sets G and H with $x_0 \in G$, $B \subset H$ and $G \cap H = \emptyset$.

Then, $A \in \mathcal{J}(G)$, $B \in \mathcal{D}(H)$ and $\mathcal{J}(G) \cap \mathcal{D}(H) = \emptyset$, hence the result.

4.13 THEOREM.- Let $(X,\mathfrak{T})$ be a T_1-space. If $(\mathfrak{F}(X),\widetilde{\mathfrak{T}}(\lambda))$ is Hausdorff, then $(X,\mathfrak{T})$ is regular.

Proof.- Let $A_0 \in \mathfrak{F}(X)$ and $x_0 \in X \setminus A_0$. Then A_0 and $A_0 \cup \{x_0\}$ are two distinct elements of $\mathfrak{F}(X)$. Since $\mathfrak{F}(X)$ is Hausdorff, there exist two basic open sets $\langle G_0,\mathcal{G}\rangle$ and $\langle H_0,\mathcal{H}\rangle$ such that $A_0 \in \langle G_0,\mathcal{G}\rangle$, $A_0 \cup \{x_0\} \in \langle H_0,\mathcal{H}\rangle$ and $\langle G_0,\mathcal{G}\rangle_{\mathfrak{F}(X)} \cap \langle H_0,\mathcal{H}\rangle_{\mathfrak{F}(X)} = \emptyset$. Since $A_0 \cup \{x_0\} \neq \emptyset$, we may assume that $H_0 = \bigcup\mathcal{H}$.

Now $x_0 \in H_0$ and so the set $\mathcal{H}' = \{H \in \mathcal{H} : x_0 \in H\}$ is nonempty. We now claim that there exists a member H of $\mathcal{H}'$ with $G_0 \cap H = \emptyset$. Indeed, suppose, to get a contradiction, that there exists a point $x_H \in G_0 \cap H$ for each $H \in \mathcal{H}'$, and let $C = \{x_H : H \in \mathcal{H}'\}$. Since $\mathcal{H}'$ is locally finite, the set C is closed, hence $A = A_0 \cup C \in \mathfrak{F}(X)$. Moreover, one easily sees that $A \in \langle G_0,\mathcal{G}\rangle \cap \langle H_0,\mathcal{H}\rangle$, a contradiction. Thus, there exists a member H of $\mathcal{H}'$ with $G_0 \cap H = \emptyset$. But then, $x_0 \in H$ and $A_0 \subset G_0$, hence the result.

If a bitopological space $(X,\mathcal{P},\mathcal{Q})$ is semi-pairwise

Hausdorff, then it remains so if $\mathcal{P}$ and/or $\mathcal{Q}$ are replaced by finer topologies; also, clearly $(X, \mathcal{P}\vee\mathcal{Q})$ is Hausdorff. Using this, together with 4.12 and 4.13, we obtain:

<u>4.14</u> COROLLARY.- If $(X,\mathfrak{T})$ is a T_1-space and $\alpha \leq \lambda$, then the following statements are equivalent:

a) $(X,\mathfrak{T})$ is regular.

b) $(\tilde{\mathfrak{F}}(X), \mathfrak{T}^*, \mathfrak{T}_*(\alpha))$ is semi-pairwise Hausdorff.

c) $(\tilde{\mathfrak{F}}(X), \hat{\tilde{\mathfrak{T}}}(\alpha))$ is Hausdorff.

If $\mathcal{P}$ and $\mathcal{Q}$ are two topologies on a set X, then $\mathcal{P}$ is said to be <u>regular</u> <u>with</u> <u>respect</u> <u>to</u> $\mathcal{Q}$ if, for each $x \in X$ and each $\mathcal{P}$-closed set $P \subset X$ with $x \notin P$, there exist a $\mathcal{P}$-open set U and a $\mathcal{Q}$-open set V such that $x \in U$, $P \subset V$ and $U \cap V = \emptyset$.

A bitopological space $(X,\mathcal{P},\mathcal{Q})$ is <u>pairwise</u> <u>regular</u> if $\mathcal{P}$ is regular with respect to $\mathcal{Q}$, and $\mathcal{Q}$ is regular with respect to $\mathcal{P}$.

<u>4.15</u> THEOREM.- If $(X,\mathfrak{T})$ is regular, then the topology $\mathfrak{T}_*(\alpha)$ of $\mathcal{B}(X)$ is regular with respect to its topology $\mathfrak{T}^*$.

Proof.- Let $A_0 \in \mathcal{B}(X)$. We show that, for each $\mathcal{G} \in \alpha(X)$ with $A_0 \in \langle\mathcal{G}\rangle$, there exists a member $\mathcal{H} \in \alpha(X)$ such that $A_0 \in \langle\mathcal{H}\rangle$ and $\overline{\langle\mathcal{H}\rangle} \subset \langle\mathcal{G}\rangle$, where the closure of the set $\langle\mathcal{H}\rangle$ is taken with respect to the topology $\mathfrak{T}^*$.

So let $A_0 \in \langle\mathcal{G}\rangle$, and choose a point $x_G \in A_0 \cap G$ for each $G \in \mathcal{G}$. Since $(X,\mathfrak{T})$ is regular, for each $G \in \mathcal{G}$ there exists an open set H_G with $x_G \in H_G$ and $\overline{H_G} \subset G$. Putting $\mathcal{H} = \{H_G\colon G \in \mathcal{G}\}$, so that $\mathcal{H} \in \alpha(X)$, we have $A_0 \in \langle\mathcal{H}\rangle$ and

$$\overline{\langle\mathcal{H}\rangle} \subset \langle\overline{\mathcal{H}}\rangle \subset \langle\mathcal{G}\rangle.$$

<u>4.16</u> THEOREM.- If $(X,\mathfrak{T})$ is normal, then the topology $\mathfrak{T}^*$ of $\tilde{\mathfrak{F}}(X)$ is regular with respect to its topology $\mathfrak{T}_*$.

Proof.- Let $A_0 \in \tilde{\mathfrak{F}}(X)$. We show that, for each open subset G of X with $A_0 \in \mathcal{B}(G)$, there exists an open subset H of X such that $A_o \in \mathcal{B}(H)$ and $\overline{\mathcal{B}(H)} \subset \mathcal{B}(G)$, where the closure of the set $\mathcal{B}(H)$ is taken with respect to the topology $\mathfrak{T}_*$.

So let $A_o \in \mathcal{B}(G)$, i.e. $A_o \subset G$. Since X is normal, there exists an open set H such that $A_o \subset H$ and $\overline{H} \subset G$. But then,

$$A_o \in \mathcal{B}(H) \text{ and } \overline{\mathcal{B}(H)} \subset \mathcal{B}(\overline{H}) \subset \mathcal{B}(G).$$

If $\mathcal{P}$, $\mathcal{Q}$ are two topologies on a set X, and $\mathcal{P}$ is regular with respect to $\mathcal{Q}$, then it is clear from the definition that $\mathcal{P}$ is regular with respect to every topology on X which is finer than $\mathcal{Q}$. Thus, the following result is a consequence of 4.15 and 4.16:

<u>4.17</u> COROLLARY.- If $(X,\mathfrak{T})$ is a T_4-space, then $(\mathfrak{F}(X),\mathfrak{T}^*,\mathfrak{T}_*(\alpha))$ is pairwise regular.

<u>4.18</u> THEOREM.- Let $(X,\mathfrak{T})$ be a T_1-space, and suppose that $\alpha \leq \lambda$. If $(\mathfrak{F}(X),\tilde{\mathfrak{T}}(\alpha))$ is regular, then $(X,\mathfrak{T})$ is normal.

Proof.- Let $A_o \in \mathfrak{F}(X)$ and G any open subset of X with $A_o \subset G$. Thus, $A_o \in \mathcal{B}(G) \cap \mathfrak{F}(X) = \langle G,\emptyset\rangle_{\mathfrak{F}(X)}$, an open subset of $\mathfrak{F}(X)$. Since $\mathfrak{F}(X)$ is regular, there exists a basic open set $\langle H_o,\mathcal{H}\rangle$ such that

$$A_o \in \langle H_o,\mathcal{H}\rangle \text{ and } \overline{\langle H_o,\mathcal{H}\rangle_{\mathfrak{F}(X)}} \subset \langle G,\emptyset\rangle_{\mathfrak{F}(X)}.$$

Then, $A_o \subset H_o$. Also, $\langle \overline{H}_o,\overline{\mathcal{H}}\rangle_{\mathfrak{F}(X)} = \overline{\langle H_o,\mathcal{H}\rangle_{\mathfrak{F}(X)}} \subset \langle G,\emptyset\rangle_{\mathfrak{F}(X)}$, so that $\overline{H}_o \subset G$.

If a bitopological space $(X,\mathcal{P},\mathcal{Q})$ is pairwise regular, then it is clear that the topological space $(X,\mathcal{P}\vee\mathcal{Q})$ is regular. Using this, together with 4.17 and 4.18, we obtain:

<u>4.19</u> COROLLARY.- If $(X,\mathfrak{T})$ is a T_1-space, and $\alpha \leq \lambda$, then the following statements are equivalent:

a) $(X,\mathfrak{T})$ is normal.

b) $(\mathfrak{F}(X),\mathfrak{T}^*,\mathfrak{T}_*(\alpha))$ is pairwise regular.

c) $(\mathfrak{F}(X),\tilde{\mathfrak{T}}(\alpha))$ is regular.

Separation by continuous functions

A topological space X is said to be a Stone space if, given any two distinct points x_0, x_1 of X, there exists a continuous function $f: X \to [0,1]$ such that $f(x_0) = 0$ and $f(x_1) = 1$. It is said to be completely regular if, for each $x_0 \in X$ and each closed set $A_0 \subset X$ with $x_0 \notin A_0$, there exists a continuous function $f: X \to [0,1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for each $x \in A_0$.

A bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is said to be semi-pairwise Stone if, given any two distinct points of X, there exists a $\mathcal{P}$-usc and $\mathcal{Q}$-lsc function $f: X \to [0,1]$ which takes the value 0 at one of the two points and the value 1 at the other. We put $\mathfrak{F}_0(X) = \mathfrak{F}(X) \setminus \{\emptyset\}$ for each topological space X.

4.20 THEOREM.- If $(X, \mathcal{T})$ is completely regular, then $(\mathfrak{F}(X), \mathcal{T}^*, \mathcal{T}_*)$ is semi-pairwise Stone.

Proof.- Let A, B be two distinct elements of $\mathfrak{F}(X)$. Then $A \setminus B \neq \emptyset$, say. Choose a point $x_0 \in A \setminus B$. Since $(X, \mathcal{T})$ is completely regular, there exists a continuous real-valued function $f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(x) = 0$ for each $x \in B$. Define $F: \mathfrak{F}(X) \to [0,1]$ by setting $F(E) = \sup \{f(x) : x \in E\}$ for each $E \in \mathfrak{F}_0(X)$ and $F(\emptyset) = 0$. By 3.30a and 3.28, F is $\mathcal{T}^*$-usc and $\mathcal{T}_*$-lsc. Also, $F(A) = 1$ and $F(B) = 0$.

4.21 THEOREM.- Let $(X, \mathcal{T})$ be a T_1-space and suppose that α is admissible. If $(\mathfrak{F}(X), \widetilde{\mathcal{T}}(\alpha))$ is a Stone space, then $(X, \mathcal{T})$ is completely regular.

Proof.- Let $A_0 \in \mathfrak{F}(X)$ and $x_0 \in X \setminus A_0$. Then A_0 and $A_0 \cup \{x_0\}$ are distinct elements of $\mathfrak{F}(X)$. Since $\mathfrak{F}(X)$ is a Stone space, there exists a continuous function $F: \mathfrak{F}(X) \to [0,1]$ such that $F(A_0) = 1$ and $F(A_0 \cup \{x_0\}) = 0$. Now define a function $f: X \to [0,1]$ by setting

$$f(x) = F(A_o \cup \{x\}) \quad \text{for each} \quad x \in X.$$

Then, $f(x_o) = F(A_o \cup \{x_o\}) = 0$ and $f(x) = F(A_o) = 1$ for each $x \in A_o$.

We now claim that f is continuous. Since F is continuous, it suffices to show that the mapping $h\colon x \mapsto A_o \cup \{x\}$ of $(X,\mathcal{T})$ into $(\mathcal{P}(X),\tilde{\mathcal{T}}(\alpha))$ is continuous. So let $x \in X$, and suppose $\langle G_o,\mathcal{U}\rangle$ is a basic open set with $h(x) \in \langle G_o,\mathcal{U}\rangle$; thus, $A_o \cup \{x\} \subset G_o$ and $(A_o \cup \{x\}) \cap G \neq \emptyset$ for each $G \in \mathcal{U}$. Now let $\mathcal{H} = \{G \in \mathcal{U}\colon A_o \cap G = \emptyset\}$. Then $\mathcal{H} \in \alpha(X)$ and so, since α is admissible, $\bigcap\mathcal{H}$ is an open subset of X. Also, $x \in G_o \cap (\bigcap\mathcal{H})$, so that $H = G_o \cap (\bigcap\mathcal{H})$ is an open neighbourhood of x in X. Finally, $h(H) \subset \langle G_o,\mathcal{U}\rangle$, which completes the proof that h is continuous.

If a bitopological space $(X,\mathcal{P},\mathcal{Q})$ is semi-pairwise Stone, then it remains so if $\mathcal{P}$ and/or $\mathcal{Q}$ are replaced by finer topologies; also, clearly $(X,\mathcal{P}\vee\mathcal{Q})$ is a Stone space. Using this, together with 4.20 and 4.21, we obtain:

<u>4.22</u> COROLLARY.- If $(X,\mathcal{T})$ is a T_1-space and α is admissible, then the following are equivalent:

a) $(X,\mathcal{T})$ is completely regular.

b) $(\mathfrak{F}(X),\mathcal{T}^*,\mathcal{T}_*(\alpha))$ is semi-pairwise Stone.

c) $(\mathfrak{F}(X),\tilde{\mathcal{T}}(\alpha))$ is Stone.

If $\mathcal{P}$, $\mathcal{Q}$ are two topologies on a set X, $\mathcal{P}$ is said to be <u>completely regular with respect to</u> $\mathcal{Q}$ if, for each $x \in X$ and each $\mathcal{P}$-closed set $P \subset X$ with $x \notin P$, there exists a $\mathcal{P}$-usc and $\mathcal{Q}$-lsc function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for each $y \in P$.

A bitopological space $(X,\mathcal{P},\mathcal{Q})$ is <u>pairwise completely regular</u> if $\mathcal{P}$ is completely regular with respect to $\mathcal{Q}$ and $\mathcal{Q}$ is completely regular with respect to $\mathcal{P}$. (Cf. [6], definition 2.3)

<u>4.23</u> THEOREM.- If $(X,\mathcal{T})$ is completely regular, then the to-

pology $\mathcal{T}_*$ of $\mathcal{K}(X)$ is completely regular with respect to its topology $\mathcal{T}^*$.

Proof.- Suppose $A_0 \in \mathcal{J}(G)$, where G is an open subset of X, and choose a point $x_0 \in A_0 \cap G$. Then there exists a continuous function $f\colon X \longrightarrow [0,1]$ such that $f(x_0) = 1$ and $f(x) = 0$ for each $x \in X \setminus G$.

Define $F\colon \mathcal{K}(X) \rightarrow [0,1]$ by setting

$$F(E) = \sup\{f(x)\colon x \in E\} \text{ for each } E \in \mathcal{K}_0(X),$$

and $F(\emptyset) = 0$. Then F is $\mathcal{T}^*$-usc and $\mathcal{T}_*$-lsc, with $F(A_0) = 1$. Also, if $E \in \mathcal{K}(X) \setminus \mathcal{J}(G) = \mathcal{K}(X \setminus G)$, then $f(x) = 0$ for each $x \in E$ and so $F(E) = 0$.

Since the sets $\mathcal{J}(G)$, with G open in X, generate $\mathcal{T}_*$, the result follows.

<u>4.24</u> THEOREM.- If $(X,\mathcal{T})$ is normal, then the topology $\mathcal{T}^*$ of $\mathfrak{F}(X)$ is completely regular with respect to its topology $\mathcal{T}_*$.

Proof.- Suppose $A_0 \in \mathcal{K}(G) \cap \mathfrak{F}(X)$, where G is an open subset of X. Then there exists a continuous real-valued function $f\colon X \longrightarrow [0,1]$ such that $f(x) = 0$ for each $x \in A_0$ and $f(x) = 1$ for each $x \in X \setminus G$. Define $F\colon \mathfrak{F}(X) \longrightarrow [0,1]$ by setting $F(E) = \sup\{f(x)\colon x \in E\}$ for each $E \in \mathfrak{F}_0(X)$, and $F(\emptyset) = 0$.

Then F is $\mathcal{T}^*$-usc and $\mathcal{T}_*$-lsc, with $F(A_0) = 0$. Also, if $E \in \mathfrak{F}(X) \setminus \mathcal{K}(G) = \mathcal{J}(X \setminus G) \cap \mathfrak{F}(X)$, then there exists a point $x \in E$ with $f(x) = 1$, and so $F(E) = 1$.

As a consequence of the last two theorems, we have:

<u>4.25</u> COROLLARY.- If $(X,\mathcal{T})$ is a T_4-space, then $(\mathfrak{F}(X), \mathcal{T}^*, \mathcal{T}_*)$ is pairwise completely regular.

<u>Separation in $\mathcal{K}(X)$</u>

If $(X,\mathcal{T})$ is a Hausdorff space, then points and compact sets in X can be separated by open sets, and so the proof of 4.12 shows that $(\mathcal{K}(X),\mathcal{T}^*,\mathcal{T}_*)$ is semi-pairwise Hausdorff. In the same way, if $(X,\mathcal{T})$ is regular, then closed sets and compact sets in X can be separated by open sets, and so the proof of 4.16 shows that the topology $\mathcal{T}^*$ of $\mathcal{K}(X)$ is regular with respect to its topology $\mathcal{T}_*$. Similar considerations apply to 4.20 and 4.24. We thus obtain the following result:

<u>4.26</u> THEOREM.- Let $(X,\mathcal{T})$ be a topological space. If α is admissible, we have:

a) $(X,\mathcal{T})$ is Hausdorff
iff $(\mathcal{K}(X),\mathcal{T}^*,\mathcal{T}_*(\alpha))$ is semi-pairwise Hausdorff
iff $(\mathcal{K}(X),\tilde{\mathcal{T}}(\alpha))$ is Hausdorff.

b) $(X,\mathcal{T})$ is regular
iff $(\mathcal{K}(X),\mathcal{T}^*,\mathcal{T}_*(\alpha))$ is pairwise regular
iff $(\mathcal{K}(X),\tilde{\mathcal{T}}(\alpha))$ is regular.

c) $(X,\mathcal{T})$ is Stone
iff $(\mathcal{K}(X),\mathcal{T}^*,\mathcal{T}_*(\alpha))$ is semi-pairwise Stone
iff $(\mathcal{K}(X),\tilde{\mathcal{T}}(\alpha))$ is Stone.

d) $(X,\mathcal{T})$ is completely regular
iff $(\mathcal{K}(X),\mathcal{T}^*,\mathcal{T}_*)$ is pairwise completely regular
iff $(\mathcal{K}(X),\tilde{\mathcal{T}})$ is completely regular.

§3. COMPARISON WITH OTHER TOPOLOGIES

<u>Comparison with the uniform topology</u>

It was shown in 2.23a that, if $(X,\mathcal{U})$ is a quasi-uniform space, then the topology $\mathcal{T}(\mathcal{U}_*)$ is finer than $(\mathcal{T}(\mathcal{U}))_*$. We now show:

<u>4.27</u> THEOREM.- Let $(X,\mathcal{U})$ be a uniform space. If $(X,\mathcal{T}(\mathcal{U}))$ is paracompact, then the topology $\mathcal{T}(\mathcal{U}_*)$ (resp. $\mathcal{T}(\tilde{\mathcal{U}})$) of $\mathcal{P}(X)$ is coarser than its topology $(\mathcal{T}(\mathcal{U}))_*(\lambda)$ (resp. $\widetilde{\mathcal{T}(\mathcal{U})}(\lambda)$).

Proof.- To show that $\mathfrak{T}(\mathcal{U}_*)$ is coarser than $(\mathfrak{T}(\mathcal{U}))_*(\lambda)$, let $A_o \in \mathcal{B}(X)$, and let U be any entourage of X. We show that the set $U_*(A_o)$ contains a neighbourhood of A_o in the topology $(\mathfrak{T}(\mathcal{U}))_*(\lambda)$.

Now there exists an open symmetric entourage V of X with $V^2 \subset U$. Since X is paracompact, the open covering $(V(x))_{x\in X}$ of X has a locally finite open refinement $\mathcal{H}$. Since each set V(x) is V^2-, and hence U-small, each set of $\mathcal{H}$ is also U-small. Let now $\mathcal{H}'$ be the set of those members of $\mathcal{H}$ which meet A_o, so that $A_o \in \langle\mathcal{H}'\rangle$. The result will follow if we can show that $\langle\mathcal{H}'\rangle \subset U_*(A_o)$.

So suppose $A \in \langle\mathcal{H}'\rangle$. To show that $A_o \subset U^{-1}(A)$, let $x_o \in A_o$. Since $\mathcal{H}'$ is a covering of A_o, there exists a set $H \in \mathcal{H}'$ such that $x_o \in H$. But $A \cap H \neq \emptyset$, so we can choose a point $x \in A \cap H$. Thus, $(x_o, x) \in H \times H \subset U$, so that

$x_o \in U^{-1}(x) \subset U^{-1}(A)$, as required.

Finally, since $\mathfrak{T}(\mathcal{U}^*)$ is coarser than $(\mathfrak{T}(\mathcal{U}))^*$, it follows that $\mathfrak{T}(\widetilde{\mathcal{U}})$ is coarser than $\widetilde{\mathfrak{T}(\mathcal{U})}(\lambda)$.

In what follows, we describe a case in which the topologies $\mathfrak{T}(\mathcal{U}_*)$ and $(\mathfrak{T}(\mathcal{U}))_*(\lambda)$ actually coincide. But first, we need some preliminaries.

A topological space X will be called <u>even</u> if every open covering $\mathcal{H}$ of X is <u>even</u>, i.e. there exists a neighbourhood U of the diagonal in $X \times X$ such that $(U(x))_{x\in X}$ refines $\mathcal{H}$.

A regular space is paracompact iff it is even (cf. [4], p. 156, theorem 28). Closely related to this result is the following lemma.

<u>4.28</u> LEMMA 1.- Let X be a topological space, $(G_i)_{i\in I}$ a locally finite family of open subsets of X and $(A_i)_{i\in I}$ a family of closed sets such that $A_i \subset G_i$ for each $i \in I$. Then there exists a neighbourhood U of the diagonal in $X \times X$ such that $U(A_i) \subset G_i$ for each $i \in I$.

Proof.- For each $i \in I$, the set

$$U_i = (G_i \times G_i) \cup ((X \setminus A_i) \times (X \setminus A_i))$$

is an open neighbourhood of the diagonal in $X \times X$. Since $U_i(x) = G_i$ for each $x \in A_i$, we have $U_i(A_i) \subset G_i$. Putting $U = \bigcap_{i \in I} U_i$, the result will follow if we can show that U is a neighbourhood of the diagonal.

For this, let $x \in X$. There exists a neighbourhood G of x in X such that $G \cap A_i = \emptyset$ for all but finitely many indices $i \in I$. Now, if $G \cap A_i = \emptyset$, then $G \subset X \setminus A_i$ and so $G \times G \subset U_i$. It follows that U contains the set

$$(G \times G) \cap (\bigcap \{U_i \colon i \in I \text{ and } G \cap A_i \neq \emptyset\}),$$

which is itself a neighbourhood of the point (x,x).

If X is even, then, by [4], p. 157, lemma 30, for every neighbourhood U of the diagonal in $X \times X$ there exists a symmetric neighbourhood V of the diagonal such that $V \cdot V \subset U$. This proves part a) of the following lemma:-

<u>4.29</u> LEMMA 2.- Let $(X, \mathcal{T})$ be a paracompact regular space.

a) The set of all neighbourhoods of the diagonal in $X \times X$ is a uniformity $\mathcal{U}$ on X.

b) The topology $\mathcal{T}'$ induced by $\mathcal{U}$ is equal to $\mathcal{T}$ itself.

Proof.- To prove part b), let $x_0 \in X$. If U is a neighbourhood of the diagonal (with respect to $\mathcal{T}$), then $U(x_0)$ is a $\mathcal{T}$-neighbourhood of x_0 in X. This shows that $\mathcal{T}'$ is coarser than $\mathcal{T}$. To show that $\mathcal{T}$ is coarser than $\mathcal{T}'$, let G be any $\mathcal{T}$-open neighbourhood of a point x_0 in X. Since X is regular, there exists a $\mathcal{T}$-open set H with $x_0 \in H$ and $\overline{H} \subset G$. Since X is even, there exists a neighbourhood U of the diagonal such that $(U(x))_{x \in X}$ refines the open covering $\{G, X \setminus \overline{H}\}$. But then, we must have $U(x_0) \subset G$, and the result follows.

If $(G_i)_{i \in I}$ is a locally finite family of subsets of X, we write $\langle G_i \rangle_{i \in I} = \{A \in \mathcal{P}(X) \colon A \cap G_i \neq \emptyset \text{ for each } i \in I\}$. Thus, putting $\mathcal{G} = \{G_i \colon i \in I\}$, we have $\langle G_i \rangle_{i \in I} = \langle \mathcal{G} \rangle$.

4.30 THEOREM.- Let X be a paracompact regular space, and let $\mathcal{U}$ be the uniformity on X for which the entourages are all the neighbourhoods of the diagonal in $X \times X$.

Then the topology $(\mathcal{T}(\mathcal{U}))_*(\lambda)$ of $\mathcal{P}(X)$ coincides with its topology $\mathcal{T}(\mathcal{U}_*)$.

Proof.- First note that $\mathcal{T}(\mathcal{U})$ is equal to the original topology of X, by 4.29. Also, by 4.27 it suffices to show that $(\mathcal{T}(\mathcal{U}))_*(\lambda)$ is coarser than $\mathcal{T}(\mathcal{U}_*)$. So let $(G_i)_{i\in I}$ be a locally finite family of open sets in $(X,\mathcal{T}(\mathcal{U}))$, and suppose that $A_0 \in \langle G_i \rangle_{i\in I}$. Choose a point $x_i \in A_0 \cap G_i$ for each $i \in I$. Since $(X,\mathcal{T}(\mathcal{U}))$ is regular, there is an open set H_i with $x_i \in H_i$ and $\overline{H_i} \subset G_i$, for each $i \in I$. By 4.28, there exists a symmetric entourage U of $(X,\mathcal{U})$ such that $U(\overline{H_i}) \subset G_i$ for each $i \in I$. We claim that $U_*(A_0) \subset \langle G_i \rangle_{i\in I}$. Indeed, if $A \in U_*(A_0)$, then $A_0 \subset U(A)$ and so, in particular, for each $i \in I$ there exists a point $y_i \in A$ such that $x_i \in U(y_i)$. Because of the symmetry of U, we thus have $y_i \in U(x_i) \subset G_i$ for each $i \in I$, hence $A \in \langle G_i \rangle_{i\in I}$.

Note that 4.29 and 4.30 still hold if X is assumed to be an even T_1-space, using the fact that singletons are then closed. Also, 4.30 should be put in connection with [8], theorem 3.4, which states that if X is normal, and if $\mathcal{U}$ is the uniformity induced on X by the Stone-Čech compactification, then $\widetilde{\mathcal{T}(\mathcal{U})}$ and $\mathcal{T}(\widetilde{\mathcal{U}})$ agree on $\mathcal{F}(X)$.

Comparison with the finite topology

In the following, the concept of pseudocompactness will play an important role. We therefore review some of the relevant facts.

A topological space X is said to be pseudocompact if every continuous (finite) real-valued function defined on X is bounded. We note:

4.31 THEOREM.- A sufficient condition for X to be pseudocompact is that every locally finite family of nonempty open subsets of X is finite.

If X is completely regular, then this condition is also necessary.

Proof.- Suppose X is not pseudocompact. Then there exists an unbounded continuous real-valued function f defined on X. Let $G_n = \{x \in X: |f(x)| > n\}$ for each natural number n. Then each G_n is nonempty and open. Moreover, $(G_n)_n$ is locally finite. Indeed, if $x_o \in X$, we can choose a natural number n_o with $|f(x_o)| < n_o$. Since f is continuous at x_o, there exists a neighbourhood U of x_o such that $|f(x)| < n_o$ for each $x \in U$. It follows that U does not meet any of the sets G_n with $n \geqslant n_o$.

Suppose X is completely regular, and that there exists a locally finite family $(G_i)_{i \in I}$ of nonempty open subsets of X which is infinite. We show that X is not pseudocompact. Now, since I is infinite, there exists an injective mapping k of the set of natural numbers into I. Choose a point x_n in $G_{k(n)}$ for each n. Since X is completely regular, there exist continuous real-valued functions $f_n \geqslant 0$ on X such that $f_n(x_n) = n$ and $f_n(x) = 0$ for each $x \in X \setminus G_{k(n)}$. Since the sequence $(G_{k(n)})_n$ is locally finite, the sum $f = \sum_n f_n$ is a well-defined continuous function. But $f(x_n) \geqslant f_n(x_n) = n$, so that f is not bounded.

We can now prove:

4.32 THEOREM.- Let $(X,\mathfrak{T})$ be a topological space, and A_o a completely regular pseudocompact subset of X. Then the systems of neighbourhoods of A_o in the topologies $\tilde{\mathfrak{T}}_*$ (resp. $\tilde{\tilde{\mathfrak{T}}}$) and $\tilde{\mathfrak{T}}_*(\lambda)$ (resp. $\tilde{\tilde{\mathfrak{T}}}(\lambda)$) coincide.

Proof.- Let $(G_i)_{i \in I}$ be a locally finite family of open subsets of X with $A_o \in \langle G_i \rangle_{i\in I}$. We shall show that $(G_i)_{i\in I}$ is finite, so that the result will trivially follow. Indeed, $(A_o \cap G_i)_{i\in I}$ is a locally finite family of nonempty open subsets of A_o, hence it must be finite, i.e. I is finite, as required.

It is clear from the proof that the same result holds for each compact subset A_o of X.

We now wish to prove a converse of 4.32. But first, we need a lemma:

<u>4.33</u> LEMMA.- Let X be an even space, A a closed subset of X and $(G_i)_{i\in I}$ a locally finite family of open subsets of A. Then there exists a locally finite family of open subsets $(H_i)_{i\in I}$ of X such that $G_i = A \cap H_i$ for each $i \in I$.

Proof.- By [4], p. 158, lemma 31, there exists an open neighbourhood U of the diagonal in $X \times X$ such that $(U(G_i))_{i\in I}$ is locally finite. Put $H_i = G_i \cup (U(G_i) \setminus A)$ for each $i \in I$; clearly $A \cap H_i = G_i$ for each $i \in I$, and the family $(H_i)_{i\in I}$ is locally finite, since $H_i \subset U(G_i)$ for each $i \in I$. It only remains to show that each set H_i is open in X. So let $i \in I$. Since G_i is open in A, there exists an open set L in X with $G_i = A \cap L$. Hence

$$H_i = (A \cap L) \cup (U(G_i) \setminus A) = (U(G_i) \cap L) \cup (U(G_i) \setminus A)$$

is open in X, as required.

<u>4.34</u> THEOREM.- Let $(X, \mathcal{T})$ be even, and let A_o be a closed subset of X. In order that every neighbourhood of A_o in the topology $\mathcal{T}_*(\lambda)$ of $\mathcal{D}(X)$ contains a neighbourhood of A_o in the topology $\tilde{\mathcal{T}}$, it is necessary that A_o be pseudocompact.

Proof.- Let $(G_i)_{i\in I}$ be a locally finite family of nonempty open subsets of A_o; we have to show that it is finite. By 4.33, there exists a locally finite family $(H_i)_{i\in I}$ of open subsets of X such that $G_i = H_i \cap A_o$ for each $i \in I$. Then, $A_o \in \langle H_i \rangle_{i\in I}$. By hypothesis, there exists a basic open set $\langle W, \mathcal{W} \rangle$ in $(\mathcal{P}(X), \tilde{\mathcal{T}})$ with $A_o \in \langle W, \mathcal{W} \rangle \subset \langle H_i \rangle_{i\in I}$. It follows that every set H_i contains a set of $\mathcal{W}$. Since $\mathcal{W}$ is finite, the family $(H_i)_{i\in I}$ must also be finite; for otherwise there would be a set of $\mathcal{W}$, and hence a point of X, which would be contained in the set H_i for infinitely many indices $i \in I$, contradicting the local finiteness of the family $(H_i)_{i\in I}$. Thus I is finite, i.e. the family $(G_i)_{i\in I}$ is finite.

This result could also have been formulated and proved for the space $\mathcal{F}(X)$ instead of $\mathcal{P}(X)$, provided it is further assumed that X is a T_1-space.

4.35 COROLLARY.- Let $(X, \mathcal{T})$ be a paracompact Hausdorff space and A_o a closed subset of X.

A necessary and sufficient condition for the systems of neighbourhoods of A_o in the topologies $\mathcal{T}_*$ (resp. $\tilde{\mathcal{T}}$) and $\mathcal{T}_*(\lambda)$ (resp. $\tilde{\mathcal{T}}(\lambda)$) to coincide is that A_o be compact.

Similarly for the space $\mathcal{F}(X)$.

Proof.- In order that the condition stated holds, it is necessary that A_o be pseudocompact. Since A_o, being closed in X, is normal, it is also countably compact, and hence compact.

We conclude with a result which stresses the differences between the finite and the locally finite topologies. If $(X, \mathcal{T})$ is separable, then $(\mathcal{F}(X), \tilde{\mathcal{T}})$ is also separable. We now show that this is not the case for $\tilde{\mathcal{T}}(\lambda)$.

4.36 LEMMA.- Let $(X, \mathcal{T})$ be a topological space, and suppose $\mathcal{H} \in \alpha(X)$ is such that the sets of $\mathcal{H}$ are nonempty and disjoint.

Then every dense subset of $(\mathcal{P}(X),\widetilde{\mathcal{T}}(\alpha))$ has cardinality at least $2^{\mathrm{card}(\mathcal{H})}$ (here, $\mathrm{card}(\mathcal{H})$ denotes the cardinality of the set $\mathcal{H}$).

Proof.- We show that the sets $\langle \bigcup \mathcal{H}, \mathcal{H} \rangle$, where $\mathcal{H}$ runs through the set of all subsets of $\mathcal{H}$, are non-empty and pairwise disjoint. The result will then follow.

To show that $\langle \bigcup \mathcal{H}, \mathcal{H} \rangle \neq \emptyset$, it suffices to choose a point $x_H \in H$ for each $H \in \mathcal{H}$ and to let $A = \{x_H : H \in \mathcal{H}\}$; for then, $A \in \langle \bigcup \mathcal{H}, \mathcal{H} \rangle$.

Next, suppose $\mathcal{H}$, $\mathcal{H}'$ are two distinct subsets of $\mathcal{H}$. Then there exists a set $H \in \mathcal{H} \setminus \mathcal{H}'$, say. Thus, $H \cap \bigcup \mathcal{H}' = \emptyset$, whence it follows that $\langle \bigcup \mathcal{H}, \mathcal{H} \rangle \cap \langle \bigcup \mathcal{H}', \mathcal{H}' \rangle = \emptyset$.

<u>4.37</u> THEOREM.- If $(X,\mathcal{T})$ is an infinite discrete space, then $(\mathcal{P}(X),\widetilde{\mathcal{T}}(\lambda))$ is not separable.

Proof.- Let $\mathcal{H} = \{\{x\} : x \in X\}$. Then $\mathcal{H} \in \lambda(X)$ clearly satisfies the conditions of the lemma. Thus, every dense subset of $\mathcal{P}(X)$ has cardinality $2^{\mathrm{card}(X)}$. Hence the result.

INDEX OF NOTATION

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