VERTEX-DECOMPOSITIONS OF GRAPHS

A thesis submitted for the degree of Doctor of Philosophy at the University of Keele, in July 1979,

by

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DECLARATION

The material in this thesis is claimed as original except where explicitly stated otherwise.This thesis has not been submitted previously for a higher degree of this or any other university.

DEDICATION

TO SALLY AND THOMAS.

ACKNOWLEDGEMENTS

My thanks are due to my supervisor,Dr. Hans Liebeck,and to the many others,both within the Mathematics Department of the University of Keele and elsewhere,who have freely given their time and energy to help and guide me.

CONTENTS

ABSTRACT

Certain partitions ("vertex-decompositions") of the vertices of a graph have the property that we can associate with them graph-like quotient structures ("generalised graphs"),and that we can define the adjacency matrix of a quotient in such a way that its eigenvalues and eigenvectors are closely related to those of the original graph.Chapters 1 and 2 give the basic definitions and results necessary to the rest of the thesis, and in addition Chapter 2 surveys most of the previous work in this field.

Chapter 3 discusses vertex-decompositions of trees,and gives methods for finding the group and characteristic polynomial of a tree from its smallest quotient.

Chapter 4 discusses vertex-decompositions of regular graphs into two classes,relating the existence of such decompositions to the possession of integer eigenvalues,and to switching classes of graphs.

Chapter 5 considers graphs for which a quotient may possess <u>all</u> the eigenvalues of the graph from which it is derived,and demonstrates that for a particular class ("singleton-regular" graphs),which includes vertex-transitive graphs,it is possible to find not only the eigenvalues but also some of their multiplicities from the adjacency matrix of the quotient.

Using this result,Chapter 6 and Appendix 1 are devoted to the construction of all possible quotients (with certain properties) of certain types of singleton-regular graph,and in Chapter 7 we decide for each quotient whether or not there are any graphs with the desired properties,sometimes utilising arguments concerning the primitivity of the action of the automorphism group or the

necessity for a graph to be a covering of a transitive graph. Exhaustive lists of graphs with the given properties are given, including a list of all symmetric trivalent graphs on ≤ 40 vertices.

Appendix 2 consists of a single proof establishing the uniqueness of the (3,12)-cage by the use of a computer program,a result used in Chapter 7.

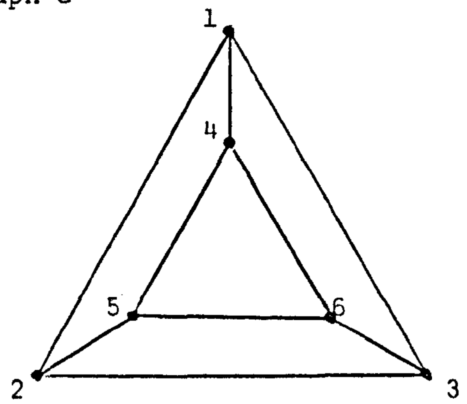
1:BASIC DEFINITIONS AND NOTATION

1.1:Generalised Graphs

We begin by discussing informally the ideas which we shall develop in Chapter 2,"vertex-decompositions" and "quotient graphs", to provide motivation for our definition of a generalised graph. We omit technical definitions at this stage for the sake of clarity of expression.

Consider the graph G

G:

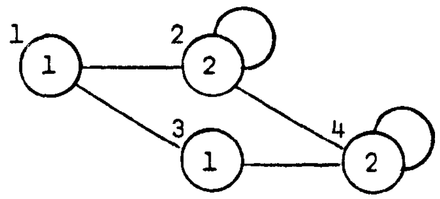


and the partition of its vertices $\mathcal{P}=\{P_1,P_2,P_3,P_4\}$ with $P_1=\{1\}$, $P_2=\{2,3\}$, $P_3=\{4\}$, $P_4=\{5,6\}$. This partition has the property that if x and y are in the same class then for each P_j $(j=1,2,3,4)$ the number of vertices in P_j adjacent to x is equal to the number of vertices in P_j adjacent to y. We shall call a partition with this property a vertex-decomposition of G (see Chapter 2).With this partition we may associate naturally the matrix

$$M=\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

whose ij-th entry is the number of vertices in class P_j adjacent to a given vertex in class P_i. We may also associate with this partition of G a "graph-like" structure G^P

G^P:



where the number contained in each vertex indicates the size of the corresponding class of the partition, and a single edge between two vertices indicates the least possible number of edges between the two corresponding classes consistent with the partition being a vertex-decomposition of G.

It is desirable to extend the usual definitions of a graph to include such structures, which we shall call generalised graphs, and also to define their adjacency matrices in such a way that M is the adjacency matrix of G^P.

1.1.1: Definition. A generalised graph G is an ordered triple (V,s,w) where

i) V is a finite set of symbols called the vertices of G,

ii) the size function $s:V \to N$ assigns to each vertex $x \in V$ a size $s(x)$,

iii) the weight function $w:V\times V \to Z^+$ assigns to the ordered pair (x,y) the weight $w(x,y)$ when $x,y \in V$.

The ordered pair (x,y) is called an edge if and only if $w(x,y) \neq 0$. If (x,y) is an edge, then x is said to be adjacent to y, which we may write as "x adj y".

$N(G,x)=\{y\in V,\ x \text{ adj } y\}$ is called the neighbourhood of x in G and $y \in N(G,x)$ is called a neighbour of x. The total weight of edges beginning at x is the outdegree of x and the weight of edges ending at x is the indegree of x. If the outdegree and indegree of x are equal, their value is said to be the valency of x. $N(x)$ will be written for $N(G,x)$ when there is no doubt as to the graph in question.

The number of vertices, $|V|$, is the *order* of G. //

1.1.2:Definition. Let $G=(V,s,w)$ be a generalised graph.

1.1.2.1: If $w(x,y)=w(y,x)$ for all $x,y\in V$ then G is *undirected*. Otherwise it is *directed*.

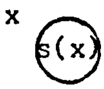
1.1.2.2: If $s(x)=1$ for all $x\in V$ then G is called a *pseudograph*.

1.1.2.3: If $s(x)=1$ for all $x\in V$, $w(x,y)=0$ or 1 and $w(x,x)=0$ for all $x,y\in V$, and G is undirected, then G is a *simple* graph. //

1.1.2.4:Notation. In the text the term "graph" without any of the modifications above will refer to a connected simple graph. //

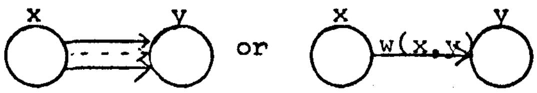
1.1.3:Diagrams. Let $G=(V,s,w)$ be a generalised graph.

1.1.3.1: If G is not a pseudograph then vertex x is represented as a circle containing s(x) thus:



The vertices of a pseudograph are represented as points.

1.1.3.2: We indicate the weight of edge (x,y) by one of the following:

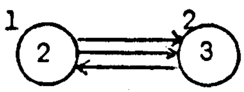


w(x,y) arrows

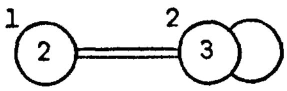
The omission of arrows from edges joining vertices x and y indicates that $w(x,y)=w(y,x)$ //

1.1.4:Examples. The following are generalised graphs:

1.1.4.1: G_1:

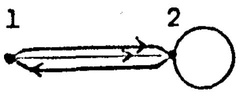


1.1.4.2: G_2:



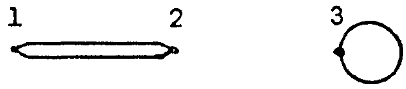
G_2 is undirected.

1.1.4.3: G_3:



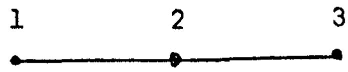
G_3 is a directed pseudograph.

<u>1.1.4.4</u>: G_4:



G_4 is an undirected pseudograph.

<u>1.1.4.5</u>: G_5:



G_5 is a simple graph. //

<u>1.1.5.1:Notation</u>.The set $\{1,2,\ldots,n\}$ is denoted by N_n. //

<u>1.1.5.2:Definition</u>.Let $G=(V,s,w)$ be a generalised graph of order n. If $V=N_n$, then G is said to be <u>properly labelled</u>. //

<u>1.1.6:Definition</u>.Let $G=(V,s,w)$ be a properly labelled graph of order n for some $n\varepsilon N$. Then the <u>adjacency matrix</u> of G, $A(G)$, is the $n\times n$ matrix (a_{ij}) with

$$a_{ij}=w(i,j)\times[s(i),s(j)]/s(i)$$

where $[a,b]$ denotes the lowest common multiple of positive integers a and b. //

<u>1.1.7:Examples</u>.

<u>1.1.7.1</u>:The adjacency matrix of G_1 in 1.1.4.1 is $\begin{pmatrix} 0 & 6 \\ 2 & 0 \end{pmatrix}$.

<u>1.1.7.2</u>:The adjacency matrix of G_5 in 1.1.4.5 is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

<u>1.1.7.3</u>:The adjacency matrix of the generalised graph G^P given at the beginning of Section 1.1 is M. //

The examples 1.1.7.2 and 1.1.7.3 together illustrate that we have defined the adjacency matrix in accordance with the aim stated at the beginning of this section without changing the common definition of the adjacency matrix of a pseudograph.

1.1.8:Notation.The characteristic polynomial of matrix A, $|A-\lambda I|$, is denoted by $\phi(A,\lambda)$. $\phi(G,\lambda)$ may be written for $\phi(A(G),\lambda)$. //

1.1.9.1:Notation.An eigenvector of a matrix A is a right-eigenvector of A unless the contrary is specifically stated. //

1.1.9.2:Definition.The spectrum of a generalised graph G is the spectrum of A(G). Two graphs are cospectral if their adjacency matrices have the same spectrum. //

With the exception of the one following piece of notation,which is used by Schwenk (37), all our other graph-theoretical terms follow Biggs (4) or Harary (21).

1.1.10:Definition.A rooted generalised graph,(G,r), is a generalised graph with one vertex, r, singled out.An isomorphism between rooted generalised graphs must take root to root. //

1.2:Families of Graphs

We shall refer repeatedly to certain well-known families of graphs,whose definitions we include here for convenience.We regard these graphs as essentially unlabelled,the labellings given being simply for ease of construction.

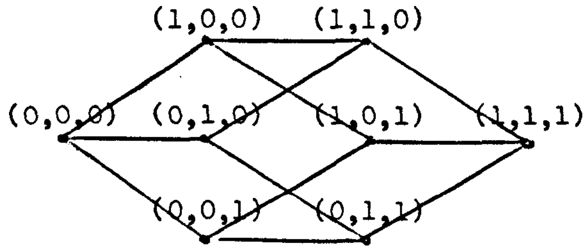
1.2.1:Notation.The complete and complete multipartite graphs will be denoted by K_n and $K_{a,b,\ldots}$ respectively,where n,a,b,... take positive integer values. //

1.2.2:Definition.A (k,g)-cage is a regular graph of valency k and girth g on the least known possible number of vertices. //

1.2.3.1:Definition.The k-cube, Q_k, is the graph defined as follows:-

The vertices of Q_k are the 2^k symbols $(x_1,x_2,\ldots,x_k)$ where $x_i=0$ or 1 for each $i \in N_k$, and two vertices are adjacent when their symbols differ in exactly one coordinate. The 3-cube is commonly simply called the cube. //

1.2.3.2: Example. The cube, Q_3:



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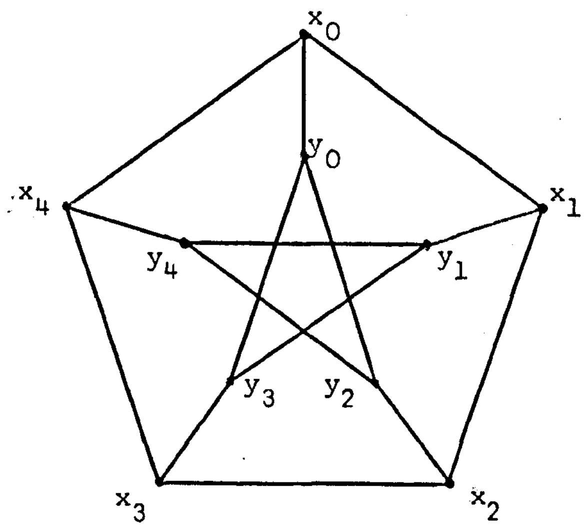
1.2.4.1: Definition. The generalised Petersen graph $P(h,t)$ has $2h$ vertices $x_0,x_1,\ldots,x_{h-1},y_0,y_1,\ldots,y_{h-1}$ and edges $\{x_i,y_i\},\{x_i,x_{i+1}\}$ and $\{y_i,y_{i+t}\}$ for $0 \leq i \leq h-1$ where the subscripts are reduced modulo h. //

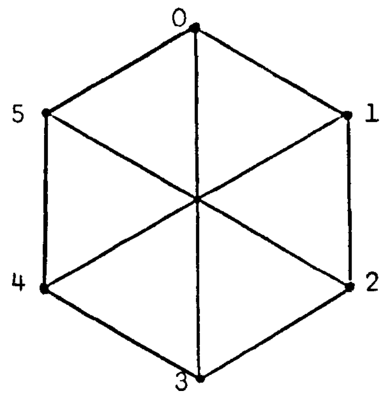
1.2.4.2: Example. Petersen's graph is $P(5,2)$:



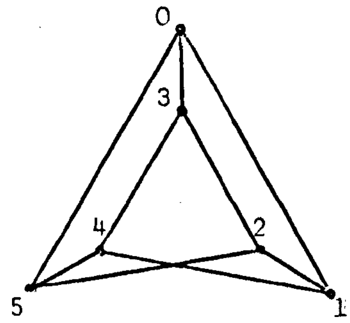
//

1.2.5.1: Definition. The Möbius ladder $M(h)$ has $2h$ vertices $0,1,\ldots,2h-1$ and vertex i is adjacent to vertices $i-1, i+1$ and $i+h$ (modulo $2h$) for $0 \leq i \leq 2h-1$. //

<u>1.2.5.2:Example</u>. M(3):



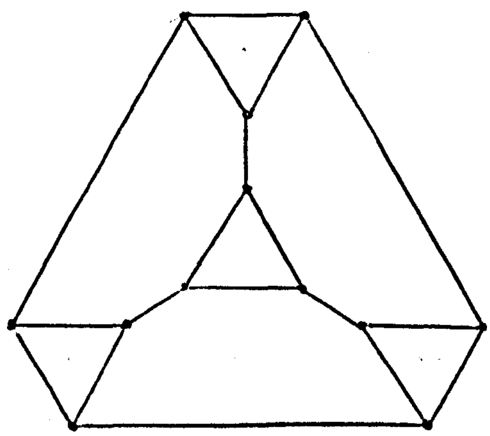
which may also be drawn



illustrating the reason for the name of this family. //

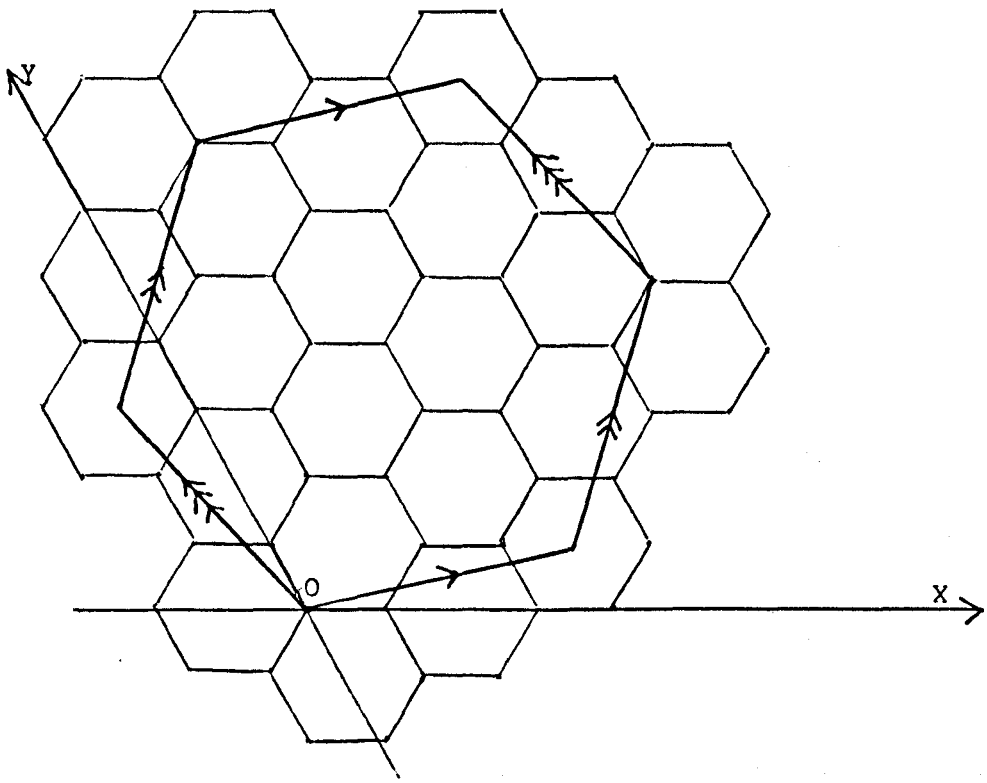
<u>1.2.6.1:Definition</u>.Let G be a regular graph of valency k.Then the <u>truncation of G</u>, T(G), is obtained by replacing each vertex of G by a copy of K_k, with one edge of G incident with each vertex of each copy of K_k. //

<u>1.2.6.2:Example</u>.The truncation of K_4, $T(K_4)$:



1.2.7.1:Definition.The hexagonal tessellation of the torus $\{6,3\}_{b,c}$ is obtained as follows:- The Euclidean plane is tessellated with hexagons whose edges have unit length relative to oblique axes OX,OY inclined at an angle of 120^0, with (0,0),(1,0),(0,1) being vertices of a hexagon.Then opposite edges of the hexagon whose vertices are (b+c,c),(0,0),(-c,b),(b-c,2b+c),(2b,2b+2c),(2b+c,b+2c) are identified.The result is a trivalent graph on $2(b^2+bc+c^2)$ vertices.Its properties are discussed in (11). //

1.2.7.2:Example. $\{6,3\}_{3,1}$:



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1.3:Automorphisms of Generalised Graphs

We follow the convention of using Greek letters for automorphisms, upper case for groups,and lower case for their elements.

1.3.1:Notation. Σ_n denotes the group of all permutations of n symbols. //

1.3.2:Definition.Let $G=(V,s,w)$ be a generalised graph.Then the permutation γ acting on V is an automorphism of G if

$$w(\gamma(x),\gamma(y))=w(x,y) \quad \text{and} \quad s(\gamma(x))=s(x) \quad \text{for all } x,y\in V.$$

//

1.3.3:Notation.The group of all automorphisms of a generalised graph $G=(V,s,w)$ is denoted by $\Gamma(G)$. The group of all automorphisms of G which fix $x\in V$ is designated $\Gamma_x(G)$ and is called the stabiliser of x in G. //

1.3.4:Definition.Let Θ be a permutation group acting on a set A. A block B is a subset of A such that B and $\theta(B)$ are either disjoint or identical for each $\theta\in\Theta$. When Θ acts transitively on A we say that Θ is primitive if the only blocks are trivial,that is of cardinality 0,1 or $|A|$. If Θ is imprimitive, A is partitioned into a disjoint union of non-trivial blocks,which are permuted by Θ. This partition is called a block system or a system of imprimitivity. A generalised graph $G=(V,s,w)$ is said to be primitive or imprimitive if $\Gamma(G)$ acting on V has the corresponding property. //

1.3.5:Definition.Let $G=(V,s,w)$ be a generalised graph.

1.3.5.1:If $\Gamma(G)$ acts transitively on V then we say G is transitive or vertex-transitive.

1.3.5.2:If $\Gamma(G)$ acts transitively on vertices and 1-arcs of G then G is symmetric.

1.3.5.3:If $\Gamma(G)$ acts transitively on t-arcs of G but not on t+1-arcs we say G is t-arc-transitive.

1.3.5.4:Let the distance $d(x,y)$ be finite for all $x,y\in V$ and let the diameter of G be d. If G is transitive and $\Gamma_x(G)$ acts transitively on vertices whose distance from x is i for each i with $1\leq i\leq d$, then we say that G is distance-transitive. //

1.4: Partitions

1.4.1: Notation. Let $P(A)=\{P_i, i\in N_m\}$ be a partition of order m of the set A. If $x,y\in P_i$ for some $i\in N_m$ then we write $x\overset{P}{\sim}y$, or just $x\sim y$ when there is no likelihood of confusion. We also write P for $P(A)$ when there is no doubt concerning the set under consideration. //

1.4.2: Definition. Let P,Q be partitions of orders p and q of the set A. If $x\overset{P}{\sim}y$ implies $x\overset{Q}{\sim}y$ then we say that

i) P is a subpartition of Q and Q is a superpartition of P,

and ii) P is finer than Q and Q is coarser than P. //

1.4.3.1: Definition. An n×n matrix has order n. //

1.4.3.2: Definition. Let $P(N_n)$ be a partition of order m and let A be a matrix of order n. Then P induces a partition of A into blocks $A^{(ij)}$, $(i,j\in N_m)$, which are submatrices of sizes $|P_i|\times|P_j|$ obtained by retaining in A only the rows corresponding to P_i and the columns corresponding to P_j. We use the notation $A=(A^{(ij)})$ and $A^{(ij)}=(a_{kr}^{(ij)})$. //

We have already used the term "block" but there is no likelihood of confusion arising and unfortunately both uses appear to be standard.

1.4.4: Definition. Let $G=(V,s,w)$ be a generalised graph and let $x\in V$ have the property that $d(x,y)$ is finite for all $y\in V$, with the maximum value of $d(x,y)$ over $y\in V$ being e, say. Then the distance partition of G with respect to vertex x, $L(G,x)$, is the set $\{P_i,\ 0\leq i\leq e\}$ where $y\in P_i$ if $y\in V$ and $d(x,y)=i$. //

2: VERTEX-DECOMPOSITIONS - GENERAL RESULTS

2.1: Decompositions, Quotient Graphs and Automorphisms

In Section 1.1 we gave an example of a vertex-decomposition of a graph. We wish to frame our definitions in such a way as to give meaning to the notion of a vertex-decomposition of a generalised graph.

2.1.1.1: Definition. Let $G=(V,s,w)$ be a generalised graph and let $\mathcal{P}=\{P_i, i\varepsilon N_m\}$ be a partition of V. The out-adjacency vector of vertex x, $\underline{\alpha}(x)$, is defined

$$\alpha(x)_i=\sum_{y\varepsilon P_i} w(x,y)\times[s(x),s(y)]/s(x) \qquad \text{for } i\varepsilon N_m,$$

where as before $[a,b]$ denotes the lowest common multiple of positive integers a and b. The in-adjacency vector of vertex x, $\underline{\beta}(x)$, is given by

$$\beta(x)_i=\sum_{y\varepsilon P_i} w(y,x)\times[s(x),s(y)]/s(x) \qquad \text{for } i\varepsilon N_m.$$

//

2.1.1.2: Proposition. Let G be a simple graph whose vertices are partitioned by $\mathcal{P}=\{P_i, i\varepsilon N_m\}$. Then for each vertex x and for all $i\varepsilon N_m$

$$\alpha(x)_i=\beta(x)_i=|N(x)\cap P_i|.$$

Proof. Trivial. //

2.1.1.3: Definition. Let $G=(V,s,w)$ be a generalised graph whose vertices are partitioned by $\mathcal{P}$, and let $\mathcal{P}$ have the property that

$x\overset{\mathcal{P}}{\sim}y$ implies that $\underline{\alpha}(x)=\underline{\alpha}(y)$ and $\underline{\beta}(x)=\underline{\beta}(y)$.

Then $\mathcal{P}$ is said to be a vertex-decomposition of G.

We denote the constant values of $\alpha(x)_j$ and $\beta(x)_j$ for $x\varepsilon P_i\varepsilon\mathcal{P}$ by s_{ij} and t_{ji}, and designate the corresponding m×m matrices by

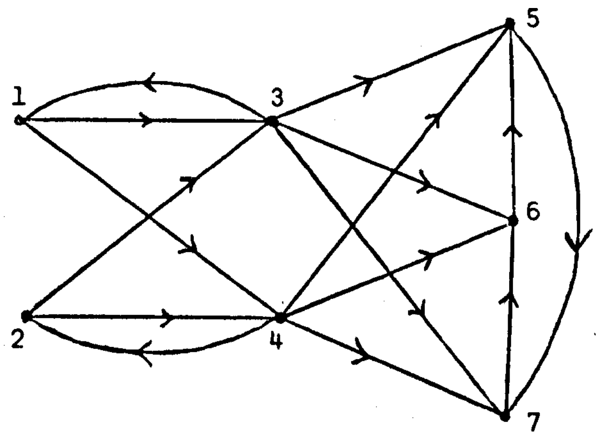
$S(G,P)=(s_{kp})$ and $T(G,P)=(t_{kp})$, for $k,p \in N_m$. //

<u>2.1.1.4:Notation.</u>As there is no likelihood of confusion with an edge-partition of a graph, we shall generally refer simply to a "decomposition" of a generalised graph. //

<u>2.1.2:Examples.</u>

<u>2.1.2.1</u>:Consider the directed pseudograph G_1.

G_1:



$$A(G_1)=\left(\begin{array}{cc|cc|ccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

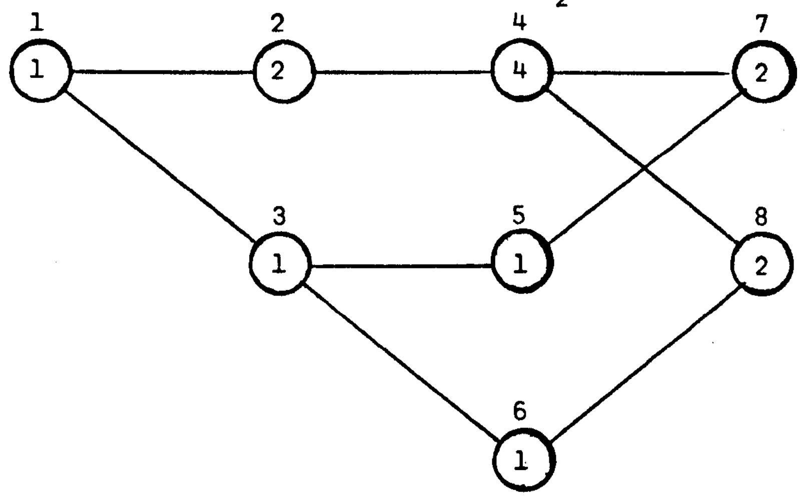
G_1 has the decomposition $\mathcal{D}_1=\{\{1,2\},\{3,4\},\{5,6,7\}\}$ with

$$S(G_1,\mathcal{D}_1)=\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

and $$T(G_1,\mathcal{D}_1)=\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

2.1.2.2: Take the undirected generalised graph G_2.

G_2:



$$A(G_2)=\left(\begin{array}{c|cc|ccc|cc} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right)$$

G_2 has the decomposition $\mathcal{D}_2=\{\{1\},\{2,3\},\{4,5,6\},\{7,8\}\}$ with

$$S(G_2,\mathcal{D}_2)=\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

and $$T(G_2,\mathcal{D}_2)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$ //

There are several elementary results relating the adjacency matrix of a generalised graph to S and T.

2.1.3:Propositions. Let $G=(V,s,w)$ be a generalised graph with decomposition $\mathcal{D}=\{D_i, i\varepsilon N_m\}$. Then:-

2.1.3.1: $\mathcal{D}$ induces a partition of $A(G)$ into blocks with the property that the sum of the entries of each row of the ij-th block is s_{ij}.

2.1.3.2: $$s_{ij}\sum_{x\varepsilon D_i} s(x)=t_{ij}\sum_{y\varepsilon D_j} s(y)$$

and if G is a pseudograph then

$$s_{ij}|D_i|=t_{ij}|D_j|.$$

2.1.3.3: If G is a pseudograph then t_{ij} is the constant column sum of the ij-th block of $A(G)$.

2.1.3.4: If G is undirected then $S(G,\mathcal{D})$ is the transpose of $T(G,\mathcal{D})$.

Proofs

1: By definition.

2: Let $x\varepsilon D_i$. then

$$s_{ij}s(x)=\sum_{y\varepsilon D_j} w(x,y)\times[s(x),s(y)].$$

Hence
$$\begin{aligned} s_{ij}\sum_{x\varepsilon D_i} s(x) &= \sum_{x\varepsilon D_i} s_{ij}s(x)=\sum_{x\varepsilon D_i}\sum_{y\varepsilon D_j} w(x,y)[s(x),s(y)] \\ &= \sum_{y\varepsilon D_j}\sum_{x\varepsilon D_i} w(x,y)[s(x),s(y)] \\ &= \sum_{y\varepsilon D_j} t_{ij}s(y)=t_{ij}\sum_{y\varepsilon D_j} s(y). \end{aligned}$$

And for a pseudograph $\sum_{x\varepsilon D_i} s(x)=|D_i|$.

3: If G is a pseudograph then

$$t_{ij}=\sum_{y\varepsilon D_i} w(y,x) \quad \text{for any } x\varepsilon D_j.$$

Hence t_{ij} is the sum of the entries of the x-th column over block $A^{(ij)}$ of $A(G)$, and t_{ij} is independent of the choice of $x\varepsilon D_j$ so that the column sum of $A^{(ij)}$ is constant.

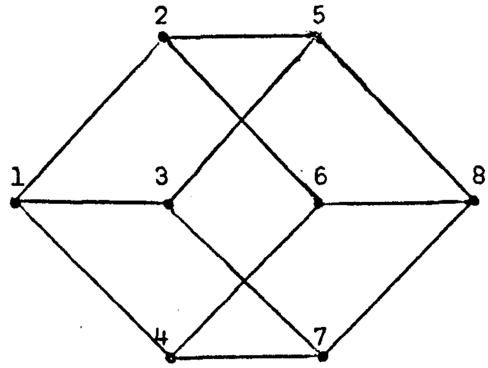
4: Let $x\varepsilon D_i$. Then

$$\begin{aligned} s_{ij} &= \sum_{y\varepsilon D_j} w(x,y)[s(x),s(y)]/s(x) \\ &= \sum_{y\varepsilon D_j} w(y,x)[s(x),s(y)]/s(x)=t_{ji}. \end{aligned}$$ //

A generalised graph may have several decompositions.

2.1.4:Examples.Consider the cube.

Q_3:



Amongst its decompositions are the following:

2.1.4.1: $\mathcal{D}_1=\{\{1,2,4,6\},\{3,5,7,8\}\}$,

2.1.4.2: $\mathcal{D}_2=\{\{1\},\{2,3,4\},\{5,6,7\},\{8\}\}=\mathcal{L}(Q_3,1)$,

2.1.4.3: $\mathcal{D}_3=\{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$. //

Note that every generalised graph has a decomposition into singleton classes,and that every regular simple graph has a decomposition of order one.

2.1.5:Definitions.

2.1.5.1:The decomposition of a generalised graph into singleton classes is said to be trivial.

2.1.5.2:The decomposition of order one of a regular simple graph is called the regular decomposition. //

At least one other vertex-partition of a graph has been studied. Weichsel (46) defines the "star-partition" thus:

2.1.6.1:Definition.Let the vertices of graph G be partitioned by $\mathcal{P}=\{P_i, i\varepsilon N_m\}$,a partition having the following properties:

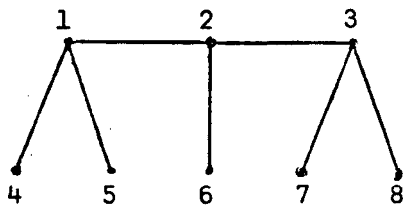
i) $x\overset{\mathcal{P}}{\sim}y$ implies that the valencies of x and y are equal;

ii)Let $x\varepsilon P_i$ and $y\varepsilon P_j$ with x adj y.Then for each $x'\varepsilon P_i$ there is a $y'\varepsilon P_j$ such that x'adj y';

Then $\mathcal{P}$ is a star-partition of G. //

It is immediately apparent that every decomposition of a simple graph is also a star-partition,though the converse is not true. Consider for example:

2.1.6.2:Example.



The partition {{1,2,3},{4,5,6,7,8}} is a star-partition but not a decomposition of this graph. //

Weichsel establishes the following propositions for star-partitions:

2.1.6.3:Proposition.Let G be a graph whose only star-partition is trivial.Then $\Gamma(G)$ is trivial. //

2.1.6.4:Proposition.A tree has trivial automorphism group if and only if it has only the trivial star-partition. //

He also provides an algorithm for the construction of the coarsest star-partition of a given graph.

We shall discuss analogues and refinements of these results using decompositions instead of star-partitions later in this chapter and in Chapter 3.

In the introduction to Section 1.1 we pointed out that a decomposition of a graph leads naturally to another generalised graph.We will call this the "quotient" graph with respect to the decomposition.

2.1.7.1:Definition.Let $G=(V,s,w)$ be a generalised graph with a decomposition $\mathcal{D}=\{D_i, i\in N_m\}$.We define the quotient graph (or simply "quotient") of G with respect to decomposition $\mathcal{D}$,$G^{\mathcal{D}}$,to be the ordered triple (N_m,s',w') where

i) $s'(i)=\sum_{x\varepsilon D_i} s(x)$

ii) $w'(i,j)=\dfrac{\sum_{x\varepsilon D_i}\sum_{y\varepsilon D_j} w(x,y)[s(x),s(y)]}{\left[\sum_{x\varepsilon D_i} s(x),\sum_{y\varepsilon D_j} s(y)\right]}$ //

To show that this is a consistent definition we need the following proposition.

<u>2.1.7.2:Proposition.</u>If $G^{\mathcal{D}}=(V,s',w')$ is the quotient of generalised graph G with respect to decomposition $\mathcal{D}=\{D_i,i\varepsilon N_m\}$,then $w'(i,j)$ is a non-negative integer for all $i,j\varepsilon N_m$.

<u>Proof.</u> $$\sum_{x\varepsilon D_i}\sum_{y\varepsilon D_j} w(x,y)[s(x),s(y)]=\sum_{x\varepsilon D_i} s_{ij}s(x)=s_{ij}\sum_{x\varepsilon D_i} s(x)$$

which is divisible by $\sum_{x\varepsilon D_i} s(x)$.

And $$\sum_{y\varepsilon D_j}\sum_{x\varepsilon D_i} w(x,y)[s(x),s(y)]=\sum_{y\varepsilon D_j} t_{ji}s(y)$$

which is divisible by $\sum_{y\varepsilon D_j} s(y)$. //

We have certain elementary results concerning quotient graphs.

<u>2.1.8.1:Proposition.</u>Let $G=(V,s,w)$ be a generalised graph with decomposition $\mathcal{D}=\{D_i,i\varepsilon N_m\}$,and let the quotient graph $G^{\mathcal{D}}=(N_m,s',w')$ as above.Then $A(G^{\mathcal{D}})=S(G,\mathcal{D})$.

<u>Proof.</u>An immediate consequence of the definition. //

<u>2.1.8.2:Proposition.</u>If $G^{\mathcal{D}}$ is a quotient graph of an undirected generalised graph G,then $G^{\mathcal{D}}$ is also undirected.

<u>Proof.</u>Trivial. //

<u>2.1.8.3:Proposition.</u>Every generalised graph is a quotient of a pseudograph,and every undirected generalised graph is a quotient of an undirected pseudograph.

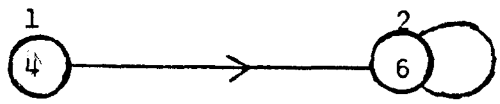
<u>Proof.</u>Consider generalised graph $G=(V,s,w)$.We construct a pseudograph $G'=(V',s',w')$ such that G is a quotient of G' thus:

i) Replace each vertex $x\epsilon V$ by a set $V(x)$ of $s(x)$ vertices of size one.

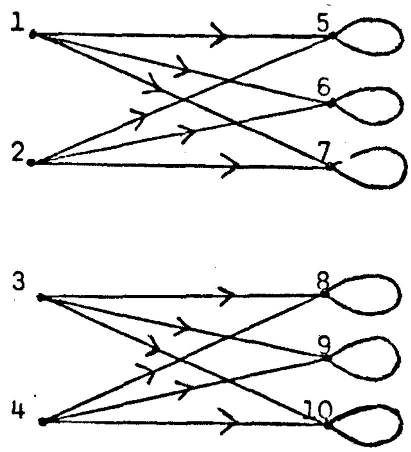
ii) For each $v\epsilon V(x)$ set $w'(v,v)=w(x,x)$.

iii) For each pair $x,y\epsilon V$ for which $w(x,y)\neq 0$, divide each of $V(x)$ and $V(y)$ into h sets of equal size, $V(x)_i$ and $V(y)_i$ for $i\epsilon N_h$, where h is the highest common factor of $s(x)$ and $s(y)$. Next construct an edge of weight $w(x,y)$ from every vertex of $V(x)_i$ to every one of $V(y)_i$ for all $i\epsilon N_h$. (In the case where G is undirected construct an undirected edge of weight $w(x,y)$). //

2.1.8.4: Example.



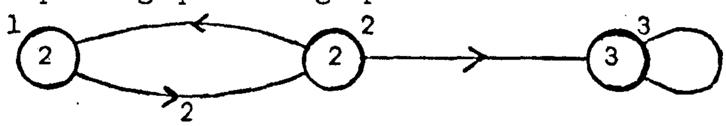
is a quotient of



//

2.1.9: Examples.

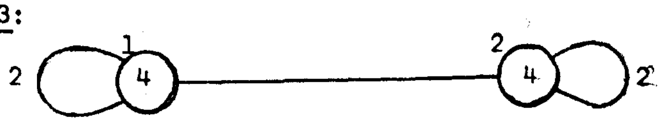
2.1.9.1: Consider the graph G_1 and decomposition D_1 in 2.1.2.1. Then the corresponding quotient graph is



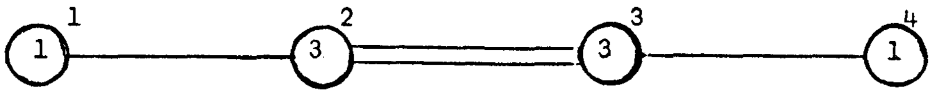
2.1.9.2: The quotient of G_2 with respect to D_2 in 2.1.2.2 is

And the quotients of the cube with respect to the decompositions given in 2.1.4 are respectively

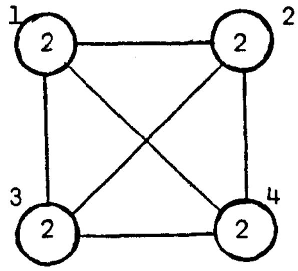
2.1.9.3:



2.1.9.4:



2.1.9.5:

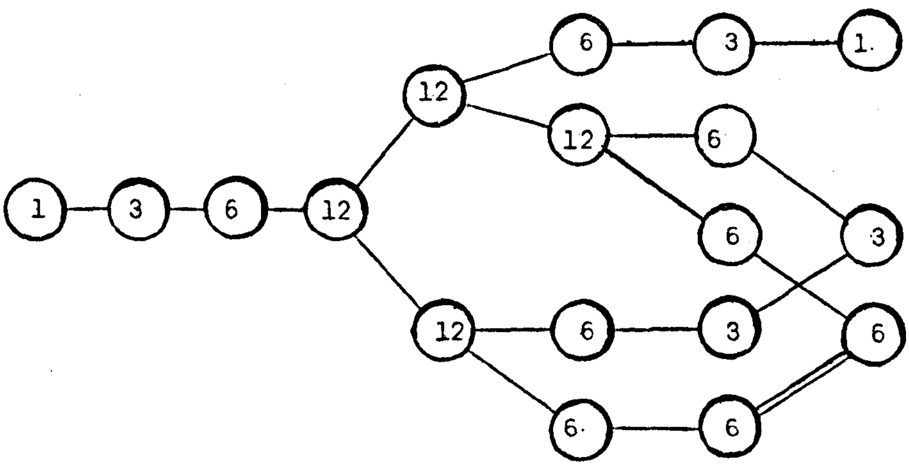


//

2.1.10:Note.A decomposition is an unordered partition of the vertices of a generalised graph,so the labelling of a quotient graph is arbitrary and the quotient is strictly only defined up to isomorphism. //

Quotients have been used as partial descriptions of graphs.For example Foster has constructed the following quotient of the Coxeter/Frucht graph on 110 vertices (10):

2.1.11:Example.



//

Harries (24) gives quotient graphs as part of his description of (3,10)-cages,and Neumann (32) uses a quotient graph to illustrate the"coloured graph" of a permutation group.

Decompositions are often closely related to automorphisms; indeed every group of automorphisms of a graph yields a decomposition as we see from the following result.

2.1.12.1:Proposition.Let $G=(V,s,w)$ be a generalised graph and let Π be a subgroup of $\Gamma(G)$.Then the partition of V into the orbits of Π,designated $\mathcal{D}(\Pi)=\{D_i,i\in N_m\}$,is a decomposition of G.

Proof.Consider $x,y\in V$ such that $x\sim y$.Then there is $\pi\in\Pi$ such that $\pi(x)=y$.So for $j\in N_m$

$$\alpha(y)_j=\sum_{z\in D_j} w(y,z)[s(y),s(z)]/s(y)$$

$$=\sum_{z\in D_j} w(\pi(x),z)[s(\pi(x)),s(z)]/s(\pi(x)).$$

Now $\pi(D_j)=D_j$ since D_j is an orbit of Π.Hence

$$\alpha(y)_j=\sum_{z\in D_j} w(\pi(x),\pi(z))[s(\pi(x)),s(\pi(z))]/s(\pi(x))$$

$$=\alpha(x)_j \text{ since } \pi \text{ is an automorphism of } G.$$

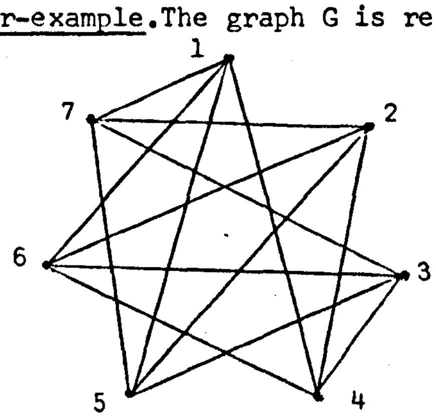
Similarly $\underline{\beta}(x)=\underline{\beta}(y)$ and hence $\mathcal{D}(\Pi)$ is a decomposition of G. //

2.1.12.2:Corollary.Let G be a generalised graph with only the trivial decomposition.Then $\Gamma(G)$ is trivial. //

The corollary is of course implied by 2.1.6.3.Except in the case of trees (see Chapter 3),the converse of Proposition 2.1.12.1 is false,which we may demonstrate by observing that there are regular graphs which are not transitive.

2.1.12.3:Counter-example.The graph G is regular but not transitive.

G:

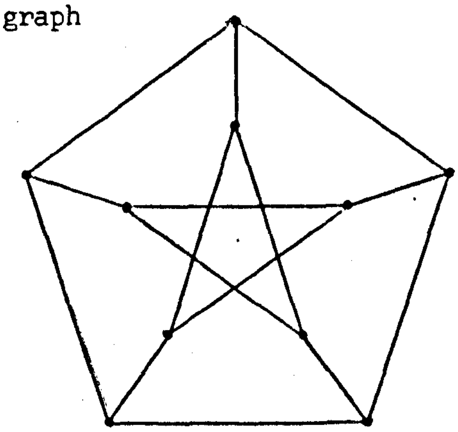


Proof.The complement of G consists of a copy of K_3 and a copy of C_4. //

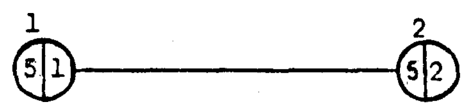
If a (connected simple) graph has an automorphism π,the orbits of $<\pi>$ give a decomposition,and the quotient provides a partial description of the graph.However when we know that such an automorphism exists,the quotient graph can be extended to give a complete description,which we call a "Frucht description" since his paper (17) proposed this method of describing graphs. Suppose π has order k and orbits $D_1,\ldots,D_m$ with $|D_i|=r(i)$ for $i\varepsilon N_m$ where of course r(i) divides k for each i.We choose a vertex arbitrarily from each orbit and label it $x_0(i)$.We give $\pi^j(x_0(i))$ the label $x_j(i)$ (where j is reduced modulo r(i)).Then it is only necessary to specify the neighbours of $x_0(i)$ for each $i\varepsilon N_m$ to describe the graph completely.This is generally done most conveniently by means of a diagram.Thus for example:

<u>2.1.13:Examples</u>.

<u>2.1.13.1</u>:Petersen's graph



has Frucht description

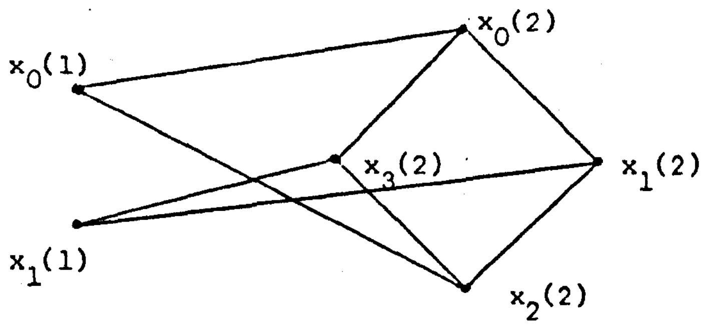


where "5|1" indicates that $x_j(1)$ adj $x_{j+1}(1)$ and $x_j(1)$ adj $x_{j-1}(1)$ (since the graph is simple) with subscripts reduced modulo 5,and "5|2" indicates that $x_j(2)$ adj $x_{j+2}(2)$ and $x_{j-2}(2)$ with subscripts reduced modulo 5.The edge joining the orbits indicates that $x_j(1)$ adj $x_j(2)$.

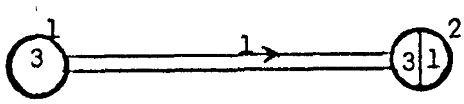
2.1.13.2: The generalised Petersen graph P(h,t) has Frucht description

2.1.13.3: The Frucht description

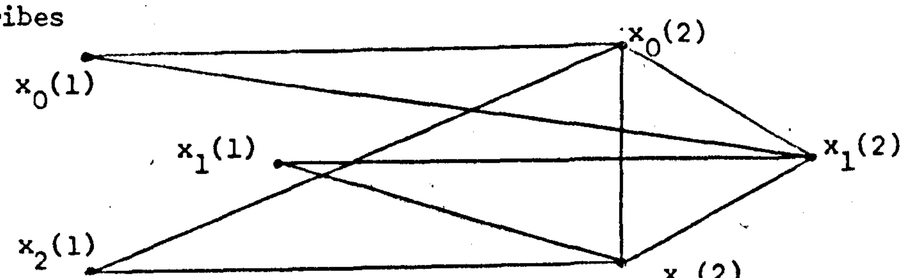
describes this graph:



2.1.13.4:



describes



the arrowed edge joining the orbits indicating that $x_j(1)$ adj $x_{j+1}(2)$. //

Similar descriptions of graphs have been used by other authors, for example Biggs and Smith (4,5), and have been employed fruitfully as an investigative tool by Evans and Wynn (14). By constructing all such descriptions obeying certain constraints up to a certain number of classes, the latter were able to identify four of the six (3,9)-cages now known, specify their automorphism groups and also obtain some negative results on the structure of (3,9)- and (3,10)-cages.

Frucht descriptions are most useful, and have generally been employed, when the automorphism gives classes of equal size. We will confine ourselves to such cases, when it is possible to derive some interesting spectral results (see Section 2.3).

2.2: Decompositions and Spectra of Graphs.

2.2.1.1: Definition (Haynsworth (25)). A partition $P=\{P_i, i\in N_m\}$ of N_n is said to be block-stochastic with respect to matrix A of order n if each block $A^{(ij)}$ of the induced partition of A has constant row sum, s_{ij}. We designate the m×m matrix (s_{ij}) by $S(A,P)$. //

2.2.1.2: Notation. In the context of graphs we may use $S(G,P)$ in place of $S(A(G),P)$, bringing our new notation into line with Definition 2.1.1.3. When there is no likelihood of confusion we will simply use S for either of the above. //

2.2.1.3: Proposition. Let G be a generalised graph with decomposition D. Then D is block-stochastic with respect to matrix A(G) and $S(G;D) =A(G^{D})$.

Proof: By Propositions 2.1.3.1 and 2.1.8.1. //

Examples.

2.2.1.4: The partition $P=\{\{1\},\{2,3,4\}\}$ is block-stochastic with respect to

$$A=\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 3 & 0 & 2 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \quad \text{and } S(A,P)=\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

2.2.1.5:The decompositions D_1 and D_2 of Examples 2.1.2 are block-stochastic with respect to $A(G_1)$ and $A(G_2)$ respectively. //

2.2.2.1:Proposition(Haynsworth (25)).If $P=\{P_i, i\epsilon N_m\}$ is block-stochastic with respect to matrix A of order n,then the zeros of $\phi(A,\lambda)$ are:

i) the m zeros of $\phi(S(A,P),\lambda)$;

ii) the n-m zeros of $\phi(C,\lambda)$,where C is the partitioned matrix $(C^{(ij)})$ with $C^{(ij)}=(a_{hk}^{(ij)}-a_{1k}^{(ij)})$ with $h=2,3,\ldots,|P_i|$ and $k=2,3,\ldots,|P_j|$.(If either of $|P_i|$ or $|P_j|$ is one the block $C^{(ij)}$ is omitted,so i,j do not necessarily take all values from 1 to m).

Proof.We perform the following similarity operations on A:

i) Replace the first column in each block by the sum of the columns in that block,and subtract the first row of each block from each subsequent row in the block;

ii) Permute the rows and columns so that the first element of class P_i becomes the i-th row and column of the matrix for each $i\epsilon N_m$,and the order is otherwise unchanged.

Then we see that A is similar to the partitioned matrix $\begin{pmatrix} S & X \\ 0 & C \end{pmatrix}$, and the result follows immediately. //

The following simple extension of this result is important in Chapter 5.

2.2.2.2:Proposition.Let partition P of order m be block-stochastic with respect to matrix A of order n.If A is diagonalisable then so is $S(A,P)$.

Proof.Using the notation of Proposition 2.2.2.1,A is similar to $M=\begin{pmatrix} S & X \\ 0 & C \end{pmatrix}$.Now A is diagonalisable,that is R^n has a basis B of left eigenvectors of M.Let the projection $\pi:R^n \to R^m$ delete the last n-m coordinates of any n-tuple.Then:

i) π has rank m,so $\pi(B)$ is a spanning set of R^m;

ii) if $\underline{b}\epsilon B$,then $\pi(\underline{b})$ is a left eigenvector of S.

So R^m is spanned by left eigenvectors of S and hence S is diagonalisable. //

2.2.2.3:Corollary.Let G be an undirected generalised graph.Then A(G) is diagonalisable.

Proof.G is a quotient of an undirected pseudograph by Proposition 2.1.8.3,and the adjacency matrix of an undirected pseudograph is real and symmetric and hence diagonalisable. //

It is often important to know the relationship not only between the eigenvalues of a graph and its quotient but also between the eigenvectors.

2.2.3.1:Notation.In this section we will consider a generalised graph $G=(N_n,s,w)$ which has decomposition $\mathcal{D}=\{D_i,i\varepsilon N_m\}$.We let $A(G)=A$ and $A(G^{\mathcal{D}})=S$.$R(G,\mathcal{D})=R$ is the m×n matrix given by

$$r_{ij}=s(j)/(\sum_{k\varepsilon D_i} s(k)) \quad \text{if } j\varepsilon D_i$$
$$=0 \quad \text{otherwise,with } i\varepsilon N_m, j\varepsilon N_n,$$

and $Q(G,\mathcal{D})=Q$ is the n m matrix given by

$$q_{ij}=1 \quad \text{if } i\varepsilon D_j$$
$$=0 \quad \text{otherwise,with } i\varepsilon N_n, j\varepsilon N_m.$$ //

2.2.3.2:Example.We will use the decomposition $\mathcal{D}_2$ of the cube, given in 2.1.4.2 as a source of examples.We will let A' denote $A(Q_3,\mathcal{D}_2)$,S' denote $S(Q_3,\mathcal{D}_2)$,and so on,to distinguish the examples from the general case.

$$A'=\left(\begin{array}{c|ccc|ccc|c}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right)$$

$$S' = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$R' = \left(\begin{array}{c:ccc:ccc:c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$Q' = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right)$$

//

<u>2.2.3.3:Lemma</u>. RAQ=S.

<u>Proof</u>. AQ is the n×m matrix obtained by summing the columns in each block of A .RAQ is the matrix obtained from AQ by the deletion of the repeated rows in each class. //

<u>2.2.3.4:Example</u>. $A'Q' = \left(\begin{array}{cccc} 0 & 3 & 0 & 0 \\ \hdashline 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ \hdashline 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ \hdashline 0 & 0 & 3 & 0 \end{array}\right)$

$$R'A'Q'=\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}=S'$$ //

<u>2.2.3.5:Lemma.</u> $RQ=I_m$.

<u>Proof</u>.Trivial. //

<u>2.2.3.6:Lemma.</u> QR commutes with A.

<u>Proof</u>.Consider $i\epsilon D_x$ and $j\epsilon D_y$.Then

$$\begin{aligned}(QRA)_{ij} &= \sum_{k=1}^{n}(QR)_{ik}a_{kj} \\ &=\Big(\sum_{k\epsilon D_x} s(k)\Big)^{-1}\sum_{k\epsilon D_x} s(k)a_{kj} \\ &=\Big(\sum_{k\epsilon D_x} s(k)\Big)^{-1}\sum_{k\epsilon D_x} w(k,j)\big[s(k),s(j)\big] \\ &=\Big(\sum_{k\epsilon D_x} s(k)\Big)^{-1}s(j)t_{xy}.\end{aligned}$$

Similarly $(AQR)_{ij}=\Big(\sum_{k\epsilon D_y} s(k)\Big)^{-1}s(j)\, s_{xy}$.

So by Proposition 2.1.3.2, $(QRA)_{ij}=(AQR)_{ij}$. //

<u>2.2.3.7:Example.</u>

$$Q'R'=\left(\begin{array}{c|ccc|ccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$Q'R'A' = \left(\begin{array}{c|ccc|ccc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 2/3 & 2/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 2/3 & 2/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 2/3 & 2/3 & 2/3 & 0 \\ \hline 0 & 2/3 & 2/3 & 2/3 & 0 & 0 & 0 & 1 \\ 0 & 2/3 & 2/3 & 2/3 & 0 & 0 & 0 & 1 \\ 0 & 2/3 & 2/3 & 2/3 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) = A'Q'R'$$ //

2.2.3.8:Proposition (Petersdorf and Sachs (34)).Let $\underline{u}$ be an eigenvector of S with eigenvalue μ.Then $Q\underline{u}$ is an eigenvector of A with eigenvalue μ.

Proof. $Q\underline{u}\neq\underline{0}$.

$S\underline{u}=\mu\underline{u}$. That is $RAQ\underline{u}=\mu\underline{u}$. So $QRAQ\underline{u}=\mu Q\underline{u}$.

Hence $AQRQ\underline{u}=\mu Q\underline{u}$, so that $AQ\underline{u}=\mu Q\underline{u}$. //

2.2.3.9:Proposition.Let $\underline{w}$ be an eigenvector of A with eigenvalue λ. Then $R\underline{w}=\underline{0}$ or $R\underline{w}$ is an eigenvector of S with eigenvalue λ.

Proof. $A\underline{w}=\lambda\underline{w}$. So $QRA\underline{w}=\lambda QR\underline{w}$ and hence $AQR\underline{w}=\lambda QR\underline{w}$.

Now $S=RAQ$ so that $SR\underline{w}=RAQR\underline{w}=\lambda RQR\underline{w}=\lambda R\underline{w}$. //

Thus we see that the relationship between the eigenvectors of a generalised graph and those of its quotients is completely determined.When G is a pseudograph it is clear that $R\underline{w}=\underline{0}$ only if the entries of $\underline{w}$ over each class of D sum to zero.If in addition A is diagonalisable,which is certainly the case when G is undirected, we have the following corollary:

2.2.3.10:Corollary.Let G be a pseudograph and let A be diagonalisable. Then R^n has a basis consisting of n eigenvectors of G which fall into two sets:

i) m in which the entries over each class are constant and for which the corresponding eigenvalues are those of G^D.

ii) n-m in which the entries over each class sum to zero and for which the corresponding eigenvalues are those of matrix C in Proposition 2.2.2.1. //

An application of this result is very well known:

<u>2.2.3.11:Corollary</u>.A regular graph of valency k has eigenvector $(1,1,\ldots,1)^t$ with eigenvalue k, and n-1 other eigenvectors whose entries sum to zero,together giving a basis of R^n. //

<u>2.2.3.12:Example</u>.

Eigenvalues of A' and S'	Eigenvectors of S'	Eigenvectors of A'
3	$(1,1,1,1)^t$	$(1,1,1,1,1,1,1,1)^t$
1	$(1,\frac{1}{3},\frac{-1}{3},-1)^t$	$(1,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{-1}{3},\frac{-1}{3},\frac{-1}{3},-1)^t$
		$(0,1,0,-1,1,0,-1,0)^t$
		$(0,1,-2,1,-1,2,-1,0)^t$
-1	$(1,\frac{-1}{3},\frac{-1}{3},1)^t$	$(1,\frac{-1}{3},\frac{-1}{3},\frac{-1}{3},\frac{-1}{3},\frac{-1}{3},\frac{-1}{3},1)^t$
		$(0,1,0,-1,-1,0,1,0)^t$
		$(0,1,-2,1,1,-2,1,0)^t$
-3	$(1,-1,1,-1)^t$	$(1,-1,-1,-1,1,1,1,-1)^t$

//

<u>2.2.3.13:Counter-example</u>.Not every block-stochastic partition of a matrix corresponds to a decomposition of a generalised graph, and while it is clear that Proposition 2.2.3.8 holds for any block-stochastic partition of a matrix,Proposition 2.2.3.9 does not.For consider partition P with respect to matrix A in Example 2.2.1.4. The eigenvalue 1 of A has an eigenvector $(0,0,0,1)^t$. Suppose G=(V,s,w) is a generalised graph with A(G)=A.Then whatever the sizes of vertices 2,3 and 4 this vector is not mapped onto $\underline{0}$ or an eigenvector of S(A,P) with eigenvalue 1 by the matrix R(G,P) so that P is not a decomposition of G. //

However if we restrict ourselves to undirected generalised graphs and their adjacency matrices we have:

2.2.3.14:Proposition.Let $G=(V,s,w)$ be an undirected generalised graph and let P be a block-stochastic partition of order m of $A(G)$. Then P is a decomposition of G.

Proof.For $x\varepsilon V$ and $i\varepsilon N_m$, $\alpha(x)_i$ is simply the sum of the entries of the i-th block of the x-th row of $A(G)$, so that if $x,y\varepsilon V$ and $x\overset{P}{\sim}y$ then $\alpha(x)_i=\alpha(y)_i$ for all $i\varepsilon N_m$.

Since G is undirected $\alpha(x)_i=\beta(x)_i$ for all $x\varepsilon V$ and $i\varepsilon N_m$. //

Proposition 2.2.3.8 has been proved and used by several authors. Petersdorf and Sachs (34) are interested in the constraints the automorphism group of a directed pseudograph imposes on its spectrum and prove Proposition 2.2.3.8,noting that the orbits of the automorphism group form a decomposition.Gardiner (20) proves the same result and refers to its use in the practical problem of determining the spectra of specific graphs.Mowshowitz (31) adopts the opposite approach to that of Petersdorf and Sachs by studying the constraints imposed on the automorphism group by the characteristic polynomial.He too notes that the orbits determine a decomposition,and concludes that if k is the number of orbits then there is a polynomial of degree k dividing the characteristic polynomial.He deduces that if the characteristic polynomial is irreducible then the automorphism group is trivial,and if it factorises into two irreducible polynomials of degrees m and n, then the number of orbits is m,n, or m+n.

Schwenk (36,37) is the only author who has employed decompositions (which he calls "equitable partitions") at all extensively.He states Proposition 2.2.3.8 for simple graphs and establishes the next two propositions.

2.2.4.1:Definition.Let G be a properly labelled simple graph of order n and let H_i be a simple graph for each $i \in N_n$.Then the generalised composition graph $G(H_1,H_2,\ldots,H_n)$ is formed by taking the disjoint union of the graphs $H_1,\ldots,H_n$ and then joining every vertex of H_i to every vertex of H_j whenever i adj j in G. //

The join and composition (21) of two graphs are special cases of the generalised composition,the join of G and H being $K_2(G,H)$ and the composition being $G(H,H,\ldots,H)$.

2.2.4.2:Proposition (Schwenk (37)).Let G be a properly labelled simple graph of order n and let $H_i=(V_i,s_i,w_i)$ be a regular simple graph of valency r_i for each $i\varepsilon N_n$.Then $D=\{V_i,i\varepsilon N_n\}$ is a decomposition of $G(H_1,\ldots,H_n)$ and

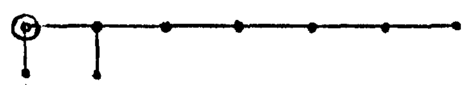
$$\phi(G(H_1,\ldots,H_n),\lambda)=\phi(G(H_1,\ldots,H_n)^D,\lambda)\prod_{i\varepsilon N_n}\phi(H_i,\lambda)/(\lambda-r_i). \qquad //$$

This result yields simple formulations for the characteristic polynomials of the composition and join of two graphs,the latter being extended to n-fold joins by Waller (43).

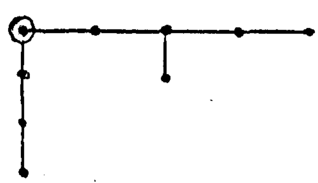
2.2.4.3:Definition.The coalescence of n rooted simple graphs (G_i,r_i) with $i\varepsilon N_n$ denoted by $G_1.G_2\ldots G_n$ is the graph formed by identifying the roots. //

2.2.4.4:Example.Let R and S be rooted trees.

R:

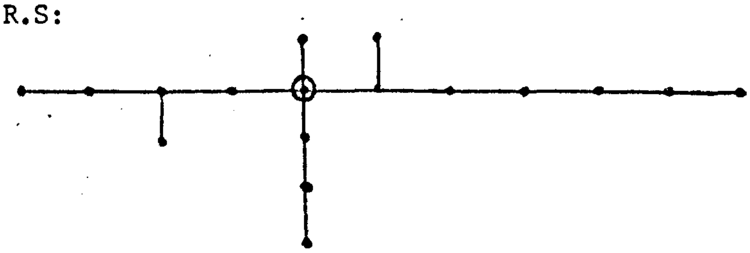


S:



Then the coalescence of R and S, R.S, is

R.S:



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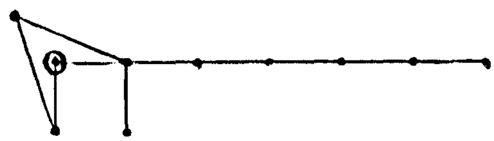
2.2.4.4:Definition.Rooted graphs (G_1,r_1) and (G_2,r_2) are cospectrally rooted if $\phi(G_1,\lambda)=\phi(G_2,\lambda)$ and $\phi(G_1-r_1,\lambda)=\phi(G_2-r_2,\lambda)$. //

2.2.4.5:Example.R and S of the previous example are cospectrally rooted trees. //

2.2.4.6:Notation.Given rooted graph (G,r),we let $G*n$ denote the graph formed by adding n-1 new vertices adjacent to the neighbours of r. //

2.2.4.7:Example.$R*2$ is the graph



//

2.2.4.8:Proposition (Schwenk (37)).If $G_1,\ldots,G_n$ are n cospectrally rooted graphs,then

$$\lambda^{n-1}\phi(G_1.G_2\ldots G_n,\lambda)=\{\phi(G_1-r_1,\lambda)\}^{n-1}\phi(G_1*n,\lambda). \qquad //$$

This proposition is useful in the construction of families of cospectral graphs.

Schwenk (36) also discusses the "main part of the spectrum",M, of a graph,originally defined by Cvetkovic (12) in connection with the number of walks in a graph.Schwenk shows that M is the set of eigenvalues of G which have an eigenvector not orthogonal to $(1,1,\ldots,1)^t$ and conjectures that M consists simply of the eigenvalues of the quotient induced by the coarsest possible decomposition of the graph.It is certainly true that M is contained in the spectrum of any quotient of the graph,but the converse remains undecided.

2.2.4.9:Proposition.If a graph G has decomposition $\mathcal{D}$ then every eigenvalue λ in the main part of the spectrum of G,M, is an eigenvalue of $G^{\mathcal{D}}$.

Proof.Corresponding to eigenvalue $\lambda\varepsilon M$ there is an eigenvector $\underline{w}$ of G whose entries do not sum to zero.Since G is a pseudograph Proposition 2.2.3.9 demonstrates that $R(G,\mathcal{D})\underline{w}$ is an eigenvector of G with eigenvalue λ. //

2.3: Circulant Decompositions.

Another result given by Haynsworth (25) simplifies the problem of finding the spectrum of a graph when it is known that the graph has a non-trivial Frucht description with all the classes having the same size.

We take Definition 2.3.1.1 and Proposition 2.3.1.2 from Biggs (4).

2.3.1.1: Definition. A $t\times t$ matrix C is said to be circulant if its entries satisfy $c_{ij}=c_{1,j-i+1}$ where the subscripts are reduced modulo t and lie in N_t. //

2.3.1.2: Proposition. Let W be the circulant matrix of order t whose first row is $(0,1,0,\dots,0)$, let C be the circulant matrix whose first row is $(c_1,c_2,\dots,c_t)$ and let $\omega=\exp(2\pi i/t)$, a t-th root of unity. Then the eigenvalues of C are

$$\lambda_r=\sum_{j=1}^{t}c_j\omega^{(j-1)r} \qquad r=0,1,\dots,t-1.$$

Proof (Biggs). $C=\sum_{j=1}^{t}c_jW^{(j-1)}$, the eigenvalues of W are $1,\omega,\omega^2,\dots,\omega^{(t-1)}$ and the result follows immediately. //

2.3.1.3: Proposition (Attributed by Haynsworth to Williamson (25)).

Let G be a generalised graph with a decomposition $\mathcal{D}$ of order m such that every block of the induced partition of $A(G)=A$ is circulant of order t with $A^{(ij)}$ having first row $(a_1^{(ij)},\dots,a_t^{(ij)})$ for $i,j\in N_m$. Then

$$\phi(A,\lambda)=\prod_{r=0}^{t-1}\phi(S_r,\lambda)$$

where S_r is the matrix of order m defined by

$$(S_r)_{ij}=\sum_{k=1}^{t}a_k^{(ij)}\omega^{(k-1)r} \qquad \text{for } i,j\in N_m.$$

Proof. Let $P^{-1}WP=D$ be the diagonal matrix with diagonal entries $1,\omega,\omega^2,\dots,\omega^{(t-1)}$. Consider the partitioned matrix Q where

$$Q=\begin{pmatrix} P & 0 & 0 & \cdot & \cdot \\ 0 & P & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

with m copies of P down the leading diagonal.Then since every block of A is a polynomial in W we have $Q^{-1}AQ=B$ a partitioned matrix in which for all $i,j\varepsilon N_m$ block $B^{(ij)}$ is a diagonal matrix of order t with entries

$$\sum_{k=1}^{t} a_k^{(ij)}\omega^{(k-1)r} \qquad r=0,1,\ldots,t-1.$$

We simply permute the rows and columns of B so that the only non-zero elements occur as the matrices S_r down the leading diagonal. //

<u>2.3.1.4:Example</u>.
$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

is similar to
$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

which can be permuted to give
$$\left(\begin{array}{cc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$ //

Note that $S_0=A(G^D)$.

2.3.2.1:Proposition. The spectrum of the generalised Petersen graph $P(h,t)$ is given by

$$\cos(2\pi r/h)+\cos(2\pi rt/h)\pm\sqrt{((\cos(2\pi r/h)-\cos(2\pi rt/h))^2+1)}$$

with $r=0,1,\ldots,h-1$.

Proof. For $P(h,t)$ the matrices S_r are

$$S_r=\begin{pmatrix} \omega^r+\omega^{(h-1)r} & 1 \\ 1 & \omega^{tr}+\omega^{t(h-1)r} \end{pmatrix} \quad \text{with } \omega=\exp(2\pi i/h)$$

$$\doteq \begin{pmatrix} 2\cos(2\pi r/h) & 1 \\ 1 & 2\cos(2\pi rt/h) \end{pmatrix}$$

and hence the eigenvalues of S_r are the solutions of

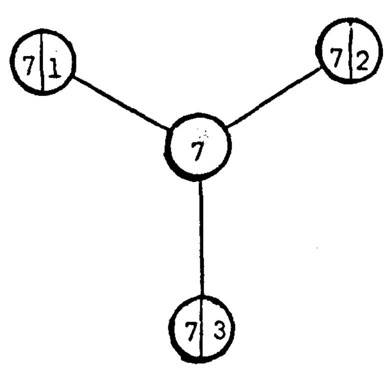
$$(2\cos(2\pi r/h)-\lambda)(2\cos(2\pi rt/h)-\lambda)-1=0 \qquad //$$

The eigenvalues of S_k are equal to those of S_{h-k} for $k\in N_{h-1}$, in this case, so it is only necessary to evaluate eigenvalues for $r=0,1,\ldots,[h/2]$ where $[a]$ denotes the integer part of a, and we have the following corollary:

2.3.2.2:Corollary. The number of distinct eigenvalues of $P(h,t)$ is bounded above by $h+1$ if h is odd, or $h+2$ if h is even. //

The spectrum of any graph with an automorphism which gives just two orbits, those orbits having equal size, may be found explicitly by the same method. As the number of orbits increases however the problem becomes more difficult to solve. Nonetheless in some cases a solution is fairly straightforward.

2.3.3.1:Example. The Tutte-Coxeter graph on 28 vertices has Frucht description (5):



So the matrices S_r are

$$S_r=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & \omega^r+\omega^{6r} & 0 & 0 \\ 1 & 0 & \omega^{2r}+\omega^{5r} & 0 \\ 1 & 0 & 0 & \omega^{3r}+\omega^{4r} \end{bmatrix}$$

where $\omega=\exp(2\pi i/7)$ and $r=0,1,\ldots,6$.

Let $s(r)=\sum_{j=1}^{6}\omega^{jr}$ for $r=0,1,\ldots,6$. Then

$$\phi(S_r,\lambda)=\lambda^4-s(r)\lambda^3+(2s(r)-3)\lambda^2+(s(r)-2)\lambda-2s(r).$$

If $r=0$ then $s(r)=6$ and we have

$$\phi(S_0,\lambda)=\lambda^4-6\lambda^3+9\lambda^2+4\lambda-12=(\lambda-3)(\lambda+1)(\lambda-2)^2.$$

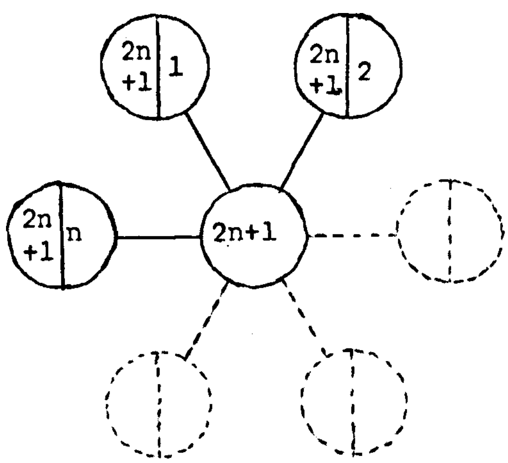
If $r\in N_6$ then $s(r)=-1$ and we have

$$\phi(S_r,\lambda)=\lambda^4+\lambda^3-5\lambda^2-3\lambda+2=(\lambda+1)(\lambda-2)(\lambda+1+\surd 2)(\lambda+1-\surd 2).$$

So the spectrum is $\begin{bmatrix} 3 & 2 & -1 & -1+\surd 2 & -1-\surd 2 \\ 1 & 8 & 7 & 6 & 6 \end{bmatrix}$ //

We can extend this technique to a family of graphs.

<u>2.3.3.2:Definition</u>.The <u>n-star-circuit graph</u> is the graph with Frucht description



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Note that the Tutte-Coxeter graph is the 3-star-circuit graph.

<u>2.3.3.3:Proposition</u>.The n-star-circuit graph has at most 2n+2 distinct eigenvalues.

<u>Proof</u>.Let $\omega=\exp(2\pi i/(2n+1))$ and let $s(r)=\sum_{j=1}^{n}\omega^{jr}+\omega^{(2n+1-j)r}$

for $r=0,1,\ldots,2n$. $\phi(S_r,\lambda)$ has coefficients which are functions of s(r). Now if $r\neq 0$, so that $\omega^r\neq 1$, then

$$s(r)=\frac{\omega^{(2n+1)r}-1}{\omega^r-1}-1=-1$$

and hence $\phi(S_a,\lambda)=\phi(S_b,\lambda)$ for $a,b\in N_{2n}$. The matrix S_r has order n+1 so there are at most 2(n+1) distinct eigenvalues. //

2.4: Finding Decompositions - an Algorithm.

Decompositions also occur in a context superficially rather different from those studied above, that of the study of algorithms to test whether two graphs are isomorphic ((9) is a survey of this subject). Any such algorithm is dependent in some way on the order n of the graphs in question. If the time taken by a computer, or the storage space it requires, to execute the algorithm increases exponentially with n, then it is clear that the algorithm will only be of practical use for relatively small values of n, and that future improvements in speed or storage capacity are unlikely to improve the situation greatly. It is desirable therefore to find a "good" algorithm, that is one for which the time and space required are bounded above by polynomials in n (see (1) for technical definitions). This problem remains unsolved, but decompositions and quotient graphs have played an important part in attempts at its solution; indeed one paper, by Corneil and Gotlieb (8), gave a good algorithm based on the (unfortunately false) conjecture that a certain sequence of quotient graphs was sufficient for the reconstruction of the original graph, up to isomorphism. Several papers employ a "refinement algorithm", which is essentially an algorithm which finds a decomposition, as part of their approach. The version we will present is an adaptation of that given by Parris and Read (33).

2.4.1.1:Definition. A classification of N_n is an ordered partition of N_n. //

2.4.1.2:Definition. Let $\underline{v}$ be an n-tuple of non-negative integers with a common upper bound 10^k for some natural number k. Then the compacted integer $\hat{v}$ is

$$\hat{v}=\sum_{i=1}^{n} v_i 10^{k(n-i)}$$ //

2.4.1.3:Algorithm. Let $\mathcal{C}=(C_j, j\varepsilon N_r)$ be a classification of the vertices of a properly labelled general graph G=(V,s,w).

Step 1. For each $i\varepsilon V$ construct the adjacency vectors, $\underline{\alpha}(i)$ and $\underline{\beta}(i)$, defined in 2.1.1.1. Sort the vertices within each class so that

i) the compacted integers $\hat{\alpha}(i)$ are monotonically decreasing and

ii) for vertices i,j in the same class with $\hat{\alpha}(i)=\hat{\alpha}(j)$ vertex i follows vertex j if $\hat{\beta}(i)<\hat{\beta}(j)$.

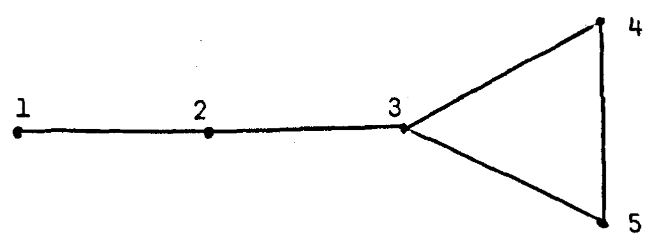
Construct a new classification $\mathcal{D}=(D_k, k\varepsilon N_s)$ with vertices i,j in the same class if and only if $i\mathcal{C}j$, $\hat{\alpha}(i)=\hat{\alpha}(j)$ and $\hat{\beta}(i)=\hat{\beta}(j)$, with classes labelled sequentially.

Step 2. If s, the order of $\mathcal{D}$, is equal to r, the order of $\mathcal{C}$, stop. Otherwise set $\mathcal{C}=\mathcal{D}$, and return to Step 1. //

2.4.1.4:Definition. The final classification of the vertices of a generalised graph G produced by Algorithm 2.4.1.3 is called a stable classification. //

2.4.1.5:Example. Consider the graph G with initial classification ({1},{2,4,5},{3}), noting that for a simple graph $\underline{\alpha}(i)=\underline{\beta}(i)$.

G:



Then we have

Vertex, i.	Class.	$\hat{\alpha}(i)$.	New class.	New $\hat{\alpha}(i)$.	New class.
1	1	010	1	0100	1
2	2	101	2	1001	2
3	3	020	4	0120	4
4	2	011	3	0110	3
5	2	011	3	0110	3

Stable classification ({1},{2},{4,5},{3}). //

2.4.2.1:Proposition. A classification of the vertices of a generalised graph G is stable if and only if it is an ordered decomposition of G.

Proof. Trivial. //

2.4.2.2:Proposition. Let G=(V,s,w) be a generalised graph and let R be a classification of V. Then the stable classification P of order p given by the application of Algorithm 2.4.1.3 to R is the coarsest decomposition of G which is a subpartition of R.

Proof. Suppose the proposition to be false and let Q be a different decomposition of G having order $q \leq p$ which is a subpartition of R. Then there are $x,y \in V$ such that $x \overset{Q}{\sim} y$ and $x \overset{P}{\nsim} y$. Furthermore for every such pair x,y there is a classification stage i at which while not in the same class of the current classification $C^{(i)}$, they were in the same class at the previous stage, for since $x \overset{Q}{\sim} y$ then $x \overset{R}{\sim} y$. Choose x,y to minimise i so that if $z,w \in V$ are in the same class of Q then they are in the same class at classification stage i-1. Now there must exist a class at this stage, $C_j^{(i-1)}$ say, such that $\alpha(x)_j \neq \alpha(y)_j$ or $\beta(x)_j \neq \beta(y)_j$. But $C_j^{(i-1)}$ is a union of classes of Q and hence, since Q is a decomposition, we have a contradiction. //

2.4.2.3:Corollary. Given any partition R of a generalised graph G, there is a unique coarsest decomposition D which is a subpartition of R. //

It is well known that Algorithm 2.4.1.3 is polynomially bounded in both its time and space requirements;Mathon (30) states that it can be executed in $O(n^3)$ time and $O(n^2)$ space.Thus we have a good algorithm for finding a decomposition from a given partition of the vertices of a generalised graph,and this can easily be extended to good algorithms for finding specific decompositions such as the coarsest possible for a given graph.

2.4.3.1:Proposition.There is a good algorithm for the construction of the coarsest decomposition of a generalised graph,and there is a canonical labelling for its classes.

Proof.We let the initial classification consist of a single class containing all the vertices ,and then apply Algorithm 2.4.1.3 to this initial classification. //

2.4.3.2:Proposition.Let $G=(N_n,s,w)$ be a generalised graph.There is a good algorithm for the construction of the coarsest decomposition of G having a specified singleton class,{i}, and the classes have a canonical labelling.

Proof.We apply Algorithm 2.4.1.3 to the initial classification $(\{i\},\{1,2,\ldots,i-1,i+1,\ldots,n\})$. //

2.4.3.3:Definition.The coarsest decomposition of generalised graph $G=(V,s,w)$ having singleton class $\{x\}$ for a specified $x\in V$ is called the singleton-decomposition of G with respect to vertex x, $\mathcal{S}(G,x)$. The corresponding quotient graph is called the singleton-quotient with respect to vertex x.If the singleton-decomposition with respect to vertex x is trivial,then G is said to be irreducible with respect to x. //

2.4.3.4:Proposition.Let (G_1,r_1) and (G_2,r_2) be rooted generalised graphs and let them be irreducible with respect to r_1 and r_2 respectively.Then there is a good algorithm to test for isomorphism between them as rooted graphs.

Proof. The canonical labelling of the singleton-quotient of a graph is for each of (G_1,r_1) and (G_2,r_2) a canonical relabelling of the vertices, with the root being given label 1. To check for isomorphism between the rooted graphs it is therefore only necessary to compare the sizes of corresponding vertices and the adjacency matrices of the relabelled graphs. //

3:DECOMPOSITIONS OF TREES

As we remarked in Section 2.1.12, it is not in general true that every decomposition of a graph is a partition of its vertices into the orbits of a group of automorphisms.Trees however do have this desirable property,and in this chapter we shall prove this result and explore the consequently very close relationship between the coarsest decomposition of a tree,its automorphism group and its characteristic polynomial.Of course the automorphism group is already well known (35) and there is also a standard approach to the problem of finding its characteristic polynomial which employs the deletion of vertices one by one (see for example (37)).

3.1:Decompositions and Automorphisms of Trees.

A useful idea is that of the "centre" of a generalised graph which we take from Harary (21).

3.1.1.1:Definition.Let $G=(V,s,w)$ be a generalised graph.The eccentricity $e(v)$ of $v\varepsilon V$ is the maximum value of $d(u,v)$ over all $u\varepsilon V$ (setting $d(u,v)=\infty$ if there is no path from u to v).The radius $r(G)$ of G is the minimum eccentricity of the vertices.A vertex v is a central point if $e(v)=r(G)$ and the centre of G is the set of all central points. //

3.1.1.2:Lemma.Let $T=(V,s,w)$ be a tree in which not all vertices are endpoints and let $\mathcal{D}$ be a decomposition of T. Then the set of all endpoints is a union of classes of $\mathcal{D}$ and the vertex-subtree T' induced by the deletion of all the endpoints of T has an induced decomposition $\mathcal{D}'$ with the property that if $x\overset{\mathcal{D}'}{\sim}y$ then $x\overset{\mathcal{D}}{\sim}y$.

Proof.$v\varepsilon V$ is an endpoint of T if and only if it has valency 1. Two vertices of a simple graph cannot be in the same decomposition class if they have different valencies. //

3.1.1.3:Lemma (Harary (21)).Let T=(V,s,w) be a tree with vertices other than endpoints and let T'=(V',s'w') be the vertex-subtree induced by the deletion of all endpoints of T. Then the eccentricity of v as a vertex of T' is exactly one less than its eccentricity as a vertex of T so that T and T' have the same centre.

Proof.The maximum of the distances from given $v \epsilon V$ to any $u \epsilon V$ clearly occurs when u is an endpoint of T. //

We shall prove the next two propositions together.

3.1.1.4:Proposition (Attributed by Harary to Jordan (21)).The centre of a tree T consists of a single point or of two adjacent points.

3.1.1.5:Proposition.Let T be a tree with decomposition D.If vertex x is a central point of T and $x \sim y$, then y is a central point of T.

Proof.If every vertex of T is an endpoint then T is K_1 or K_2 and the propositions are proved.Otherwise we consider T' defined as in the lemmas.We repeat the process until we are left with K_1 or K_2. //

3.1.1.6:Corollary.Every decomposition of a singleton-centred tree has a singleton class consisting of the central point. //

The property of singleton-centred trees given in the corollary above is very useful.Fortunately every doubleton-centred tree is closely related to a tree with a singleton centre,so we can generally extend results obtained using 3.1.1.6 to trees with doubleton centres.

3.1.2.1:Definition.Let T=(V,s,w) be a doubleton-centred tree with central points a,b. We define the extension of T, ext(T), to be the tree with vertex set $V \cup \{c\}$ and with the same edges as

T except for the replacement of the edge joining a and b by edges joining a and c, b and c. //

3.1.2.2:Proposition. If $T=(V,s,w)$ is a doubleton-centred tree then ext(T) is a singleton-centred tree with central point c.

Proof. The eccentricity of every $v \in V$ is exactly one greater in ext(T) than it is in T, since the longest path in T from any $v \in V$ clearly includes the edge joining the two central points. //

3.1.2.3:Proposition. If T is a doubleton-centred tree with decomposition D, then $D' = D \cup \{\{c\}\}$ is a decomposition of ext(T).

Proof. Trivial. //

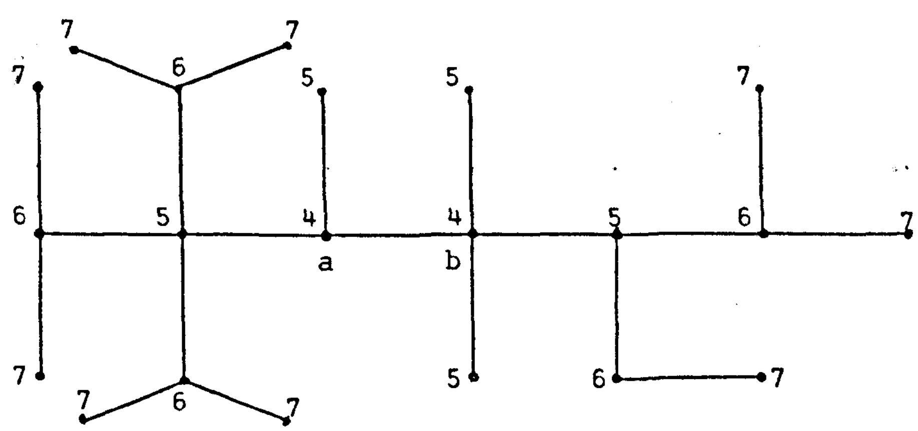
3.1.2.4:Proposition. Let $T=(V,s,w)$ be a doubleton-centred tree. Then $\Gamma(T)$ is the restriction of $\Gamma(\text{ext}(T))$ to V.

Proof. $\Gamma(T)$ clearly stabilises the centre of T setwise and $\Gamma(\text{ext}(T))$ clearly stabilises c, the additional vertex. //

3.1.3:Examples.

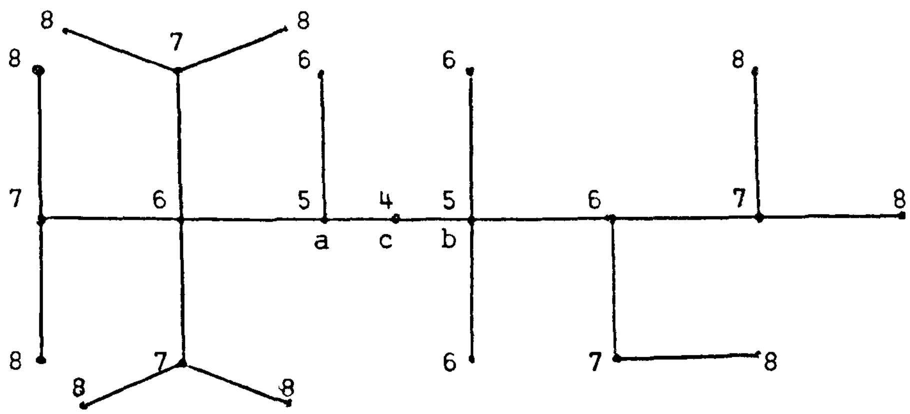
3.1.3.1:The eccentricities of the vertices of a tree.

T_1:



The two vertices of eccentricity 4 are the central points of T_1, labelled a and b. Thus ext(T_1) with its eccentricities is:

3.1.3.2:



//

We have established that a vertex can only be in the same decomposition class as a central point if it is also a central point.This result is easily extended.

3.1.4.1:Proposition.Let $T=(V,s,w)$ be a singleton-centred tree with central point c and let T have decomposition D. Consider $x\varepsilon V$. For all $y\varepsilon V$, if $y\sim x$ then $d(c,y)=d(c,x)$.

Proof.By induction.

i) If x adj c and $x\sim y$ then y adj c since $\{c\}$ is a singleton class of D.

ii) Suppose the proposition true for all $z\varepsilon V$ with $d(c,z)\leqq k$, let $d(c,x)=k+1$, and let $y\sim x$. Then firstly $d(c,y)>k$ since otherwise by hypothesis $d(c,x)\leqq k$ which is not the case.Secondly x adj x' for some $x'\varepsilon V$ with $d(c,x')=k$ and y adj y' for some $y'\varepsilon V$ with $x'\sim y'$. So $d(c,y')=k$. Hence $d(c,y)=k-1,k$ or $k+1$. But we have already seen that $d(c,y)\neq k-1$ or k. Thus $d(c,y)=k+1$. //

3.1.4.2:Corollary.Let $T=(V,s,w)$ be a tree with decomposition D. Consider $x,y\varepsilon V$. If $x\sim y$ then $e(x)=e(y)$.

Proof.If T is singleton-centred the result follows immediately from 3.1.4.1. If not consider the decomposition $D\cup\{\{c\}\}$ of ext(T). //

3.1.4.3:Corollary.Let $T=(V,s,w)$ be a tree with decomposition D and let $x,y \in V$. Then $x \sim y$ and x adj y implies that $\{x,y\}$ is the centre of T.

Proof. It is clear that two adjacent vertices of a tree only have the same eccentricity if they are both central points. //

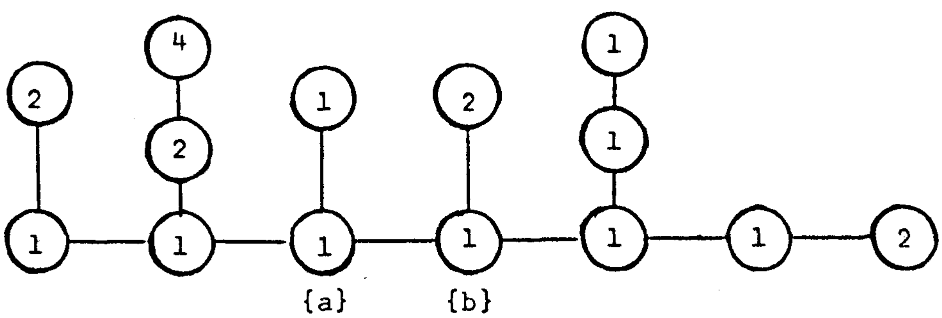
Given a tree T, consider any vertex x which is not a central point. Then clearly x is adjacent to one vertex whose eccentricity is $e(x)-1$, this being the vertex lying on the path from x to the centre, and every other neighbour of x has eccentricity $e(x)+1$.

3.1.5.1:Definition. Let $T=(V,s,w)$ be a tree. For every $x \in V$ such that x is not a central point of T we define the father of x to be that neighbour of x whose eccentricity is $e(x)-1$, and any other neighbours to be the sons of x. //

3.1.5.2:Proposition. Let $T=(V,s,w)$ be a tree with decomposition $\mathcal{D}=\{D_i, i \in N_m\}$. Consider $x,y \in V$ such that $x \in D_i, y \in D_j$ and x is the father of y. Then $|D_i|$ divides $|D_j|$ and every vertex of D_j is adjacent to exactly one vertex of D_i.

Proof. Trivial. //

3.1.5.3:Example. A quotient of the tree T_1 given in Example 3.1.3.1:



The vertices corresponding to the central points of T_1 are labelled {a} and {b}. //

3.1.6.1:Definition. Let $G=(V,s,w)$ be a generalised graph with the following properties:

i) G is connected and undirected;

ii) the weight of every edge is 0 or 1;

iii) $w(x,x)=0$ for all $x \in V$;

iv) G has no circuits.

Then G is said to be a generalised tree. //

In other words we define a generalised tree to be a generalised graph with the property that if the size of every vertex were reduced to 1, then it would become a tree.

3.1.6.2:Proposition.Any quotient of a singleton-centred tree is a generalised tree,and any quotient of a doubleton-centred tree is a generalised tree with the possible addition of one loop.

Proof.Every quotient of a tree is connected and undirected.By 3.1.5.2 the weight of an edge of a quotient cannot exceed 1, and by 3.1.4.3 there is at most one loop,which can occur only when the tree has a doubleton centre.

Suppose we have a quotient with a circuit.Then there is a path of arbitrarily great length in the tree passing in turn through each of the classes corresponding to the vertices of the circuit. But in any tree there is a path of maximum length and so we have a contradiction. //

The converse of this proposition is clearly false.Consider the following generalised tree:

3.1.6.3:Counter-example.

G:

G is not a quotient of any tree. //

The next sections will relate the automorphisms and the decompositions of a tree.We shall use Harary's nomenclature for permutation groups.

3.1.7:Definitions (Harary (21)).Let Γ be a permutation group of order m and degree d acting on the set $X=\{x_1,x_2,\ldots,x_d\}$ and let

Π be a permutation group of order n and degree e acting on the set $Y=\{y_1,y_2,\dots,y_e\}$.

3.1.7.1: The sum, $\Gamma+\Pi$, sometimes also called the product or direct product, is the permutation group which acts on $X\cup Y$ and whose elements are all the ordered pairs of permutations $\gamma\varepsilon\Gamma$ and $\pi\varepsilon\Pi$, written $\gamma+\pi$. Any element $z\varepsilon X\cup Y$ is permuted by $\gamma+\pi$ according to the rule

$$\begin{aligned}(\gamma+\pi)(z)&=\gamma(z) \quad \text{if } z\varepsilon X\\ &=\pi(z) \quad \text{if } z\varepsilon Y.\end{aligned}$$

3.1.7.2: The composition, $\Gamma[\Pi]$, sometimes called the wreath product (29), acts on $X\times Y$. For each $\gamma\varepsilon\Gamma$ and any sequence $(\pi_1,\pi_2,\dots,\pi_d)$ of d (not necessarily distinct) permutations in Π, there is a unique permutation in $\Gamma[\Pi]$ written $(\gamma;\pi_1,\pi_2,\dots,\pi_d)$ such that for (x_i,y_j) in $X\times Y$

$$(\gamma;\pi_1,\pi_2,\dots,\pi_d)(x_i,y_j)=(\gamma(x_i),\pi_i(y_j)).$$

A member of the composition may be thought of as first permuting each copy of Y within itself and then permuting the $|X|$ copies of Y amongst themselves. //

The next step is to associate a permutation group with every quotient $T^{\mathcal{D}}$ of a tree T, in such a way that the group produced is a group of automorphisms of T which stabilises the classes of $\mathcal{D}$ setwise and acts transitively on them, so that the classes are its orbits.

3.1.8.1: Lemma. A tree can be reconstructed up to isomorphism from any of its quotients.

Proof. Trivial. //

3.1.8.2: Algorithm. Let $T=(V,s,w)$ be a singleton-centred tree with decomposition $\mathcal{D}=\{D_i, i\varepsilon N_m\}$ and quotient $T^{\mathcal{D}}=(N_m,s',w')$. We associate a permutation group with $T^{\mathcal{D}}$ thus:

Step 1.Calculate the eccentricity of every vertex of T, and assign a level i to each class of $\mathcal{D}$ according to the eccentricity of the vertices it contains,assigning the lowest level, 0, to the classes containing vertices of the greatest eccentricity.Then assign to vertex j of $T^{\mathcal{D}}$ the level given to class D_j of $\mathcal{D}$.

Step 2.Assign the group Σ_1 to every endpoint of $T^{\mathcal{D}}$ except that on the highest level.Every vertex on level 0 is thus assigned a group.Set i to be 1.

Step 3.Every vertex on level i-1 has been assigned a group.For each vertex x on level i which has not yet been given a group do the following:- let Y be the set of sons* of x and for each $y \in Y$ let $\Gamma^{(y)}$ be the group assigned to y. Then assign to vertex x the permutation group

$$\Sigma_1 + \sum_{y \in Y} \Sigma_{s'(y)/s'(x)}\left[\Gamma^{(y)}\right].$$

If there are any vertices on level i+1,set i to be i+1 and return to Step 2. Otherwise the group assigned to the vertex on level i is the group we associate with $T^{\mathcal{D}}$. //

3.1.8.3:Lemma.Let a graph G be constructed from k_i copies of the rooted graph (G_i, r_i) for $i \in N_m$ by making every root adjacent to a single additional vertex v. Let

$$\Pi = \Sigma_1 + \Sigma_{k_1}\left[\Gamma_{r_1}(G_1)\right] + \ldots + \Sigma_{k_m}\left[\Gamma_{r_m}(G_m)\right].$$

Then Π is a subgroup of $\Gamma(G)$, and $\Pi = \Gamma(G)$ if and only if $i \neq j$ implies that (G_i, r_i) and (G_j, r_j) are not isomorphic as rooted graphs.

Proof.Any permutation of the vertices of G consisting of an automorphism of each copy of G_i fixing r_i followed by any permutation of the copies of G_i amongst themselves is clearly an automorphism of G, for all $i \in N_m$. If $i \neq j$ implies that $(G_i, r_i)$is not isomorphic to (G_j, r_j) then every automorphism of G consists of permutations of

* "sons" - neighbours on level i-1.

this form, since v is the unique central point of G and hence is stabilised by $\Gamma(G)$. //

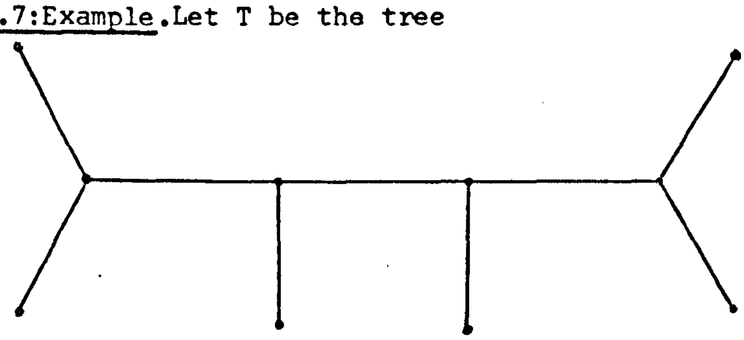
3.1.8.4:Proposition. Let T be a singleton-centred tree with a decomposition $\mathcal{D}$. Then the permutation group associated with $T^{\mathcal{D}}$ by Algorithm 3.1.8.2 is a group of automorphisms of T which stabilises the classes of $\mathcal{D}$ setwise and acts transitvely on them. If $\mathcal{D}$ is the coarsest decomposition of T then the group associated with $T^{\mathcal{D}}$ is $\Gamma(T)$.

Proof. For each vertex x of $T^{\mathcal{D}}$ we apply Lemma 3.1.8.3 noting that $s(y)/s(x)$ is the number of copies of isomorphic rooted subtrees adjacent to each vertex of T in D_x. When $\mathcal{D}$ is the coarsest decomposition of T there clearly cannot be two sons of x for which the corresponding rooted subtrees are isomorphic. //

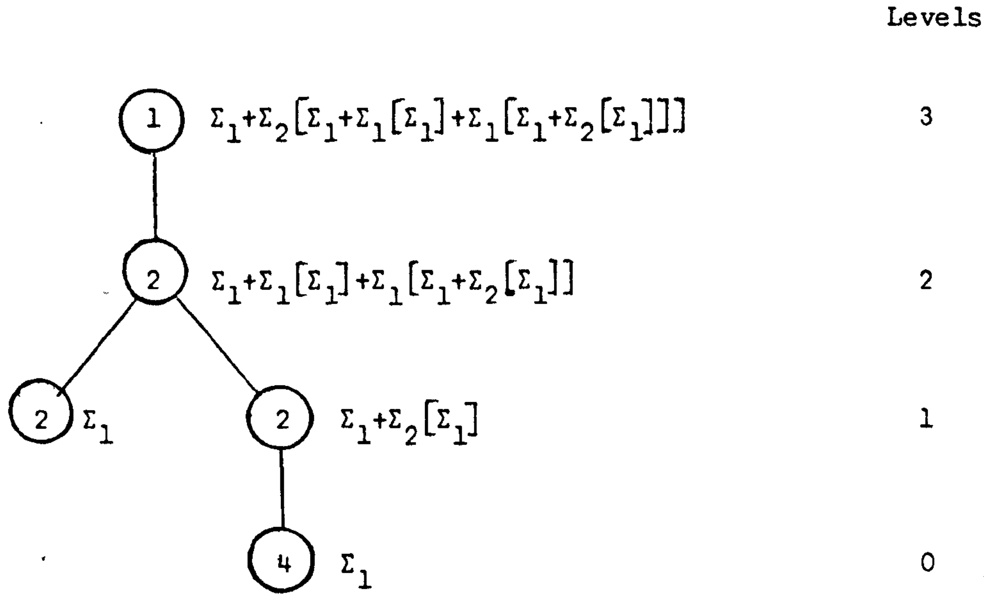
3.1.8.5:Corollary. Let $T=(V,s,w)$ be a doubleton-centred tree with coarsest decomposition $\mathcal{C}$. Then $\Gamma(T)$ is the group associated with $(\mathrm{ext}(T))^{\mathcal{C}\cup\{\{c\}\}}$ restricted to V. //

3.1.8.6:Note. The groups we have constructed act on certain specified sets of objects, so that we have only constructed the automorphism groups of particular graphs with respect to a certain labelling of the vertices. We call these groups the automorphism groups in the same sense as Σ_n is said to be the symmetric group on n objects. //

3.1.8.7:Example. Let T be the tree



Then $(\mathrm{ext}(T))^{C\cup\{\{c\}\}}$ and the groups assigned to its vertices are



Hence the automorphism group of T is $\Sigma_2[\Sigma_1+\Sigma_1[\Sigma_1]+\Sigma_1[\Sigma_1+\Sigma_2[\Sigma_1]]]$.

//

Thus we have shown that for a tree every decomposition corresponds to the orbits of a group of automorphisms. An analogy of Weichsel's proposition concerning star-partitions (2.1.6.4) follows immediately.

<u>3.1.8.8:Corollary</u>. A tree has trivial automorphism group if and only if its only decomposition is trivial. //

Another simple corollary is the well-known result:

<u>3.1.8.9:Corollary</u> (Polya (35)). Let P denote the class of permutation groups constructed according to the following rules:

i) P contains all the symmetric groups;

ii) P is closed under the operations of taking sums and compositions.

Then T is a tree implies that $\Gamma(T)\varepsilon P$. //

3.2: The Characteristic Polynomial of a Tree.

We begin by considering the characteristic polynomial of a generalised graph of the form described in Lemma 3.1.8.3.

3.2.1.1: Proposition. Let a generalised graph $G=(V,s,w)$ consist of k_i copies of the rooted generalised graphs (G_i,r_i) for $i\varepsilon N_m$ and a single additional vertex v joined to each root by an undirected edge of weight 1. Let E be the decomposition of G given thus:- Let $x,y\varepsilon V$. Then $x\overset{E}{\sim}y$ if and only if x and y are corresponding vertices in two copies of G_j for some $j\varepsilon N_m$.

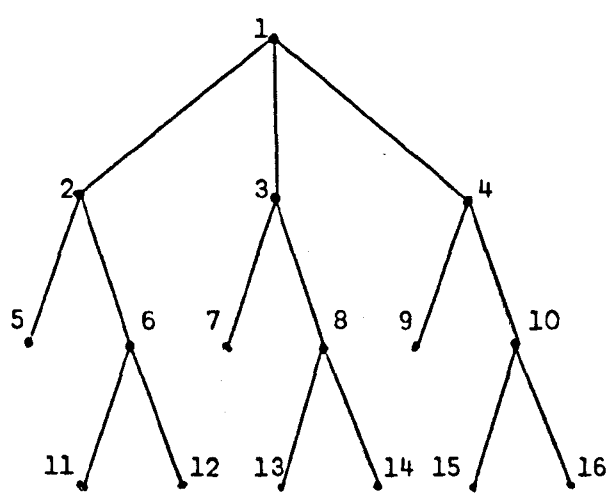
Then $$\phi(G,\lambda)=\phi(G^E,\lambda)\prod_{i\varepsilon N_m}\phi(G_i,\lambda)^{k_i-1}$$

Proof. The vertices of G can be labelled in such a way that $A(G_i)$ appears as a block k_i times down the leading diagonal of $A(G)$ for each $i\varepsilon N_m$. So the result follows immediately from Proposition 2.2.2.1 when we note that the matrix C of that proposition consists simply of $A(G_i)$ k_i-1 times down the leading diagonal for each $i\varepsilon N_m$. //

This result enables us to simplify the evaluation of the characteristic polynomial of a tree.

3.2.1.2: Example. Consider the tree T_1:

T_1:

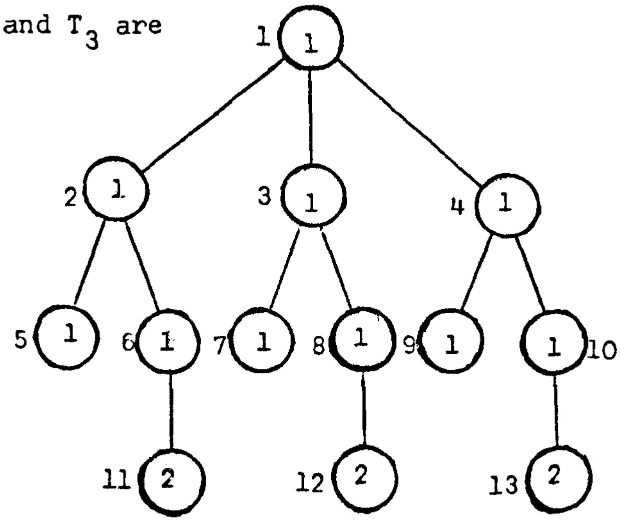


Vertices 6,8 and 10 are each joined to two isomorphic rooted subtrees consisting of a single vertex, so by repeated application of the proposition

$$\phi(T_1,\lambda)=\phi(T_2,\lambda)\{\phi(T_3,\lambda)\}^3$$

where T_2 and T_3 are

T_2:



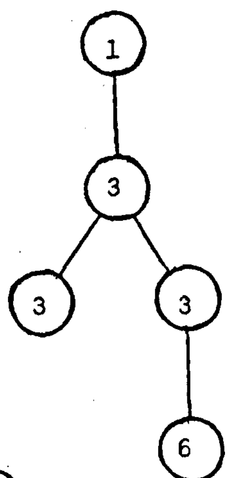
T_3:

Now $\phi(T_2,\lambda)$ can be simplified since vertex 1 in T_2 is joined to three isomorphic rooted subtrees. Thus

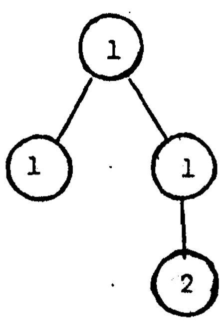
$$\phi(T_2,\lambda)=\phi(T_4,\lambda)\{\phi(T_5,\lambda)\}^2$$

where T_4 and T_5 are

T_4:



T_5:



Hence $\phi(T_1,\lambda)=\phi(T_4,\lambda)\{\phi(T_5,\lambda)\}^2\{\phi(T_3,\lambda)\}^3.$ //

Since a tree T can be reconstructed from any of its quotients it is a trivial remark that it must be possible to find the simplified formulation for $\phi(T,\lambda)$ directly from the quotient corresponding to its coarsest decomposition. If the coarsest decomposition has a singleton class we can base an algorithm on the method exemplified above without any modifications.

3.2.2.1:Definition. Let $T=(V,s,w)$ be a generalised tree and let $p:V\to Z^+$ assign a level to every vertex of T. Then $y\in V$ is a descendant of $x\in V$ if the path $(a_1,a_2,\ldots,a_k)$ from x to y in T with $a_1=x$ and $a_k=y$ has the property that if $1\leq i<j\leq k$ then $p(a_j)<p(a_i)$. The descendant subtree of vertex x, $T^{(x)}$, is the vertex subtree of T induced by the descendants of vertex x. //

3.2.2.2:Algorithm. Let $T=(V,s,w)$ be a tree whose coarsest decomposition $\mathcal{C}$ has a singleton class, and let $T^{\mathcal{C}}=(V',s',w')$.

Step 1. Assign levels to the vertices of $T^{\mathcal{C}}$ as in Algorithm 3.1.8.2.

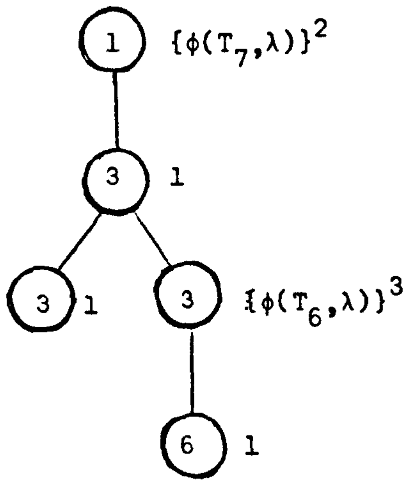
Step 2. For each vertex $x\in V'$ do the following:- Let Y be the set of sons of x in $T^{\mathcal{C}}$ and assign to x the polynomial

$$P(x,\lambda)=1\times\prod_{y\in Y}\{\phi(T^{(y)},\lambda)\ s'(x)(s'(y)/s'(x)-1)$$

Step 3. The polynomial associated with $T^{\mathcal{C}}$ is

$$\phi(T^{\mathcal{C}},\lambda)\prod_{x\in V'}P(x,\lambda)\ .$$ //

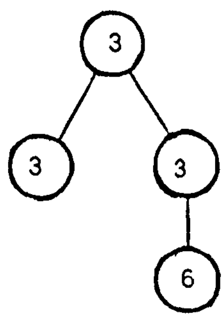
3.2.2.3:Example. Consider the tree T_1 with quotient T_4 in Example 3.2.1.2. The polynomials assigned to its vertices are



where T_6 and T_7 are

T_6:

T_7:



So the polynomial associated with T_4 is

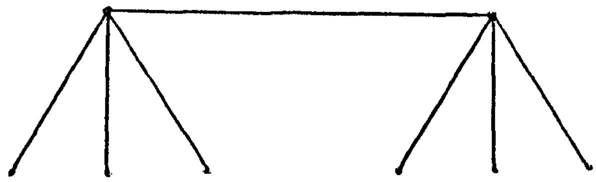
$$\phi(T_4,\lambda)\{\phi(T_7,\lambda)\}^2\{\phi(T_6,\lambda)\}^3.$$

T_6 and T_7 may be obtained from T_3 and T_5 by multiplying the sizes of their vertices by 6 and 3 respectively.So $A(T_6)=A(T_3)$ and $A(T_7)=A(T_5)$. Thus the polynomial associated with T_4 is the characteristic polynomial of T_1. //

It is clear that as long as the coarsest decomposition C of a tree T has a singleton class,so that any vertex at the highest level of T^C has size 1, the polynomial associated with T^C will be $\phi(T,\lambda)$. We shall now investigate the only remaining possibility.

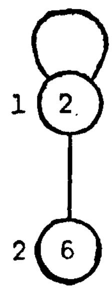
3.2.3.1:Example.Consider the following tree T.

T:



The quotient of T corresponding to its coarsest decomposition C is

T^C:

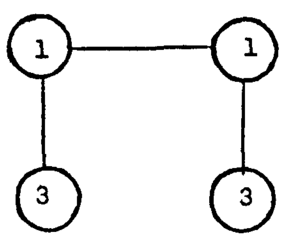


If we were to apply Algorithm 3.2.2.2 to T , the polynomial constructed would be

$$\phi(T^C,\lambda)\{\phi((T^C)^{(2)},\lambda)\}^4,$$

a polynomial of degree 6, whereas the characteristic polynomial of T has degree 8. The method used in 3.2.1.2 requires that $\phi(T^C,\lambda)$ be replaced by $\phi(T^D,\lambda)$ where T^D is

T^D:



Then $\phi(T^D,\lambda)\{\phi((T^C)^{(2)},\lambda)\}^4$ is indeed $\phi(T,\lambda)$. //

Fortunately it is always possible to express $\phi(T^D,\lambda)$ very simply in terms of $\phi(T^C,\lambda)$.

3.2.3.2:Proposition.Let T be a tree, D be its coarsest decomposition with a singleton class, C be its coarsest decomposition,the order of C be m and C and D be distinct.Then

$$\phi(T^D,\lambda)=\phi(T^C,\lambda)\phi(T^C,-\lambda)\times(-1)^m.$$

Proof. Let E be the matrix of order m with

$$e_{ij}=1 \qquad \text{if } i=j=1$$
$$=0 \qquad \text{otherwise for } i,j\varepsilon N_m.$$

Since the coarsest decomposition must have both central points in one class there is an automorphism of T interchanging them. Hence $T^{\mathcal{D}}$ consists of two copies of a rooted generalised tree (T',r) with the roots joined by an undirected edge of weight 1. So with suitable labelling $A(T^{\mathcal{D}})$ can be written as the partitioned matrix

$$\begin{pmatrix} A(T') & E \\ E & A(T') \end{pmatrix} \text{ which is similar to } \begin{pmatrix} A(T')+E & E \\ 0 & A(T')-E \end{pmatrix}.$$

Thus $$\phi(T^{\mathcal{D}},\lambda)=|A(T')+E-\lambda I|\times|A(T')-E-\lambda I|$$
$$=\phi(T^{\mathcal{C}},\lambda)\times|A(T')-E-\lambda I|.$$

Now T' is a generalised tree and thus bipartite. If m is even, then $|A(T')-\lambda I|$ consists of even powers of λ only, so that $|A(T')+E-\lambda I|$ and $|A(T')-E-\lambda I|$ differ only in that odd powers of λ are of opposite sign. Thus $|A(T')-E-\lambda I|$ is $\phi(T^{\mathcal{C}},-\lambda)$. Similarly, if m is odd, then $|A(T')-E-\lambda I|$ is $-\phi(T^{\mathcal{C}},-\lambda)$. //

As a consequence of this proposition it is only necessary to modify Algorithm 3.2.2.2 slightly to deal with trees whose coarsest decompositions have no singleton classes.

3.2.3.3: Algorithm. Let T=(V,s,w) be a tree with coarsest decomposition $\mathcal{C}$ and let $\mathcal{C}$ have no singleton classes. Let $T^{\mathcal{C}}=(V',s',w')$.

Step 1. As Algorithm 3.2.2.2.

Step 2. As algorithm 3.2.2.2.

Step 3. The polynomial associated with $T^{\mathcal{C}}$ is

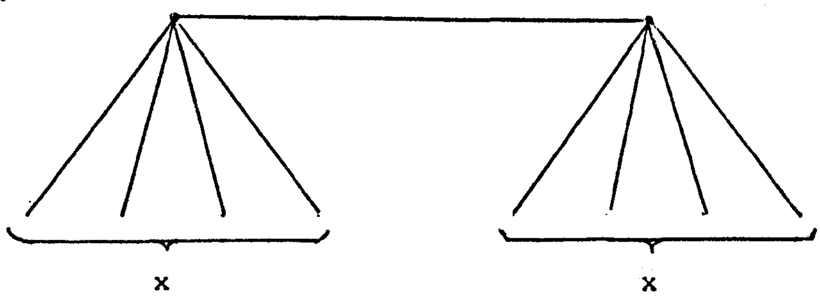
$$(-1)^{|V'|}\phi(T^{\mathcal{C}},\lambda)\phi(T^{\mathcal{C}},-\lambda)\prod_{x\varepsilon V'}P(x,\lambda). \qquad //$$

Thus considering all the cases we have the proposition:

3.2.4.1: Proposition. Let T=(V,s,w) be a tree, let $\mathcal{C}$ be its coarsest decomposition and let $T^{\mathcal{C}}=(V',s',w')$. THen the polynomial associated with T^C by Algorithm 3.2.2.2 if C has a singleton class or by Algorithm 3.2.2.3 otherwise is the characteristic polynomial of T. //

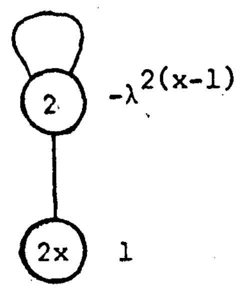
<u>3.2.4.2:Example</u>.Let T be the tree obtained by taking two copies of $K_{1,x}$ and adding an edge joining their centres thus:

T:



The quotient corresponding to the coarsest decomposition of T is T^C and the polynomials associated with its vertices are

T^C:



$$\phi(T^C,\lambda)= \begin{vmatrix} 1-\lambda & x \\ 1 & -\lambda \end{vmatrix} =\lambda^2-\lambda-x$$

with zeros $1\pm\sqrt{(1+4x)}/2$. Hence the zeros of $\phi(T^C,-\lambda)$ are $-1\pm\sqrt{(1+4x)}/2$, and

$$\text{Spec } T= \begin{pmatrix} \pm 1\pm\sqrt{(1+4x)}/2 & 0 \\ 1 & 2(x-1) \end{pmatrix}$$

Expressing the spectrum in this form (compare Schwenk (36,p107)) enables us to answer a question raised in (22)- when are the eigenvalues of T integers? It is clear that Spec T is integral if and only if 1+4x is a square,that is if and only if $x=r(r+1)$ for some $r\in N$. //

4:DECOMPOSITIONS OF ORDER TWO OF REGULAR SIMPLE GRAPHS

4.1:General Results.

4.1.1.1:Definition. Let G be a regular simple (but not necessarily connected) graph with a decomposition $\mathcal{D}$ of order 2. Let $(A(G^{\mathcal{D}}))_{12}=a$ and $(A(G^{\mathcal{D}}))_{21}=b$. Then we say that $\mathcal{D}$ is a decomposition of type a,b of G. //

4.1.1.2:Example. $\mathcal{D}_1$ in Example 2.1.4.1 is a decomposition of type 1,1 of the cube. //

4.1.2:Propositions. Let G be a regular simple graph of valency k with a decomposition $\mathcal{D}=\{D_1,D_2\}$ of type a,b for some $a,b\in Z^+$. Then

4.1.2.1: The induced vertex-subgraphs $\langle D_1\rangle$ and $\langle D_2\rangle$ are regular of valencies k-a and k-b respectively;

4.1.2.2: $|D_1|\times a=|D_2|\times b$.

Proofs. Trivial. //

4.1.3.1:Proposition. Let G be a regular (connected) graph of valency k. G has an eigenvector with exactly two distinct entries if and only if G has a decomposition of order 2.

Proof. i) Suppose G has a decomposition $\mathcal{D}$ of type a,b for some $a,b\in N$. (We cannot have a=b=0 since G is connected). $A(G^{\mathcal{D}})=\begin{pmatrix}k-a & a\\ b & k-b\end{pmatrix}$; hence $G^{\mathcal{D}}$ has an eigenvector $(-a,b)^t$ with eigenvalue k-(a+b) and G has a corresponding eigenvector with exactly two distinct entries, -a and b, by Proposition 2.2.3.8.

ii) Conversely, suppose G has an eigenvector with two distinct entries, say $(x,x,\ldots,x,y,y,\ldots,y)^t$ where x occurs m times and y occurs n times with $m,n\neq 0$, and let the corresponding eigenvalue be λ. Then A(G) can be partitioned thus:

$$A(G)=\begin{array}{cc} \overbrace{}^{m} & \overbrace{}^{n} \end{array}\begin{pmatrix} B & C\\ D & E\end{pmatrix}\begin{array}{l}\}m\\ \}n\end{array}$$

For any $i\varepsilon N_m$ we have

$$\sum_{j=1}^{m} b_{ij}+\sum_{p=1}^{n} c_{ip}=k \quad \text{since G is regular, and}$$

$$x\sum_{j=1}^{m} b_{ij}+y\sum_{p=1}^{n} c_{ip}=\lambda x.$$

But $x\neq y$, so $\sum_{j=1}^{m} b_{ij}$ and $\sum_{p=1}^{n} c_{ip}$ are independent of the choice of $i\varepsilon N_m$; in other words B and C have constant row sums. The same is true of D and E and thus G has a decomposition of order 2, by Proposition 2.2.3.14. //

4.1.3.2:Corollary. If a transitive graph G of valency k has a simple eigenvalue $\lambda\neq k$, then G has a decomposition of type a,a for some $a\varepsilon N_k$.

Proof. It is well known (see Biggs (4,p109)) that the eigenvector of G corresponding to λ is $(\pm 1,\pm 1,\ldots,\pm 1)^t$ with exactly half the entries negative. //

4.1.3.3:Definition. Let G be a properly labelled regular simple graph of valency k and order n with a decomposition $\mathcal{D}=\{D_1,D_2\}$ of type a,b for some $a,b\varepsilon N_k$. The eigenvector of G corresponding to $\mathcal{D}$, $\underline{d}(G,\mathcal{D})$ or simply $\underline{d}$ when there is no likelihood of confusion, is the n-tuple defined thus:

$$d_i=-a \quad \text{if } i\varepsilon D_1$$
$$\quad =b \quad \text{if } i\varepsilon D_2 \quad \text{for } i\varepsilon N_n.$$

k-(a+b) is the eigenvalue of G corresponding to $\mathcal{D}$. //

Looking for decompositions of order 2 is often an easy way of locating integer eigenvalues of a graph. Having found an eigenvalue λ of a graph G by this method, it is natural to enquire whether we can say anything about its multiplicity. If we know some or all of the automorphisms of G we can determine a lower bound for the multiplicity by calculating the dimension of the eigenspace spanned by the images of the associated eigenvector $\underline{d}$ under the natural action of the automorphisms. An alternative method which is usually equivalent to the one just outlined is to search for other decompositions of G of the same type and then examine the space spanned by the corresponding eigenvectors. In this case there is a simple sufficiency condition for the linear independence of the eigenvectors produced.

4.1.4.1: Proposition. Let $G=(V,s,w)$ be a properly labelled regular simple graph with decompositions $\mathcal{D}^{(i)}=\{D_1^{(i)},D_2^{(i)}\}$ for $i=1,2,\ldots,r$ all of type a,b for some $a,b,r\in N$ and let $\underline{d}^{(i)}$ denote the eigenvector of G corresponding to $\mathcal{D}^{(i)}$. Suppose that

i) there is a vertex $x\in V$ such that $x\in D_1^{(i)}$ for all $i\in N_r$,

ii) for each $i\in N_r$ there is a vertex $y(i)\in V$ such that

either $y(i)\in D_1^{(i)}$ and $y(i)\notin D_1^{(j)}$ for $j\neq i$

or $y(i)\notin D_1^{(i)}$ and $y(i)\in D_1^{(j)}$ for $j\neq i$.

Then the set of vectors $\{\underline{d}^{(i)}\}$ is independent.

Proof. Suppose we have $\mu_i\in R$ for $i=1,2,\ldots,r$ such that $\sum_{i=1}^{r}\mu_i\underline{d}^{(i)}=\underline{0}$.
Consider the x-th entry of this sum. Then $\sum_{i=1}^{r}-\mu_i a=0$.
Now for each $i\in N_r$ consider the y(i)-th entry of the sum. We have

$$-\mu_i a+\sum_{j\neq i}\mu_j b=0$$

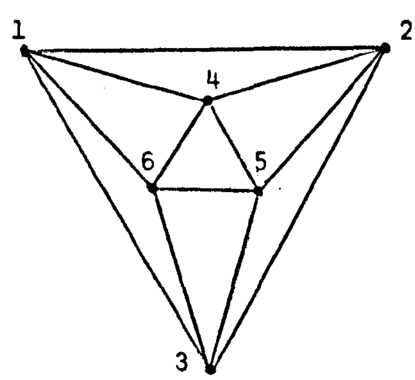
or $$\sum_{j\neq i}-\mu_j a+\mu_i b=0.$$

In either case $\mu_i=0$. //

It is possible to determine the spectra of some graphs completely using this proposition alone.

4.1.4.2: Example. The octahedron, $K_{2,2,2}$.

$K_{2,2,2}$:



First class of decomposition	Corresponding eigenvalue
	4
{1,2,3}	0
{1,2,4}	
{1,2,6}	
{1,3,4,5}	-2
{1,2,5,6}	

//

4.1.4.3:Definition.A regular graph of order n with n-1 decompositions of order 2 whose corresponding eigenvectors are independent is called determinable. //

4.1.4.4:Proposition.A determinable graph has an integral spectrum.

Proof.Trivial. //

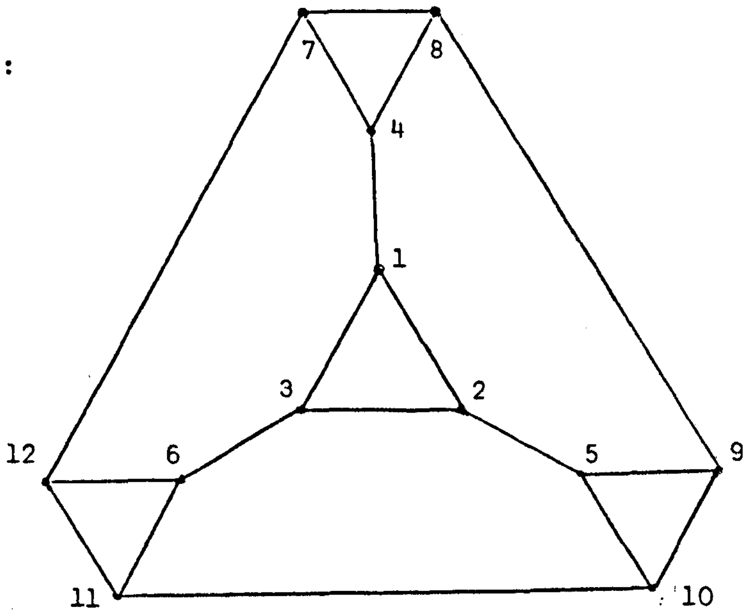
4.1.4.5:Example.K_n is determinable for all $k \in N$.

Proof.K_2 is determinable by inspection,so we suppose that $n>2$. Let K_n have vertex set V. Take a fixed vertex $x \in V$ and consider the decompositions of the form $\{\{x,y\},V\setminus\{x,y\}\}$ for $y \in V$ with $y \neq x$. There are n-1 such,and the corresponding eigenvectors are independent by Proposition 4.1.4.1. //

If a graph G of valency k has a decomposition $D=\{D_1,D_2\}$ of type a,b for some $a,b \in N_k$, then the ratio a:b must equal $|D_2|:|D_1|$. In addition a,b are small if k is small so that for graphs of low valency it is very easy to decide the possible natures of decompositions of order 2.

4.1.5 :Example.Consider the truncated tetrahedron $T(K_4)$, which has valency 3 and order 12.

$T(K_4)$:



Possible a and b			Corresponding $\lvert D_1 \rvert$ and $\lvert D_2 \rvert$.	
i)	3	3	6	6
ii)	3	2	Not possible	
iii)	3	1	3	9
iv)	2	2	6	6
v)	2	1	4	8
vi)	1	1	6	6.

We consider each possibility in turn.

i) can be discarded immediately since the graph is not bipartite.

ii) is impossible since the order,12, can not be split into two integers in the ratio 2:3.

iii) yields three decompositions satisfying Proposition 4.1.4.1 so that the eigenvalue -1 has multiplicity at least 3 in G.

iv) also gives three decompositions satisfying the proposition so that -1 again has multiplicity at least 3 in G, but we discover that the eigenspaces spanned in iii) and iv) are the same.

v) gives two decompositions satisfying the proposition so that 0 has multiplicity at least 2 in G.

vi) is soon found to be impossible.

This is as far as decompositions of order 2 will take us for this graph. We may complete the calculation of Spec $T(K_4)$ as follows. We note the decomposition of order 4

$\mathcal{D}=\{\{1,2,3\},\{4,5,6\},\{7,9,11\},\{8,10,12\}\}$

whose classes are the orbits of the obvious "rotation" automorphisms. The eigenvalues of the quotient are $3,-1,\pm 2$ and from the eigenvectors with eigenvalues ± 2 we construct corresponding eigenvectors of $T(K_4)$ as in Proposition 2.2.3.8. Considering the action of the same automorphisms on these eigenvectors we find that the images of the eigenvectors span spaces of dimension 3 each. Thus

$$\text{Spec } T(K_4) = \begin{pmatrix} 3 & 2 & 0 & -1 & -2 \\ 1 & 3 & 2 & 3 & 3 \end{pmatrix} \qquad //$$

Note that in the above example it is immediately apparent that $T(K_4)$ is not a determinable graph. For suppose that there is a decomposition of type a,b for some a,b in N_3 whose corresponding eigenvalue is -2. Then a+b would be 5 and since 5 and 12, the order of the graph, are coprime, it is clear that we cannot choose a,b,D_1,D_2 so that a:b equals $|D_2|:|D_1|$. We end this section by proving a generalisation of this remark.

<u>4.1.6.1:Proposition</u>. Let G be a determinable graph of valency k and order n, and let $\lambda \neq k$ be an eigenvalue of G. Then $k-\lambda$ divides rn for some $r \in N_k$.

<u>Proof</u>. Suppose λ is the eigenvalue corresponding to the decomposition $\mathcal{D}$ of type a,b. The ratio a:b can be reduced to its least integer form by the division of both terms by some $r \in N_k$. If the least integer form is a':b', then n is divisible by a'+b', that is $(k-\lambda)/r$. //

<u>4.1.6.2:Corollary</u>. Let G be a regular graph of prime order $p>2$ and valency $k<p/2$. Then G is not determinable.

<u>Proof</u>. G has a negative eigenvalue since the zeros of the characteristic

polynomial of G sum to 0 and G has a positive eigenvalue k. Let λ be such a negative eigenvalue and suppose that λ corresponds to some decomposition of order 2. Then $(k-\lambda)/r>1$ for all $r\epsilon N_k$ and hence $(k-\lambda)/r=p$ for some $r\epsilon N_k$. $k<p/2$ by hypothesis, so $-\lambda>k$. This is not possible and so we have a contradiction. //

This result cannot be extended to all k less than p since we know by Example 4.1.4.5 that K_p is determinable.

4.2: Switching and Decompositions.

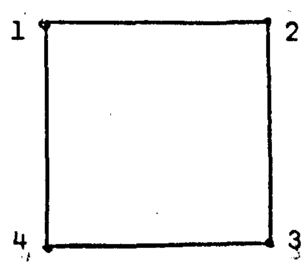
4.2.1.1: Definition. Let $G=(V,s,w)$ be a simple (but not necessarily connected) graph. We construct a new graph $G'=(V,s,w')$ by switching G with respect to $B\subset V$ thus:- For every pair $x,y\epsilon V$ set

$$w'(x,y)=w(x,y) \quad \text{if } x,y\epsilon B \text{ or } x,y\epsilon V\setminus B,$$
$$=w(x,y)+1 \text{ (modulo 2)} \quad \text{otherwise.}$$

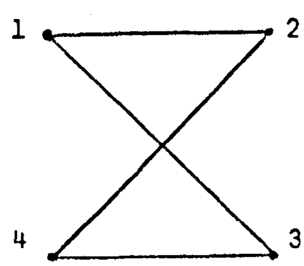
We say that G and G' are in the same switching-class, they are equivalent under switching or they are switching-equivalent. //

4.2.1.2: Example. Consider C_4:

C_4:



Then the graph obtained by switching with respect to vertex set {1,2} is:



//

The concept of switching arose initially in the study of families of graphs whose parameters almost determine the graphs uniquely. Seidel (38) found that some of the exceptional graphs are switching-equivalent to graphs in the families.More recently switching has been studied in other contexts and for its own sake (see Harries (23) for example).Considering the definition, it is not surprising that there is a close connection between switching and decompositions of order 2.

4.2.2.1:Proposition.Let $G=(V,s,w)$ be a regular simple graph of valency k. G is switching-equivalent to a regular simple graph $G'=(V,s,w')$ of valency k' if and only if G has a decomposition $\mathcal{D}=\{D_1,D_2\}$ of type a,b for some $a,b\in Z^+$ with $|D_1|-|D_2|=2(b-a)$. In this case $\mathcal{D}$ is a decomposition of G' of type $|D_2|-a,|D_1|-b$, $k'=k+|D_2|-2a=k+|D_1|-2b$, and $k'=k$ if and only if $|D_2|=2a$ and $|D_1|=2b$.

Proof.Sufficiency:Suppose G has a decomposition $\mathcal{D}=\{D_1,D_2\}$ of type a,b for some $a,b\in Z^+$ such that $|D_1|-|D_2|=2(b-a)$. Then let G' be the result of switching G with respect to D_1. Every vertex in D_1 has valency $k+|D_2|-2a$ as a vertex of G', every vertex in D_2 has valency $k+|D_1|-2b$ as a vertex of G', and the result follows immediately.

Necessity:Suppose that G' is obtained by switching G with respect to $B\subset V$ and that G' is regular while $\langle B\rangle$ is not.Then B contains vertices x,x^* such that

$$|N(G,x)\cap B|=c$$
$$|N(G,x^*)\cap B|=c^* \qquad \text{with } c\neq c^*.$$

Now the valency of x in G' is $2c+|V\setminus B|-k$ and the valency of x^* in G' is $2c^*+|V\setminus B|-k$, contradicting the regularity of G'. Hence $\langle B\rangle$ is regular with $|N(G,x)\cap\langle V\setminus B\rangle|=k-c=a$, say,for all $x\in B$.

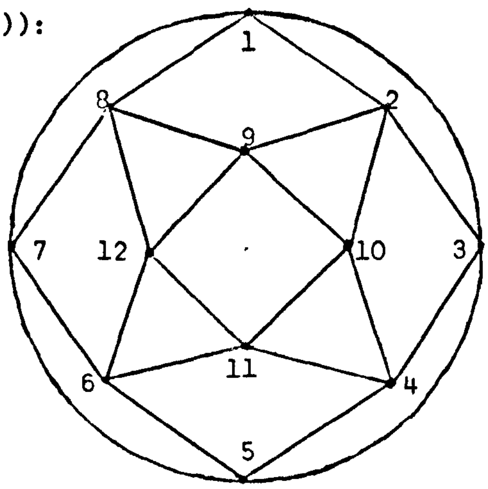
Similarly <V\B> is regular with $|N(G,y)\cap\langle B\rangle|=b$, say, for all $y\varepsilon V\setminus B$. Finally $|B|-|V\setminus B|=2(b-a)$ since $x\varepsilon B$ and $y\varepsilon V\setminus B$ have equal valencies as vertices of G'. //

4.2.2.2:Note. As in Example 4.2.1.2 the switching-equivalent graphs may be isomorphic. //

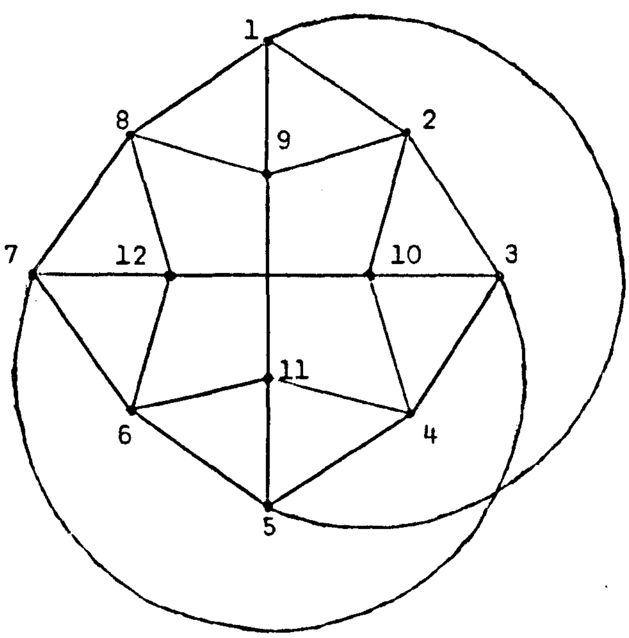
We can apply Proposition 4.2.2.1 and the results of Section 4.1 to the problem which originally motivated the study of switching.

4.2.2.3:Example. Hoffman and Ray-Chaudhuri (27) asked whether the distinct eigenvalues of the line graph of a symmetric balanced incomplete block-design were sufficient to determine the graph uniquely. They found that they were, with one exception. There are two graphs with the eigenvalues of $L(\Pi(4,3,2))$:

$L(\Pi(4,3,2))$:



G:



In order to decide whether these two graphs are equivalent under switching we consider the decompositions of order 2 of $L(\Pi(4,3,2))$ which might correspond to a switched graph of equal valency.

a and b		$\|D_1\|$ and $\|D_2\|$.		
4	4	6	6	Not possible- graph not bipartite.
4	3	Not possible.		
4	2	4	8.	
4	1	Not possible.		
3	3	6	6.	
3	2	Not possible.		
3	1	3	9	Not possible- $\|D_1\|-\|D_2\|\neq 2(b-a)$.
2	2	6	6	Not possible- $\|D_1\|\neq 2b$.
2	1	4	8	Not possible- $\|D_1\|-\|D_2\|\neq 2(b-a)$.
1	1	6	6	Not possible $\|D_1\|\neq 2b$.

We investigate the two remaining possibilities and find almost immediately that switching $L(\Pi(4,3,2))$ on the vertex set $\{1,4,6,9\}$ gives a graph isomorphic to G. //

In general we can split the choices of a and b into two distinct classes:- those for which a=b, and others. The next proposition shows that if two switching-equivalent graphs have different valencies then the switch is of the first type.

4.2.2.4:Proposition. Let $G=(V,s,w)$ be a regular simple graph of valency k and order n with a decomposition $\mathcal{D}=\{D_1,D_2\}$ of type a,b for some $a,b\in Z^+$, and let $G'=(V,s,w')$ be a regular simple graph of valency k' obtained by switching G with respect to D_1. Then either $|D_1|=n/2$ or $k=k'$.

Proof. $\quad |D_1|-|D_2|=2(b-a)$ by 4.2.2.1. (1)

$|D_1|\times a=|D_2|\times b$ by 4.1.2.2. (2)

Suppose $b=0$. Then $a=0$ and $|D_1|=|D_2|=n/2$.

Otherwise substitute for $|D_2|$ into (1) to give

$$|D_1|(1-a/b)=2b(1-a/b).$$

Thus <u>either</u> $a=b$ and $|D_1|=|D_2|=n/2$

<u>or</u> $|D_1|=2b$ and $k=k'$ by 4.2.2.1. //

A simple spectral result also follows.

<u>4.2.2.5:Proposition</u>.Let G and G' be switching-equivalent regular simple graphs of order n and valencies k and k' respectively. Then G has an eigenvalue $\lambda=k'-n/2$.

<u>Proof</u>.G has a decomposition $D=\{D_1,D_2\}$ of type a,b for some $a,b\varepsilon Z^+$ with $|D_1|-|D_2|=2(b-a)$, and hence has a corresponding eigenvalue $\lambda=k-(a+b)$. But $k'=k+|D_2|-2a=k+|D_1|-2b$ by 4.2.2.1, and so $2k'=2k+|D_1|+|D_2|-2(a+b)=2\lambda+n$. That is $\lambda=k'-n/2$. //

<u>4.2.2.6:Corollary</u>.A regular simple graph of odd order is not switching-equivalent to any other regular simple graph.

<u>Proof</u>.Suppose the contrary.By Proposition 4.2.2.5 the graph would have a rational but not integral eigenvalue,a contradiction. //

This corollary is a special case of a general result concerning switching-classes proved by Harries (23) using different methods.

There is in fact a very close connection between the complete spectra of switching-equivalent regular simple graphs.

<u>4.2.3.1:Notation</u>.Let G be a simple graph of order n and let J denote the matrix of order n whose every entry is 1. Then we define $A^*(G)$ to be the matrix $J-I-2A(G)$. //

$A^*(G)$ is the (1,0,-1)-adjacency matrix used by Seidel (38).

<u>4.2.3.2:Example</u>.Consider C_4 in Example 4.2.1.2. Then the matrices $A(C_4)$ and $A^*(C_4)$ are

$$A(C_4) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$A^*(C_4) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

//

<u>4.2.3.3:Lemma</u>.Let G be a regular simple graph of order n, valency k, with c components and spectrum $\begin{pmatrix} k & \lambda_i \\ c & m_i \end{pmatrix}$ with $i=1,2,\ldots,r$ for some $r \in N$. Then the spectrum of $A^*(G)$ is

$$\text{Spec } A^*(G) = \begin{pmatrix} n-1-2k & -1-2k & -1-2\lambda_i \\ 1 & c-1 & m_i \end{pmatrix}$$

(where $n-1-2k$ may equal $-1-2\lambda_i$ for some $i \in N_r$).

<u>Proof</u>.Since G is regular A(G) commutes with J so that they are simultaneously diagonalisable.It is well known that Spec J is $\begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}$ and clearly Spec I is $\begin{pmatrix} 1 \\ n \end{pmatrix}$.

Corresponding to eigenvalue n of J there is an eigenvector $(1,1,\ldots,1)^t$ which is also an eigenvector of A(G) with eigenvalue k. There is in addition an (n-1)-dimensional eigenspace of J corresponding to the eigenvalue 0 which is spanned by eigenvectors of A(G) with eigenvalues k and λ_i for $i=1,2,\ldots,r$. //

<u>4.2.3.4:Lemma</u> (Seidel (38)).Let $G=(V,s,w)$ and $G'=(V,s,w')$ be properly labelled switching-equivalent simple graphs.Then $A^*(G)$ and $A^*(G')$ are cospectral matrices.

Proof. The process of switching with respect to $B\subset V$ is equivalent to a similarity operation on $A^*(G)$, for $A^*(G')=MA^*(G)M$ where

$$m_{ij}=0 \quad \text{for } i\neq j$$
$$=-1 \quad \text{for } i\in B$$
$$=1 \quad \text{for } i\notin B \qquad \text{with } i,j\in V$$

and clearly $M^{-1}=M$. //

An immediate consequence of these two lemmas is the following proposition.

4.2.3.5: Proposition. Let the regular simple graphs G and G' of valencies k and k' and with c and c' components respectively be switching-equivalent. Let Spec $G=\begin{pmatrix} k & \lambda_i \\ c & m_i \end{pmatrix}$ with $i=1,2,\ldots,r$ for some $r\in N$ and let Spec $G'=\begin{pmatrix} k' & \lambda'_j \\ c' & m'_j \end{pmatrix}$ with $j=1,2,\ldots,s$ for some $s\in N$. Then

$$\begin{pmatrix} n-1-2k & -1-2k & -1-2\lambda_i \\ 1 & c-1 & m_i \end{pmatrix} = \begin{pmatrix} n-1-2k' & -1-2k' & -1-2\lambda'_j \\ 1 & c'-1 & m'_j \end{pmatrix}.$$ //

We can apply this result directly to the calculation of the spectra of graphs.

Examples.

4.2.3.6: N_4 is switching-equivalent to C_4. Now Spec $N_4=\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ so Spec $A^*(N_4)=\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ = Spec $A^*(C_4)$. But the valency of C_4 is 2 and so Spec $C_4=\begin{pmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}$.

4.2.3.7: Let G be the graph constructed by taking two labélled copies of K_r and making corresponding vertices of the two copies adjacent.Then the complete r-partite graph $K_{2,2,\ldots,2}$ is switching-equivalent to G.

Now Spec $K_{2,2,\ldots,2} = \begin{Bmatrix} 2r-2 & 0 & -2 \\ 1 & r & r-1 \end{Bmatrix}$ (see Biggs (4,p17)),

so Spec $A^*(K_{2,2,\ldots,2}) = \begin{Bmatrix} -2r+3 & -1 & 3 \\ 1 & r & r-1 \end{Bmatrix} = \text{Spec } A^*(G)$.

Hence Spec $G = \begin{Bmatrix} r & r-2 & 0 & -2 \\ 1 & 1 & r-1 & r-1 \end{Bmatrix}$.

//

In addition we have the following corollaries.

4.2.3.8:Corollary.Switching-equivalent regular simple graphs are cospectral if and only if their valencies are the same. Otherwise their spectra differ in just two values. //

4.2.3.9:Corollary.Let G be a regular simple graph of order n with vertex set V and let G' be a regular simple graph obtained from G by switching with respect to $B\subset V$ where $|B|\neq n/2$. Then G and G' are cospectral. //

Considering this corollary it is reasonable to ask whether it is possible to switch a regular graph with respect to half its vertex set and obtain a cospectral graph.We already have a trivial example of this in 4.2.1.2. Seidel (38) gives an example in the case of the L_2-graphs,where the graphs are cospectral and not isomorphic.

We end this chapter with a brief investigation of the k-cubes (see Definition 1.2.3.1) and a family of graphs derived from the k-cubes by switching.It is necessary to single out certain decompositions,which we do inductively.

4.2.5.1:Definition.We label the k-cubes and define their special decompositions thus:

i) $Q_1=K_2$ defined on vertex set $V_1=\{(0),(1)\}$ with special decomposition $\{\{(0)\},\{(1)\}\}$.

ii) Let $\underline{e}$ be an r-tuple for some $r\varepsilon N$. We use the notation $(\underline{e},x)$ to denote the (r+1)-tuple whose first r entries are those of $\underline{e}$ and whose last entry is x.

The vertex set V_{k+1} of Q_{k+1} is $\{(\underline{e},0),\underline{e}\varepsilon V_k\}\cup\{(\underline{e},1),\underline{e}\varepsilon V_k\}$.

In Q_{k+1} $(\underline{e},i)$ adj $(\underline{f},i)$ if $\underline{e}$ adj $\underline{f}$ in Q_k for i=0,1, and $(\underline{e},0)$ adj $(\underline{e},1)$ for all $\underline{e}$ in V_k.

Let A(i) denote $\{(\underline{e},i),\underline{e}\varepsilon A\subseteq V_k\}$, with i=0,1.

Let $\mathcal{D}=\{D_1,D_2\}$ be a partition of V_k. We define the <u>$\mathcal{E}$-extension of $\mathcal{D}$</u> to be $\mathcal{E}=\{D_1(0)\cup D_1(1),D_2(0)\cup D_2(1)\}$ and the <u>$\mathcal{F}$-extension of $\mathcal{D}$</u> to be $\mathcal{F}=\{D_1(0)\cup D_2(1),D_2(0)\cup D_1(1)\}$.

Suppose $\{\mathcal{D}^{(i)},i\varepsilon N_r\}$ for some $r\varepsilon N$ is the set of special decompositions of Q_k. Then the set of <u>special decompositions of Q_{k+1}</u> is defined to be

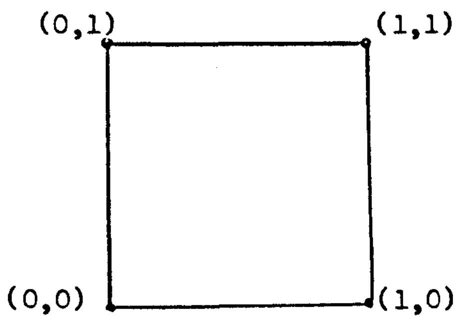
$$\{\mathcal{E}^{(i)},i\varepsilon N_r\}\cup\{\mathcal{F}^{(i)},i\varepsilon N_r\}\cup\{\{V_k(0)\ ,\ V_k(1)\}\}. \qquad //$$

<u>4.2.5.2:Example.</u> Q_1 and Q_2.

Q_1:

Special decomposition of Q_1: $\mathcal{D}=\{\{(0)\},\{(1)\}\}$.

Q_2:



Special decompositions of Q_2:

$\mathcal{E}=\{\{(0,0),(0,1)\},\{(1,0),(1,1)\}\}$,

$\mathcal{F}=\{\{(0,0),(1,1)\},\{(1,0),(0,1)\}\}$

$\{V_1(0),V_1(1)\}=\{\{(0,0),(1,0)\},\{(0,1),(1,1)\}\}.$ //

<u>4.2.5.3:Lemma.</u>Let D be a decomposition of Q_k of type i,i for some $i\epsilon N_k$. Then the E-extension and F-extension of D are decompositions of Q_{k+1} of types i,i and (i+1),(i+1) respectively.

<u>Proof.</u>$\langle D_1(0)\cup D_1(1)\rangle$ and $\langle D_1(0)\cup D_2(1)\rangle$ are regular of valencies k-i+1 and k-i respectively. //

<u>4.2.5.4:Lemma.</u>The special decompositions of Q_k are decompositions of Q_k. There are $\binom{k}{i}$ special decompositions of type i,i for each $i\epsilon N_k$ and none of type a,b for any a≠b. ($\binom{k}{i}$ denotes the binomial coefficient.)

<u>Proof.</u>By induction on k. //

<u>4.2.5.5:Lemma.</u>If the eigenvectors corresponding to the special decompositions of Q_k form a linearly independent set,then so do those corresponding to the special decompositions of Q_{k+1}.

<u>Proof.</u>Let the set of special decompositions of Q_k be $\{D^{(j)},j\epsilon N_r\}$ where $r=2^k-1$ and $D^{(j)}=\{D_1^{(j)},D_2^{(j)}\}$, and let the eigenvector corresponding to $D^{(j)}$ be $\underline{d}^{(j)}$ for $j\epsilon N_r$. Then $\underline{d}^{(j)}$ can be written $(\pm1,\pm1,\ldots,\pm1)^t$ where the negative entries correspond to vertices of $D_1^{(j)}$, so we see that,using an obvious extension of the notation of Definition 4.2.5.1, the eigenvectors of Q_{k+1} corresponding to $E^{(j)}$ and $F^{(j)}$ are $((\underline{d}^{(j)})^t,(\underline{d}^{(j)})^t)^t$ and $((\underline{d}^{(j)})^t,(-\underline{d}^{(j)})^t)^t$ and the eigenvector corresponding to the decomposition $\{V_k(0),V_k(1)\}$ is $(-1,-1,\ldots,-1,1,1,\ldots,1)^t$. In what follows we shall suppress t.

Consider any $\mu_j,\nu_j,\kappa\epsilon R$ with $j=1,2,\ldots,r$ such that

$$\sum_{j=1}^{r}\mu_j(\underline{d}^{(i)},\underline{d}^{(i)})+\sum_{j=1}^{r}\nu_j(\underline{d}^{(j)},-\underline{d}^{(j)})$$

$$+\kappa(-1,-1,\ldots,-1,1,1,\ldots,1)=\underline{0}$$

Then $\sum_{j=1}^{r}(\mu_j+\nu_j)\underline{d}^{(j)}-\kappa(1,1,\ldots,1)=\underline{0}$

and $\sum_{j=1}^{r}(\mu_j-\nu_j)\underline{d}^{(j)}+\kappa(1,1,\ldots,1)=\underline{0}$.

So $\sum_{j=1}^{r}\mu_j\underline{d}^{(j)}=\underline{0}$ and thus $\mu_j=0$ for all $j\epsilon N_r$.

Also $\sum_{j=1}^{r}\nu_j\underline{d}^{(j)}=\kappa(1,1,\ldots,1)$.

But the sum of the entries of each $\underline{d}^{(j)}$ is zero and hence $\nu_j=0$ for all $j\epsilon N_r$ and $\kappa=0$. //

4.2.5.6:Proposition. Q_k is determinable with spectrum $\begin{pmatrix} k & k-2i \\ 1 & \binom{k}{i} \end{pmatrix}$ where i takes values $1,2,\ldots,k$.

Proof.By induction on k, using the lemmas. //

4.2.6.1:Lemma.Consider two special decompositions $\mathcal{D}^{(1)}=\{D_1^{(1)},D_2^{(1)}\}$ and $\mathcal{D}^{(2)}=\{D_1^{(2)},D_2^{(2)}\}$ of Q_k of the same type i,i for some $i\epsilon N_k$. Let $G^{(1)}$ and $G^{(2)}$ be the simple graphs obtained by switching Q_k with respect to $D_1^{(1)}$ and $D_1^{(2)}$ respectively.Then $G^{(1)}$ is isomorphic to $G^{(2)}$.

Proof.Consider any special decomposition $\mathcal{D}=\{D_1,D_2\}$ of type i,i of Q_k, and choose some $x\epsilon D_1$. Then $\mathcal{D}$ is determined by $D_1\cap N(x)$. For consider the vertex $y\epsilon N(x)$ which differs from x only in its last entry.If $y\epsilon D_1$,then $\mathcal{D}$ is an E-extension of a special decomposition of Q_{k-1}. If $y\notin D_1$, then either $\mathcal{D}=\{V_{k-1}(0),V_{k-1}(1)\}$ * or $\mathcal{D}$ is an F-extension of Q_{k-1}. The neighbour of x which differs from x in its penultimate entry determines the nature of the special decomposition of Q_{k-1} and so on.

Now $\Gamma_x(Q_k)$ acts as Σ_k on $N(x)$ and $\Gamma(Q_k)$ is transitive so that there is an automorphism of Q_k taking any special decomposition to any other of the same type. //

* which is the case if every other neighbour of x is in D_1

4.2.6.2: Definition. The i-switched k-cube, $SQ_k^{(i)}$, with $i\in N_k$, is the graph constructed by switching Q_k with respect to a class of any special decomposition of type i,i of Q_k. //

By Lemma 4.2.6.1 $SQ_k^{(i)}$ is well-defined up to isomorphism.

4.2.6.3: Lemma. $\operatorname{Spec} SQ_k^{(i)} = \begin{pmatrix} k+2^{k-1}-2i & k-2^{k-1} & k-2i & k-2j \\ 1 & 1 & \binom{k}{i}-1 & \binom{k}{j} \end{pmatrix}$

with $j=1,2,\dots,i-1,i+1,\dots,k$, and where the eigenvalues given are not necessarily all distinct.

Proof. $SQ_k^{(i)}$ is regular of valency $k+2^{k-1}-2i$. We apply Proposition 4.2.3.5. $\operatorname{Spec} Q_k = \begin{pmatrix} k & k-2j \\ 1 & \binom{k}{j} \end{pmatrix}$ with $j=1,2,\dots,k$.

So $\operatorname{Spec} A^*(Q_k) = \begin{pmatrix} 2^k-1-2k & -1-2(k-2j) \\ 1 & \binom{k}{j} \end{pmatrix} = \operatorname{Spec} A^*(SQ_k^{(i)})$

and $\operatorname{Spec} SQ_k^{(i)}$ is as stated. //

4.2.6.4: Proposition. Let $k\in N$ and $i\in N_k$, and suppose that the pair (k,i) does not have any of the following values: (1,1),(2,1),(2,2),(3,2),(3,3),(4,4). Then

i) For any $k'\in N$ with $k'\neq k$ there is no $j\in N_{k'}$ such that $SQ_k^{(i)}$ and $SQ_{k'}^{(j)}$ are isomorphic.

ii) There is no $j\in N_k$ with $j\neq i$ such that $SQ_k^{(i)}$ and $SQ_k^{(j)}$ are isomorphic.

iii) $SQ_k^{(i)}$ is connected.

iv) $SQ_k^{(i)}$ is not isomorphic to Q_k.

Proof. i) The two simple graphs have different orders.

ii) In this case they have different valencies.

iii) $k-2j\leq k-2$ for $j\in N_k$ and $k+2^{k-1}-2i\geq 2^{k-1}-k$. Then $2^{k-1}-k\leq k-2$ implies that $k\leq 3$. Hence for $k>3$ $k+2^{k-1}-2i$ is a simple eigenvalue of $SQ_k^{(i)}$ which is therefore connected.

iv) If $SQ_k^{(i)}$ were isomorphic to Q_k then we would have

$k+2^{k-1}-2i=k$, that is $2^k=4i$. Since $i\varepsilon N_k$ the only solutions of this equation for (k,i) are $(2,1)$, $(3,2)$ and $(4,4)$.

By inspection we find that:

$SQ_1^{(1)}$ is not connected;

$SQ_2^{(1)}$ is isomorphic to Q_2;

$SQ_2^{(2)}$ is not connected;

$SQ_3^{(2)}$ is isomorphic to Q_3;

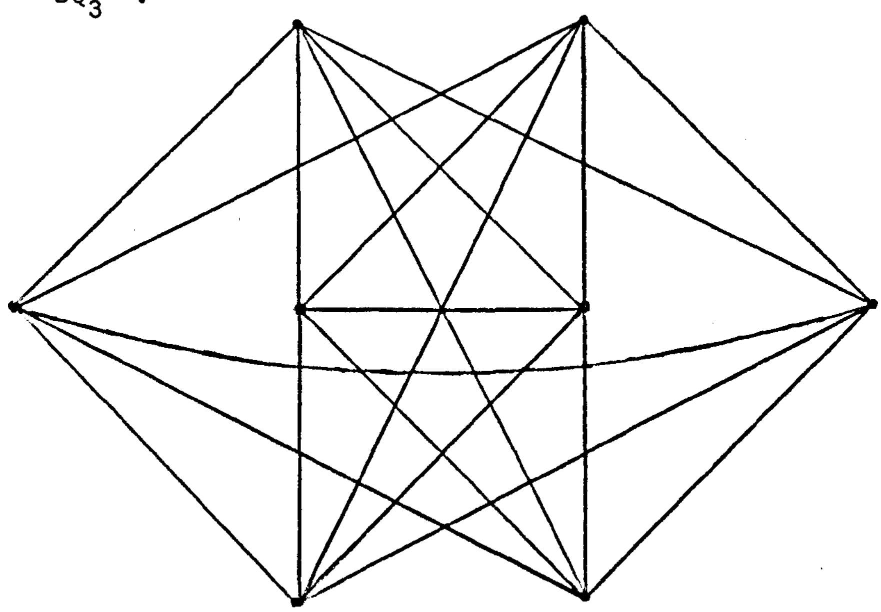
$SQ_3^{(3)}$ is not connected;

$SQ_4^{(4)}$ is isomorphic to Q_4. //

So we have shown that the i-switched k-cubes constitute a "proper" family of graphs - with the exceptions given they are connected regular simple graphs isomorphic neither to each other nor to the graphs from which they are derived.

4.2.6.5:Example. The smallest connected i-switched k-cube not isomorphic to a k-cube is $SQ_3^{(1)}$.

$SQ_3^{(1)}$:



//

5:FAITHFULNESS AND INTEGRITY

5.1:Faithful Decompositions.

5.1.1:Definition. Let D be a decomposition of a generalised graph G with the property that every eigenvalue of G is an eigenvalue of G^D. Then D is a faithful decomposition of G and G^D is a faithful quotient of G. //

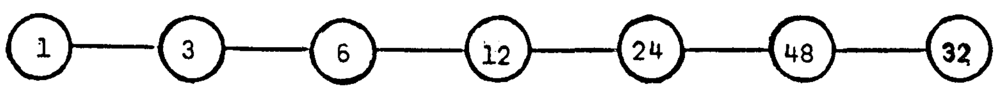
5.1.2.1:Definition. Let G=(V,s,w) be a generalised graph. G is singleton-regular if its singleton-quotient with respect to vertex $x \varepsilon V$ (Definition 2.4.3.3) is independent of the choice of x. If G is singleton-regular we let G^S denote the common singleton-quotient and adopt the convention (in line with Proposition 2.4.3.2) that the vertex of G^S corresponding to {x} is labelled vertex 1. //

5.1.2.2:Proposition. A transitive generalised graph is singleton-regular.

Proof. Trivial. //

The converse of this proposition is not true.

5.1.2.3:Counter-example. Benson's graph is singleton-regular with singleton-quotient



but it is not transitive. In fact its automorphism group partitions the vertices into two orbits (7). //

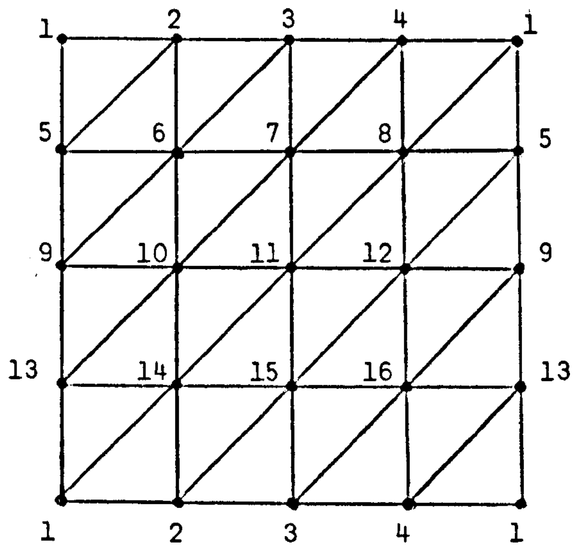
Important classes of singleton-regular graphs are the distance-transitive graphs (1.3.5.4) and the distance-regular graphs, which are defined as follows:

5.1.2.4:Definition.A singleton-regular graph $G=(V,s,w)$ is distance-regular if its singleton-decomposition with respect to vertex x coincides with its distance partition with respect to x (1.4.4.1) for every choice of x. If in addition the diameter of G is 2, then the graph is strongly-regular. //

Examples.

5.1.2.5:The k-cubes are distance-transitive.

5.1.2.6:The following triangular tessellation of the torus is transitive and strongly-regular but not distance-transitive:



5.1.2.7:Benson's graph is distance-regular but not transitive. //

The next section gives a sufficient condition for a decomposition to be faithful,investigates the consequences and in particular shows that the singleton-decomposition of a singleton-regular graph satisfies this condition.

5.1.3.1:Notation.For any $n\in N$ let $\underline{e}_i^n$ with $i\in N_n$ denote the n-tuple with entry 1 in the i-th position and 0 elsewhere. //

5.1.3.2:Proposition.Let $G=(V,s,w)$ be a graph of order n with a series of r decompositions for some $r\in N$, $\mathcal{D}^{(1)},\mathcal{D}^{(2)},\ldots,\mathcal{D}^{(r)}$ all of the same order m for some $m\in N_n$, with $\mathcal{D}^{(i)}=\{D_1^{(i)},\ldots,D_m^{(i)}\}$

for $i=1,2,\ldots,r$, such that

i) for any $x\varepsilon V$ there is an $i\varepsilon N_r$ and $j\varepsilon N_m$ with $D_j^{(i)}=\{x\}$,

ii) the quotient graphs $G^{\mathcal{D}^{(i)}}$ for $i=1,2,\ldots,r$ are isomorphic.

Using $G^{\mathcal{D}}$ to denote the common quotient we have:- $G^{\mathcal{D}}$ is a faithful quotient of G.

<u>Proof</u>.We shall show that for every $x\varepsilon V$ there is $s\varepsilon N$ and a set $A(x)=\{\underline{v}^{(k)},k\varepsilon N_s\}$ of eigenvectors of G with the properties that

i) if $\underline{v}^{(k)}$ has eigenvalue $\lambda^{(k)}$, then $\lambda^{(k)}$ is an eigenvalue of $G^{\mathcal{D}}$,

ii) there are $\mu_k\varepsilon R$ for $k\varepsilon N_s$ such that $\sum_{k=1}^{s}\mu_k\underline{v}^{(k)}=\underline{e}_x^n$.

It will then follow that the union of the sets A(x) taken over all $x\varepsilon V$ spans R^n, so that every eigenvalue of G is an eigenvalue of $G^{\mathcal{D}}$.

Let $G^{\mathcal{D}}=(N_m,s',w')$. Now since $G^{\mathcal{D}}$ is undirected, $A(G^{\mathcal{D}})$ is diagonalisable (Corollary 2.2.2.3).Hence for each $j\varepsilon N_m$ there is $s\varepsilon N$, a set of eigenvectors of $G^{\mathcal{D}}$, $B(j)=\{\underline{u}^{(k)},k\varepsilon N_s\}$, where $\underline{u}^{(k)}$ has eigenvalue $\lambda^{(k)}$, and $\mu_k\varepsilon R$ for $k=1,2,\ldots,s$ such that $\sum_{k=1}^{s}\mu_k\underline{u}^{(k)}=\underline{e}_j^m$.

Take any $x\varepsilon V$. We consider a decomposition $\mathcal{D}^{(i)}$ of G with $D_j^{(i)}=\{x\}$ for some $j\varepsilon N_m$, the matrix $Q(G,\mathcal{D}^{(i)})=Q$ (Notation 2.2.3.1), and the set B(j). Then $Q\underline{u}^{(k)}$ is an eigenvector $\underline{v}^{(k)}$ of G for each $k\varepsilon N_s$ with eigenvalue $\lambda^{(k)}$ by Proposition 2.2.3.8. Thus

$$\sum_{k=1}^{s}\mu_k\underline{v}^{(k)}=\sum_{k=1}^{s}\mu_k Q\underline{u}^{(k)}=Q\sum_{k=1}^{s}\mu_k\underline{u}^{(k)}=Q\underline{e}_j^m=\underline{e}_x^n\ ,$$

so that $\{Q\underline{u}^{(k)},k\varepsilon N_s\}$ is the desired set A(x). //

<u>5.1.3.3:Corollary</u>.If G is a singleton-regular graph then $G^{\mathcal{S}}$ is a faithful quotient and the number of distinct eigenvalues of G is bounded above by the order of $G^{\mathcal{S}}$. //

<u>5.1.3.4:Corollary</u>.If G is a transitive graph then the number

of distinct eigenvalues of G is bounded above by the number of orbits of the group of automorphisms of G stabilising a vertex,x say, $\Gamma_x(G)$. //

5.1.3.5:Example.The Coxeter/Frucht graph (Example 2.1.11) is transitive.Hence it has at most 18 distinct eigenvalues.By evaluating the spectrum of the given quotient we find that its distinct eigenvalues are in fact ±3, ±2.5243, ±2.2361, ±1.6180 ±0.7923, ±0.6180, a total of 12 different values. //

5.1.3.6:Corollary.Let $G=(V,s,w)$ be a graph with a decomposition $\mathcal{D}=\{D_1,D_2,\ldots,D_m\}$ for some $m\varepsilon N$ which has the following property:- For every $x\varepsilon V$ there is $\gamma\varepsilon\Gamma(G)$ and $D_i=\{y\}\varepsilon\mathcal{D}$ for some $y\varepsilon V$ such that $\gamma(y)=x$. Then $\mathcal{D}$ is a faithful decomposition of G.

Proof.We consider the decompositions of G induced by the natural action of $\Gamma(G)$ on $\mathcal{D}$. These decompositions satisfy the conditions of Proposition 5.1.3.2. //

5.1.3.7:Example.The generalised Petersen graph $P(h,t)$ has a "reflection automorphism" γ stabilising vertices x_0,y_0 (using the notation of Definition 1.2.4.1) and a "rotation automorphism" π taking x_i to x_{i+1} and y_i to y_{i+1} for $i=0,1,\ldots,h-1$, subscripts reduced modulo h. Hence $\mathcal{D}(\langle\gamma\rangle)$ is a faithful decomposition of $P(h,t)$ and the number of distinct eigenvalues of $P(h,t)$ is bounded above by $h+1$ if h is odd or $h+2$ if h is even (a result already established in Corollary 2.3.2.2). //

5.1.3.8:Example.It is easily verified directly that a circulant graph of order n has at most $[n/2]+1$ distinct eigenvalues (where $[x]$ denotes the integer part of x).However it may also be demonstrated as follows:- Kagno (28) proves that a circulant graph of order n has D_n, the dihedral group on n objects,as a group of automorphisms.

So there is again a "reflection automorphism" generating a subgroup with $[n/2]+1$ orbits, whose associated decomposition is faithful. //

5.1.3.9:Note. Not every faithful decomposition has a singleton class. For example the decomposition $\{\{1,5\},\{2,3\},\{4,8\},\{6,7\}\}$ of the cube (labelled as in Examples 2.1.4) is faithful. //

5.2: Multiplicities.

In "Algebraic Graph Theory" Biggs (4) presents the following remarkable result.

Let G be a distance-regular graph of order n and let λ be an eigenvalue of G^{S} with left- and right-eigenvectors $\underline{u}$ and $\underline{v}$ respectively. $\underline{u}$ and $\underline{v}$ can be chosen so that $u_1=v_1=1$ and in this case the multiplicity of λ in G is $n/(\underline{u},\underline{v})$ where $(\underline{u},\underline{v})$ denotes the inner product of $\underline{u}$ and $\underline{v}$.

It is the purpose of this section to prove a similar proposition for all singleton-regular graphs and to discuss the consequences. We follow the method of proof employed by Biggs in "Finite Groups of Automorphisms" (3).

5.2.1.1:Definition. Consider a graph $G=(V,s,w)$ and let $q(x,r)$ be the number of walks of length r beginning and ending at vertex $x\varepsilon V$. We call G return-regular if for each $r\varepsilon N$, $q(x,r)$ is independent of the choice of x. //

5.2.1.2:Lemma. Let $G=(V,s,w)$ be a graph with a decomposition $\mathcal{D}=\{D_i, i\varepsilon N_m\}$ of order m. Consider a vertex $x\varepsilon V$ and let $x\varepsilon D_i$ for some $i\varepsilon N_m$. Then the number of walks of length r from x to vertices of class D_j is $(A(G^{\mathcal{D}})^r)_{ij}$ for all $r\varepsilon N$ and $j\varepsilon N_m$.

Proof. By induction on r.

i) Since G is simple $(A(G^{\mathcal{D}}))_{ij}$ is the number of walks of length 1 from x to vertices in class D_j, that is the number of neighbours of x in D_j.

ii) Suppose the proposition is true when $r=s$ for some $s\in N$. Then, for each $k\in N_m$, the number of walks of length $s+1$ from x to the vertices of D_k is

$$\sum_{j=1}^{m}(A(G^{\mathcal{D}})^s)_{ij}(A(G^{\mathcal{D}}))_{jk}=(A(G^{\mathcal{D}})^{s+1})_{ik} \quad . \qquad //$$

<u>5.2.1.3:Proposition</u>. A singleton-regular graph $G=(V,s,w)$ is return-regular.

<u>Proof</u>. $q(x,r)=(A(G^{\mathcal{S}})^r)_{11}$ for all $x\in V$. //

<u>5.2.2.1:Notation</u>. $\mathcal{A}(G)$ denotes the algebra of polynomials of the adjacency matrix $A(G)$ of graph G. //

<u>5.2.2.2:Lemma</u>. Let G be a graph of order n with a decomposition $\mathcal{D}$ of order m for some $m\in N_n$. Then the mapping $X\to R(G,\mathcal{D})XQ(G,\mathcal{D})=\overline{X}$ (Notation 2.2.3.1) of real matrices of order n into real matrices of order m is a homomorphism of $\mathcal{A}(G)$ into $\mathcal{A}(G^{\mathcal{D}})$.

<u>Proof</u>. Let $A(G)=A$, $R(G,\mathcal{D})=R$, and $Q(G,\mathcal{D})=Q$.

i) Clearly $\overline{X}+\overline{Y}=\overline{X+Y}$ and $a\overline{X}=(\overline{aX})$ for all real matrices X,Y of order n and $a\in R$.

ii) $\overline{X}\overline{Y}=\overline{XY}$ for all $X,Y\in\mathcal{A}(G)$ if and only if $\overline{A}^i\overline{A}^j=\overline{A^iA^j}$ for all $i,j\in N$. But

$$\begin{aligned}\overline{A}^i\overline{A}^j&=RA^iQRA^jQ=RA^iA^jQRQ \qquad (2.2.3.6)\\ &=RA^iA^jQI_m \qquad (2.2.3.5)\\ &=\overline{A^iA^j} \quad .\end{aligned} \qquad //$$

<u>5.2.2.3:Lemma</u>. Let G be a graph with a faithful decomposition $\mathcal{D}$. Then the mapping $X\to\overline{X}$ defined as above is a monomorphism of $\mathcal{A}(G)$.

<u>Proof</u>. $\mathcal{A}(G)$ and $\mathcal{A}(G^{\mathcal{D}})$ have the same minimum polynomial and hence

their algebras have the same dimension. //

5.2.2.4:Notation. In 5.2.2.5-7 we shall consider a return-regular properly labelled graph G of order n for some $n\varepsilon N$ with a faithful decomposition $\mathcal{D}=\{D_1,D_2,\ldots,D_m\}$ for some $m\varepsilon N_n$ with $D_1=\{1\}$. Let $A(G)=A$ and $A(G^{\mathcal{D}})=B$. We suppose λ to be a simple eigenvalue of $G^{\mathcal{D}}$ and an eigenvalue of G with multiplicity $m(\lambda)$, and let the corresponding eigenvector of B be $\underline{v}$. We define $q(t)$ to be the common minimum polynomial of $\mathcal{A}(G)$ and $\mathcal{A}(G^{\mathcal{D}})$, set $q(t)=(t-\lambda)\bar{q}(t)$ and write $Z=\bar{q}(A)$. //

5.2.2.5:Lemma. $m(\lambda)tr(\bar{Z})=tr(Z)$, and $tr(Z)\neq 0$.

Proof. For any polynomial $f(t)$ we have $tr(f(A))=\sum_{\mu\varepsilon Spec\ G} m(\mu)f(\mu)$. $\bar{q}$ vanishes for $\mu\neq\lambda$, so $tr(\bar{q}(A))=m(\lambda)\bar{q}(\lambda)\neq 0$ since $m(\lambda)\geq 1$ and $\bar{q}(\lambda)\neq 0$.

Now $tr(\bar{q}(B))=\bar{q}(\lambda)$ so that $tr(\bar{q}(A))=m(\lambda)tr(\bar{q}(B))$. But $X\to\bar{X}$ is a homomorphism of $\mathcal{A}(G)$ into $\mathcal{A}(G^{\mathcal{D}})$, so $\bar{q}(B)=\bar{Z}$, that is

$$tr(Z)=m(\lambda)tr(\bar{Z})\neq 0.$$ //

5.2.2.6:Lemma. $\mathcal{A}(G)$ has a basis $\{A_f, f\varepsilon N_r\}$ for some $r\varepsilon N$, with the property that $A_1=I_n$ and every other A_f has diagonal entries zero.

Proof. For each $s\varepsilon N$, $(A^s)_{ii}$ is the number of walks of length s beginning and ending at vertex i. Thus since G is return-regular every member of $\mathcal{A}(G)$ has constant diagonal entries. //

5.2.2.7:Proposition. Let the m-tuple $\underline{u}$ be defined by $u_i=|D_i|v_i$ for $i\varepsilon N_m$. Then $m(\lambda)=nv_1^2/(\underline{u},\underline{v})$, where $(\underline{u},\underline{v})$ denotes the inner product of $\underline{u}$ and $\underline{v}$.

Proof. $Z=\sum_{j=1}^{r} z_jA_j$ for some $z_j\varepsilon R$ with $j=1,2,\ldots,r$ and hence

$$tr(Z)=z_1tr(A_1)=nz_1 \quad \text{where } z_1\neq 0 \text{ by 5.2.2.5.}$$

$q(A)=0$ and $q(A)=(A-\lambda I)Z$, so that $AZ=\lambda Z$. Since $Z\varepsilon\mathcal{A}(G)$, Z commutes with A so that we also have $ZA=\lambda Z$. Hence $\bar{A}\bar{Z}=\lambda\bar{Z}$ and $\bar{Z}\bar{A}=\lambda\bar{Z}$. Thus every column of $\bar{Z}$ is a right-eigenvector of B and every row of $\bar{Z}$ is a left-eigenvector of B with eigenvalue λ. Since the multiplicity of λ in B is 1, the rows and columns are therefore determined to within a multiplicative constant. Every column is a multiple of $\underline{v}$. Let K be the diagonal matrix of order m whose ii-th entry is $|D_i|$. By Propositions 2.1.3.2 and 2.1.3.4 KB is symmetrical,that is $(KB)^t=KB$. So

$$B^tK\underline{v}=(KB)^t\underline{v}=KB\underline{v}=K\lambda\underline{v}=\lambda K\underline{v},$$

$K\underline{v}=\underline{u}$ is a right-eigenvector of B^t and $\underline{u}^t$ is a left-eigenvector of B. Hence every row is a multiple of $\underline{u}^t$.

Thus $(\bar{Z})_{jf}=\alpha_j u_f=\beta_f v_j$ for some $\alpha_j,\beta_f\varepsilon R$ for each $j,f\varepsilon N_m$. But $\bar{Z}=\sum_{j=1}^{r} z_j\bar{A}_j$ and so,since $|D_1|=1$, $(\bar{Z})_{11}=z_1$. Thus $\alpha_1 u_1=\beta_1 v_1=z_1\neq 0$ so that $\beta_1=z_1/v_1$. And since $u_1=v_1$, $\alpha_j v_1=\beta_1 v_j$ so that $\alpha_j=\beta_1 v_j/v_1=z_1 v_j/v_1^2$. Hence $(\bar{Z})_{jf}=z_1 v_j u_f/v_1^2$, and

$$tr(\bar{Z})=z_1(\underline{u},\underline{v})/v_1^2 .$$

So by Lemma 5.2.2.5

$$m(\lambda)=nv_1^2/(\underline{u},\underline{v}) . \qquad //$$

We have established in passing that $v_1\neq 0$, so $\underline{v}$ can be chosen with $v_1=1$. We usually adopt the convention that this has been done and give the above result as $m(\lambda)=n/(\underline{u},\underline{v})$.

5.2.2.8:Corollary.Let G be a singleton-regular graph of order n, and let λ be a simple eigenvalue of G^S with associated eigenvector $\underline{v}$ chosen so that $v_1=1$. Then the multiplicity of λ in G is $n/(\underline{u},\underline{v})$ where $\underline{u}$ is defined as above. //

5.2.2.9:Corollary (Biggs).Let G be a distance-regular graph and let λ be any eigenvalue of G^S. Then using the notation above

$m(\lambda)=n/(\underline{u},\underline{v})$.

Proof. If G has diameter d, then S has order d+1, and every graph has at least d+1 distinct eigenvalues (4,p13), so that every eigenvalue of G^S is simple. //

5.2.2.10:Example. The truncated tetrahedron $T(K_4)$ of Example 4.1.5 is transitive and thus singleton-regular with $S(1)=\{\{1\},\{2,3\},\{4\},\{5,6\},\{7,8\},\{9,12\},\{10,11\}\}$, n=12 and

$$A((T(K_4))^S)=\begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Eigenvalues λ	Multiplicities in $T(K_4)$, $m(\lambda)$	in $(T(K_4))^S$	Eigenvectors of $(T(K_4))^S$, $\underline{v}$	$n/(\underline{u},\underline{v})$
3	1	1	$(1,1,1,1,1,1,1)^t$	1
2	3	2	$(1,\frac{1}{2},1,\frac{-1}{2},\frac{1}{2},\frac{-1}{2},-1)^t$	2
			$(1,1,0,0,\frac{-1}{2},\frac{-1}{2},\frac{-1}{2})^t$	$2\frac{2}{3}$
0	2	1	$(1,\frac{-1}{2},1,\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},1)^t$	2
-1	3	2	$(1,1,-3,-3,1,1,1)^t$	$\frac{4}{11}$
			$(1,0,-1,-1,0,1,0)^t$	2
-2	3	1	$(1,\frac{-1}{2},-1,\frac{1}{2},\frac{1}{2},\frac{-1}{2},0)^t$	3

//

The main value of Corollary 5.2.2.8 is that it provides a strong necessary condition which must be satisfied if a generalised graph is to be the singleton-quotient of a singleton-regular graph.

5.2.3.1:Definition. Let H=(V,s,w) be a properly labelled generalised graph of order m with s(1)=1. For any m-tuple $\underline{v}$ we define $\underline{u}(\underline{v},H)=\underline{u}$ by $u_i=s(i)v_i$ for $i=1,2,\ldots,m$.

Suppose that for every simple eigenvalue λ of H with corresponding eigenvector $\underline{v}$, $(v_1^2 \sum_{i=1}^{m} s(i)/(\underline{u},\underline{v}))$ is a positive integer. Then H satisfies the <u>integrity condition</u>. //

Of course if H is to be the singleton-quotient of a singleton-regular graph then it must satisfy other more obvious conditions. For example the row-sum of A(H) must be constant.

<u>5.2.3.2:Definition</u>.Let C be a class of singleton-regular graphs. Then any condition other than integrity which must be satisfied by a generalised graph H if it is to be the singleton-quotient of a member of C is called a <u>simple feasibility condition with respect to C</u>. If H satisfies all known simple feasibility conditions with respect to C then H is said to be <u>simply feasible with respect to C</u>. If in addition it satisfies the integrity condition H is <u>strictly feasible with respect to C</u>. If there is a graph $G\varepsilon C$ such that H is the singleton-quotient of G, then H is said to be <u>realisable with respect to C</u>. //

In practice the simple feasibility conditions are often most easily expressed as conditions on the adjacency matrix of the generalised graph.

<u>5.2.4:Proposition</u>.Let H=(V,s,w) be a properly labelled generalised graph of order m and let A(H)=A. The following are simple feasibility conditions on H with respect to all singleton-regular graphs.

<u>5.2.4.1</u>: s(1)=1, and H is irreducible (Definition 2.4.3.3) with respect to vertex 1.

<u>5.2.4.2</u>: H is connected.

<u>5.2.4.3</u>: H is undirected,so that $a_{ij}s(i)=a_{ji}s(j)$.

<u>5.2.4.4</u>:The row-sum of A is a constant k independent of the

row chosen.

5.2.4.5: $k\sum_{i=1}^{m} s(i)$ is even.

5.2.4.6: $a_{ii}s(i)$ is even for all $i\varepsilon N_m$.

5.2.6.7: $a_{ii}<s(i)$ for all $i\varepsilon N_m$.

5.2.6.8: $a_{ij}\leq s(j)$ for all $i,j\varepsilon N_m$.

Proofs.

1. By definition.

2,3,4. Trivial.

5,6. A regular simple graph or subgraph of order r and valency s has rs even since rs/2 is the number of edges.

7,8. A simple graph does not have loops or multiple edges. //

It follows from 5.2.4.1-3 that

5.2.4.9:Corollary.Let H be a simply feasible properly labelled generalised graph with respect to singleton-regular graphs, with H=(V,s,w) and A(H)=H as above.Then $a_{11}=0$, $a_{1j}=0$ or $s(j)$, and every s(j) may be determined from A(H). //

In other words if we are given that the size of vertex 1 is 1, then it is only necessary to examine the adjacency matrix of a generalised graph to determine whether or not it is strictly feasible with respect to a class of singleton-regular graphs.

5.2.5.1:Definition.We define the decomposition rank of a singleton-regular graph to be the order of its singleton-quotient. //

5.2.5.2:Proposition.The only graphs of decomposition rank 2 are the complete graphs K_n for $n\varepsilon N$.

Proof.If H is to be simply feasible with respect to singleton-regular graphs of decomposition rank 2 then A(H) must have the form $\begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix}$. Since the row sums are equal y=x-1 and H is trivially the singleton-quotient of K_x. //

5.2.5.3:Proposition.Every graph of decomposition rank 3 is strongly-regular.

Proof.If H is simply feasible with respect to singleton-regular graphs of decomposition rank 3, then for a suitable labelling of H, A(H) has first row

i) $(0 \quad x \quad 0)$

or ii) $(0 \quad x \quad y)$.

In case i) A(H) has form $\begin{pmatrix} 0 & x & 0 \\ 1 & y & z \\ 0 & w & u \end{pmatrix}$

and the graph is singleton-regular.In case ii) A(H) has the form

$$\begin{pmatrix} 0 & x & y \\ 1 & x-1 & y \\ 1 & x & y-1 \end{pmatrix}$$

and H is the singleton-quotient of K_{x+y}, which has decomposition rank 2. //

5.2.5.4:Definition.Let G be a transitive graph.The rank of G is the rank of its automorphism group,that is the number of orbits of the group of automorphisms stabilising a vertex. //

5.2.5.5:Corollary (Hestenes (26)).Every transitive graph of rank 3 is strongly-regular.

Proof.If G is a transitive graph its decomposition rank is less than or equal to its rank.So the decomposition rank of a graph of rank 3 is ≤ 3. But the only graphs with decomposition rank <3 are K_n whose ranks are 2. //

In the next chapter we shall investigate the problem of feasibility conditions and the determination of all strictly feasible generalised graphs with respect to certain classes of trivalent graphs. We end this chapter with a brief survey of the feasibility conditions applicable to classes of distance-regular graphs.

5.2.6: Proposition. The following are simple feasibility conditions on a generalised graph $H=(V,s,w)$ with respect to distance-regularity.

5.2.6.1: H has a labelling with respect to which $A(H)$ is tridiagonal, with $s(1)=1$.

5.2.6.2: (Biggs (4)). When H is labelled so that $A(H)$ is tridiagonal the above-diagonal entries are monotonically decreasing and the below-diagonal entries are monotonically increasing.

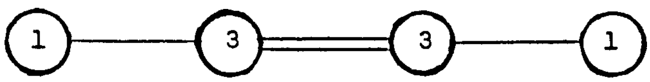
Proofs.

1. Trivial.

2. See Biggs (4,p135). //

5.2.6.3: Example. The following generalised graph H is realisable with respect to trivalent distance-regular graphs (it is the singleton-quotient of the cube).

H:

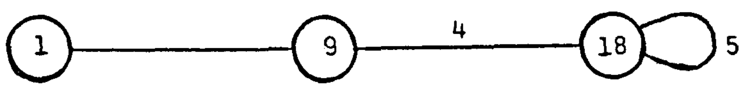


$$A(H)=\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$ //

In the case of distance-transitive graphs of given valency it is sometimes possible to put a bound on the diameter of the graph and thus on the order of its singleton-quotient. Biggs and Smith (5) demonstrate that a trivalent distance-transitive graph has diameter at most 15. This and a similar resùlt on distance-transitive graphs of valency 4 enable Smith to find all strictly feasible generalised graphs with respect to trivalent and tetravalent distance-transitive graphs, and hence to construct all such graphs (40). In (41) Smith shows that it is possible to bound the diameter of bipartite distance-transitive graphs of valency p+1, where p is a prime.

It is noteworthy that while it is easy to construct a simply feasible and not realisable generalised graph with respect to distance-regularity, it is rare for a strictly feasible generalised graph not to be realisable.

5.2.7:Example (Biggs (3)).



is strictly feasible but not realisable with respect to distance-regularity. //

Feasibility conditions are a very powerful tool in the study of distance-regular and in particular strongly-regular graphs. Among their applications are the study of

i) cages (see Biggs (4)),

ii) simple groups (for example Biggs (3) and Hestenes (26)),

iii) designs and partial geometries (for example Bose (6) and Seidel (38)).

6:THE CONSTRUCTION OF FEASIBLE GENERALISED GRAPHS

In this chapter we shall concern ourselves with the problem of constructing all strictly feasible generalised graphs with respect to three classes of trivalent singleton-regular graphs:

i) transitive graphs of fixed decomposition rank;

ii) t-arc-transitive graphs of fixed decomposition rank and arc-transitivity;

iii) symmetric graphs on a fixed number of vertices.

6.1:Necessary and Forbidden Subgraphs.

If a generalised graph H is simply feasible with respect to trivalent singleton-regular graphs, the value of k in 5.2.4.4, the row sum of A(H), is 3. This fact enables us to make some simple observations on the sizes of the vertices of H.

6.1.1:Proposition. Let $H=(V,s,w)$ be a simply feasible generalised graph with respect to trivalent singleton-regular graphs, and suppose that vertex 1 is not adjacent to a vertex of size 3.

6.1.1.1:Each vertex of H has size 2^i for some $i\varepsilon Z^+$.

6.1.1.2:If there is a vertex of size 2^k with $k\varepsilon N$, then there is a vertex of size 2^{k-1}.

Proof. Consider $x\varepsilon V$ and a path from vertex 1 to x on vertices $1,y_1,y_2,\ldots,y_r,x$. Then $s(y_1)\leqq 2$, $\frac{1}{2}s(y_{i-1})\leqq s(y_i)\leqq 2s(y_{i-1})$ for $i=2,3,\ldots,r$ and $\frac{1}{2}s(y_r)\leqq s(x)\leqq 2s(y_r)$. //

6.1.2:Proposition. Let H be as above, but suppose that vertex 1 is adjacent to a vertex of size 3.

6.1.2.1:Each vertex has size $3^j\times 2^i$ for $j=0$ or 1 and for some $i\varepsilon Z^+$.

6.1.2.2:Every vertex of size 2^i with $i\varepsilon Z^+$ has valency 1 and is adjacent to a vertex of size 3×2^{i} or $3\times2^{i-1}$.

6.1.2.3: If there is a vertex of size 3×2^{k} with $k\varepsilon N$, then there is a vertex of size $3\times2^{k-1}$.

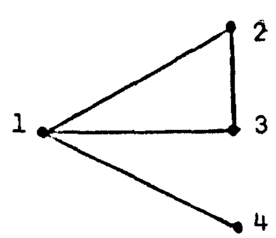
Proof. As in 6.1.1 consider a path from 1 to $x\varepsilon V$ on vertices $1,y_1,y_2,\ldots,y_r,x$. Then $s(y_1)=3$, $\frac{1}{2}s(y_{i-1})\leq s(y_i)\leq 2s(y_{i-1})$ for $i=2,3,\ldots,r$ and $s(y_r)/3\leq s(x)\leq 2s(y_r)$. If $s(x)=s(y_r)/3$ or $2s(y_r)/3$ then x has valency 1. //

Consider a properly labelled transitive trivalent graph G. If $N(G,1)\varepsilon S(1)$ (as is the case when G is symmetric), then certain edge-subgraphs can be excluded in general, so that further simple feasibility conditions can be stated in addition to those of Proposition 6.1.2.

6.1.3.1; Notation. In Section 6.1.3 we shall consider a properly labelled trivalent transitive graph $G=(V,s,w)$ with $N(1)\varepsilon S(1)$. //

6.1.3.2: Proposition. Suppose that G contains a triangle as an edge-subgraph. Then G is K_4.

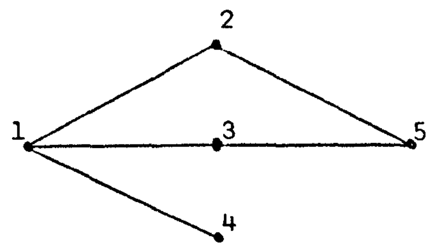
Proof. G may be taken to have edge-subgraph



But $\{2,3,4\}\varepsilon S(1)$, so that the vertex-subgraph $<\{2,3,4\}>$ is regular. It must have valency 2 and hence $G=K_4$. //

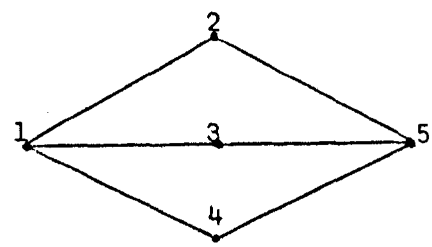
6.1.3.3: Proposition. Suppose that G contains a square as an edge-subgraph. Then G is K_4, $K_{3,3}$ or the cube Q_3.

Proof. K_4 contains a square. If G is not K_4 then it may be taken to have the edge-subgraph

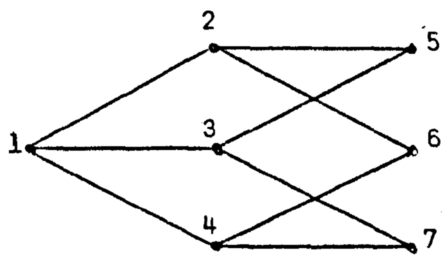


Then since $\{2,3,4\}\varepsilon S(1)$, G has (up to isomorphism) one of the edge-subgraphs

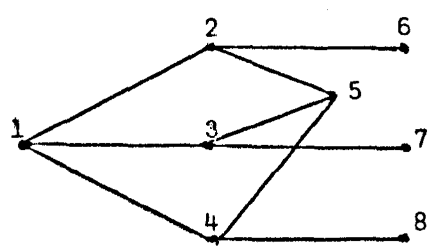
i)



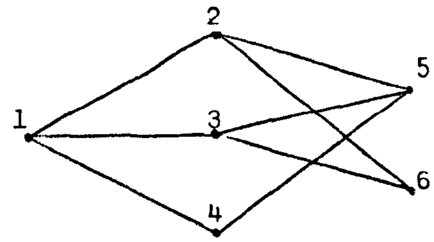
or ii)



In case i) vertex 2 is adjacent to another vertex,say 6. Suppose neither 3 nor 4 is adjacent to 6.Then G has edge-subgraph

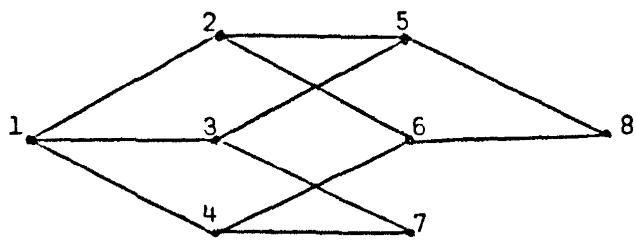


Vertex 1 is on three squares whereas vertex 2 is only on two, so that G cannot be transitive and we have a contradiction.So we may suppose without loss of generality that vertex 3 is adjacent to 6.



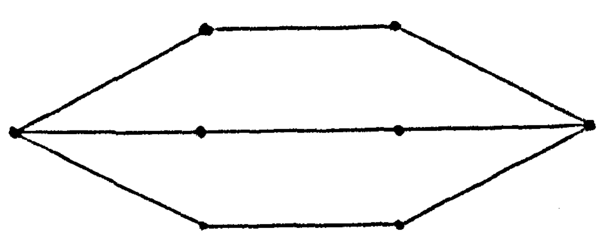
Consider the class of $S(1)$ containing vertex 6. Every member is adjacent to at least two of 2,3,4 and since G is trivalent the class contains no new vertices. Hence 6 adj 4 and $G=K_{3,3}$.

In case ii) every 2-arc beginning at vertex 1 can be extended to become a square in exactly one way. Since G is transitive this is true of every 2-arc in G. Consider the 2-arc (5,2,6). Then 5 and 6 must have a common neighbour other than 2, say vertex 8.



Similarly vertices 6 and 7 share a neighbour other than 4. But since 6 is already adjacent to three vertices this common neighbour must be 8 and G is Q_3. //

<u>6.1.3.4:Proposition</u>. Suppose G contains the edge-subgraph

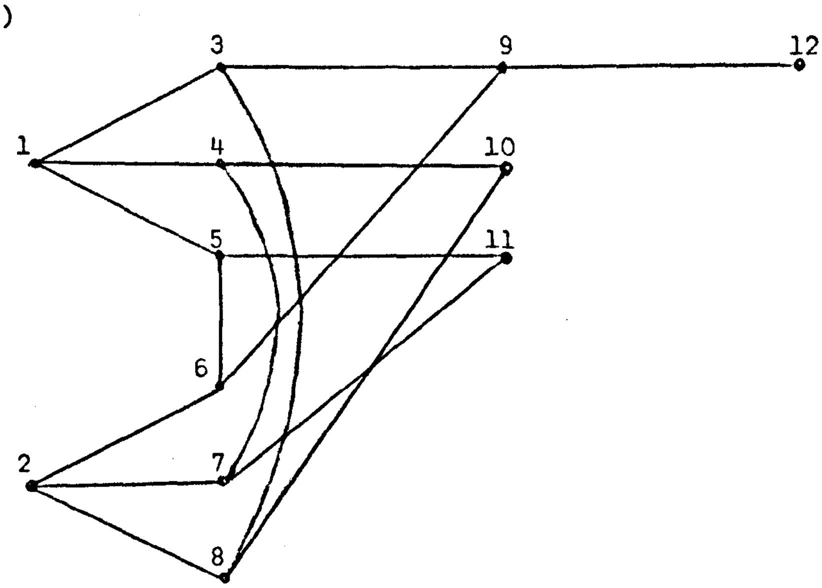


Then G is Q_3 or P(8,3) or Heawood's graph (21,p173).

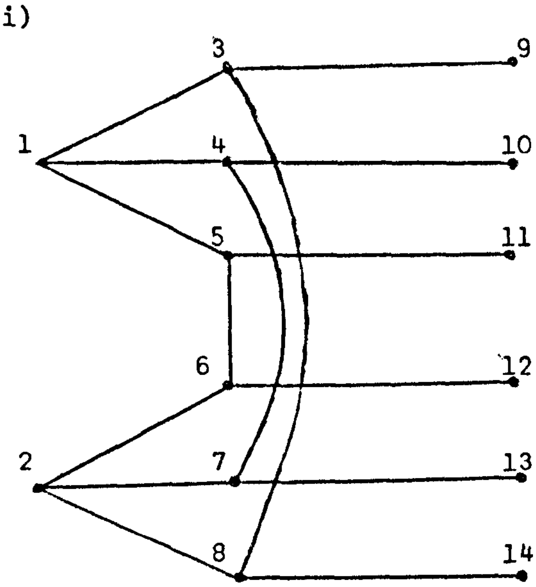
<u>Proof</u>. G has at least 8 vertices so that it is not K_4 or $K_{3,3}$. Q_3 contains the subgraph. Hence we need not consider graphs

containing triangles or squares.The possibilities are these (up to isomorphism):

i)



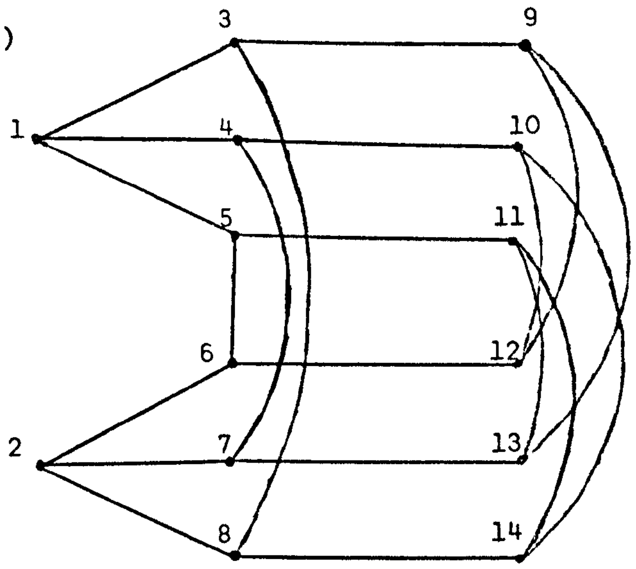
and ii)



Consider in case i) the 2-arcs beginning at vertex 3. Since G is transitive there is $a\varepsilon\{6,12\}$, $b\varepsilon\{4,5\}$ and $c\varepsilon\{2,10\}$ such that a,b and c have a common neighbour and no two of a,b and c are adjacent. 6 and 4 do not share a neighbour and 6 adj 5. So a=12. 2 and 12 have no common neighbour so c=10. But 10 adj 4 and 10 does not share a neighbour with 5, giving a contradiction. Hence G cannot have the edge-subgraph given in i).

In case ii) we must consider six distinct further possible developments:

a)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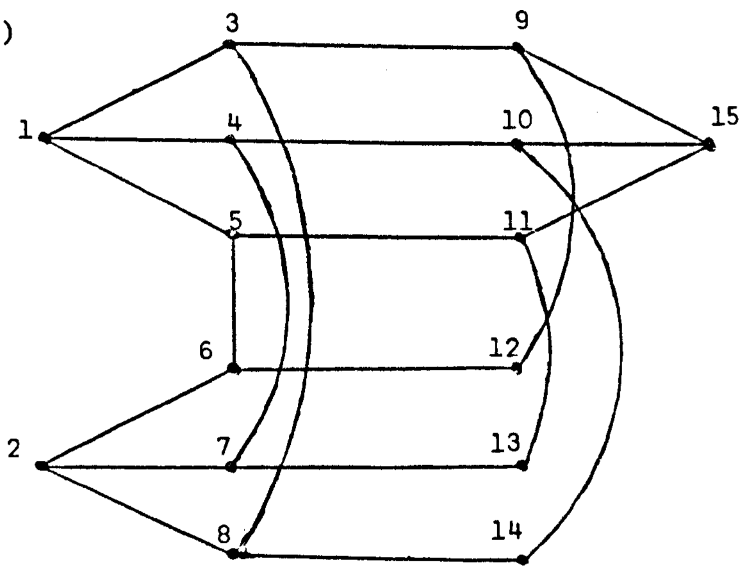
b)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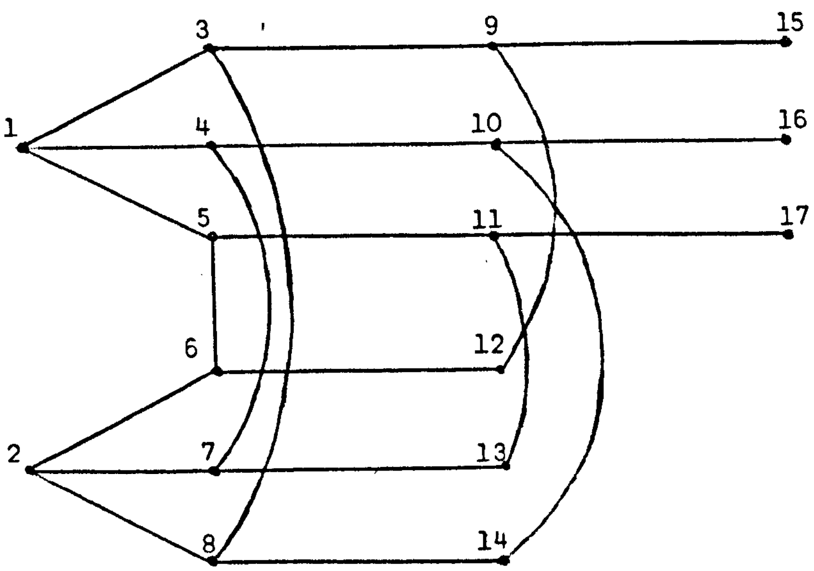
c)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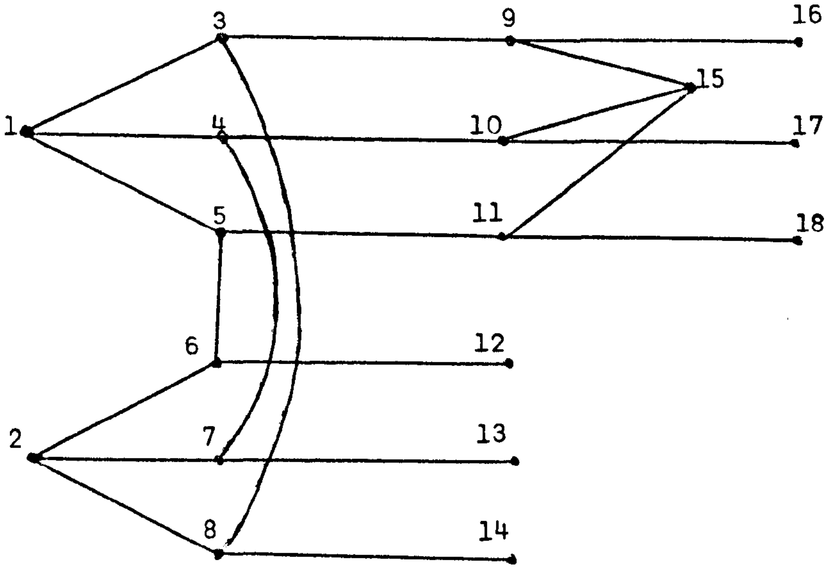
d)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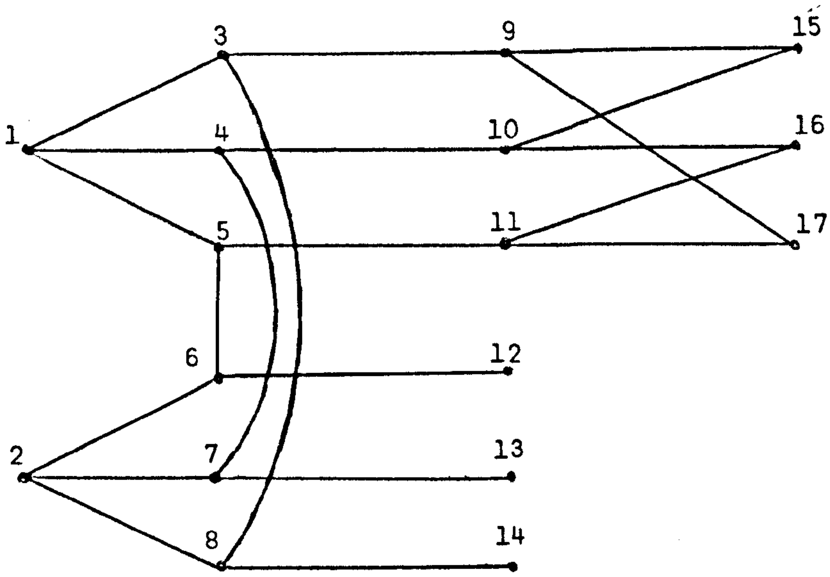
e)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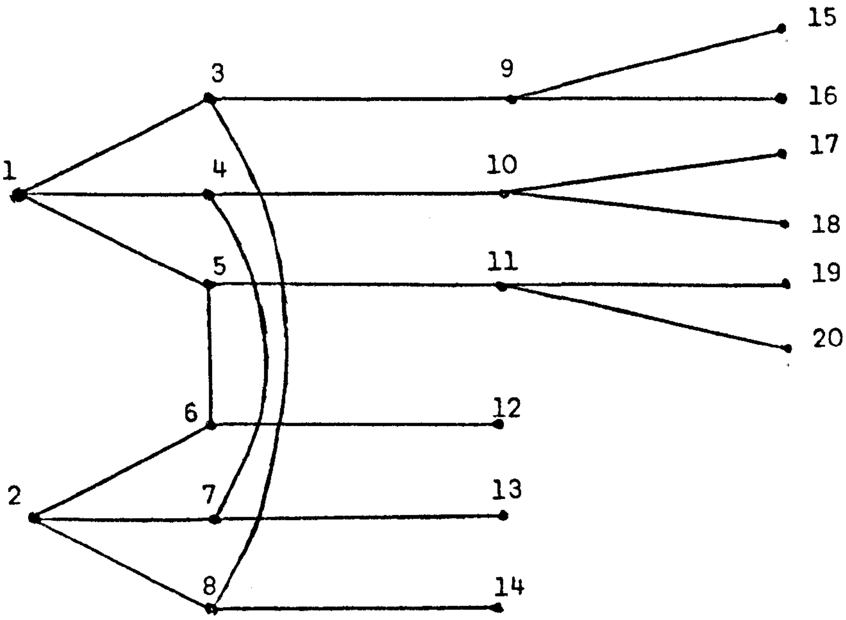
f)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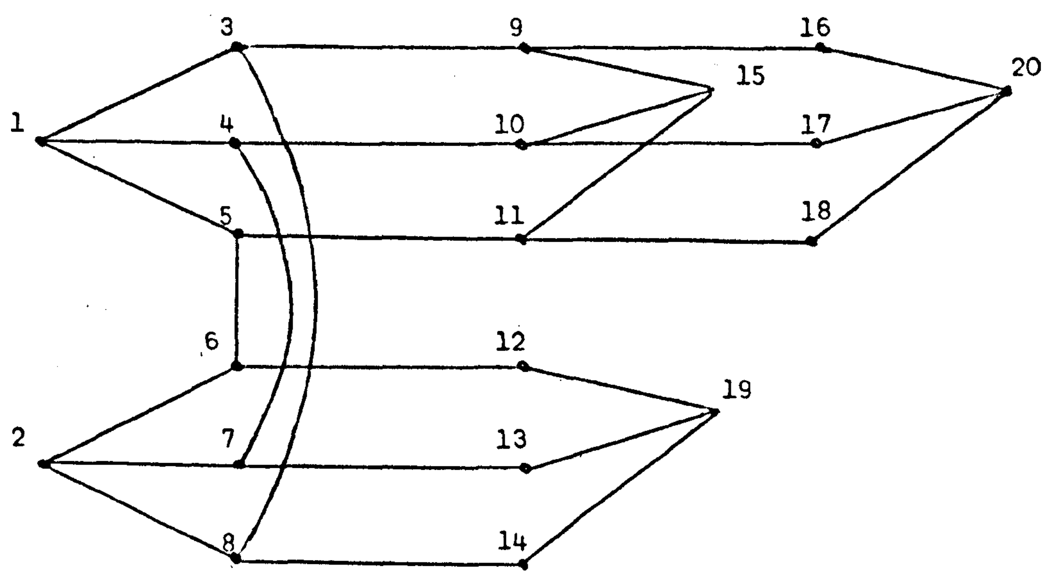
We consider the possibilities in turn.

a) The construction is complete and G is Heawood's graph.

b) The ends of the 2-arcs beginning at vertex 1 may be partitioned into two sets of three vertices {6,7,8} and {9,10,11} such that the three vertices in each set have a common neighbour. Hence so may the ends of those originating at vertex 2. Thus 12,13 and 14 have a common neighbour, 16 say.We have constructed P(8,3).

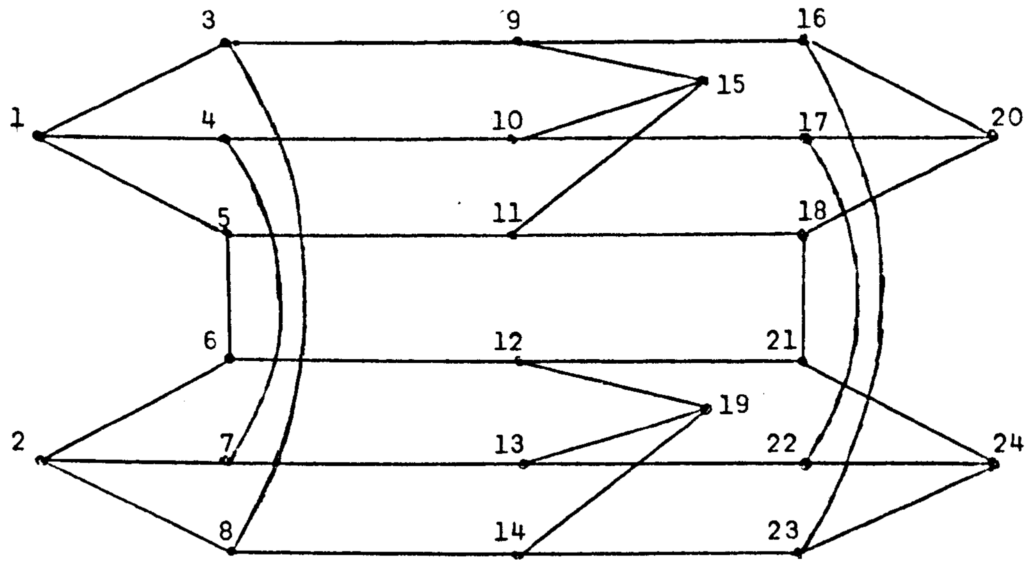
c) Consider 2-arcs beginning at vertex 3. There must be $a\epsilon\{12,15\}$, $b\epsilon\{2,14\}$ and $c\epsilon\{4,5\}$ such that a,b,c have a common neighbour and no two are adjacent.By inspection a=12, b=2, c=5 and their shared neighbour is 6. Now consider the vertices whose distance from 3 is two and which do not have a common neighbour, 4,14 and 15, and those whose distance from 6 is two which are not adjacent to vertex 3, that is 7,11 and x where x>14. Then since G is transitive each of 7,11 and x must be adjacent to just one of 4,14 and 15. But 11 is not adjacent to any of them, we have a contradiction and G cannot contain the edge-subgraph c).

d) As in case b) 12,13 and 14 must have a common neighbour, 19 say, and similarly so must 16,17 and 18, say 20. So we have



Every 2-arc originating at vertex 1 extends into a hexagon and since G is transitive,every 2-arc in G must do so.The shortest circuit through vertex 1 is a hexagon.Thus since G is transitive, its girth is six,and vertex 18 is not adjacent to vertex 12. If 18 adj 13 or 14, then the arc (12,6,5) does not extend into a hexagon.Hence 16,17 and 18 are not adjacent to any of 12,13 and 14, and so they are adjacent to three new vertices 21,22,23 say, which again must have a common neighbour,24 say.

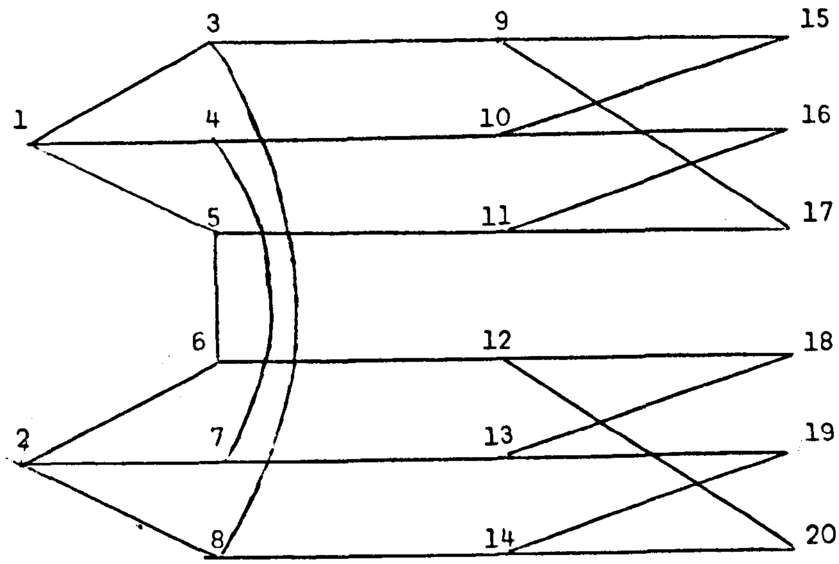
Finally in order that 2-arcs (12,6,5),(13,7,4) and (14,8,3) may be extended into hexagons,we must have 18 adj 21, 17 adj 22 and 16 adj 23. The resulting graph is



Considering vertices whose distance from vertex 3 is two we find that they do not form two sets of three vertices,each set being the neighbourhood of some vertex of G. So G is not transitive and we have a contradiction.

e) Using a similar argument to that in case b) we may assert that vertices 12,13 and 14 are adjacent to three vertices in such a way that any two of 12,13 and 14 share a neighbour.Since the graph is trivalent these vertices must be new.Take them to be 18,19 and 20 with 18 adjacent to 12 and 13, 19 adjacent to

13 and 14, and 20 adjacent to 12 and 14. We have



As in c) we consider the ends of 2-arcs beginning at vertex 3. Then there must be $a\varepsilon\{4,5\}$, $b\varepsilon\{15,17\}$, $c\varepsilon\{2,14\}$ such that a,b and c have a common neighbour. Now neither 15 nor 17 shares a neighbour with 2. Hence c=14. But neither 4 nor 5 shares a neighbour with 14 and we have a contradiction.

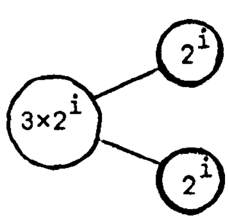
f) Again considering the ends of 2-arcs beginning at vertex 3 we find that there must be $a\varepsilon\{4,5\}$, $b\varepsilon\{2,14\}$, $c\varepsilon\{15,16\}$ such that a,b and c have a common neighbour. But neither 15 nor 16 shares a neighbour with 4 or 5 and so we have a contradiction.

//

As a result of these propositions we can specify some forbidden subgraphs of simply feasible generalised graphs.

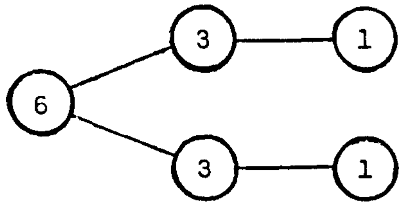
6.1.4:Proposition. Let H be a simply feasible properly labelled generalised graph with respect to transitive trivalent graphs. H does not contain the following edge-subgraphs:

6.1.4.1:



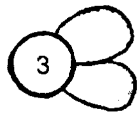
for $i\varepsilon N$;

<u>6.1.4.2</u>:

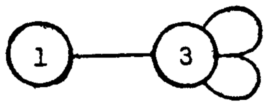


unless one of the vertices of size 1 is vertex 1;

<u>6.1.4.3</u>:

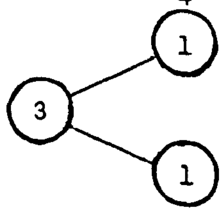


unless H is

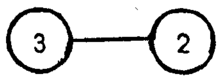


the singleton-quotient of K_4;

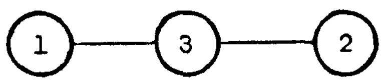
<u>6.1.4.4</u>:



<u>6.1.4.5</u>:

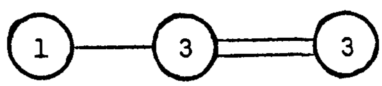


unless H is



the singleton-quotient of $K_{3,3}$;

<u>6.1.4.6</u>:



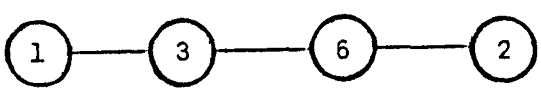
unless H is

the singleton-quotient of Q_3;

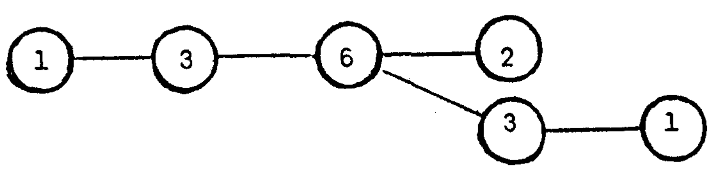
<u>6.1.4.7</u>:

unless H is

<u>6.1.4.8</u>:



unless H is



the singleton-quotient of P(8,3).

Proof.If vertex 1 of H is not adjacent to a vertex of size 3, none of the subgraphs occur by 6.1.1.1. Otherwise the proofs are as follows:

1,2. H is irreducible with respect to vertex 1;

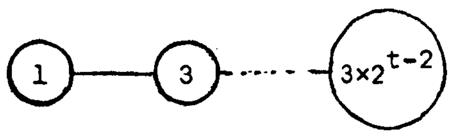
3. By 6.1.3.2;

4,5,6. By 6.1.3.3;

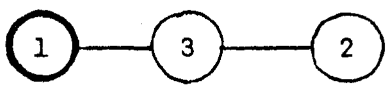
7,8. By 6.1.3.4. //

If H is to be simply feasible with respect to symmetric trivalent graphs,then it is a trivial observation that vertex 1 must be adjacent to a vertex of size 3. A symmetric graph is at least 1-arc transitive.For t-arc-transitive graphs with $t>1$ we can go further.Note that it is well known that for trivalent graphs $t\leq 5$ (see for example Biggs (4)) so we need only deal with $2\leq t\leq 5$.

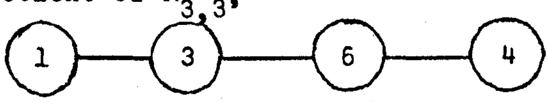
6.1.5:Proposition.Let H be a simply feasible properly labelled generalised graph with respect to trivalent t-arc-transitive graphs for some t, $2\leq t\leq 5$. Then H contains the edge-subgraph



or H is one of the following:

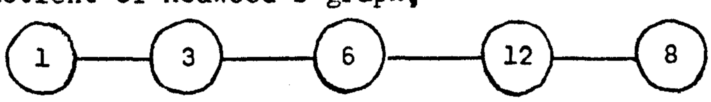


the singleton-quotient of $K_{3,3}$,



the singleton-quotient of Heawood's graph,

or

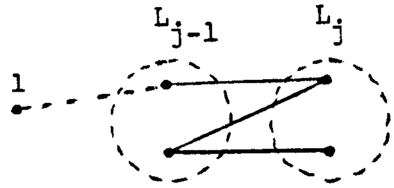


the singleton-quotient of Tutte's graph (42).

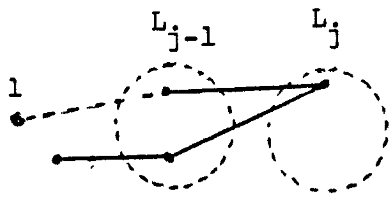
Proof.Let G=(V,s,w) be a properly labelled trivalent t-arc-transitive graph of diameter d for some $d\in N$ and $2\leq t\leq 5$, let $L_i\subset V$ denote the set $\{x,\ d(1,x)=i\}$ for $0\leq i\leq d$ and let min(a,b) with $a,b\in R$ denote the lesser of a and b.

Clearly $L_i \in S(1)$ for $i \leq \min(d,t)$. Suppose that there is a j with $2 \leq j \leq \min(d,t)$ and $|L_j| < 2|L_{j-1}|$ and choose the least such. Then every vertex of L_j has at least two neighbours in L_{j-1} and every vertex in L_{j-1} has at least two neighbours in L_j. So there exist $(j+2)$-arcs of the forms

i)



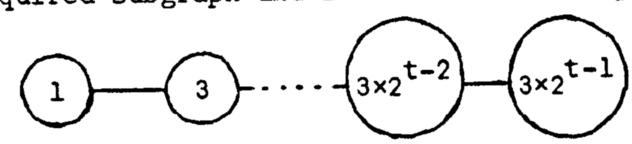
and ii)



Hence $t < j+2$, that is $j \geq t-1$.

Suppose $j = t-1$. Then every vertex of L_j has three neighbours in L_{j-1} so that $d = t-1$ and G is a cage of even girth g where the number of vertices attains the lower bound $2(2^{g/2}-1)$. These graphs are well-known to exist for $t=3,4,5$, being $K_{3,3}$, Heawood's graph and Tutte's graph respectively (Tutte (42)).

Otherwise, if $j=t$ then the singleton-quotient of G has the required subgraph. If there is no j then the singleton-quotient of G has the required subgraph and indeed has the subgraph



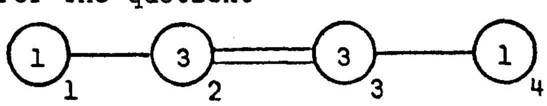
. //

6.2: Algorithms for the Construction of Strictly Feasible Generalised Graphs.

We wish to construct every strictly feasible generalised graph with respect to the classes given at the beginning of this chapter, and since the construction will have to be carried out on a computer we must use a suitable representation of the generalised graphs in question. As we are dealing only with quotients of trivalent graphs we may represent a singleton-quotient of order m by two arrays, a vector of order m, SIZE, containing the sizes of the vertices (whose first entry is of course always 1), and an $m \times 3$ matrix, GRM, from which the adjacency matrix of the quotient may be constructed:-

6.2.1.1: Definition. Let H be a properly labelled simply feasible generalised graph of order m with respect to trivalent singleton-regular graphs. A graph representation matrix, GRM, of H is a matrix of size $m \times 3$ with the property that the number of entries of value j in row i is equal to the ij-th entry of A(H) for $i, j \in N_m$. //

6.2.1.2: Example. For the quotient



the arrays may be

$$\text{SIZE} \quad \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} \qquad\qquad \text{GRM} \quad \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 3 \end{pmatrix}$$ //

Notes.

6.2.1.3: If we further impose the condition that the entries of GRM must be in ascending order in each row then GRM is uniquely defined.However this would introduce practical complications into the construction of the graph representation matrices.

6.2.1.4:If we are given GRM the array SIZE is strictly unnecessary by the arguments put forward in Corollary 5.2.4.9. However it saves considerable computation to store the sizes of the vertices as well as GRM,and we shall see later that in the case of symmetric graphs of fixed order,SIZE will be determined completely before the application of the construction algorithm for GRM. //

The algorithm used is essentially the same for all the classes of graphs that we investigate.We shall give an outline applicable to all these classes and then deal with the necessary specialisations. An example of the computer program used in one case is given in Appendix 1.

6.2.2.1:Notation.A stack is a list with the property that items may only be added to or removed from its end.Items are pushed onto or popped from the stack (see for example (1) for a fuller description).

In the construction algorithm we shall use a series of stacks, one for each value of a non-negative integer variable STEP called STACK(STEP),and the items in the stacks will be $m \times 4$ arrays holding the entries of GRM and SIZE so far determined at each step,which we shall call GRM/SIZE. //

6.2.2.2:Algorithm.

Stage 1.Set the order of the generalised graph to be constructed, the initial value of GRM/SIZE according to the class of graph being considered,set STEP=1 and push GRM/SIZE onto STACK(STEP).

Stage 2.If STEP=0 then stop.If there are no entries in STACK(STEP) reduce STEP by 1 and restart Stage 2. Otherwise pop GRM/SIZE from STACK(STEP).

Stage 3.If there is no vertex with the properties that

i) it is already incident with an edge

and ii) another edge may be added incident with it,

then either the construction of a graph representation matrix is complete in which case go to Stage 4, or the generalised graph being constructed cannot be connected in which case return to Stage 2. Otherwise choose the first such vertex and go to Stage 5.

Stage 4.Check for forbidden subgraphs.If there are any go to Stage 2. Otherwise apply Algorithm 2.4.1.3 to find the singleton-quotient $S(1)$ of the generalised graph constructed.If it is reducible go to Stage 2. If not,the canonical labelling of the classes of $S(1)$ is a canonical labelling of the generalised graph and from it we may derive a canonical form of GRM.Check whether this canonical form of GRM has already been stored.If so return to Stage 2. Otherwise store it and test the adjacency matrix of the generalised graph for the integrity condition.If this is satisfied output GRM/SIZE.Return to Stage 2.

Stage 5.Increase STEP by 1 and construct all possible GRM/SIZE arrays corresponding to the addition of one edge incident with the vertex chosen (avoiding obvious isomorphisms and reducible constructions) - this operation may involve assigning a size to a vertex whose size was previously undetermined.Store these arrays in STACK(STEP).Return to Stage 2. //

We first apply this algorithm for the class of transitive trivalent graphs.

6.2.3:Algorithm.To construct all strictly feasible generalised graphs with respect to all transitive trivalent graphs of fixed decomposition rank,m.

Apply Algorithm 6.2.2.2 in three cases.

i) Set the initial value of GRM/SIZE thus:-The first row of GRM has entries 2,3,4, the next three rows have first entry 1 and all the other entries of GRM are undetermined.The first four entries of SIZE are 1 and the other entries are undetermined.

ii) Set the initial value thus:-The first row of GRM has entries 2,2,3, the next two rows have first entry 1 and the other entries are undetermined.The first three entries of SIZE are 1,2,1; the rest are undetermined.

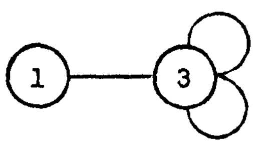
iii) Set the initial values thus:-The first row of GRM is 2,2,2, the next row has first entry 1 and the other entries are undetermined.The first two entries of SIZE are 1 and 3; the remainder are again undetermined.Exclude forbidden subgraphs given in 6.1.3.2-4. //

The application of this algorithm for m≦9 gives the next proposition.

<u>6.2.4:Proposition</u>.The only strictly feasible generalised graphs with respect to transitive trivalent graphs of decomposition rank≦9 are the following:

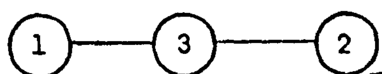
<u>m=2:</u>

<u>6.2.4.1:</u>

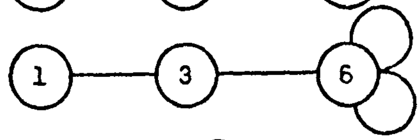


<u>m=3:</u>

<u>6.2.4.2:</u>

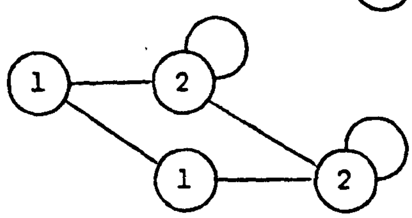


<u>6.2.4.3:</u>

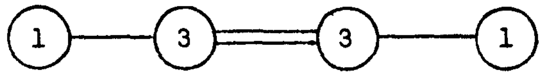


<u>m=4:</u>

<u>6.2.4.4:</u>



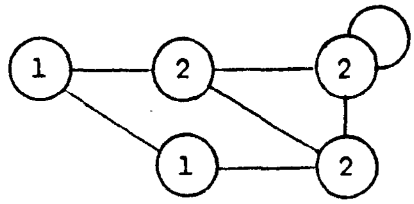
6.2.4.5:



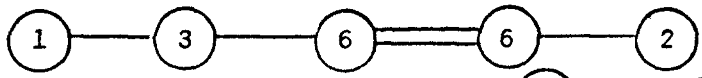
6.2.4.6:

m=5:

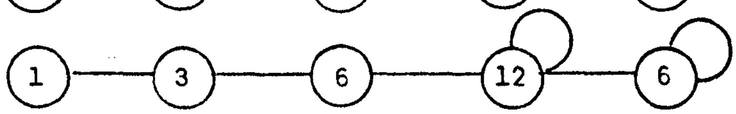
6.2.4.7:



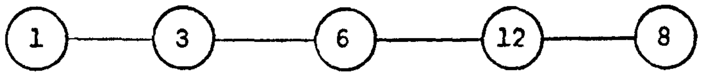
6.2.4.8:



6.2.4.9:

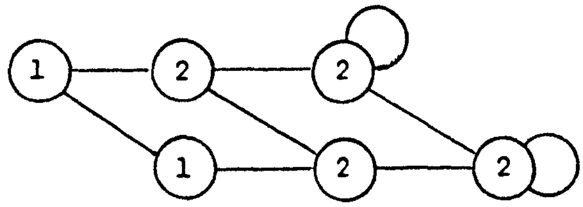


6.2.4.10:



m=6:

6.2.4.11:



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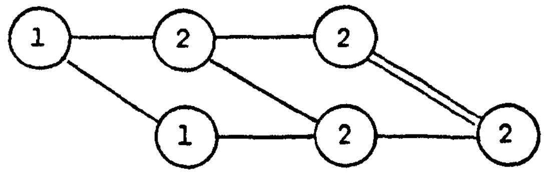

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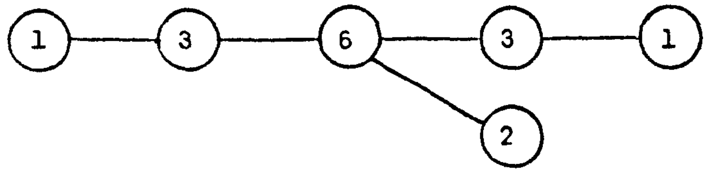
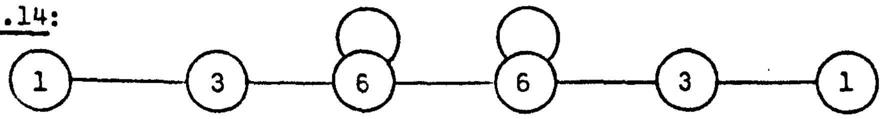


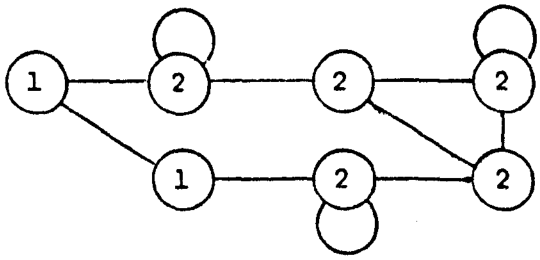
6.2.4.14:



6.2.4.15:

m=7:

6.2.4.16:



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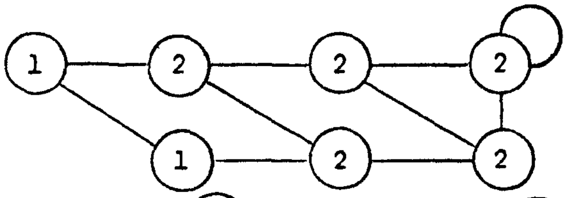

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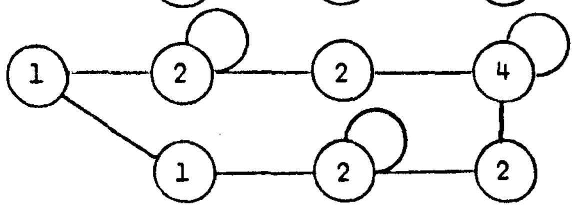

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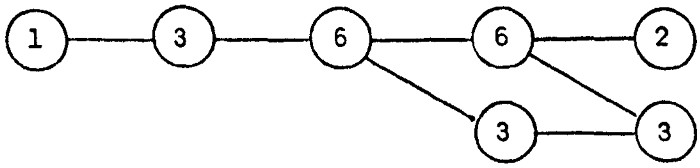

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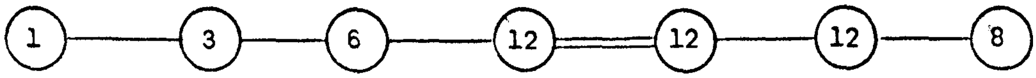

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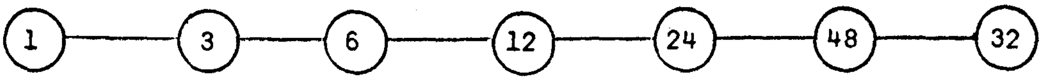
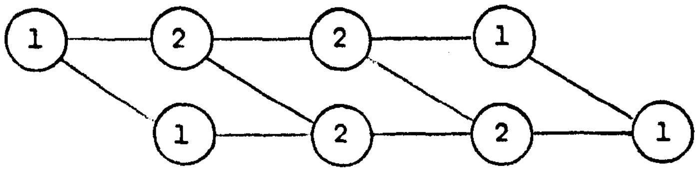


m=8:

6.2.4.22:



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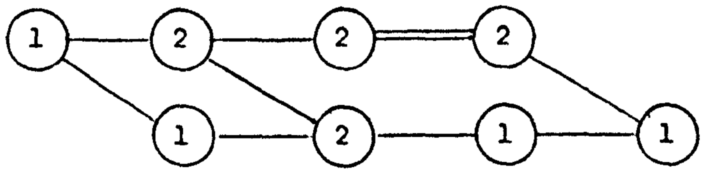

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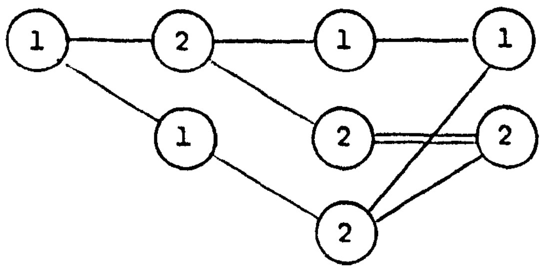

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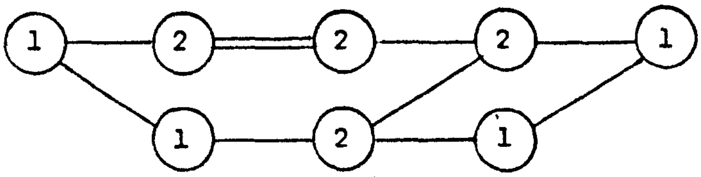

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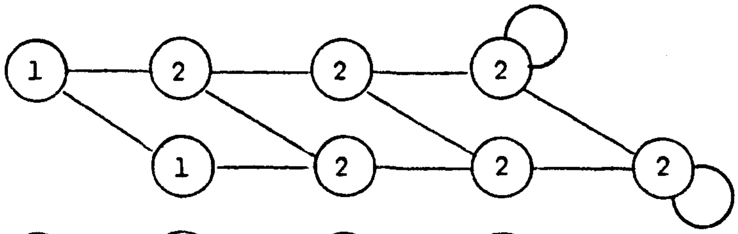

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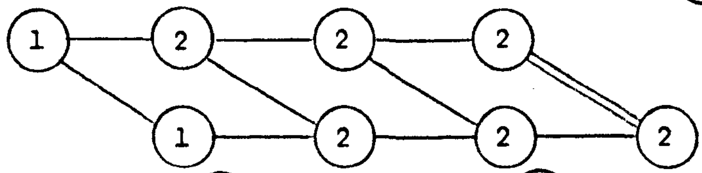

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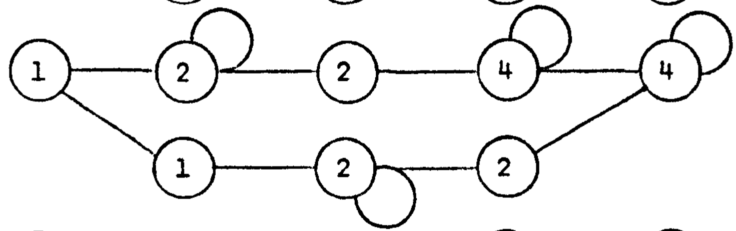

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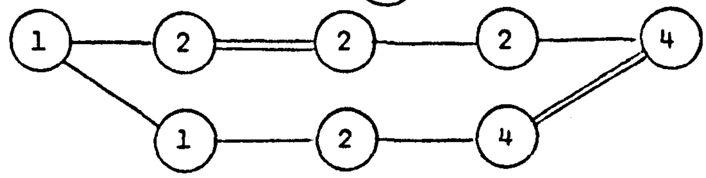

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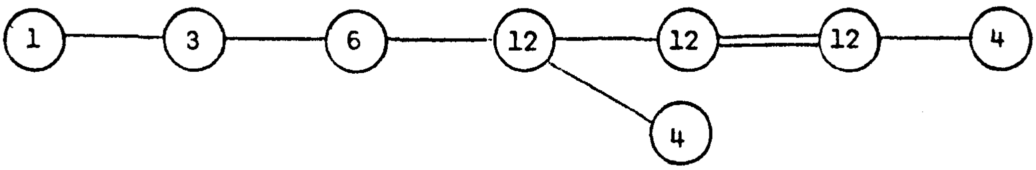

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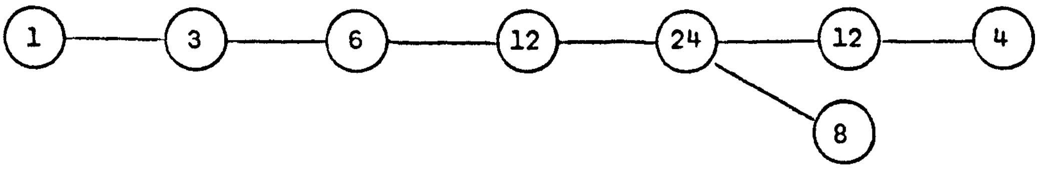

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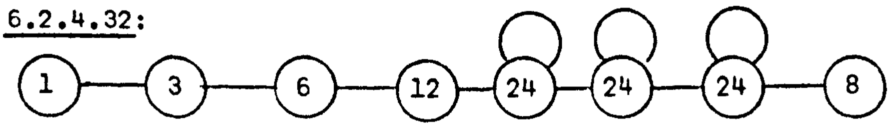

m=9:

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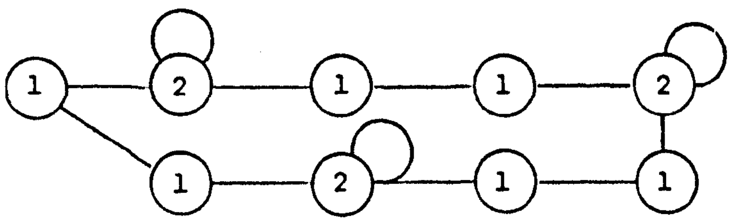

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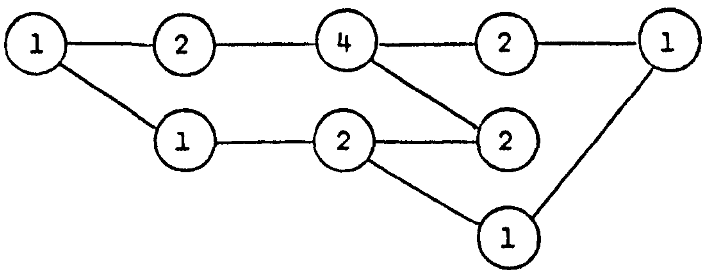

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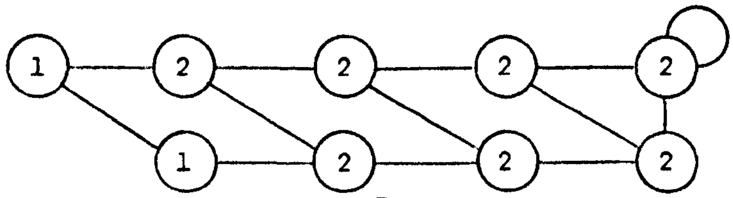

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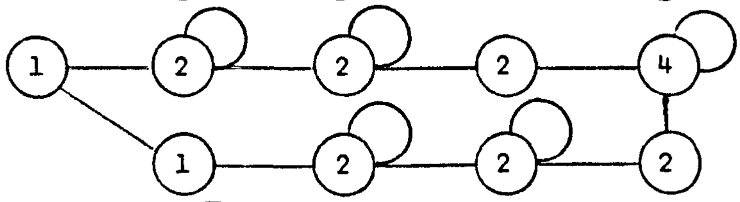

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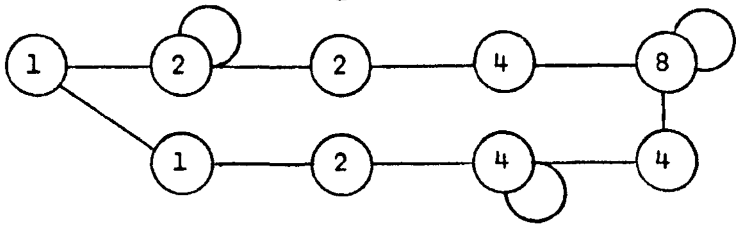

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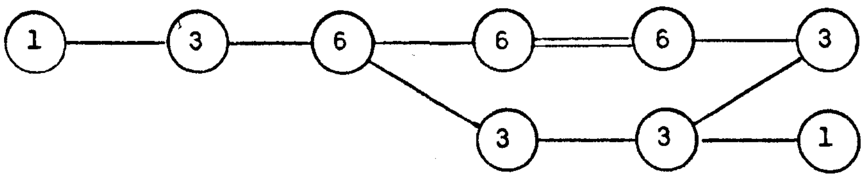

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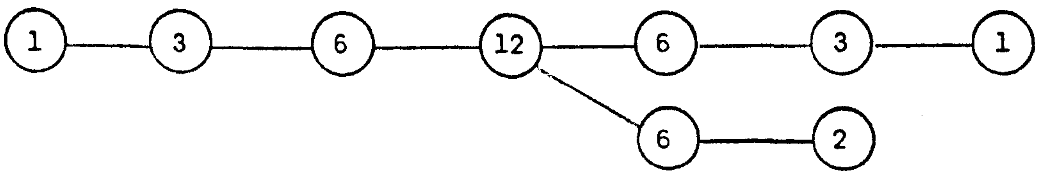

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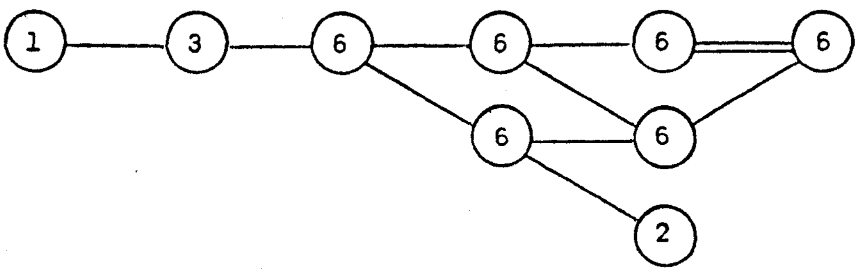

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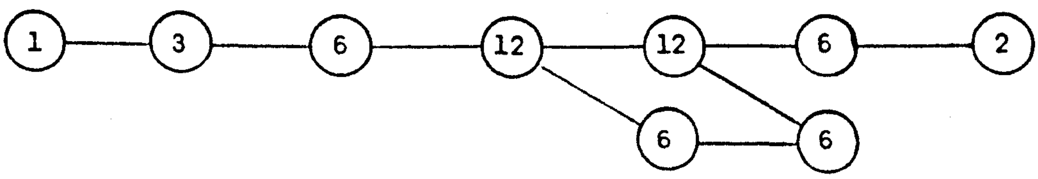

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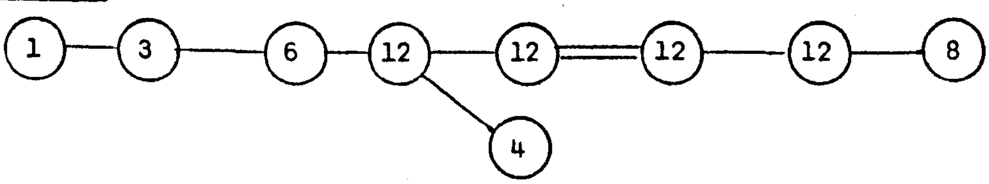

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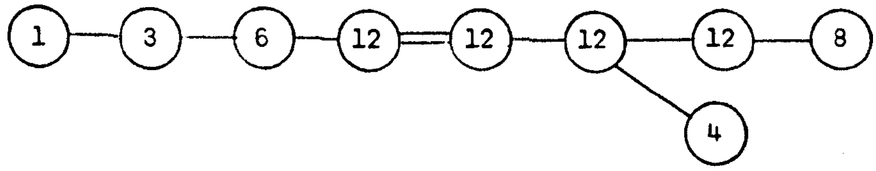
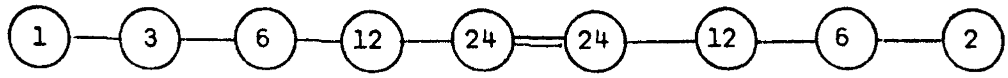


6.2.4.44:



//

Secondly we consider t-arc-transitive trivalent graphs for given t ($1 \leq t \leq 5$) and given decomposition rank m. With the exceptions given their singleton-quotients have the edge-subgraphs specified in Proposition 6.1.5 and the forbidden subgraphs given by Propositions 6.1.3.2-4 are excluded.Note that a generalised graph which is strictly feasible with respect to t-arc-transitive graphs with t=i is also strictly feasible for graphs with t=j when j<i. The application of Algorithm 6.2.2.2 with the above constraints yields the next proposition.

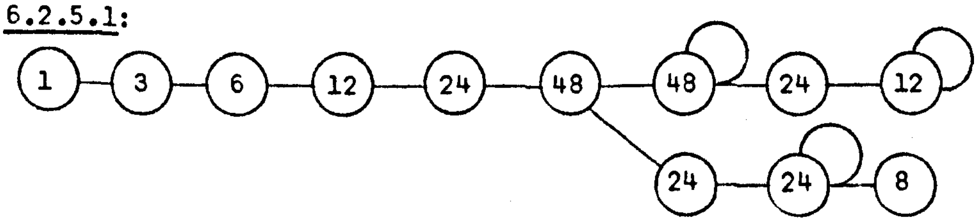
6.2.5:Proposition.*The only strictly feasible generalised graphs with respect to t-arc-transitive trivalent graphs of decomposition rank m for $1 \leq t \leq 5$ and $1 \leq m \leq t+8$ are the following:*

i) Generalised graphs strictly feasible for t=5:

m=5: 6.2.4.10. m=7: 6.2.4.21. m=9: 6.2.4.44.

m=12:

6.2.5.1:

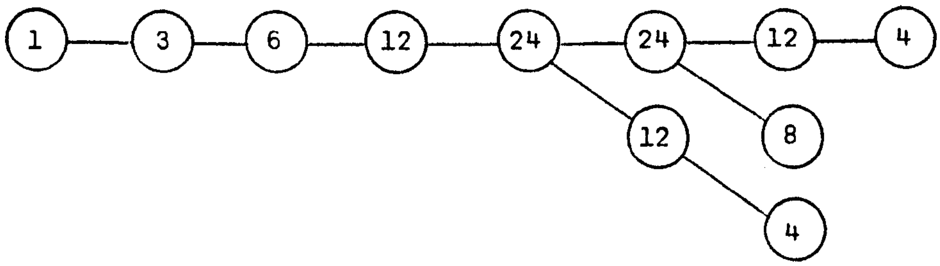


ii) Generalised graphs strictly feasible for t=4 but not for t=5:

m=4: 6.2.4.6. m=7: 6.2.4.20. m=8: 6.2.4.31,32. m=9: 6.2.4.43.

m=11:

6.2.5.2:



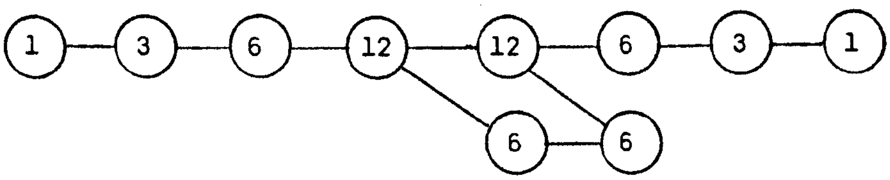
iii) Generalised graphs strictly feasible for t=3 but not for t=4:

m=3: 6.2.4.2,3. m=5: 6.2.4.8,9. m=6: 6.2.4.15. m=8: 6.2.4.30.

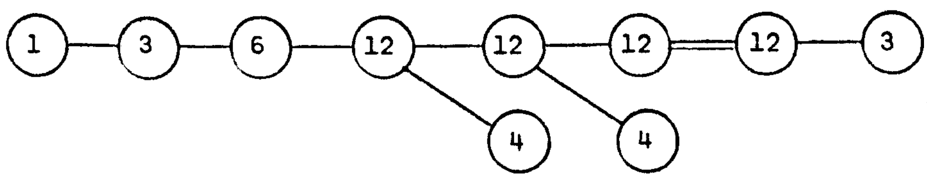
m=9: 6.2.4.39,41,42.

m=10:

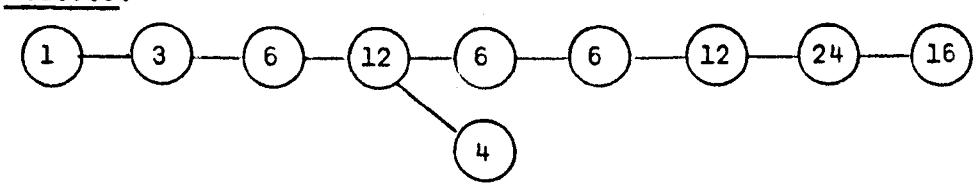
6.2.5.3:



6.2.5.4:

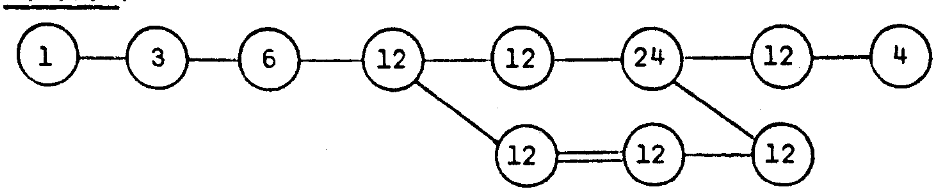


6.2.5.5:



m=11:

6.2.5.6:



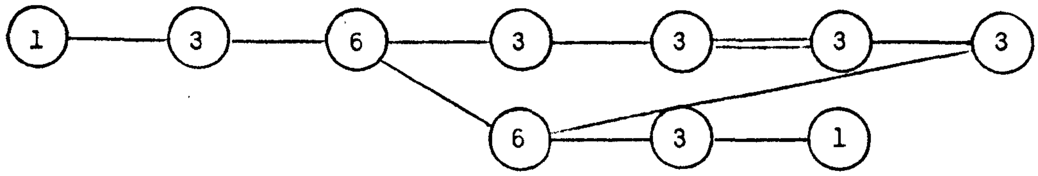
iv) Generalised graphs strictly feasible for t=2 but not for t=3:

m=2: 6.2.4.1. m=4: 6.2.4.5. m=6: 6.2.4.13,14. m=7: 6.2.4.19.

m=9: 6.2.4.38,40.

m=10:

6.2.5.7:



v) Generalised graphs strictly feasible for t=1 but not t=2:

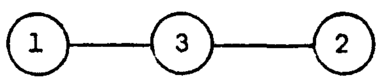
None for $m \leq 9$. //

Finally we turn our attention to symmetric trivalent graphs of fixed order.The first step is to calculate all possible values of SIZE for singleton-quotients of members of this class when the order is given.Clearly the results of 6.1.2 are applicable in this case.

6.2.6.1:Notation.Let $G=(V,s,w)$ be a generalised graph.Then $n(a)$ denotes the number of vertices $x \in V$ such that $s(x)=a$. //

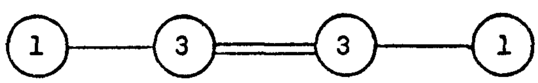
Proposition.Let H be a simply feasible generalised graph with respect to symmetric trivalent graphs.Then the following conditions hold on H:

6.2.6.2: $n(6) \geq n(2)$ unless H is



6.2.6.3: $n(3) \geq n(1)$.

6.2.6.4:Unless H is



the following holds:-

$3n(3)+n(6)+1 \geq 4n(1)$ and if H has vertices of sizes 1 and 3 only then $3n(3) \geq 4n(1)$.

6.2.6.5:If H has no vertices of size 12 then $n(4)+n(2) \leq n(6)$.

Proof.

2. By 6.1.2.2 a vertex of size 2 is adjacent to a vertex of size 3 or 6. But by 6.1.3.3 no vertex of size 2 is adjacent to a vertex of size 3 unless H is the singleton-quotient of $K_{3,3}$. Further by 6.1.4.1 only one vertex of size 2 may be adjacent to any given vertex of size 6.

3. This result follows from 6.1.2.2 and 6.1.4.4.

4. Every vertex of size 1 is adjacent to a vertex of size 3, and so is part of a vertex-subgraph of the form

J: 

Unless H is the singleton-quotient of $K_{3,3}$, for which the proposition is true,the vertex of size 3 is adjacent to a vertex of size 3 or 6. If H has the edge-subgraph

then by 6.1.3.4 H is the singleton-quotient of the cube,so that except in this case subgraphs isomorphic to J cannot be attached to each other.There are at most three copies of J attached to a vertex of size 3 and at most one attached to a vertex of size 6 unless one of the vertices of size 1 is vertex 1 in which case there may be two,by 6.1.4.2. Hence

$3(n(3)-n(1))+n(6)+1\geq n(1)$ and if there are vertices of sizes 1 and 3 only then $3(n(3)-n(1))\geq n(1)$.

5. This follows from 6.1.2.2 and 6.1.4.1. //

Using this proposition and Proposition 6.1.2 it is relatively easy to work out the possible values of SIZE corresponding to a given order.

6.2.7.1:Example.For order 20 the possible values of SIZE are as follows:

(1,3,6,6,4)	- We begin with the largest possible sizes
(1,3,6,6,3,1)	- and reduce from the right.
(1,3,6,6,2,2)	- (1,3,6,6,2,1,1) is excluded by 6.2.6.3.
(1,3,6,4,3,3)	- $n(4)\leq n(6)$ by 6.2.6.5.
(1,3,6,3,3,3,1)	- (1,3,6,4,3,2,1) and (1,3,6,4,3,1,1,1) are excluded by 6.2.6.5 and 6.2.6.3 respectively.
(1,3,3,3,3,3,3,1)	- (1,3,6,3,3,2,2) is excluded by 6.2.6.2, (1,3,6,3,3,2,1,1) and so on are excluded by 6.2.6.3.
	- (1,3,3,3,3,3,2,2),(1,3,3,3,3,3,1,1,1,1) and so on are excluded by 6.2.6.2 and 6.2.6.4. //

6.2.7.2:Note.In the algorithm which follows,the ordering of the entries of SIZE is immaterial except in that the first two must be 1 and 3 respectively. //

6.2.8:Algorithm.To construct all strictly feasible generalised graphs with respect to symmetric trivalent graphs on a fixed number, n, of vertices.

Firstly construct all possible values of SIZE for the given n. Then for each value of SIZE apply Algorithm 6.2.2.2 with the forbidden subgraphs specified in 6.1.3.2-4 and the following initial value of GRM:- The first row has entries 2,2,2, the second has first entry 1 and all other entries are undetermined. //

This algorithm has been executed for $n\leq 40$ with the results given in the next proposition.

6.2.9:Proposition.The only strictly feasible generalised graphs with respect to symmetric trivalent graphs are the following:

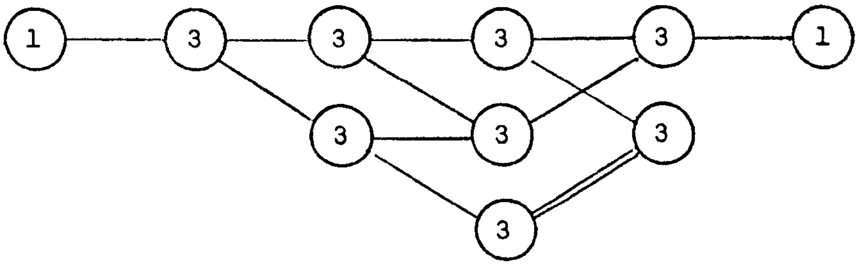
n=4: 6.2.4.1. n=6: 6.2.4.2. n=8: 6.2.4.5. n=10: 6.2.4.3.

n=14: 6.2.4.6. n=16: 6.2.4.13. n=18: 6.2.4.8. n=20: 6.2.4.14,15.

n=24: 6.2.4.19.

n=26:

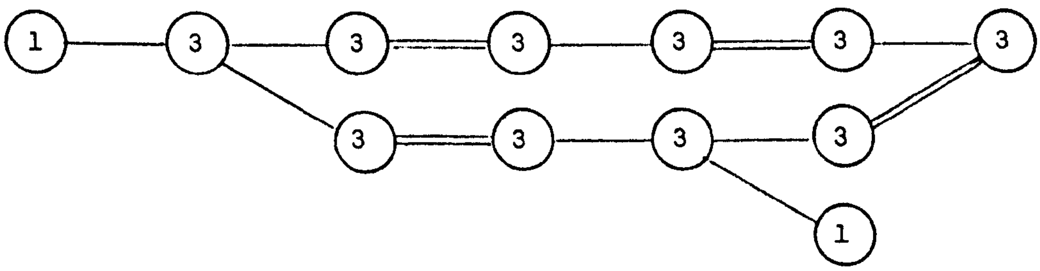
6.2.9.1:



n=28: 6.2.4.9. n=30: 6.2.4.10.

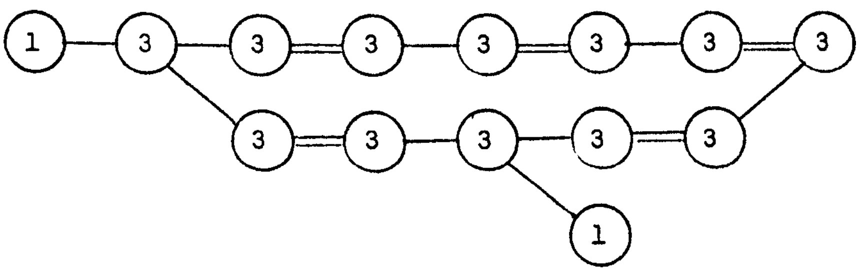
n=32: 6.2.4.38,6.2.5.7 and

6.2.9.2:

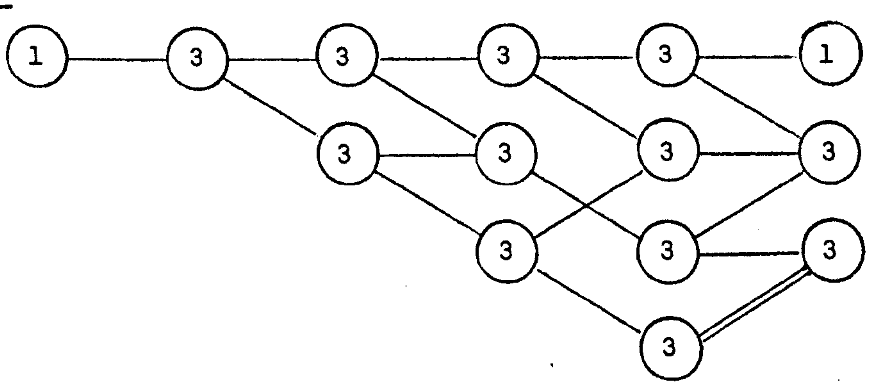


n=38:

6.2.9.3:



6.2.9.4:



n=40: 6.2.4.39. //

Clearly the next step is to decide for each of the strictly feasible generalised graphs whether or not it is realisable and if so which graphs in the given class have it as their singleton-quotient.This is the problem to which we shall address ourselves in the next chapter.

7:EXISTENCE AND UNIQUENESS

7.1:Covering Graphs.

Consider a decomposition $\mathcal{D}$ of a graph G. Unless $\mathcal{D}$ is trivial the quotient $G^{\mathcal{D}}$ is not a simple graph.However in some cases $G^{\mathcal{D}}$ is very closely related to a simple graph,and the connections between the properties of G and those of $G^{\mathcal{D}}$ can aid us greatly in determining whether there is a unique transitive graph whose singleton-quotient is a particular one of the generalised graphs constructed in Chapter 6.

7.1.1.1:Definition.Let $G=(V,s,w)$ be a generalised graph with the property that the size of every vertex is divisible by r for some $r\varepsilon N$. Then the reduced generalised graph G/r is the generalised graph (V,s',w) where $s'(x)=s(x)/r$ for all $x\varepsilon V$. //

7.1.1.2:Definition.Let $G=(V,s,w)$ be a simple graph with a decomposition $\mathcal{D}$ of order m where $\mathcal{D}=\{D_i,i\varepsilon N_m\}$. $\mathcal{D}$ is said to be a (0,1)-decomposition of G if

i) $|N(x)\cap D_i|=0$ or 1

and ii) if $x\varepsilon D_i$ then $|N(x)\cap D_i|=0$

for all $x\varepsilon V$ and $i\varepsilon N_m$. //

7.1.1.3:Proposition.Let $G=(V,s,w)$ be a graph with a (0,1)-decomposition of order m, $\mathcal{D}=\{D_i,i\varepsilon N_m\}$. Then all the decomposition classes are of the same size and $G^{\mathcal{D}}/|D_1|$ is also a simple connected graph.

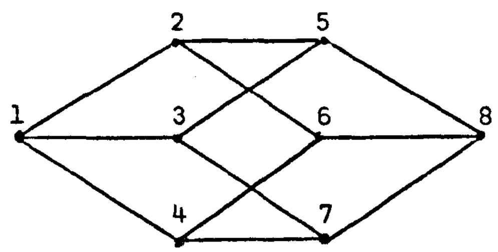
Proof.Consider $x\varepsilon V$ and $D_i,D_j\varepsilon\mathcal{D}$ such that $x\varepsilon D_i$ and $|N(x)\cap D_j|\neq 0$. Then every vertex in D_i is adjacent to exactly one vertex of D_j and vice versa,so that $|D_i|=|D_j|$. Since G is connected, every decomposition class has the same size. $G^{\mathcal{D}}$ has no loops or multiple edges by definition. //

7.1.1.4:Definition. Let D be a (0,1)-decomposition of a graph G and let the size of every decomposition class be r for some $r\varepsilon N$. Then G is called an r-fold covering of G^D/r, the graph covered by G. //

The mapping of G onto G^D/r is a graph homomorphism in the sense defined by Harary (21), and covering graphs have been studied extensively by (among others) Waller (44,45), Farzan (15,16), Gardiner (20), Biggs (4), Smith (39), and Djoković (13). Covering graphs in which the classes of the (0,1)-decomposition consist of "antipodal" vertices have been particularly thoroughly studied.

7.1.2.1:Definition. Let G=(V,s,w) be a graph of diameter d. Vertices $u,v\varepsilon V$ are antipodal if d(u,v)=d. If d(u,v)=d(u,w)=d for $u,v,w\varepsilon V$ implies that v=w or d(v,w)=d, then G is an antipodal graph. //

7.1.2.2:Example. Consider the decomposition D={{1,8},{2,7},{3,6},{4,5}} of the cube



This is a (0,1)-decomposition where the classes consist of antipodal pairs of vertices and $G^D/2$ is K_4, so that the cube is an "antipodal double-covering of K_4". //

7.1.2.3:Proposition. If a transitive graph is antipodal then the classes of antipodal vertices form a system of imprimitivity.

Proof. Trivial. //

We take the following straight-forward propositions concerning the properties of covering graphs from (16),(20) and (4).

7.1.3.1:Proposition.If G is a regular covering graph of H then H is regular with the same valency as G. //

7.1.3.2:Proposition.Let G be a t-arc-transitive covering of H for some $t\varepsilon Z^+$ with the property that the classes of the (0,1)-decomposition form a system of imprimitivity of G. Then H is at least t-arc-transitive. //

7.1.3.3:Definition.Consider graphs $G_1=(V_1,s_1,w_1)$ and $G_2=(V_2,s_2,w_2)$. The Kronecker product $G_1\wedge G_2$ is a simple graph with vertex set $V_1\times V_2$ and adjacency defined by (v_1,v_2) adj (w_1,w_2) with $v_1,w_1\varepsilon V_1$ and $v_2,w_2\varepsilon V_2$, if and only if v_1 adj w_1 in G_1 and v_2 adj w_2 in G_2. //

7.1.3.4:Proposition.Suppose a graph G is not bipartite.Then $G\wedge K_2$ is a bipartite double-covering of G and if G is t-arc-transitive for some $t\varepsilon Z^+$ then $G\wedge K_2$ is also at least t-arc-transitive. //

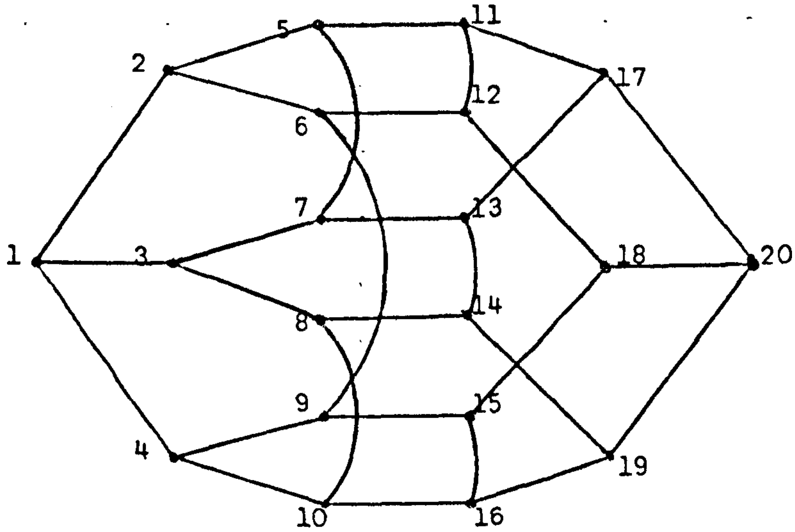
7.1.4:Remarks.Covering graphs have in general been used in existence and uniqueness proofs in the context of distance-transitive graphs,so it is worth noting two properties of distance-transitive covering graphs which fail to hold for symmetric covering graphs.

7.1.4.1:Suppose a distance-transitive graph is imprimitive.Then any block system is either a bipartition or an antipodal system and in the latter case the graph is an antipodal covering graph (4). Now consider the quotient of the Coxeter/Frucht graph illustrated in 2.1.11. It is clear that every vertex of this graph has a unique "opposite" vertex,though the graph is not antipodal, and so the graph has a block system which is neither of the types given above.Further it is not a covering graph with respect to this system since if it were,the covered graph would have

55 vertices and valency 3, a contradiction. //

7.1.4.2: Gardiner, (20), establishes that a distance-transitive r-fold covering G of a graph H where the classes of the (0,1)-decomposition are a block system may exist only for certain values of r not exceeding the valency of H. Consider the dodecahedron P(10,2):

P(10,2):



This graph is an antipodal double-covering of Petersen's graph (and they are both distance-transitive). It is not bipartite and so it in turn has a symmetric double-covering $P(10,2)\wedge K_2$. This latter graph is a 4-covering of Petersen's graph and the (0,1)-decomposition is a block system. Note however that it is not an antipodal system. //

7.2: Superdecompositions and Imprimitivity.

In this section our aim is to establish results which will under certain circumstances enable us to decide whether a given generalised graph is necessarily a quotient of a covering graph, and if it is to identify the covered graph. To do this we must relate the decompositions of a graph to those of its quotients.

7.2.1.1:Definition. Let $G=(V,s,w)$ be a generalised graph with vertex-partitions P and Q. P is a superdecomposition of Q if P is a decomposition of G and P is a superpartition of Q, that is for all $x,y\epsilon V$ if $x\overset{Q}{\sim}y$ then $x\overset{P}{\sim}y$. //

7.2.1.2:Proposition. Consider an undirected generalised graph G with decompositions $D=\{D_i, i\epsilon N_n\}$ and $E=\{E_j, j\epsilon N_m\}$ such that E is a superdecomposition of D. Then there is a decomposition E' of G^D with $(G^D)^{E'}$ isomorphic to G^E.

Proof. Define E', a partition of N_n, thus:- For all $i,j\epsilon N_n$ $i\overset{E'}{\sim}j$ if and only if D_i and $D_j \subset E_k$ for some $k\epsilon N_m$. The proposition follows immediately, for the row sums of the E-blocks of $A(G)$ are equal to the row sums of the E'-blocks of $A(G^D)$, the sizes of the corresponding vertices of $(G^D)^{E'}$ and G^E are clearly equal, and E' is a decomposition of G^D by Proposition 2.2.3.14. //

7.2.2.1:Definition. Suppose E and F are decompositions of a generalised graph G. Then D, the closure of E and F, is the finest decomposition of G which is a superdecomposition of both E and F. //

7.2.2.2:Proposition. Let $G=(V,s,w)$ be a generalised graph, let E and F be decompositions of G, and let $D=\{D_i, i\epsilon N_m\}$ be the partition of V defined thus:- $x\overset{D}{\sim}y$ if and only if there is an $r\epsilon N$ and a finite chain of vertices of G $v_0(=x), u_1, v_1, u_2, \ldots, v_n(=y)$, not necessarily all distinct, such that $u_i\overset{E}{\sim}v_{i-1}$ and $v_i\overset{F}{\sim}u_i$ for all $i\epsilon N_r$. Then D is the closure of E and F.

Proof. D is clearly a superpartition of E and F, for any D_i is a union of members of E and also a union of members of F. Suppose $x\overset{D}{\sim}y$ for some $x,y\epsilon V$. Then there is a chain of vertices $v_0(=x), u_1, v_1, u_2, \ldots, v_r(=y)$ such that $u_i\overset{E}{\sim}v_{i-1}$ and $v_i\overset{F}{\sim}u_i$ for all $i\epsilon N_r$. But since the members of D are unions of members of E, and E is a decomposition of G,

$\alpha(v_{i-1})_j=\alpha(u_i)_j$ and $\beta(v_{i-1})_j=\beta(u_i)_j$ for all $i\varepsilon N_r$ and $j\varepsilon N_m$ (see Definition 2.1.1.1). Similarly since F is a decomposition of G, $\alpha(u_i)_j=\alpha(v_i)_j$ and $\beta(u_i)_j=\beta(v_i)_j$ for all $i\varepsilon N_r$ and $j\varepsilon N_m$. Hence $\underline{\alpha}(x)=\underline{\alpha}(y)$ and $\underline{\beta}(x)=\underline{\beta}(y)$ and D is a decomposition of G.

Furthermore every superdecomposition of both E and F must have the property that any such chain of vertices is contained in a single class, since a class of a superdecomposition is a union of members of each of E and F. Thus D is the finest superdecomposition of E and F. //

Thus if D is the closure of E and F, both of which are decompositions of an undirected generalised graph G, we have the following diagram:

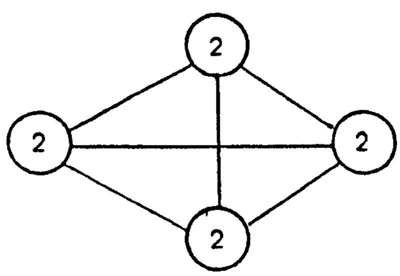
$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] \arrow[dd] & \\ G^{E} \arrow[dr] & & G^{F} \arrow[dl] \\ & G^{D} & \end{tikzcd}$$

<u>7.2.2.3:Example</u>. Let G be the cube, labelled as in 7.1.2.2.

Consider the decompositions

i) $E=\{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$

with G^E:

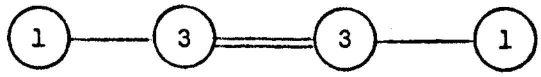


and

$$A(G^E)=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

ii) $F=\{\{1\},\{2,3,4\},\{5,6,7\},\{8\}\}$

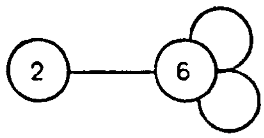
with G^F:



and $$A(G^F)=\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

iii) $D=\{\{1,8\},\{2,3,4,5,6,7\}\}$, the closure of E and F,

with G^D:



and $$A(G^D)=\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}.$$

G^D is clearly a quotient of both G^E and G^F. //

7.2.3.1:Lemma.Let G be an imprimitive graph and let B be a member of the block system.Then G has a decomposition B with $B\varepsilon B$.

Proof.By definition $\Gamma(G)$ has a subgroup Π which stabilises B setwise and acts transitively on the members of B. $D(\Pi)$ is the required decomposition B. //

7.2.3.2:Proposition.Let G be a properly labelled imprimitive graph.Consider $D(\Gamma_1(G))$, the decomposition of G into the orbits of the stabiliser of vertex 1, which contains a class {1}. Then $G^{D(\Gamma_1(G))}$ has a non-trivial decomposition E' of order >1 with the property that $\{1\}\notin E'$.

Proof.Let vertex 1 be contained in a block B of a block system of G. Then G has a decomposition B with $B\varepsilon B$, by the lemma.Let E be the closure of $D(\Gamma_1(G))$ and B. $|B|\neq 1$, so that $\{1\}\notin E$

and the order of E is strictly less than that of $D(\Gamma_1(G))$. Further vertex $1\varepsilon B$ so $\Gamma_1(G)$ stabilises B setwise and hence B is a union of classes of $D(\Gamma_1(G))$, and since $B\varepsilon B$, $B\varepsilon E$. Thus the order of E is strictly greater than 1. The required decomposition E' of $G^{D(\Gamma_1(G))}$ is that corresponding to the decomposition E of G as in Proposition 7.2.1.2. //

7.2.3.3:Corollary. Let G be a properly labelled imprimitive graph with a (0,1)-decomposition of order m, $B=\{B_1,B_2,\ldots,B_m\}$, which is also a block system, so that G is a $|B_1|$-fold covering of a graph H, say. Then $G^{D(\Gamma_1(G))}$ has a non-trivial decomposition E' with the property that $(G^{D(\Gamma_1(G))})^{E'}/|B_1|$ is a quotient of H with a vertex of size 1 corresponding to block $B_1\varepsilon B$. //

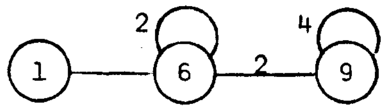
7.2.3.4:Example. Consider Example 7.2.2.4 with $H=G^E/2$ and $G^D/2$ the singleton-quotient of H. //

We have shown that if a graph is imprimitive then the quotient corresponding to a vertex-stabiliser has a non-trivial decomposition of order greater than 1 and it follows that if a quotient has no such decomposition then it may only be the quotient corresponding to a vertex-stabiliser of a graph if that graph is primitive. It would be very convenient if this result held for singleton-quotients, that is if we could make the following statement:-

If a strictly feasible generalised graph with respect to transitive graphs has no non-trivial decomposition of order greater than 1, then any transitive graph of which it is the singleton-quotient is primitive.

However this assertion is unfortunately not true.

7.2.3.5:Counter-example. The triangular tessellation of the torus given in 5.1.2.6 has singleton-quotient



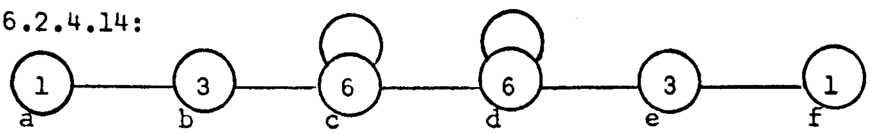
which has no non-trivial decomposition of order greater than 1, and it has a block system {{1,3,9,11},{2,4,10,12},{5,7,13,15},{6,8,14,16}}. //

7.3:Construction.

We shall use three techniques in identifying transitive trivalent graphs whose singleton-quotients are among those strictly feasible generalised graphs constructed in Chapter 6. The first is to employ the elementary consequences of the transitivity of the graph we are seeking.These consequences fall into three categories which we shall illustrate by means of the following example:

7.3.1.1:Example.Consider the generalised graph 6.2.4.14:

6.2.4.14:



Suppose it is the singleton-quotient of a properly labelled transitive graph G, with singleton-decomposition $\{S_a,S_b,\ldots,S_f\}$ as indicated,with $S_a=\{1\}$.

i) The shortest circuit in G containing vertex 1 must pass from 1 to S_b,S_c,S_c again,S_b, and back to 1, and so has length 5. Since the graph is transitive,the girth of G is 5.

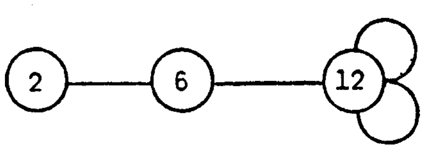
ii) Every 2-arc beginning at vertex 1 in G can be extended to become two 3-arcs with the property that one can be extended into a circuit of length 5 while the other cannot.Since G is transitive,this is true of every 2-arc in G.

iii) Every vertex of distance 3 from vertex 1 is adjacent to exactly one other vertex which is the same distance from vertex 1 and is distance 2 (via S_e) from exactly one other vertex of distance 3 from 1. This configuration must exist for the set of vertices whose distance is 3 from any given vertex of G. //

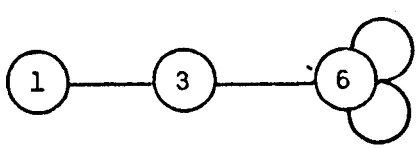
The second technique is to show that any transitive graph with a particular singleton-quotient must have a (0,1)-decomposition which is also a block system, and to demonstrate that the closure of the (0,1)-decomposition with the singleton-decomposition has order greater than 1. We then identify the covered graph from the quotient corresponding to the closure.

7.3.1.2:Example. Consider again the generalised graph given in 6.2.4.14, labelled as above, and again suppose it to be the singleton-quotient of a properly labelled transitive graph G. G is clearly antipodal, and an antipodal system is a block system. To show that this system is a (0,1)-decomposition we consider the vertex 2' antipodal to vertex $2 \in S_b$. It is easily established that any vertex of distance 5 from 2 lies in S_e. Thus vertex 1', antipodal to vertex 1, is adjacent to vertex 2'. Thus the vertex antipodal to each neighbour of 1 is adjacent to 1'. Since the graph is transitive, this is true for every vertex, the antipodal system is a (0,1)-decomposition and G is a double-covering. Further it is clear that the closure of the antipodal system with the singleton-quotient of G is $\{S_a \cup S_f, S_b \cup S_e, S_c \cup S_d\}$ and hence the corresponding quotient of G, H say, is

H:



then H/2 is



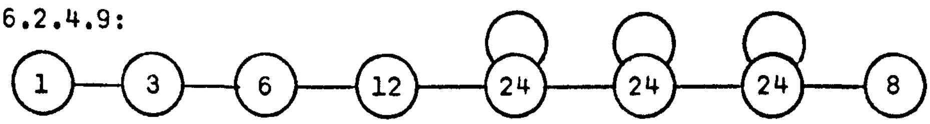
which must be the singleton-quotient of the covered graph by

7.2.2.3. Since G is transitive and the (0,1)-decomposition is a block system,the covered graph is also transitive.Thus the problem of identifying G is reduced to that of identifying transitive graphs with the smaller singleton-quotient and then constructing their transitive double-coverings. //

The third technique is to demonstrate that any transitive graph with the given singleton-quotient is primitive,subject to a restriction on its rank.All primitive trivalent graphs may be easily constructed from the list of primitive permutation groups with a suborbit of length three given by Wong (47).

7.3.1.3:Example.Consider the generalised graph 6.2.4.9:

6.2.4.9:



Any transitive graph G of rank less than or equal to 9 with this singleton-quotient is primitive.For

i) Suppose the rank is 8. Then the singleton-quotient corresponds to the stabiliser of a vertex.But it has no non-trivial decomposition of order greater than 1. Hence G is primitive by 7.2.3.2.

ii) Suppose the rank is 9. Then one of the classes corresponds to two orbits of the stabiliser of a vertex.But it is easily seen that if one class consists of two orbits,then so does one of the neighbouring classes,a contradiction. //

The last of these three techniques is the least satisfactory in that it forces us to impose a stricter condition on the classes of graphs we are concerned with.We must replace the decomposition rank by the rank in Propositions 6.2.4 and 6.2.5. Of course the lists of strictly feasible generalised graphs produced are unchanged.

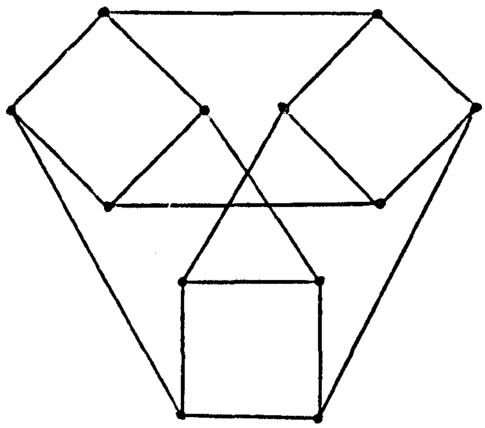
7.3.1.4:Notation.We shall label the vertices of the strictly feasible generalised graphs with lower case letters.If such a generalised graph H is the singleton-quotient of a properly labelled transitive trivalent graph G we shall take the singleton-decomposition of G to be indexed with the same letters,so that for example vertex x of H corresponds to class S_x of the singleton-decomposition of G. In addition we shall take S_a to be {1} and we shall denote an unspecified member of a class, say S_x by $\hat{x}$, so that the circuit given in Example 7.3.1.1 could be written $<1,\hat{b},\hat{c},\hat{c},\hat{b}>$. //

7.3.2:Proposition.The following strictly feasible generalised graphs have these realisations with respect to transitive trivalent graphs and no others:

Generalised graph.	Realisation	Order,	rank,	arc-transitivity.
6.2.4.1	K_4	4	2	2
6.2.4.2	$K_{3,3}$	6	3	3
6.2.4.3	P(5,2)	10	3	3
6.2.4.4	P(3,1)	6	4	0
6.2.4.5	Q_3	8	4	2
6.2.4.6	Heawood's graph,H	14	4	4
6.2.4.7	M(4)	8	5	0
6.2.4.8	The Pappus graph,P	18	5	3
6.2.4.10	Tutte's graph,T	30	5	5
6.2.4.11	P(5,1)	10	6	0
6.2.4.12	M(5)	10	6	0
6.2.4.13	P(8,3)	16	6	2
6.2.4.14	P(10,2)	20	6	2
6.2.4.15	P(10,3)	20	6	3

Generalised graph	Realisation	Order,rank,arc-transitivity		
6.2.4.16	$T(K_4)$	12	7	0
6.2.4.17	M(6)	12	7	0
6.2.4.18	None			
6.2.4.22	P(6,1)	12	8	0
6.2.4.23	None			
6.2.4.24	R	12	8	0

R:



6.2.4.25	None			
6.2.4.26	P(7,1)	14	8	0
6.2.4.27	M(7)	14	8	0
6.2.4.28	$T(K_{3,3})$	18	8	0
6.2.4.29	None			
6.2.4.30	None			
6.2.4.33	None			
6.2.4.34	None			
6.2.4.35	M(8)	16	9	0
6.2.4.36	None			
6.2.4.37	None			
6.2.4.38	None			
6.2.4.42	None			
6.2.5.4	None			
6.2.5.5	None			

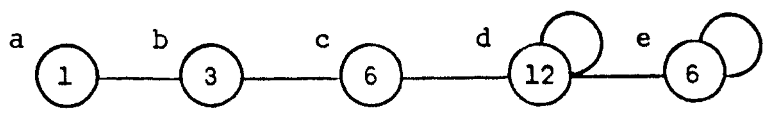
Generalised graph	Realisation
6.2.5.6	None
6.2.9.2	None
6.2.9.3	None

Proof. In each of the cases above the non-existence or uniqueness of a realisation is established without difficulty using elementary methods (as in Example 7.3.1.1). //

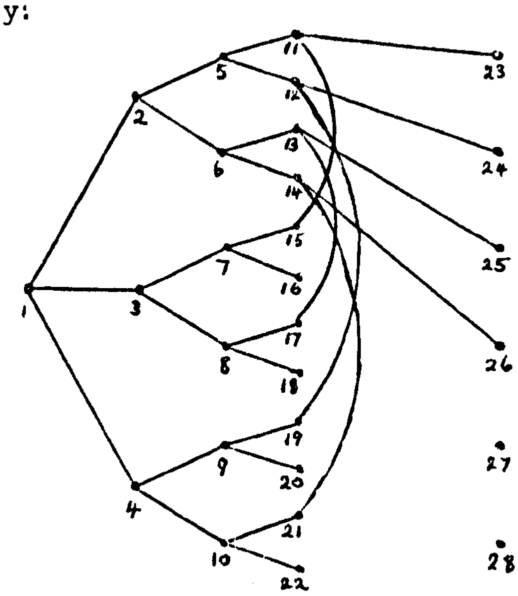
We shall consider the remaining strictly feasible generalised graphs in turn. The proofs in some cases will only involve elementary techniques but if so they will be less straightforward than in the proposition just given.

7.3.3 : Proposition. The only transitive graph whose singleton-quotient is 6.2.4.9 is the Tutte-Coxeter graph, TC, on 28 vertices, whose rank is 5 and which is 3-arc-transitive.

6.2.4.9:

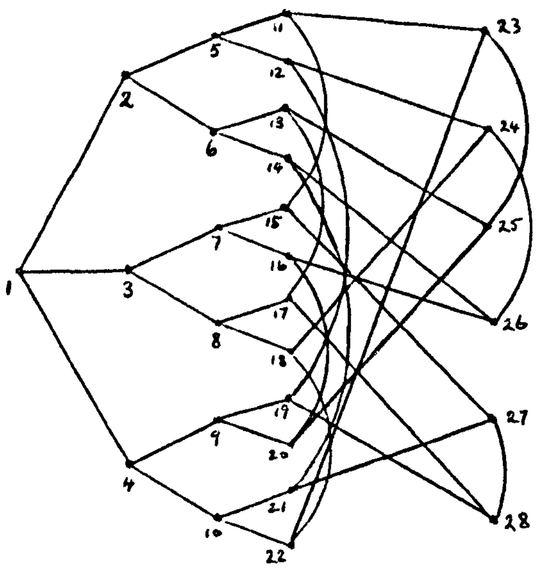


Proof. Let G be a transitive graph with this singleton-quotient. Then G has girth 7 and every 3-arc extends to exactly one heptagon and one octagon. G may be constructed so far without loss of generality:

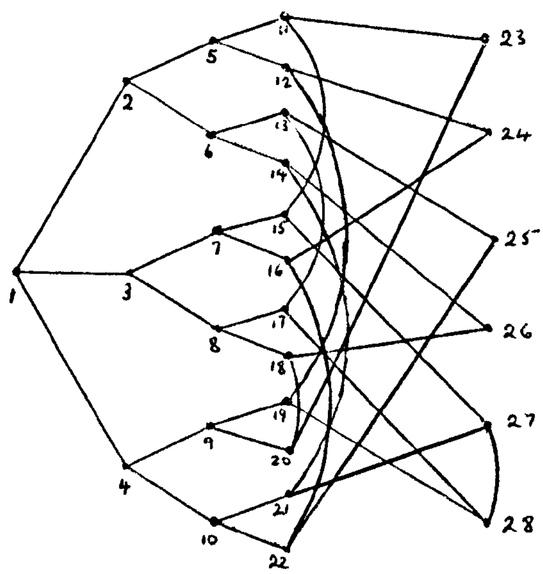


Vertex 16 is adjacent to 20 or 22.

i) Suppose 16 adj 20. Then 18 adj 22. Consider vertices whose distance is 4 from vertex 2. These are 16,18,20,22,27,28. 16 adj 20, 18 adj 22, and so 27 adj 28. Hence 23,24,25 and 26 are adjacent in pairs.Now consider the 3-arc (2,5,11,15). It must extend to an octagon via S_e so that without loss of generality 15 adj 27 and 27 is adjacent to one of 17,19 and 21. But 27 not adj 17 or 19 by girth so 27 adj 21. Considering the extensions of (2,5,12,19),(3,7,15,11,23),(3,8,17,13,25) and (4,9,19,12,24) we find that 19 adj 28, 28 adj 17, 23 adj 22, 25 adj 20, 24 adj 18 and 26 adj 16. Finally 23 not adj 24 or 26 by girth,so 23 adj 25 and 24 adj 26. Thus we have



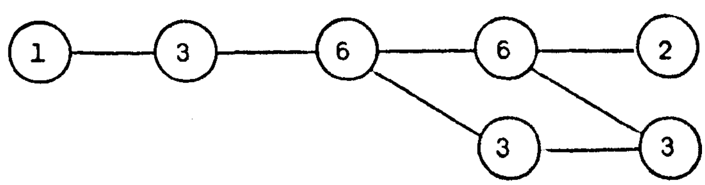
ii) Suppose 16 adj 22. Then 18 adj 20. As above 27 adj 28 and 23,24,25 and 26 are adjacent in pairs.Also as above 15 adj 27, 27 adj 21, 17 adj 28, 28 adj 19, 23 adj 20, 25 adj 22, 24 adj 16 and 26 adj 18. Thus we have



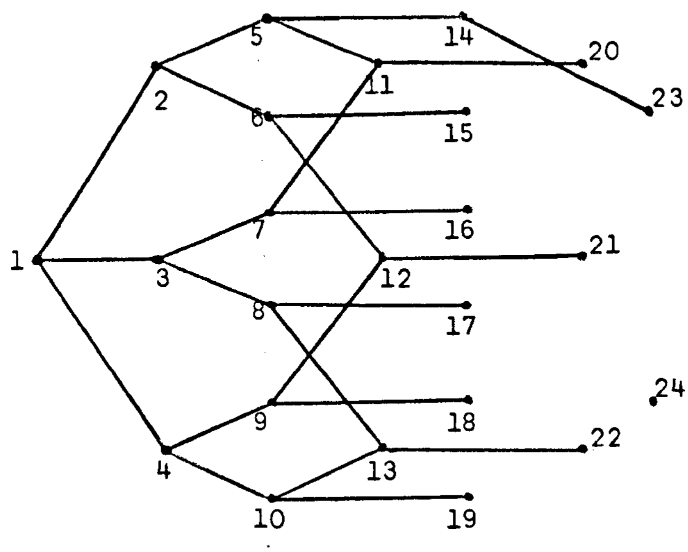
Now vertex 9 is distance 3 from vertex 5, and is adjacent to two other vertices, 20 and 4, whose distance from 5 is 3. This is a contradiction and hence the construction of i) is the only one possible. TC is well known to have the correct singleton-quotient. //

<u>7.3.4 :Proposition</u>.The only transitive graph whose singleton-quotient is 6.2.4.19 is P(12,5) which has order 24, rank 7, and is 2-arc-transitive.

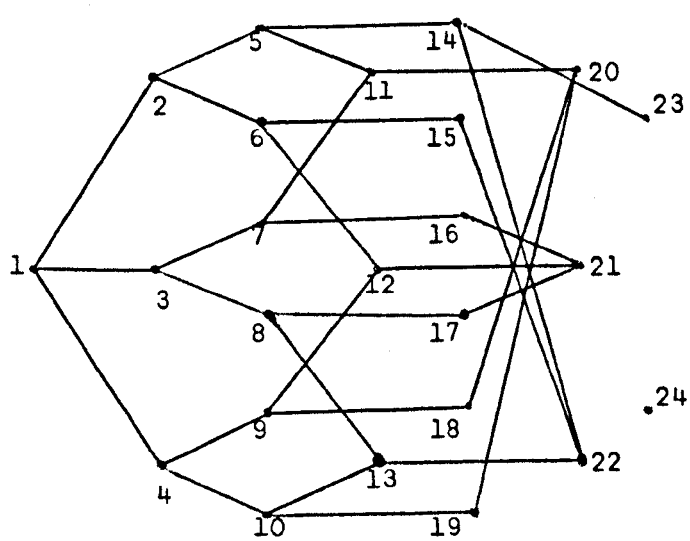
6.2.4.19:



<u>Proof</u>.The girth of any transitive graph with this singleton-quotient is 6. Hence the graph may be constructed so far without loss of generality:



Every 2-arc beginning at vertex 1 extends to exactly one hexagon, so this is true of every 2-arc in the graph. By consideration of girth we see that 20 is adjacent to neither 14 nor 16, 21 adj neither 15 nor 18, and 22 adj neither 17 nor 19. (2,6,12) extends to a hexagon <2,6,12,9,4,1>. Hence 21 not adj 14, since if it were, (2,6,12) would also extend to the hexagon <2,6,12,21,14,5>. Similarly 21 not adj 19, 20 adj neither 15 nor 17, 22 adj neither 16 nor 18. Hence 21 adj 16 and 17, 20 adj 18 and 19, 22 adj 14 and 15.

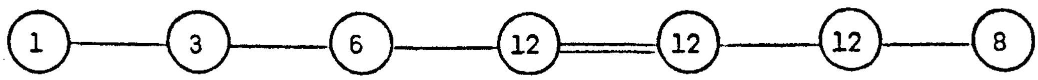


23 not adj 15 since the girth is 6. Hence 24 adj 15. Again considering girth 23 is adjacent to exactly one of 18 and 19. Thus (23,14,5) extends to a hexagon <23,14,5,11,20,18 or 19>. Suppose 16 adj 23. Then (23,14,5) also extends to the hexagon <23,14,5,11,7,16>. So 16 not adj 23, and hence 16 adj 24, and 17 adj 23. Similarly 19 not adj 23 so 18 adj 23 and 19 adj 24. Thus the construction is unique.

Now P(12,5) is known to be a transitive trivalent graph on 24 vertices,and it is easily shown that it has the desired singleton-quotient. //

<u>7.3.5 :Proposition</u>.There is no transitive graph whose singleton-quotient is 6.2.4.20.

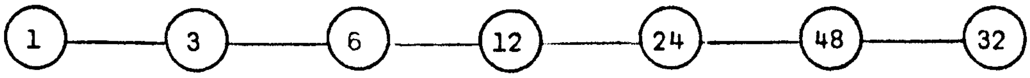
6.2.4.20:



<u>Proof</u>.Any transitive graph with this singleton-quotient would be distance-regular.But the above diagonal entries in the tridiagonal form of the adjacency matrix of this quotient would not decrease monotonically,contradicting Proposition 5.2.6.2. //

<u>7.3.6 :Proposition</u>.There is no transitive graph whose singleton-quotient is 6.2.4.21.

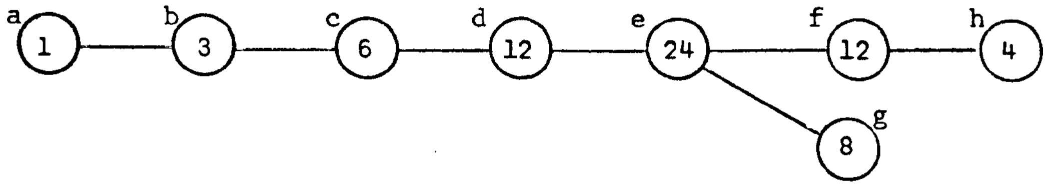
6.2.4.21:



<u>Proof</u>.Any transitive graph with this singleton-quotient would be a (3,12)-cage.But there is only one (3,12)-cage,Benson's graph (see Appendix 2 for a proof of uniqueness) and this graph is known not to be transitive (7). //

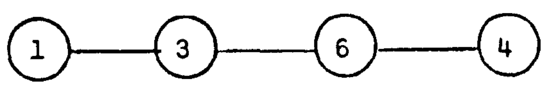
<u>7.3.7,1:Lemma</u>.Any transitive graph whose singleton-quotient is 6.2.4.31 is a 5-covering of Heawood's graph.

6.2.4.31:



<u>Proof</u>.The diameter of any transitive graph with this singleton-quotient is 6. Consider any two vertices in S_h. Then the shortest path from one to the other is via S_d or S_g and has length 6. Since the graph is transitive it is antipodal.

Consider a vertex ,2 say,in S_b. Now the vertices antipodal to 2 clearly do not lie in S_a,S_b,S_c,S_d,S_e, or S_h. The girth of the graph is 10 and so every vertex in S_g must be adjacent to two vertices whose distance from 2 is 5 and to one vertex whose distance from 2 is 3. Hence the vertices antipodal to 2 do not lie in S_g. So they lie in S_f. No vertex of S_h is adjacent to two vertices antipodal to vertex 2 since the graph is antipodal and has diameter 6. Hence every vertex antipodal to vertex 1 is adjacent to exactly one vertex antipodal to each neighbour of 1 and since the graph is transitive the antipodal system is a (0,1)-decomposition.The antipodal system is,of course, a block system.Thus any transitive graph with this singleton-quotient is a 5-covering of a transitive graph.The reduced quotient corresponding to the closure of the singleton-decomposition and the antipodal system is easily seen to be

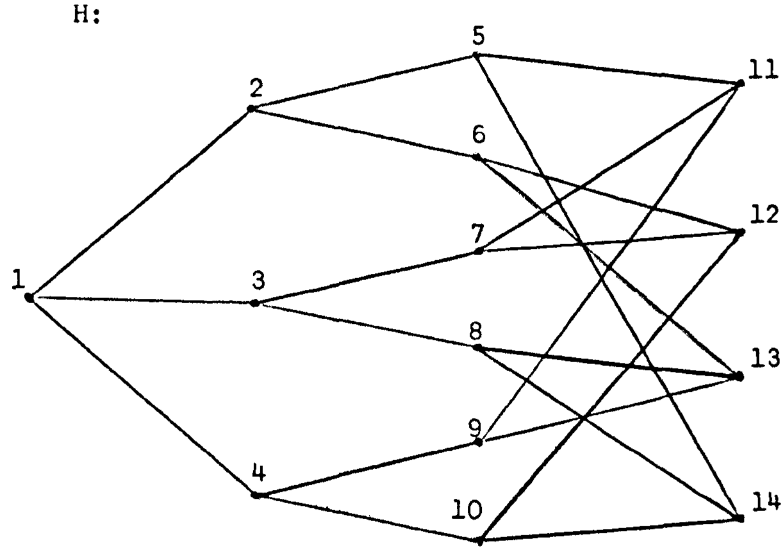


and the only transitive graph with this singleton-quotient is Heawood's graph by Proposition 7.3.2. //

<u>7.3.7.2:Proposition</u>.There is no transitive graph whose singleton-quotient is 6.2.4.31.

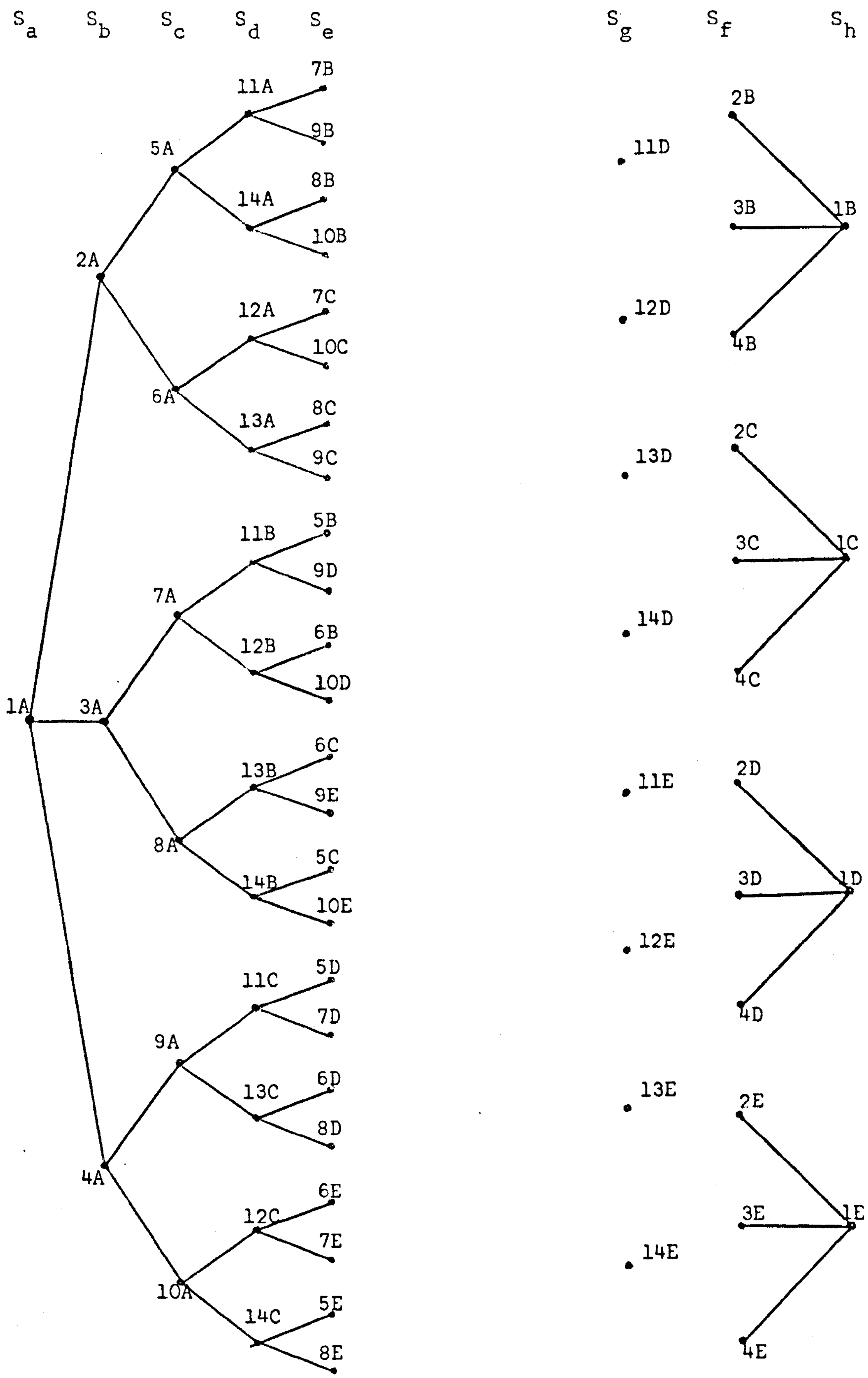
Proof. By the lemma any such graph is a transitive 5-covering of Heawood's graph, H.

H:



We assume there is such a covering and attempt to construct it, denoting the antipodal blocks by the integers 1 to 14, their members by the upper case letters A to E, and as before the classes of the singleton-decomposition by the lower case letters a to h.

The edges joining classes S_a, S_b, S_c, S_d, and S_e and those joining classes S_f and S_h may be constructed without loss of generality as follows:



S_a
S_b
S_c
S_d
S_e
S_g
S_f
S_h
1A
2A
3A
4A
5A
6A
7A
8A
9A
10A
11A
14A
12A
13A
11B
12B
13B
14B
11C
13C
12C
14C
7B
9B
8B
10B
7C
10C
8C
9C
5B
9D
6B
10D
6C
9E
5C
10E
5D
7D
6D
8D
6E
7E
5E
8E
11D
12D
13D
14D
11E
12E
13E
14E
2B
3B
1B
4B
2C
3C
1C
4C
2D
3D
1D
4D
2E
3E
1E
4E

In Heawood's graph 11 adj 5,7 and 9. It follows that in the graph we wish to construct each of 11D and 11E is adjacent to three of 7C,9C,9E,5C,7E and 5E, with no common adjacencies. Without loss of generality let 11D adj 5C. Then 11E adj 5E, and since the girth is 10,11D not adj 9E, so that 11D adj 9C. Similarly 11D not adj 7C so that 11D adj 7E.

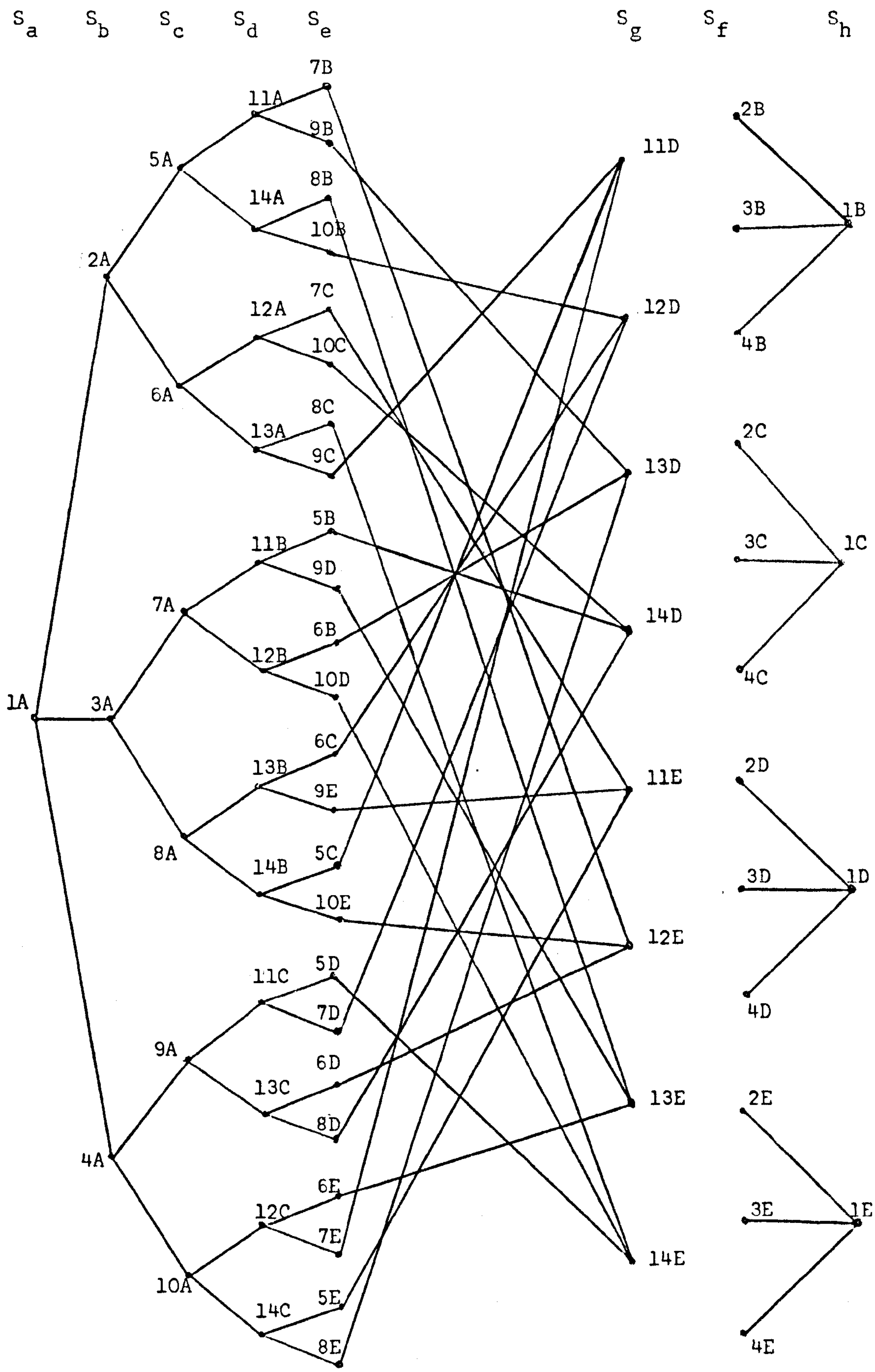
In the same way,choosing 12D adj 6C, 13D adj 6B and 14D adj 5B without loss of generality,the edges between classes S_e and S_g are uniquely determined:

Vertex	Neighbours
11D	5C,9C,7E
11E	5E,9E,7C
12D	6C,10B,7D
12E	6D,10E,7B
13D	6B,8E,9B
13E	6E,8B,9D
14D	5B,8D,10C
14E	5D,8C,10D

(See diagram overleaf).

Now in Heawood's graph 2 adj 1,5 and 6. Without loss of generality let 2B adj 5B. Then 2B not adj 6B by girth (<2B,5B,11B,7A,12B,6B>), 2B not adj 6C by girth (<2B,5B,11B,7A,3A,8A,13B,6C>), 2B not adj 6D by girth (<2B,5B,14D,8D,13C,6D>) and 2B not adj 6E by girth (<2B,6E,13E,9D,11B,5B>). So we have a contradiction and the proposition is proven. //

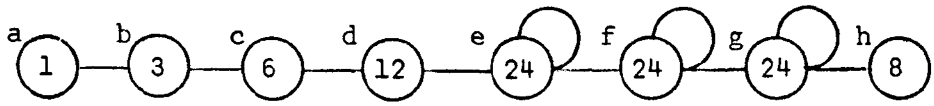
It is worth noting that a graph of order 70 and girth 10,having the correct singleton-quotient with respect to some vertices does exist;the discovery of this graph is due to Harries (24).



S_a S_b S_c S_d S_e S_g S_f S_h
1A
2A
3A
4A
5A
6A
7A
8A
9A
10A
11A
14A
12A
13A
11B
12B
13B
14B
11C
13C
12C
14C
7B
9B
8B
10B
7C
10C
8C
9C
5B
9D
6B
10D
6C
9E
5C
10E
5D
7D
6D
8D
6E
7E
5E
8E
11D
12D
13D
14D
11E
12E
13E
14E
2B
3B
4B
1B
2C
3C
4C
1C
2D
3D
4D
1D
2E
3E
4E
1E

7.3.8.1:Lemma.Any transitive graph of rank $r \leq 12$ whose singleton-quotient is 6.2.4.32 is primitive.

6.2.4.32:



Proof.Suppose G is a transitive graph with this singleton-quotient. The generalised graph has no decomposition of order greater than 1, so if it is the quotient corresponding to the stabiliser of a vertex of G,then G is primitive.If not,at least one class consists of at least two orbits of $\Gamma_1(G)$, and the rank r is easily seen to be at least 12. Now if any class consists of more than two orbits then r>12 clearly.Further if any class consists of two orbits of unequal size it is clear that S_b must consist of two orbits and r>12. r may only be 12 if S_a,S_b,S_c,S_d consist of one orbit each and S_e,S_f,S_g,S_h each consists of two orbits of equal size.We suppose this to be the case,and further suppose that the graph is imprimitive with block B containing vertex 1.

i) Suppose B contains S_b. Then every neighbour of vertex 1 is a member of B, and so,since the graph is symmetric,every neighbour of every vertex in S_b is a member of B, and so on, so that B contains every vertex of G, a contradiction.

ii) Suppose B contains S_c. Then every vertex of distance 2 from vertex 1 is in B, so every vertex of distance 2 from a vertex of S_c is in B, so $S_e \subset B$. But then adjacent vertices are members of B so that B contains S_b and we again have a contradiction.

iii) If B contains S_d then it also contains S_g and we have the same contradiction as above.

iv) Suppose half or all of S_e is contained in B. There are paths of length 3 containing three members of S_e, and since unless adjacent vertices are members of B every vertex in S_e not contained in B is adjacent to a member of B, there are such paths containing two members of B. Hence B contains S_b, S_c or S_d, a contradiction.

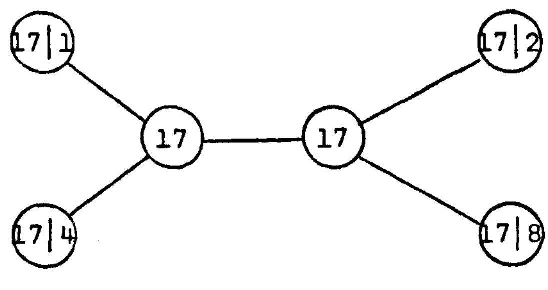
v) Suppose half or all of S_f is contained in B. Then in the same way as in iv) there are paths of length 4 containing two members of B so that B contains members of S_b, S_c, S_d or S_e, a contradiction.

vi) A similar argument applies to S_g.

vii) Finally neither 5 nor 9 divides 102, the order of G, so B cannot consist solely of vertex 1 and half or all of S_h.

Our assumption of imprimitivity is contradicted, and thus any transitive graph with this singleton-quotient is either primitive or has rank greater than 12. //

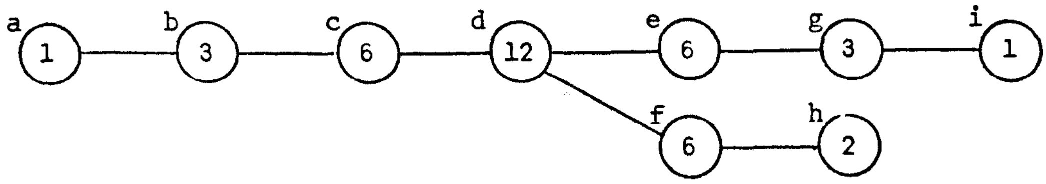
7.3.8.2: Proposition. The only transitive graph with rank less than or equal to 12 whose singleton-quotient is 6.2.4.32 is the graph with Frucht description



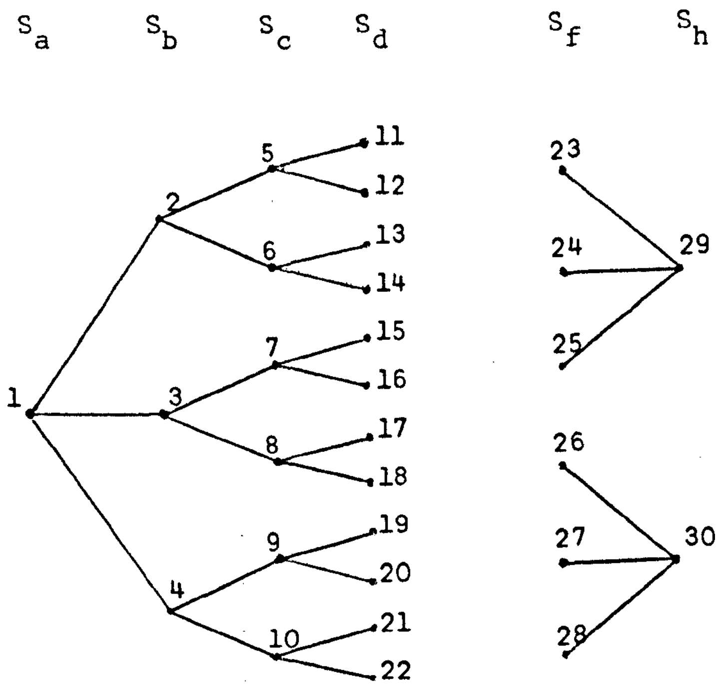
Proof. Smith (39) establishes that there is only one primitive graph with this singleton-quotient, and this graph is identified by Biggs and Smith (5) as having the description given. //

7.3.9.1: Lemma. Any transitive graph whose singleton-quotient is 6.2.4.39 is a double-covering of the Desargue graph P(10,3).

6.2.4.39:



Proof.Let G be a transitive graph with this singleton-quotient. Then the girth of G is 8, and G is clearly antipodal with diameter 6. We may construct part of G without loss of generality as follows:



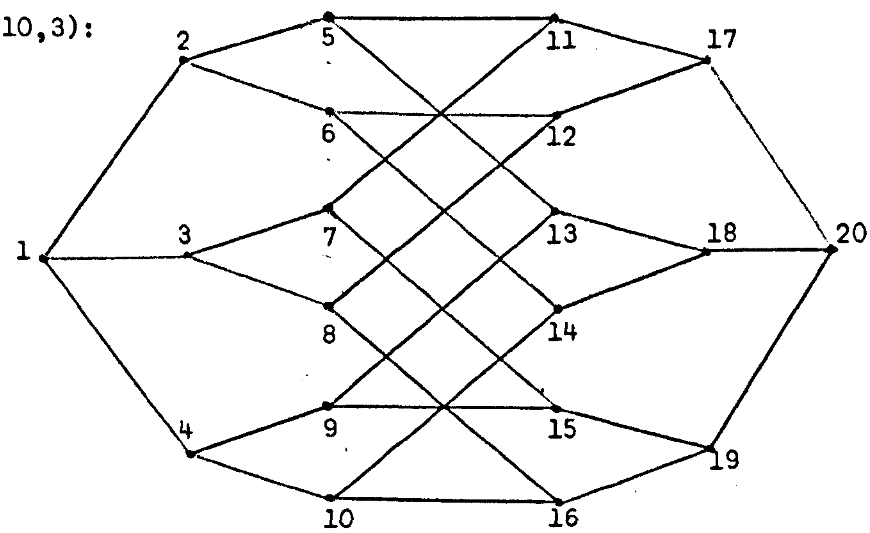
We consider vertex 2 in S_b. Suppose vertex 29 is antipodal to 2, that is d(2,29)=6. Then each of 23,24,25 is adjacent to two of the vertices 15-22. Now the vertices 26,27,28 are each adjacent to two vertices of S_d. Trivially therefore one of them is adjacent to two of the vertices 11-14, so that the girth of G is 6, a contradiction.Similarly vertex 30 is not antipodal to vertex 2, and hence the vertex antipodal to vertex 2 lies in S_g.

Using the same arguments as in Lemma 7.3.7.1 we see that any transitive graph with the desired singleton-quotient must be a double-covering of a transitive graph whose singleton-quotient is 6.2.4.15. But there is only one such graph and that is P(10,3). //

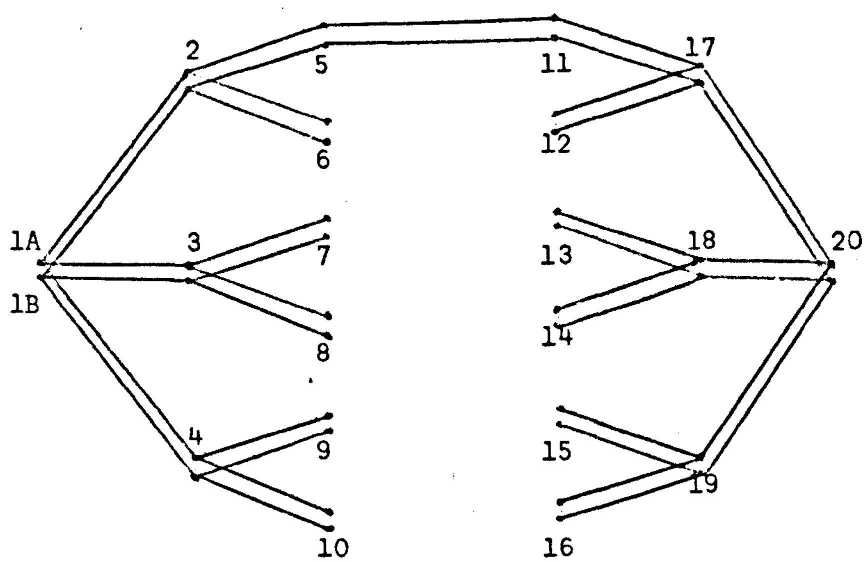
7.3.9.2:Proposition.The only transitive graph whose singleton-quotient is 6.2.4.39 is the Kronecker product $P(10,2)\wedge K_2$, which has order 40, rank 9 and is 3-arc-transitive.

Proof.By the lemma any such graph is a double-covering of P(10,3).

P(10,3):

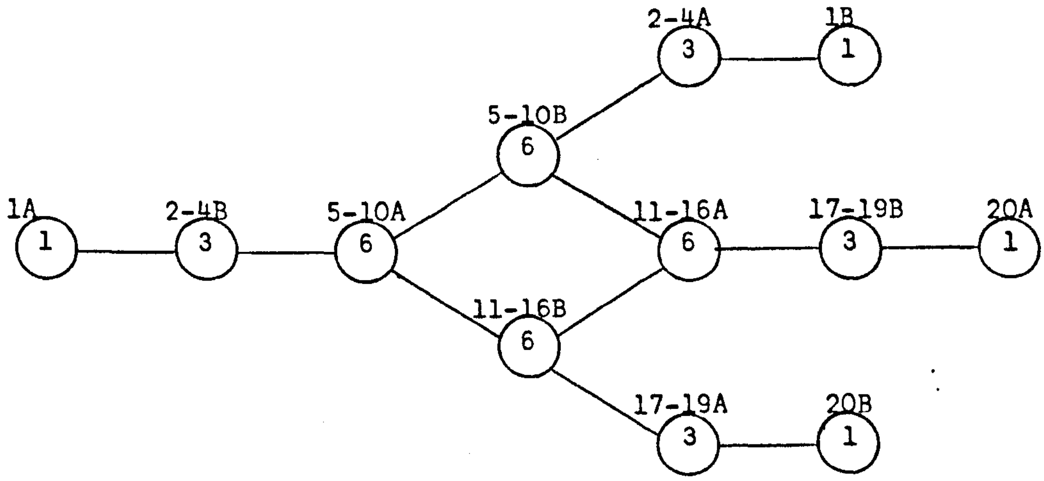


Let G be such a double-covering.Then we may construct it so far without loss of generality (labelling the vertices of each pair A and B, with the upper labelled A):



Now in P(10,3) 5 adj 13. Since the girth of G is 8, 5A adj 13B and 5B adj 13A. Similarly considering the girth 13A adj 9A, 9A adj 15B, 15A adj 7A, 7A adj 11B, 6A adj 12B, 12A adj 8A, 8A adj 16B, 16A adj 10A, 10A adj 14B, 14A adj 6A and the construction is unique.

Now $P(10,2)\wedge K_2$ is known to be a symmetric graph on 40 vertices. With the labelling of 7.1.4.2 the adjacencies of this double-covering of P(10,2) are indicated by the following diagram:



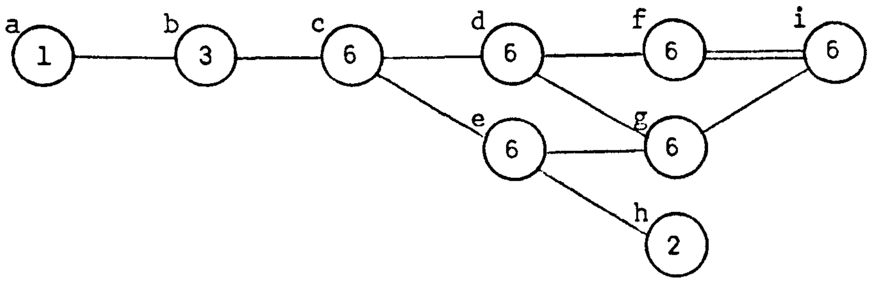
This graph clearly has singleton-quotient 6.2.4.39. Further it may be shown that there is an automorphism stabilising 1A and exchanging 5-10B with 11-16B so that the rank and arc-transitivity are as stated. //

7.3.9.3:Note.In the double-covering of P(10,2) the (0,1)-decomposition is not a block system. //

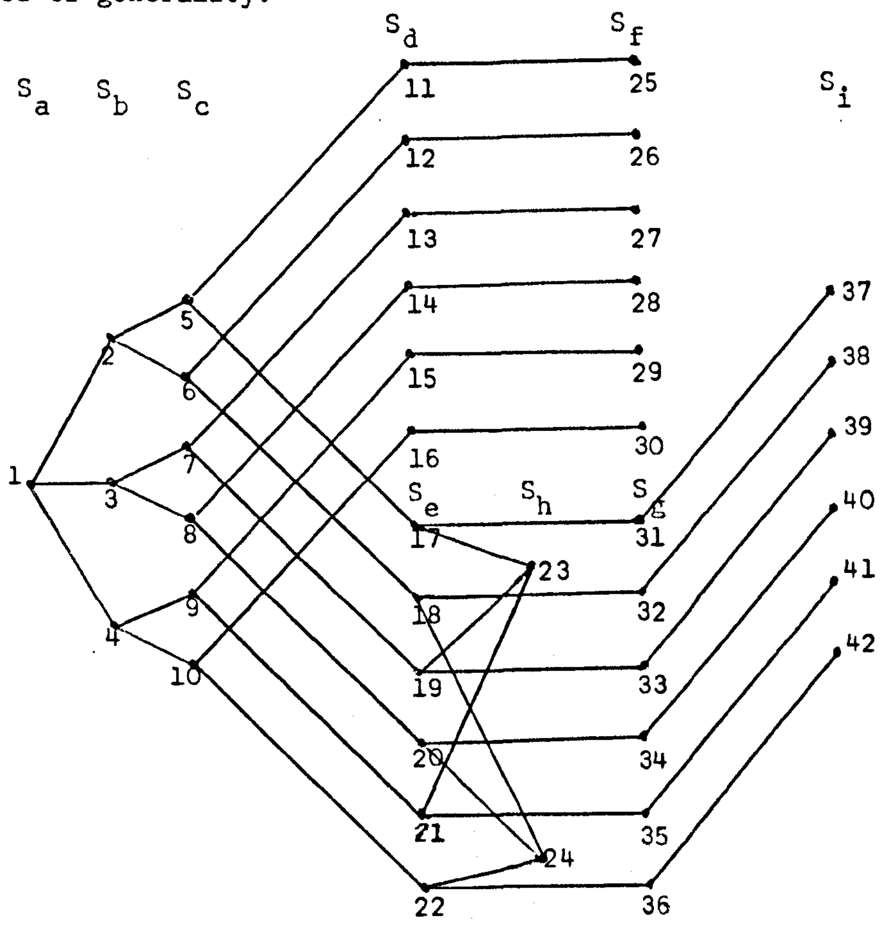
7.3.9.4:Note.The graph constructed is interesting in that it is simultaneously a double-covering of P(10,2), a double-covering of P(10,3) and a 4-covering of P(5,2). //

7.3.10:Proposition.There is no transitive graph whose singleton-quotient is 6.2.4.40.

6.2.4.40:



Proof.Let G be a transitive graph with this singleton-quotient. Then G has girth 8 and thus may be constructed so far without loss of generality:

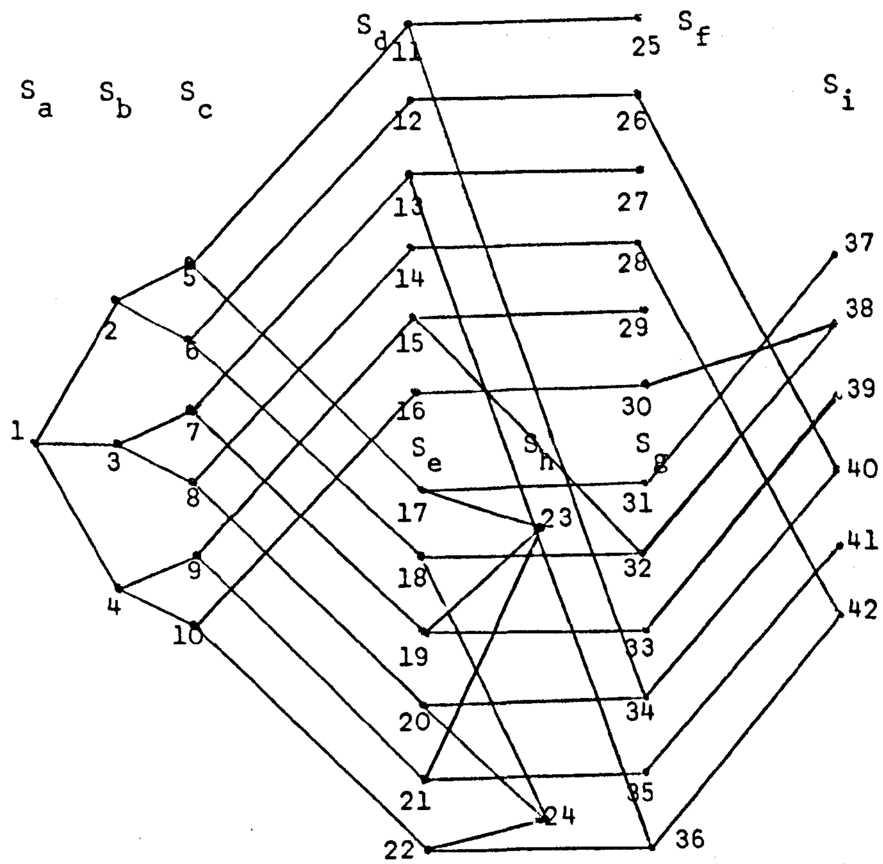


Now vertex 11 is not adjacent to vertices 31,32,33,35 by girth considerations and so without loss of generality we may set 11 adjacent to 34. It follows that 15 adj 32 and 13 adj 36.

Consider 2-arcs beginning at vertex 1, for example (1,2,5). The two extensions to 3-arcs behave differently.One,of type A say,extends to exactly one octagon.In the example (1,2,5,11) extends to <1,2,5,11,34,20,8,3>. The other,in the example

(1,2,5,17), extends to one 4-arc of type B which extends to two octagons,(here (1,2,5,17,23) extends to <1,2,5,17,23,19,7,3> and <1,2,5,17,23,21,9,4>) and to another 4-arc,of type C,which extends to one octagon,(here (1,2,5,17,31) extends to <1,2,5,17,31,$\hat{d}$,$\hat{c}$,$\hat{b}$ >). Every 2-arc in the graph may be extended in these ways.

Consider the 2-arc (2,5,11). Then (2,5,11,34,20) extends to two octagons <2,5,11,34,20,24,18,6> and <2,5,11,34,20,8,3,1> . Hence (2,5,11,34,40) is of type C, and so 40 adj 26. Similarly 38 adj 30 and 42 adj 28. Thus we have

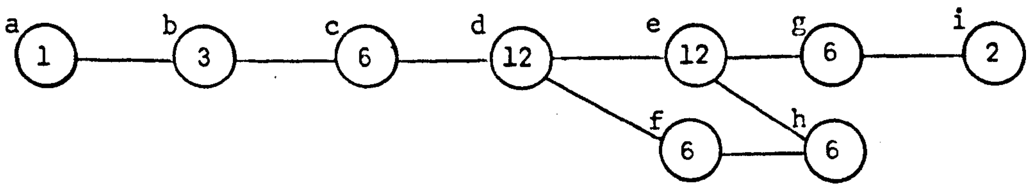


Now consider vertices whose distance from vertex 1 is 2. They split into two sets of three, {5,7,9},{6,8,10}, each of which has a common vertex at distance 2 other than vertex 1, vertices 23 and 24 respectively.So the same must be true of those vertices whose distance from vertex 23 is 2. Vertices 5,7 and 9 clearly form one set,since each of them is distance

2 from vertex 1. The other set must therefore consist of vertices 31,33 and 35 and their common vertex, x say, clearly lies in S_f. Now there must be two arcs of the form $(x,\hat{i},y)$ for some $y\varepsilon\{31,33,35\}$, and one arc of the form $(x,\hat{d},y)$ for some $y\varepsilon\{31,33,35\}$. But for the first two arcs to exist x must be one of 25,27 and 29, and for the third to exist x must be one of 26,28 and 30, a contradiction. //

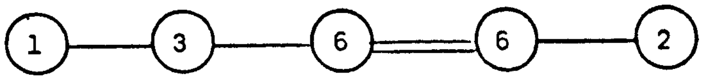
7.3.11.1: Lemma. Any transitive graph whose singleton-quotient is 6.2.4.41 is a 3-covering of the Pappus graph.

6.2.4.41:



Proof. Let G be a transitive graph with this singleton-quotient. Then the girth of G is 8 and the diameter is 6. A path from one vertex in S_i to the other is of the form $(\hat{i},\hat{g},\hat{e},\hat{h},\hat{e},\hat{g},\hat{i})$ if it is to be as short as possible, so that the vertices of S_i have distance 6 from each other and the graph is antipodal.

Consider a vertex, 2 say, in S_b. There are four vertices of distance 3 from vertex 2 in S_e and so at most four of distance 4 from vertex 2 in S_g. The others must have distance 6 from 2. Thus the vertices antipodal to vertex 2 lie in S_g, and the antipodal system is a (0,1)-decomposition. The reduced quotient corresponding to the closure of the singleton-decomposition and the antipodal system of G is easily seen to be

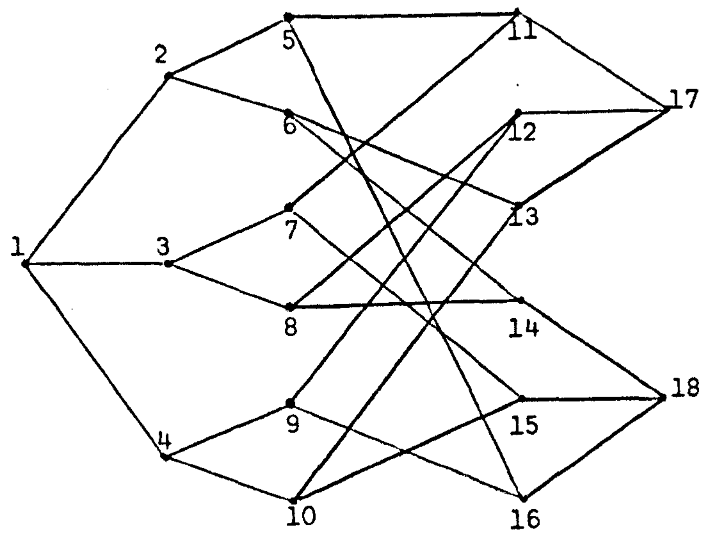


and the only transitive graph with this singleton-quotient is the Pappus graph by Proposition 7.3.2. //

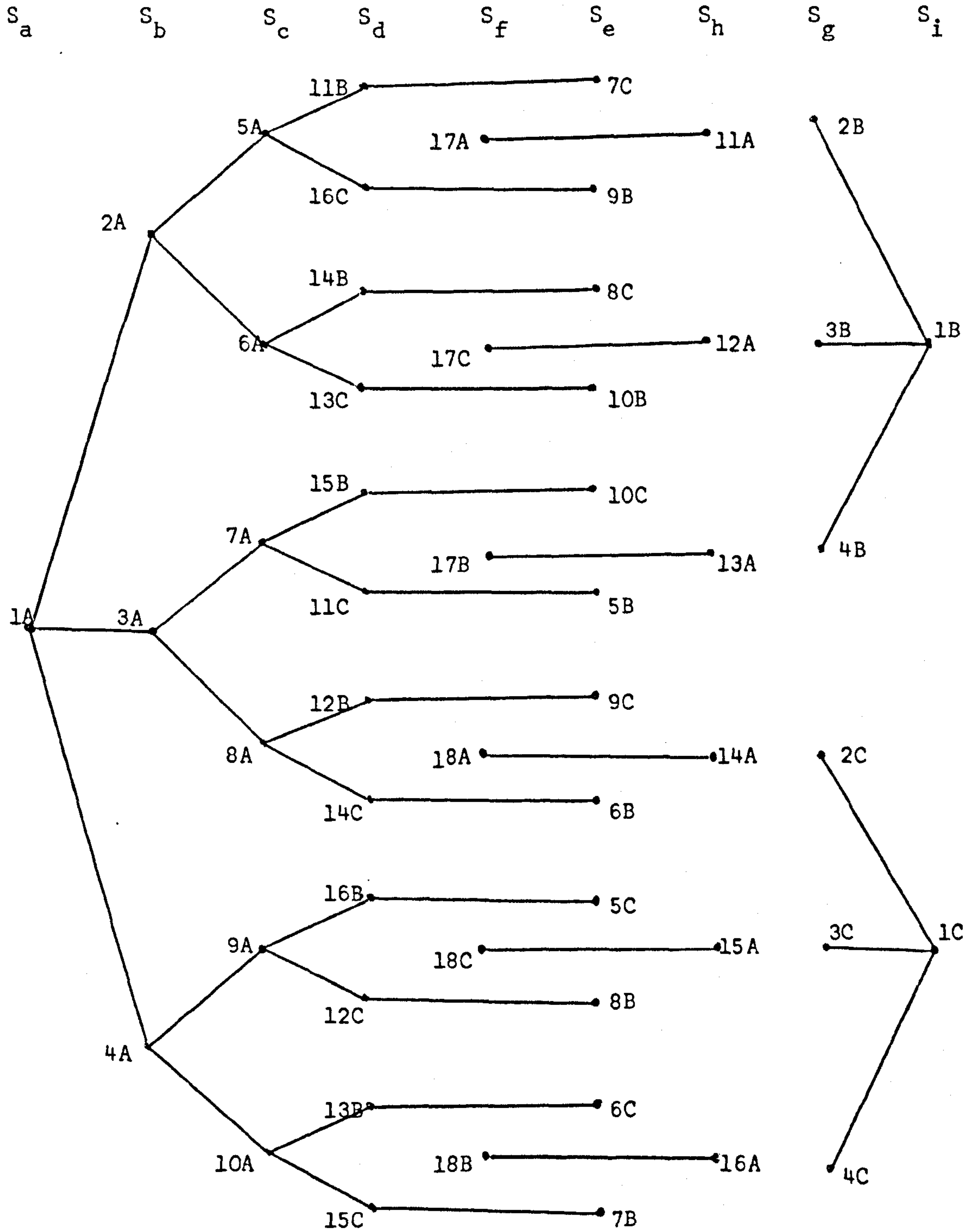
7.3.11.2:Proposition. There is no transitive graph whose singleton-quotient is 6.2.4.41.

Proof. Suppose G is such a graph. Then by the lemma G is a 3-covering of the Pappus graph, P.

P:



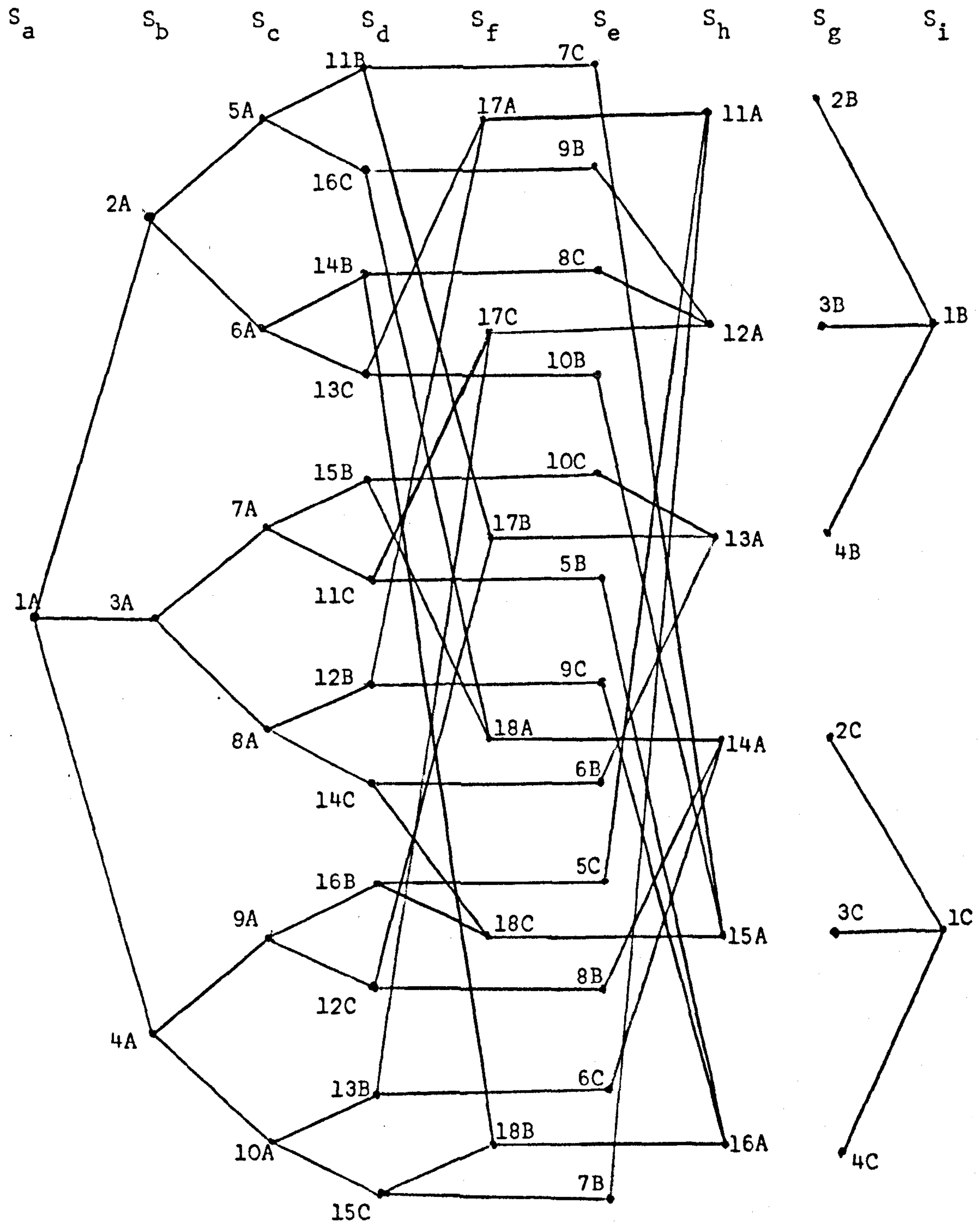
Let S_a={1A}, S_b consist of 2-4A, S_c of 5-10A, S_d of 11-16B and C, S_g of 2-4B and C, and S_i of 1B and C. Then 2B and 2C lie in S_g and are adjacent to 5B,5C,6B and 6C. Thus 5B,5C,6B,6C lie in S_e, and similarly so do 7-10B and C. In the same way S_f consists of 17 and 18A-C and S_h consists of 11-16A. Thus without loss of generality we may construct G so far as follows:



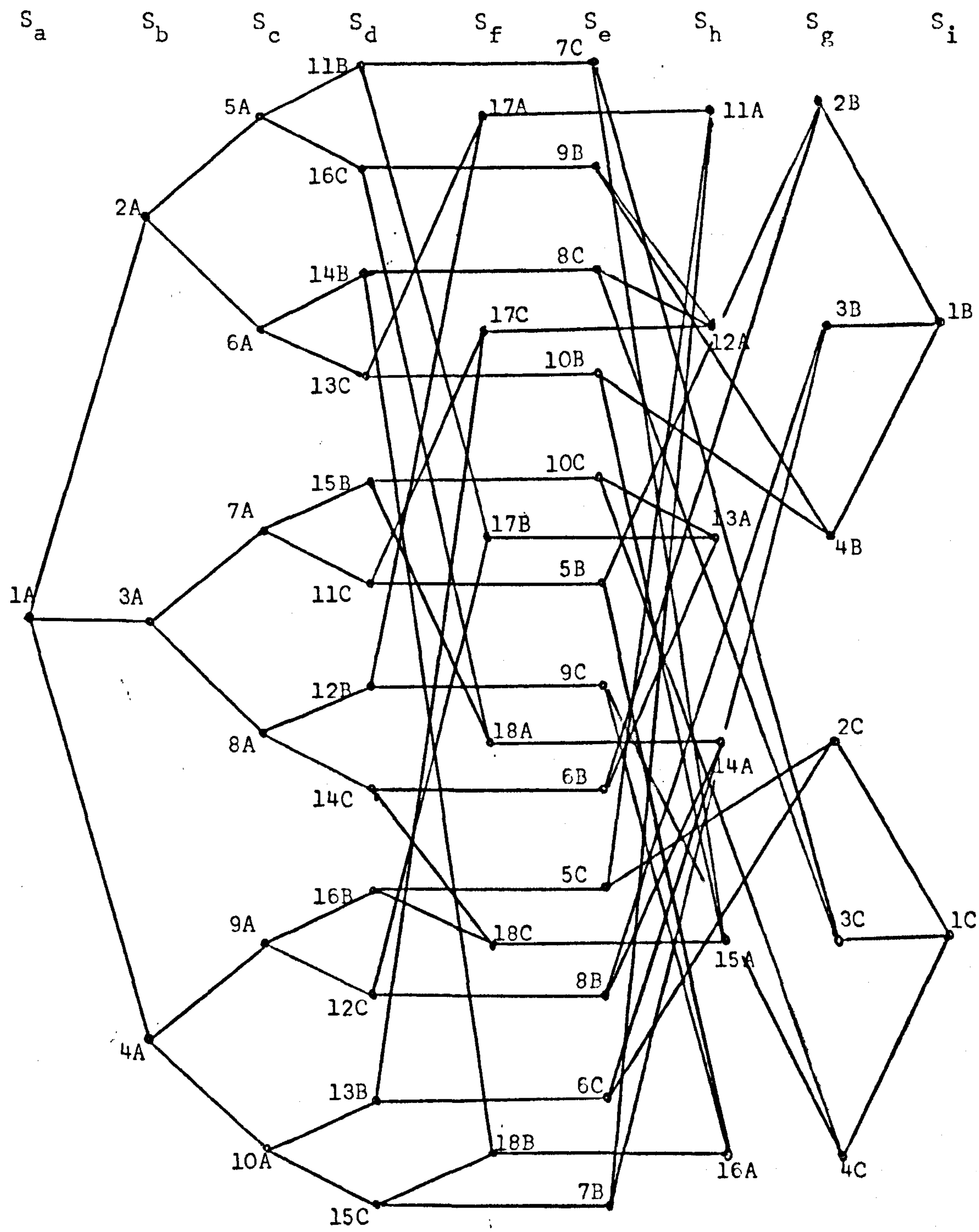
S_a
S_b
S_c
S_d
S_f
S_e
S_h
S_g
S_i
11B
7C
5A
17A
11A
2B
16C
9B
2A
14B
8C
6A
17C
12A
3B
1B
13C
10B
15B
10C
7A
17B
13A
4B
11C
5B
1A
3A
12B
9C
8A
18A
14A
2C
14C
6B
16B
5C
9A
18C
15A
3C
1C
12C
8B
4A
13B
6C
10A
18B
16A
4C
15C
7B

In P, 7 adj 3,11,15. So 7C adj 15A and so on, determining all the edges between S_e and S_h.

Suppose 17A adj 13B. Then <13B,17A,11A,7B,15C,10A> is a hexagon, which is a contradiction. Hence 17A adj 13C and so on, determining all the edges between S_d and S_f. At this stage we have:



Without loss of generality 2B adj 5B. Then 2B not adj 6C by girth considerations (there would otherwise be a hexagon <2B,6C,13B,17C,11C,5B>), and so 2B adj 6B, and 2C is adjacent to 5C and 6C. 3C not adj 7B (to avoid the hexagon <3C,7B,11A, 5C,2C,1C>). So 3C adj 7C and so on,and the rest of the edges are determined.So the construction is unique up to isomorphism.

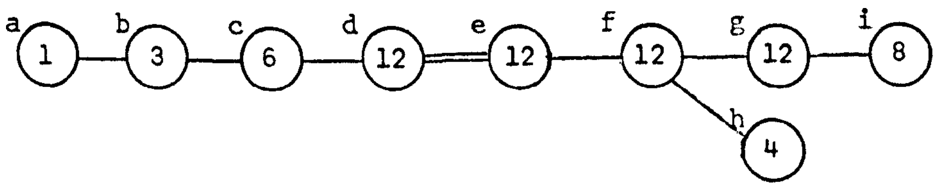


The graph constructed is not transitive. For suppose it is. Then the antipodal system is a block system, so that if an automorphism γ of G takes 1A to 2A, then 1B and 1C are taken by γ to 2B and 2C in some order. We consider the images of 17 and 18A-C under the action of γ. These are the vertices whose distance from 2A is 4 and which lie on octagons containing 2A, that is to say 15B,12B,12C,15C,4B,3C. Now 2B has distance 2 from 4B, and since γ is an automorphism, 1B or 1C must have distance 2 from one of 17 and 18A-C. But this is not the case and we have a contradiction. //

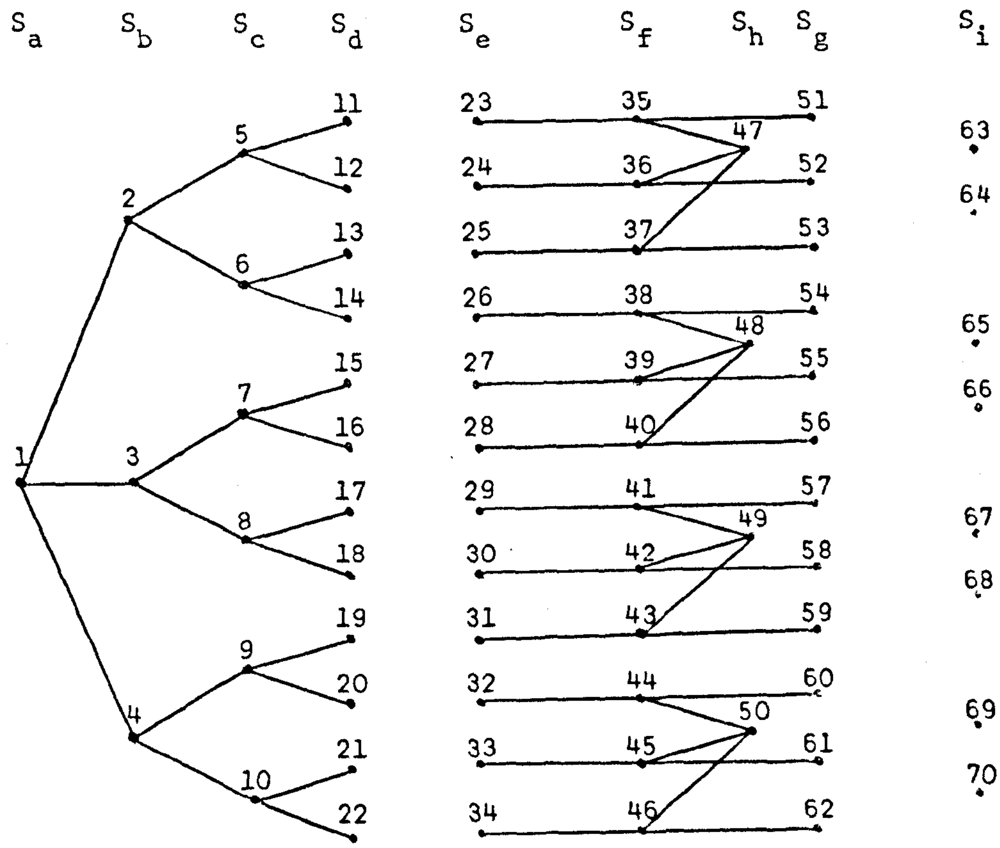
7.3.11.3:Note. The graph constructed above is not even singleton-regular. //

7.3.12:Proposition. There is no transitive graph whose singleton-quotient is 6.2.4.43.

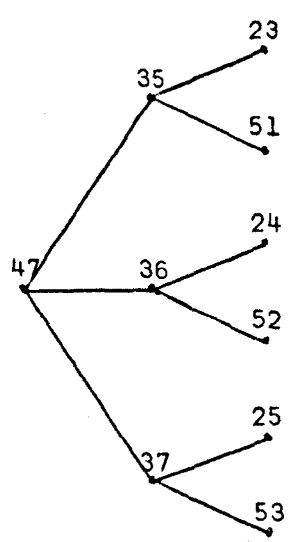
6.2.4.43:



Proof. Let G be a transitive graph with this singleton-quotient. Then G has girth 8 and any 4-arc in G extends to exactly one octagon. The graph may be constructed so far without loss of generality:



Since G is transitive,it may be redrawn (in part) without loss of generality:



Now 23,24 and 25 are each adjacent to two vertices of S_d,whereas 51,52 and 53 are each adjacent to two vertices of S_i.Further the distance between a vertex of S_d and a vertex of S_i is greater than 2. Hence,returning to the original diagram,we see that the vertices of S_d may be partitioned into two sets $A=\{11,12,15,16,19,20\}$ and $B=\{13,14,17,18,21,22\}$ without loss of generality so that if $a\varepsilon A$ and $b\varepsilon B$, then $d(a,b)>2$.

Now there may be no vertex of S_e adjacent to both 11 and 12 by consideration of the girth of G, so without less of generality the distance between vertices 11 and 15 is 2. Considering the 4-arc (7,3,1,2,5) we see that 12 cannot have distance 2 from vertex 15 or 16, so the distance of 12 from say 19 is 2. Similarly 16 shares a neighbour with one of 19 and 20. Now vertex 20 has at least one other neighbour in S_e, say vertex x. But x is adjacent to neither 15 nor 16 by consideration of the 4-arc (9,4,1,3,7), and similarly x is adjacent to neither of 11 and 12 by consideration of (9,4,1,2,5). Thus we have a contradiction. //

<u>7.3.13:Proposition</u>.The only transitive graph whose singleton-quotient is 6.2.4.44 is the 3-covering of Tutte's graph constructed by Smith (39).

<u>Proof</u>.Any such graph is easily shown to be a 3-covering of Tutte's graph by the method of Example 7.3.1.2. The existence and uniqueness of a transitive covering graph is established by Smith (39). //

Proposition 6.2.4 and Propositions 7.3.2-13 give us all transitive trivalent graphs whose rank is 9 or less.

<u>7.3.14:Proposition</u>.The only transitive trivalent graphs of rank ≤ 9 are the following:

Graph	Rank,	order,	arc-transitivity
K_4	2	4	2
$K_{3,3}$	3	6	3
P(5,2)	3	10	3

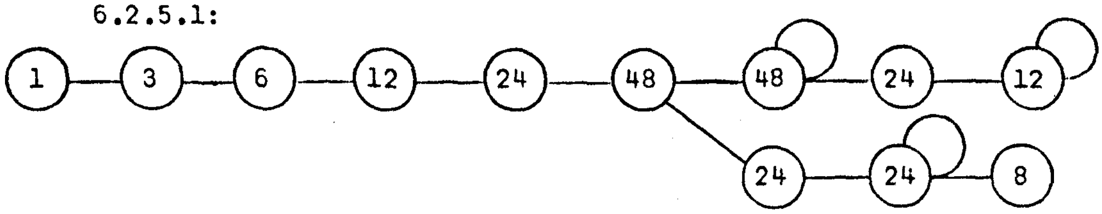
Graph	Rank	order	arc-transitivity
P(3,1)	4	6	0
Q_3	4	8	2
H	4	14	4
M(5)	5	8	0
P	5	18	3
TC	5	28	3
T	5	30	5
P(5,1)	6	10	0
M(5)	6	10	0
P(8,3)	6	16	2
P(10,2)	6	20	2
P(10,3)	6	20	3
$T(K_4)$	7	12	0
M(6)	7	12	0
P(12,5)	7	24	2
P(6,1)	8	12	0
R (see 7.3.2)	8	12	0
P(7,1)	8	14	0
M(7)	8	14	0
$T(K_{3,3})$	8	18	0
Graph given in 7.3.8.2	8	102	4
M(8)	9	16	0
$P(10,2)\wedge K_2$	9	40	3
3-covering of T	9	90	5

//

We now consider the remaining generalised graphs constructed in 6.2.5.

7.3.15:Proposition. There is only one t-arc-transitive graph with rank $r \leq t+8$ and singleton-quotient 6.2.5.1.

6.2.5.1:



This graph is constructed as follows:- The vertices correspond to the 234 triangles in PG(2,3) and two vertices are adjacent whenever the corresponding triangles have one common point and their remaining four points are distinct and collinear.The graph is primitive with order 234, rank 12 and is 5-arc-transitive.

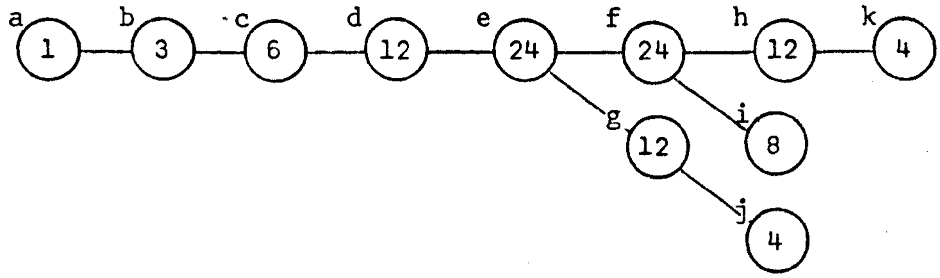
Proof. 6.2.5.1 has no non-trivial decomposition of order greater than 1. Suppose G is a transitive graph with this singleton-quotient and suppose that a class of the singleton-decomposition consists of more than one orbit of the stabiliser of vertex 1. Then it is clear that at least two classes consist of more than one orbit and so the rank $r>13$. Now $t\leq 5$, so that if $r\leq t+8$ then every class consists of a single orbit.Hence by 7.2.3.2 G is primitive.

There is exactly one primitive trivalent graph of the desired order,since there is only one primitive permutation group of degree 234 where the stabiliser of one symbol has an orbit of length 3, and this stabiliser has only one orbit of length 3 (Wong (47)).

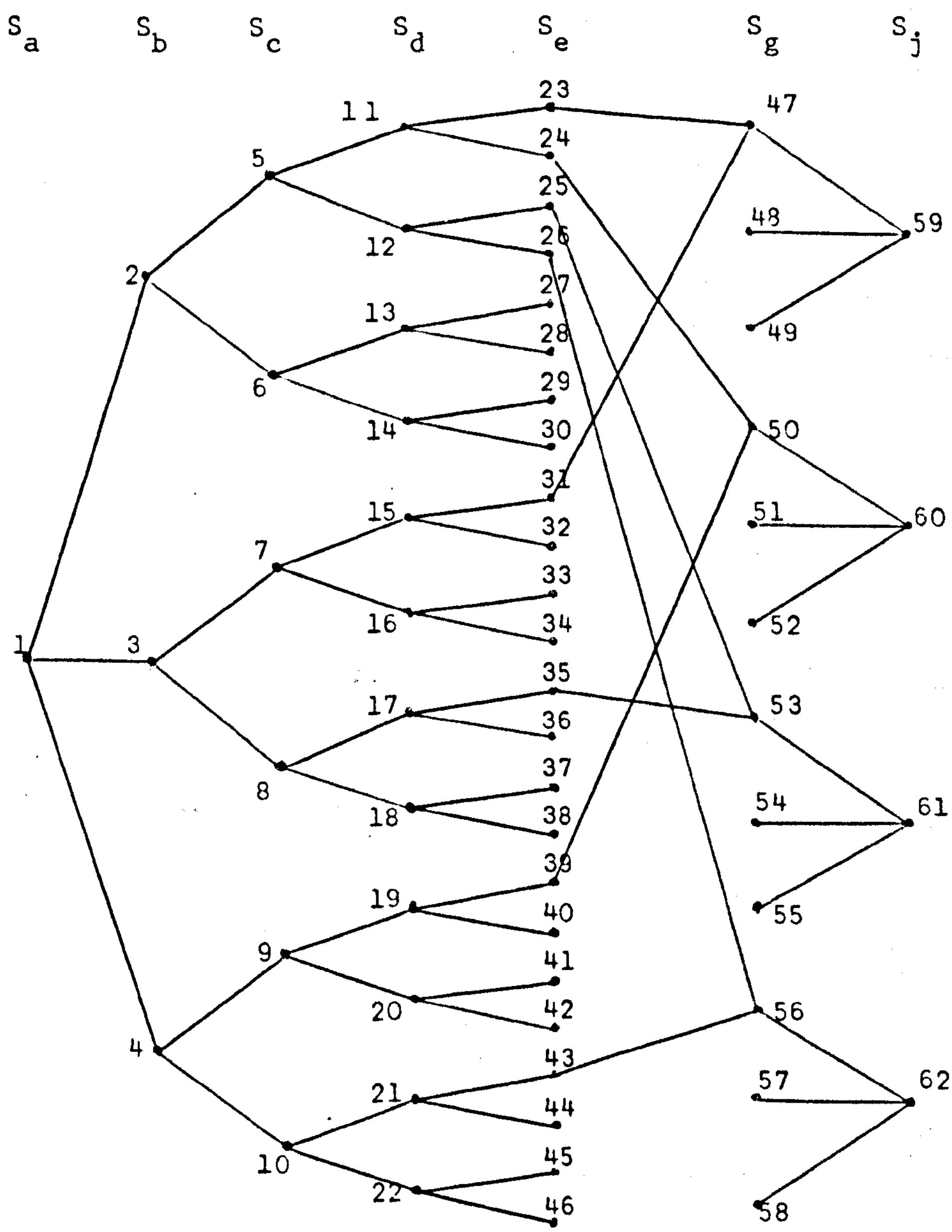
The construction of this graph is given by Biggs (4,p125) and it may be verified directly that it has the desired singleton-quotient. //

7.3.16:Proposition.There is no transitive graph whose singleton-quotient is 6.2.5.2.

6.2.5.2:



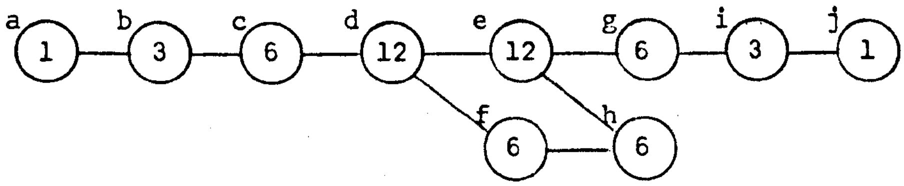
Proof.Let G be a transitive graph with this singleton-quotient. Then G has girth 10 and every 4-arc can be extended to exactly one decagon.We may construct G so far without loss of generality:



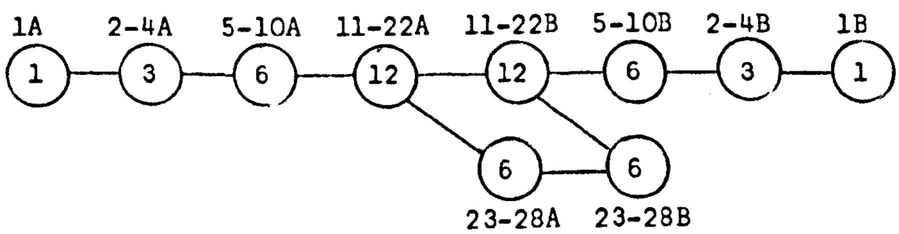
Without loss of generality 27 adj 48. But then the 4-arc (2,5,11,23,47) can be extended to two decagons, <2,5,11,23, 47,31,15,7,3,1> and <2,5,11,23,47,59,48,27,13,6>, so we have a contradiction. //

<u>7.3.17:Proposition</u>.The only transitive graph whose singleton-quotient is 6.2.5.3 is the double-covering of the Tutte-Coxeter graph, $TC\wedge K_2$, which has order 56, rank 10 and is 3-arc-transitive.

6.2.5.3:



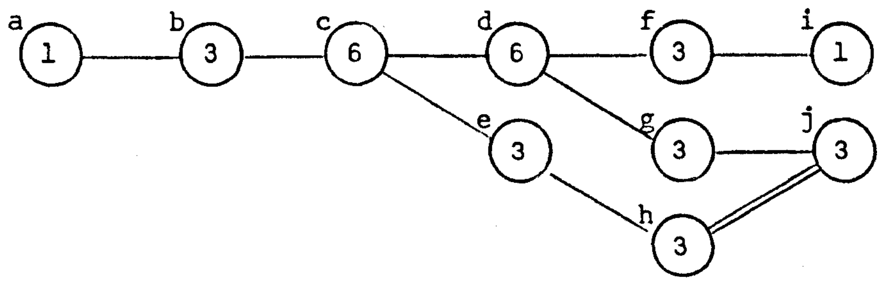
<u>Proof</u>.Let G be a transitive graph with this singleton-quotient. Then it is easily established using the methods of Example 7.3.1.2 that G is a double-covering of TC. We consider TC to be labelled as in Proposition 7.3.3 and label the vertices of G 1-28A and B as in Propositions 7.3.7.2 ,7.3.9.2 and 7.3.11.2. We may say that the vertices of S_a,S_b,S_c,S_d are 1-22A. Now the girth of G is 8 so 11-22A are adjacent to 11-22B. Without loss of generality 11-22A are adjacent to 23-28A and then 23-28A are adjacent to 23-28B and 23-28B are adjacent to 11-22B in order that G has the desired singleton-quotient.So the construction is unique:



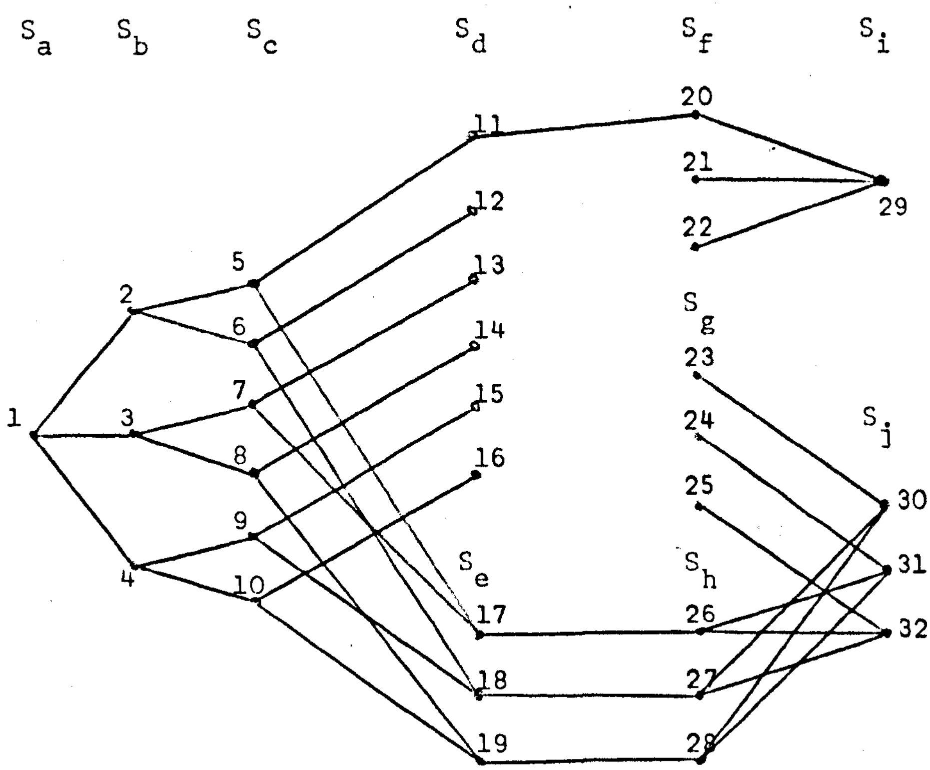
There is known to be a 3-arc-transitive double-covering of TC, that is $TC\wedge K_2$ and it is easily shown that this graph has the desired singleton-quotient. //

7.3.18:Proposition.The only transitive graph whose singleton-quotient is 6.2.5.7 is the hexagonal tessellation $\{6,3\}_{4,0}$ which has order 32, rank 10 and is 2-arc-transitive.

6.2.5.7:



Proof.Let G be a transitive graph with this singleton-quotient. Then G has girth 6 and every 2-arc extends to exactly one hexagon. We may construct G so far without loss of generality:

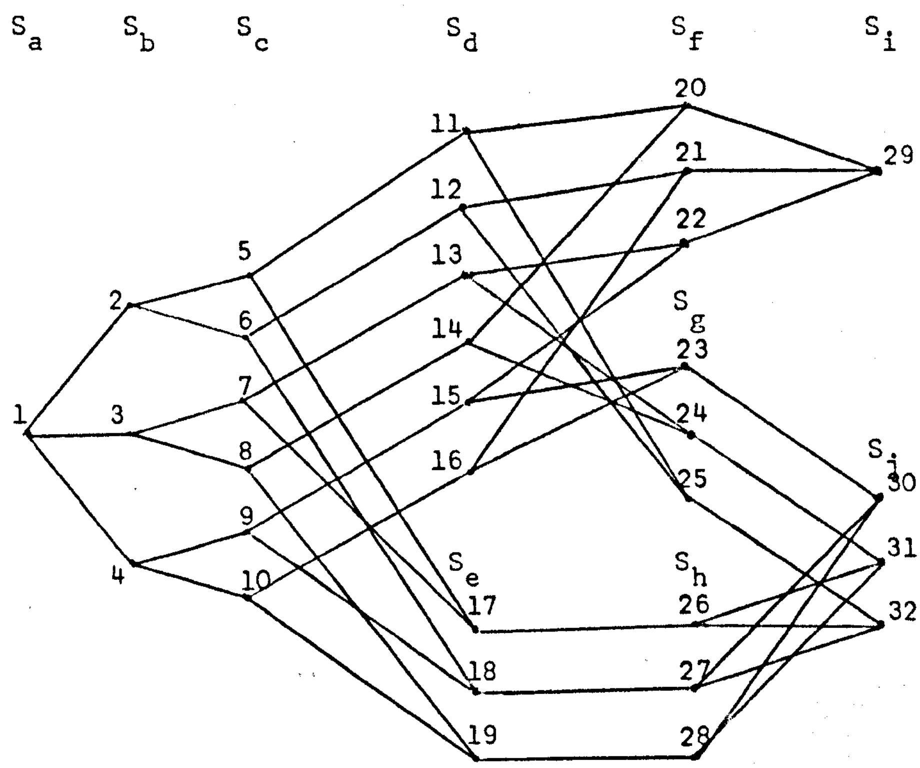


In order that (17,5,11) may extend to a hexagon,11 not adj 23, and similarly 13 not adj 23, 24 adj neither 12 nor 15, and 25 adj neither 14 nor 16.

In order that (5,17,7) should not extend to two hexagons, 11 and 13 do not have a common neighbour.Similarly 12 and 15 do not share a neighbour, and neither do 14 and 16.

Consider the neighbours of vertex 20. We know that 20 not adj 13. Suppose that 20 adj 14. Then so that (2,5,11) extends to a hexagon, 11 and 12 have a common neighbour in S_g, which must be 25. By similar arguments 13 and 14 adj 24, and 15 and 16 adj 23.

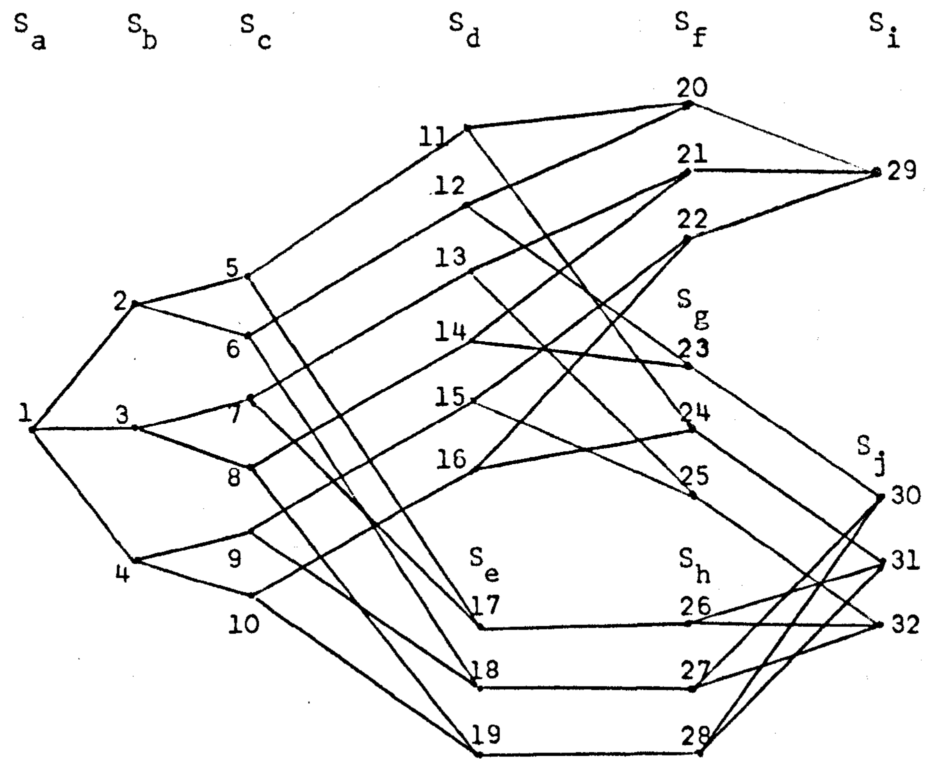
Without loss of generality 21 adj 12, and then by girth and since 12 and 15 do not have a common neighbour, 21 adj 16, and 22 adj 13 and 15. So we have:



It is easily seen that the graph so constructed is not transitive, since if we consider the vertices of distance 5 from vertex 2, that is 22,23,24 and 28, we find that each of them has distance 2 from two of the others,which is not the case for vertices of distance 5 from vertex 1. So 20 not adj 14. In exactly the same way 20 adj neither 15 nor 16.

Thus 20 adj 12, and it follows that 13 and 14 must have a common neighbour in S_f, so without loss of generality 21 adj

13 and 14, and 15 and 16 adj 22. Now we note that the vertices whose distance from vertex 2 is 5 are 21,22,28 and one of S_g. d(21,22)=2 and both d(21,28) and d(22,28) are at least 4. Hence they must each have distance 2 from the member of S_g wanted,which must in turn have distance at least 4 from 28. Thus the vertex in S_g is 25. Hence 25 adj 13 and 15. Then by girth 12 adj 23, and 11 adj 24. Finally considering vertices of distance 5 from vertex 3, 20,22,27 and a member of S_g,this member of S_g must be 24. Hence 24 not adj 14 (whose distance from vertex 3 is 2), so 24 adj 16 and23 adj 14. Thus the construction is unique.



Now there is known to be a 2-arc-transitive graph on 32 vertices and of valency 3, that is the tessellation $\{6,3\}_{4,0}$.(Coxeter and Moser (11)). Referring to 6.2.9 we see that any symmetric graph on 32 vertices must have singleton-quotient 6.2.4.38, 6.2.5.7 or 6.2.9.2. But 6.2.4.38 and 6.2.9.2 have no transitive realisations by 7.3.2. thus the tessellation is a realisation of 6.2.5.7. By inspection the rank is 10. //

To summarise the results of these propositions we have:

7.3.19:Proposition. The only t-arc-transitive trivalent graphs of rank r with $r \leq t+8$ not included in 7.3.14 are

i) the primitive trivalent graph on 234 vertices whose rank is 12 and which is 5-arc-transitive,

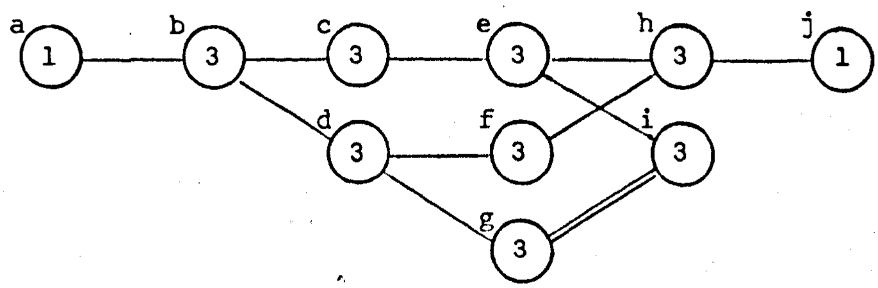
ii) $TC \wedge K_2$ which has order 56, rank 10 and is 3-arc-transitive,

iii) the hexagonal tessellation $\{6,3\}_{4,0}$ which has order 32, rank 10 and is 2-arc-transitive. //

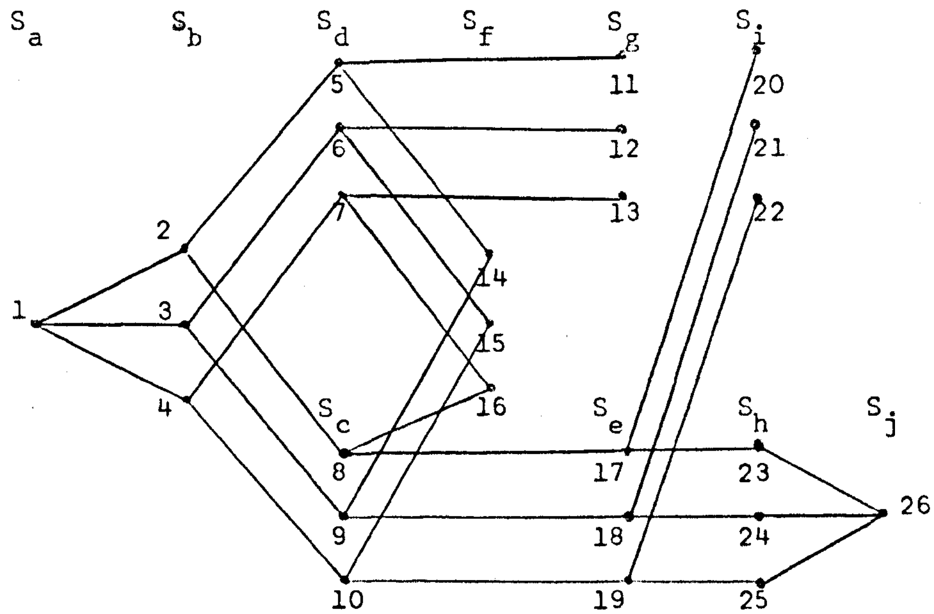
Finally we consider the remaining generalised graphs which appear in 6.2.9.

7.3.20:Proposition. The only transitive graph whose singleton-quotient is 6.2.9.1 is the hexagonal tessellation $\{6,3\}_{3,1}$ which has order 26, rank 10 and is 1-arc-transitive.

6.2.9.1:



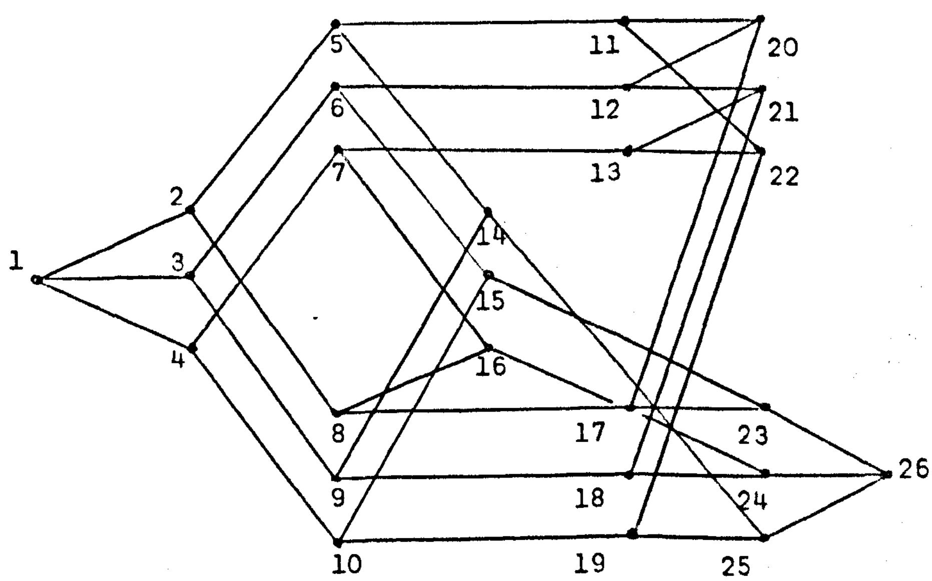
Proof. Let G be a transitive graph with this singleton-quotient. Then G has girth 6 and every 2-arc extends to exactly one hexagon. G may be constructed so far without loss of generality:



By consideration of the girth 14 not adj 24. So that (2,5,14) does not extend to two hexagons, 14 not adj 23. Hence 14 adj 25, 15 adj 23 and 16 adj 24.

Consider the arc (5,14,25). It can only be extended to a hexagon via S_i. Hence 22 adj 11, and similarly 20 adj 12 and 21 adj 13.

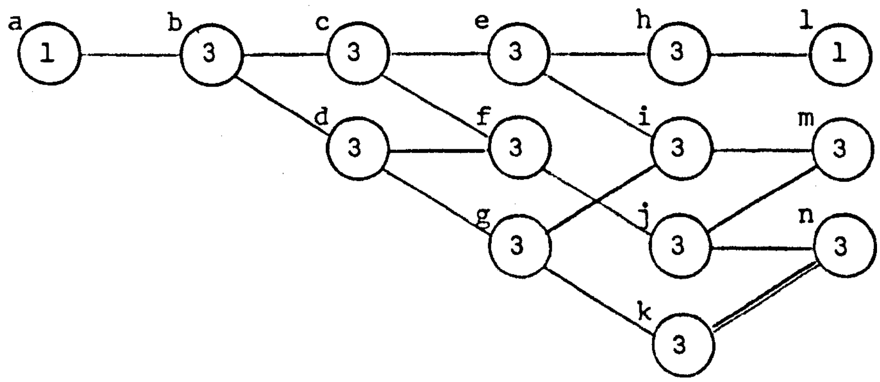
Suppose that 11 adj 21, from which it follows that 12 adj 22 and 13 adj 20. Then the arc (2,5,11) does not extend to a hexagon. Hence 11 adj 20, 12 adj 21 and 13 adj 22, so that the construction is unique:



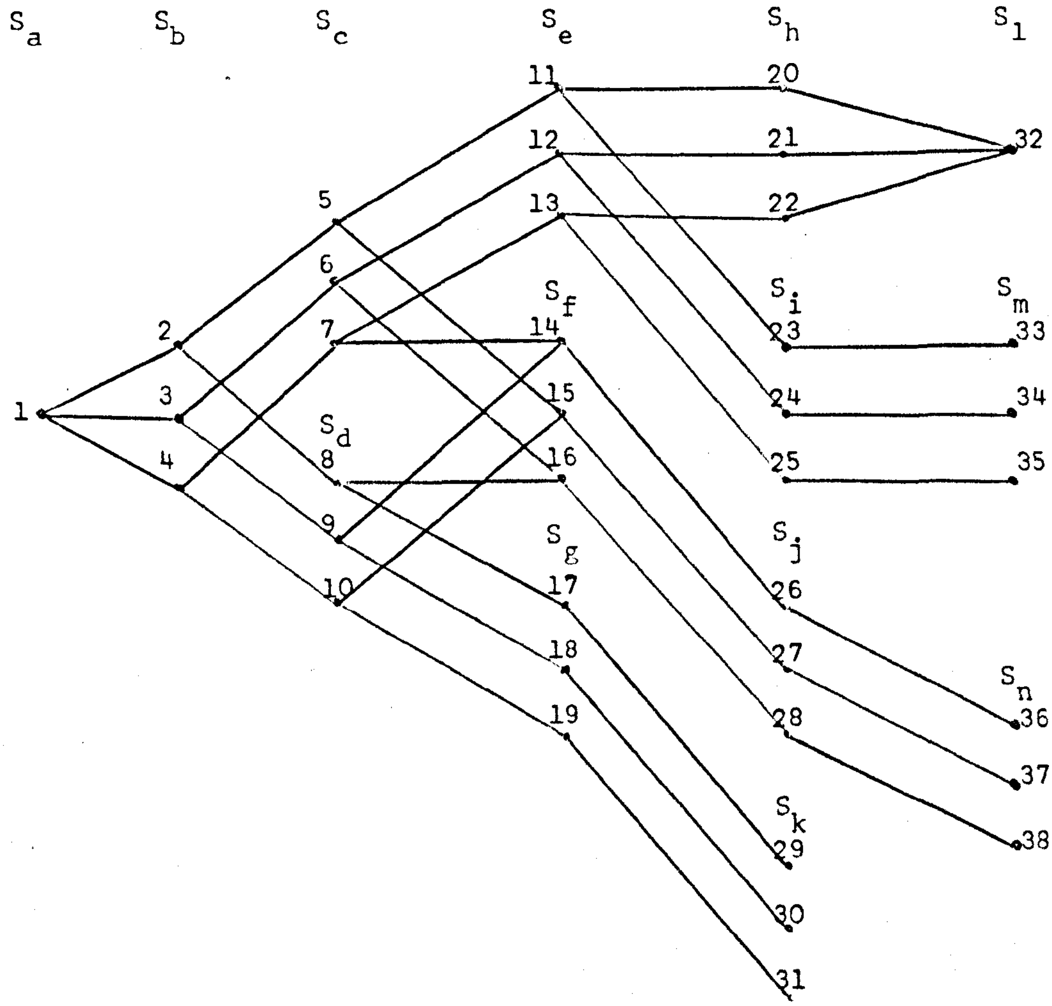
Considering 6.2.9, no other graph can be a symmetric trivalent graph on 26 vertices. But $\{6,3\}_{3,1}$ is known to be symmetric and to have the desired order, and hence is the graph we have constructed. Its rank is 10 by inspection. //

7.3.21: Proposition. The only transitive graph whose singleton-quotient is 6.2.9.4 is the hexagonal tessellation $\{6,3\}_{3,2}$ which has order 38, rank 14 and is 1-arc-transitive.

6.2.9.4:



Proof. Let G be a transitive graph with this singleton-quotient. Then G has girth 6 and every 2-arc extends to exactly one hexagon. G may be constructed so far without loss of generality:

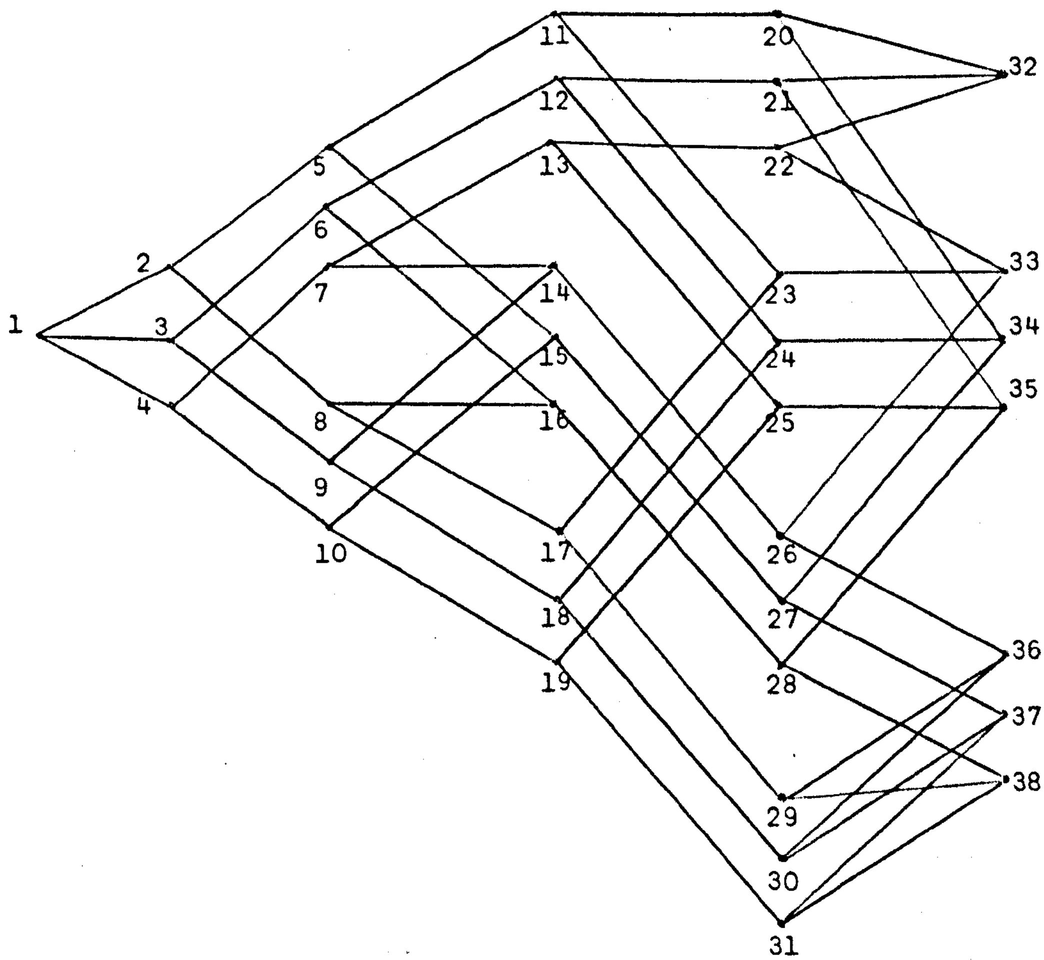


There are 9 vertices whose distance from vertex 1 is 3. We already have 9 vertices of distance 3 from vertex 2, and 17 is adjacent to a vertex of S_i. Hence 17 adj 23. Similarly 18 adj 24, and 19 adj 25.

Consider the arc (5,11,20). If it is to extend to a hexagon 20 and 27 must have a common neighbour in S_m, and similarly so must the pairs 21,28 and 22,26.

By consideration of girth 20 not adj 33, 21 not adj 34 and 22 not adj 35, so that also 27 not adj 33, 28 not adj 34 and 26 not adj 35.

Consider the 2-arcs (17,29,$\hat{n}$). Both must extend to a hexagon, one via vertices 28,16 and 8, so that for this arc $\hat{n}$=38, and one via vertices $\hat{j}$,33 and 23. But 27 not adj 33 so that in this case $\hat{n}=36$ and $\hat{j}=26$. Thus 29 adj 36 and 38, and 26 adj 33. Similarly 30 adj 36 and 37, 31 adj 37 and 38, 21 adj 34, and 28 adj 35. And from above 20 adj 34, 21 adj 35, and 22 adj 33. So the construction is unique:



Considering 6.2.9, there can be no other symmetric trivalent graph on 38 vertices.But as before $\{6,3\}_{3,2}$ is known to be symmetric and to have the desired order,and hence is the graph we have constructed.Its rank is 14 by inspection. //

To summarise our results on symmetric trivalent graphs whose order is ≤ 40, we have the following proposition:

7.3.22:Proposition.The only symmetric trivalent graphs on ≤ 40 vertices are the following:

Graph	Order, rank, arc-transitivity		
K_4	4	2	2
$K_{3,3}$	6	3	3
Q_3	8	3	2
P(5,2)	10	3	3
H	14	4	4
P(8,3)	16	6	2
P	18	5	3
P(10,2)	20	6	2
P(10,3)	20	6	3
P(12,5)	24	7	2
$\{6,3\}_{3,1}$	26	10	1
TC	28	5	3
T	30	5	5
$\{6,3\}_{4,0}$	32	10	2
$\{6,3\}_{3,2}$	38	14	1
$P(10,2)\wedge K_2$	40	9	3

//

It is worth noting that the list given above agrees exactly, so far as it goes, with the non-exhaustive list prepared by Foster (19) to whom my gratitude is due for identifying some of the graphs constructed as hexagonal tessellations.

APPENDIX 1:A FORTRAN PROGRAM FOR ALGORITHM 6.2.3.

We give a program for the implementation of Algorithm 6.2.3, case iii),since this program is the most complex and those for the other related algorithms can easily be constructed by simple deletions or substitutions.The program was written in extended FORTRAN for the University of Manchester Regional Computer Centre ICL1906/CDC7600. Non-ANSI FORTRAN and special instructions are marked with a "*". Nearly all such involve the use of an expression as an array subscript,an extension of FORTRAN implemented on most modern machines.

```
*     PROGRAM SYMM(INPUT,OUTPUT,TAPE7=INPUT,TAPE2=OUTPUT,
     1DEBUG=OUTPUT)
*     IMPLICIT INTEGER(A-Z)
      DIMENSION SIZE(30),GRM(30,3),DIST(3,3),GRO(20,20,60)
      DIMENSION CAN(30),NCPM(20),TREE(2500,22),TIP(20)
      COMMON GRO,TREE,TREENO
*     LEVEL 2,GRO,TREE,TREENO
```

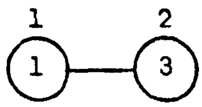
The first step is to read the number of different orders for which we wish to construct all strictly feasible generalised graphs.In the modified version for Algorithm 6.2.8, we read instead the number of different values of SIZE to be investigated.

```
      READ(7,105) NUMBER
  105 FORMAT(I2)
      DO 400 NREP=1,NUMBER
```

For each case we must read the order and set the initial values.

```
      STEP=1
```

```
      IFLAG=0
      DO 10 I=1,20
      TIP(I)=0
      NCPM(I)=0
      DO 10 J=1,20
      DO 10 K=1,60
   10 GRO(I,J,K)=0
      READ(7,105) ORDER
      DO 25 I=1,2500
      DO 25 J=1,22
   25 TREE(I,J)=0
      TREENO=1
```

Next we set the first member of the first stack. 

is represented by

SIZE	GRM		
1	2	2	2
3	1	0	0
0	0	0	0
.	.	.	.

```
      GRO(1,1,1)=1
      GRO(1,1,2)=2
      GRO(1,1,3)=2
      GRO(1,1,4)=2
      GRO(1,1,5)=3
      GRO(1,1,6)=1
      TIP(1)=1
```

If there are entries in the stack for the current value of STEP we move onto the next stage (label 35). If there are none, we reduce STEP by 1. If STEP is zero we have constructed every strictly feasible generalised graph for the current value of ORDER and we move on to its next value.If STEP is not zero we again check whether there are any entries in the current stack and repeat the procedure.

```
   30 IF(TIP(STEP).NE.0) GO TO 35
      STEP=STEP-1
      IF(STEP.EQ.0) GO TO 400
      GO TO 30
```

Given that the current stack has an entry we now pop the stack to give us the current values of SIZE and GRM.

```
   35 DO 50 I=1,ORDER
*     SIZE(I)=GRO(STEP,TIP(STEP),4*I-3)
      DO 50 J=1,3
*  50 GRM(I,J)=GRO(STEP,TIP(STEP),4*I-3+J)
      TIP(STEP)=TIP(STEP)-1
```

We look for the first vertex which is connected to at least one other vertex but which still has neighbours to be determined. If we find one we move on to the stage beginning at label 140. If there is no such vertex then there are two possibilities. The first is that the generalised graph we are constructing has to be disconnected,so we reject it by returning to label 30. The second is that we have completed the construction of a so far satisfactory generalised graph.We test it thus:-

i) It must not contain subgraphs

or

ii).It must be irreducible and it must not be isomorphic to any (rooted) generalised graph already constructed.Subroutine

CODE checks these points.

iii) It must satisfy the integrity condition,which is tested by subroutine FEAS.

If the generalised graph passes all these tests it is strictly feasible and we output GRM/SIZE and other pertinent information (this is accomplished as part of FEAS). Anyway having made the tests we return to label 30 to look for the next GRM/SIZE in the current stack.

```
      K=0
      DO 120 I=1,ORDER
      IF(GRM(I,1).EQ.0) GO TO 120
      K=K+1
      DO 130 J=2,3
      IF(GRM(I,J).EQ.0) GO TO 140
  130 CONTINUE
  120 CONTINUE
      IF(K.EQ.ORDER) GO TO 160
      GO TO 30
  160 CONTINUE
      DO 560 I=1,ORDER
      IF(SIZE(I).NE.1) GO TO 560
      DO 550 J=1,3
      DO 550 K=1,3
*     DIST(J,K)=GRM(GRM(I,J),K)
      DO 550 L=1,3
*     IF(SIZE(GRM(DIST(J,K),L)).EQ.1) GO TO 30
*     IF(SIZE(GRM(DIST(J,K),L)).EQ.2) GO TO 30
  550 CONTINUE
  560 CONTINUE
```

```
      CALL CODE(GRM,CAN,ORDER)
      IF(CAN(1).EQ.0) GO TO 30
      CALL FEAS(GRM,SIZE,ORDER)
      GO TO 30
```

When there is a vertex which is connected to at least one other vertex but which still has neighbours to be determined we choose the first such,and the purpose of the rest of the main program is to store in the stack for the next value of STEP all possible GRM/SIZE resulting from the addition of one edge incident with this vertex to the generalised graph so far constructed.

The vertex in question is relabelled NCP (or "next connection point") and the number of zeros in the NCP-th row of GRM is labelled NOC (or "number of connections").

```
  140 NCP=I
      NOC=4-J
      NCPOLD=NCPM(STEP)
      STEP=STEP+1
      NCPM(STEP)=NCP
      IF(STEP.EQ.21) WRITE(2,290)
  290 FORMAT(1X,16HOVERFLOW IN STEP)
      TIP(STEP)=0
      NUM=2
      IF(NCPOLD.EQ.NCP) NUM=GRM(NCP,2)
```

The next DO-loop,from 200 to 211 finds all the possible additions of one edge which will remove one zero from the NCP-th row of GRM. However there are several sorts of repetition to be avoided:

i) If the stack for STEP-1 results from the addition of edges beginning at the same vertex NCP then we need not consider the possibility of edges to vertices with lower labels than that to which the edge was joined at the previous step.This is dealt with by NCPM(STEP) and NCPOLD;

ii) We only wish to consider one vertex whose size is as yet undetermined.This is dealt with by IFLAG;

iii) If two vertices have identical adjacencies we need only choose one of them.This is dealt with by the DO-loop "DO 245".

```
 200 DO 211 I=NUM,ORDER
     IF(IFLAG.EQ.1) GO TO 212
```

If the vertex to which we are considering joining NCP has undetermined size we give it its three possible sizes,SIZE(NCP) divided by 1,2 and 3 in turn.If any of these is not an integer it is rejected.

```
     KK=1
     IF(SIZE(I).NE.0) GO TO 505
     IFLAG=1
     KK=3
 505 DO 210 JJ=1,KK
     IF(KK.EQ.1) GO TO 500
     IF(SIZE(NCP)/JJ*JJ.NE.SIZE(NCP)) GO TO 210
     SIZE(I)=SIZE(NCP)/JJ
 500 CONTINUE
```

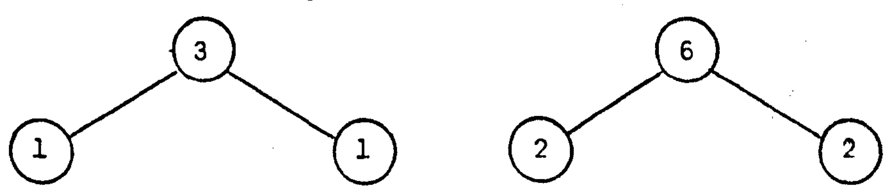
As implied above we cannot join NCP to any vertex whose size is not compatible with that of NCP, nor to any vertex which does not have enough zeros in the corresponding row of GRM.

```
      IF(SIZE(NCP)/SIZE(I)*SIZE(I).NE.SIZE(NCP)) GO TO 210
      MULT=1
      DO 220 J=1,3
      MULT=MULT*GRM(I,J)
      IF(MULT.NE.0) GO TO 220
      EDGES=(4-J)*SIZE(I)
      GO TO 230
  220 CONTINUE
      GO TO 210
  230 IF(EDGES.LT.SIZE(NCP)) GO TO 210
```

The next DO-loop excludes the consideration of vertices whose adjacencies are identical to those of a previous choice.

```
      M=I-1
      DO 245 L=1,M
      IF(L.EQ.NCP) GO TO 245
      IF(SIZE(L).NE.SIZE(I)) GO TO 245
      IDIFF=0
      DO 255 K=1,3
      IDIFF=IDIFF+IABS(GRM(I,K)-GRM(L,K))
  255 CONTINUE
      IF(IDIFF.EQ.0) GO TO 210
  245 CONTINUE
```

We exclude simple forbidden subgraphs,that is loops on vertices of size 1 or 3 and the subgraphs



```
      IF(SIZE(NCP).EQ.1) GO TO 251
      IF(SIZE(NCP).NE.3) GO TO 247
  251 IF(I.EQ.NCP) GO TO 210
      IF(SIZE(NCP).NE.3) GO TO 247
      IF(SIZE(I).NE.1) GO TO 247
      DO 249 L=1,2
      IF(GRM(NCP,L).EQ.0) GO TO 249
*     IF(SIZE(GRM(NCP,L)).EQ.1) GO TO 210
  249 CONTINUE
  247 IF(SIZE(NCP).NE.6) GO TO 253
      IF(SIZE(I).NE.2) GO TO 253
      DO 257 L=1,2
      IF(GRM(NCP,L).EQ.0) GO TO 257
*     IF(SIZE(GRM(NCP,L)).EQ.2) GO TO 210
  257 CONTINUE
  253 CONTINUE
```

Having found that it is possible to add an edge from NCP to I we do so and store the resulting GRM/SIZE in the stack for the current value of STEP.

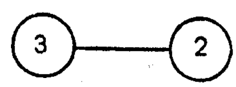
```
      TIP(STEP)=TIP(STEP)+1
      DO 235 M=1,ORDER
*     GRO(STEP,TIP(STEP),4*M-3)=SIZE(M)
      DO 235 L=1,3
*     GRO(STEP,TIP(STEP),4*M-3+L)=GRM(M,L)
  235 CONTINUE
      K=3
      IF(GRM(NCP,2).EQ.0) K=2
*     GRO(STEP,TIP(STEP),4*NCP-3+K)=I
      NN=J+SIZE(NCP)/SIZE(I)-1
```

```
      DO 240 LL=J,NN
*     GRO(STEP,TIP(STEP),4*I-3+LL)=NCP
  240 CONTINUE
      IF(TIP(STEP).EQ.21) WRITE(2,280)
  280 FORMAT(1X,17HOVERFLOW IN STACK)
  210 CONTINUE
      IF(IFLAG.EQ.1) SIZE(I)=0
  211 CONTINUE
  212 CONTINUE
      IFLAG=0
```

If NOC=2 we must consider the addition of an edge which removes both zeros from the NCP-th row of GRM, that is an edge joining NCP to a vertex whose size is two-thirds or twice that of NCP. This is the purpose of the DO-loop "DO 311" which is generally similar to "DO 211".

```
      IF(NOC.EQ.1) GO TO 30
  300 DO 311 I=2,ORDER
      IF(IFLAG.EQ.1) GO TO 312
      KK=1
      IF(SIZE(I).NE.0) GO TO 605
      IFLAG=1
      KK=3
  605 DO 310 JJ=1,KK,2
      IF(KK.EQ.1) GO TO 600
      IF(SIZE(NCP)*2/JJ*JJ.NE.SIZE(NCP)*2) GO TO 310
      SIZE(I)=SIZE(NCP)*2/JJ
  600 CONTINUE
      IF(SIZE(NCP)*2/SIZE(I)*SIZE(I).NE.SIZE(NCP)*2) GO TO 310
      IF(SIZE(NCP)/SIZE(I)*SIZE(I).EQ.SIZE(NCP)) GO TO 310
```

```
      MULT=1
      DO 320 J=1,3
      MULT=MULT*GRM(I,J)
      IF(MULT.NE.0) GO TO 320
      EDGES=(4-J)*SIZE(I)
      GO TO 330
  320 CONTINUE
      GO TO 310
  330 IF(EDGES.LT.SIZE(NCP)*2) GO TO 310
      M=I-1
      DO 345 L=1,M
      IF(L.EQ.NCP) GO TO 345
      IF(SIZE(L).NE.SIZE(I)) GO TO 345
      IDIFF=0
      DO 355 K=1,3
      IDIFF=IDIFF+IABS(GRM(I,K)-GRM(L,K))
  355 CONTINUE
      IF(IDIFF.EQ.0) GO TO 310
  345 CONTINUE
```

The only subgraph we exclude here is 


```
      IF(SIZE(NCP).NE.3) GO TO 347
      IF(SIZE(I).EQ.2) GO TO 310
  347 TIP(STEP)=TIP(STEP)+1
      DO 335 M=1,ORDER
*     GRO(STEP,TIP(STEP),4*M-3)=SIZE(M)
      DO 335 L=1,3
*     GRO(STEP,TIP(STEP),4*M-3+L)=GRM(M,L)
  335 CONTINUE
*     GRO(STEP,TIP(STEP),4*NCP-1)=I
```

```
*     GRO(STEP,TIP(STEP),4*NCP)=I
      NN=J+2*SIZE(NCP)/SIZE(I)-1
      DO 340 LL=J,NN
*     GRO(STEP,TIP(STEP),4*I-3+LL)=NCP
  340 CONTINUE
      IF(TIP(STEP).EQ.21) WRITE(2,280)
  310 CONTINUE
      IF(IFLAG.EQ.1) SIZE(I)=0
  311 CONTINUE
  312 CONTINUE
      IFLAG=0
      GO TO 30
  400 CONTINUE
      STOP
      END
```

Once we have constructed a simply feasible generalised graph we much check whether it is irreducible with respect to vertex 1, and whether a rooted generalised graph isomorphic to the present one has already been constructed.These tests are accomplished by subroutine CODE.

```
      SUBROUTINE CODE(GRM,CAN,ORDER)
      IMPLICIT INTEGER(A-Z)
      DIMENSION GRM(30,3),PC(30),IN(30),NC(30),CAN(30)
      DIMENSION GRO(20,20,60),TREE(2500,22),COM(32)
      COMMON GRO,TREE,TREENO
*     LEVEL 2,GRO,TREE,TREENO
```

The first stage is to find the singleton-decomposition of the generalised graph constructed,with its canonical labelling, as in Proposition 2.4.3.2. The function COMP arranges three integers between 0 and 99 in ascending order and compacts them into a single 6-digit integer.

```
      PC(1)=1
      DO 10 I=2,ORDER
   10 PC(I)=2
   90 DO 20 J=1,ORDER
*  20 IN(J)=COMP(PC(GRM(J,1)),PC(GRM(J,2)),PC(GRM(J,3)))
     1+PC(J)*1000000
      COUNT=0
   30 LEAST=IN(1)
      DO 40 I=2,ORDER
   40 IF(IN(I).LT.LEAST) LEAST=IN(I)
      IF(LEAST.EQ.100000000) GO TO 50
      COUNT=COUNT+1
      DO 60 I=1,ORDER
      IF(IN(I).NE.LEAST) GO TO 60
      NC(I)=COUNT
      IN(I)=100000000
   60 CONTINUE
      GO TO 30
   50 IDIFF=0
      DO 80 I=1,ORDER
      IDIFF=IDIFF+IABS(NC(I)-PC(I))
   80 PC(I)=NC(I)
      IF(IDIFF.NE.0) GO TO 90
```

If the constructed generalised graph is not irreducible we reject it.

```
      IF(COUNT.EQ.ORDER) GO TO 100
  110 DO 150 I=1,ORDER
  150 CAN(I)=0
      RETURN
```

If the generalised graph is irreducible,the algorithm has found a canonical labelling of its vertices and,putting the entries of each row in ascending order,a canonical form of GRM, which we call CAN, representing each row by a single compacted integer.We compare CAN with the entries on a binary tree to check whether or not the current generalised graph is isomorphic as a rooted graph to any earlier construction.If so it is rejected. If not it is stored on the tree and we return to the main program. It should be noted that this stage is so written that the tree can overflow without affecting the correct execution of the rest of the program.

```
  100 DO 120 J=1,ORDER
* 120 CAN(PC(J))=COMP(PC(GRM(J,1)),PC(GRM(J,2)),PC(GRM(J,3)))
      NPOINT=1
      WIDTH=ORDER+2
  200 POINT=NPOINT
      DO 210 I=1,WIDTH
  210 COM(I)=TREE(POINT,I)
      DO 220 I=1,ORDER
      IF(CAN(I)-COM(I)) 230,220,240
  220 CONTINUE
      DO 250 I=1,ORDER
  250 CAN(I)=0
      RETURN
```

```
230 NPOINT=COM(ORDER+2)
    IF(NPOINT.NE.0) GO TO 200
    IF(TREENO.GT.2500) GO TO 280
    TREE(POINT,ORDER+2)=TREENO
    GO TO 260
240 NPOINT=COM(ORDER+1)
    IF(NPOINT.NE.0) GO TO 200
    IF(TREENO.GT.2500) GO TO 280
    TREE(POINT,ORDER+1)=TREENO
260 DO 270 I=1,ORDER
270 TREE(TREENO,I)=CAN(I)
    TREE(TREENO,ORDER+1)=0
    TREE(TREENO,ORDER+2)=0
280 TREENO=TREENO+1
    IF(TREENO.EQ.2500) WRITE(2,135)
135 FORMAT(1X,16HOVERFLOW IN TREE)
    RETURN
    END
```

Subroutine FEAS decides whether the generalised graph we have constructed satisfies the integrity condition.If so the subroutine outputs SIZE,GRM, the adjacency matrix,its eigenvalues,and the multiplicities of the eigenvalues in any singleton-regular graph of which we have the singleton-quotient (where determinable).

```
    SUBROUTINE FEAS(GRM,SIZE,ORDER)
    INTEGER INTGER(30),ORDER,SIZE(30),GRM(30,3),B(30,30),TOT
    REAL A(30,30),RR(30),RI(30),VR(30,30),VI(30,30)
    REAL SUM(30),RM(30),SM(30)
    DO 10 J=1,ORDER
    DO 10 K=1,ORDER
```

```
   10 B(J,K)=0
```

We construct the adjacency matrix of the generalised graph.

```
      DO 15 J=1,ORDER
      DO 15 L=1,3
*  15 B(J,GRM(J,L))=B(J,GRM(J,L))+1
```

The order of any graph of which we have constructed the singleton-quotient is called TOT.

```
      TOT=0
      DO 19 I=1,ORDER
   19 TOT=TOT+SIZE(I)
```

To find the eigenvalues and eigenvectors of the generalised graph we use a Nottingham Algorithms Group (NAG) routine.This program, FO2AGF, reduces the adjacency matrix to upper triangular form by means of elementary row and column operations,and finds simple eigenvalues (we are not concerned with multiple eigenvalues) and the entries of the corresponding eigenvectors correct to approximately 9 decimal places.The eigenvalues are held in RR and the eigenvectors are the columns of VR.

```
      DO 20 I=1,ORDER
      DO 20 J=1,ORDER
   20 A(I,J)=B(I,J)
      IFAIL=1
      IA=30
      IVR=30
      IVI=30
      CALL FO2AGF(A,IA,ORDER,RR,RI,VR,IVR,VI,IVI,INTGER,IFAIL)
```

```
      IF(IFAIL.EQ.0) GO TO 25
      WRITE(2,99996)
99996 FORMAT(1X,23HERROR IN FO2AGF IFAIL=1)
      DO 24 I=1,ORDER
      WRITE(2,99987) (B(I,J),J=1,ORDER)
   24 CONTINUE
      GO TO 500
```

Finally we test the eigenvectors for the integrity condition. If they fail we reject the generalised graph constructed.Otherwise we output the relevant information.

```
   25 DO 40 J=1,ORDER
      SUM(J)=0
      DO 30 I=1,ORDER
*  30 SUM(J)=SUM(J)+SIZE(I)*VR(I,J)*VR(I,J)
*  40 RM(J)=VR(1,J)*VR(1,J)*TOT/SUM(J)
      NN=ORDER-1
      DO 41 I=1,NN
      KK=I+1
      DO 41 J=KK,ORDER
      IF(ABS(RR(I)-RR(J)).GT.1E-6) GO TO 41
      RM(I)=0
      RM(J)=0
   41 CONTINUE
      DIFF=0
      DO 50 J=1,ORDER
      TM=RM(J)-FLOAT(INT(RM(J)))
      IF(TM.EQ.0.) GO TO 43
      SM(J)=FLOAT(INT(RM(J)+0.5))
      GO TO 44
```

```
   43 SM(J)=RM(J)
   44 DIFF=DIFF+ABS(RM(J)-SM(J))
      IF(DIFF.GT.1E-3) GO TO 500
   50 CONTINUE
      WRITE(2,110) (SIZE(I),I=1,ORDER)
      DO 180 J=1,3
  180 WRITE(2,110) (GRM(K,J),K=1,ORDER)
      WRITE(2,110) (I,I=1,ORDER)
  110 FORMAT(1X,40I2)
      WRITE(2,190)
  190 FORMAT(1X,//)
      DO 60 I=1,ORDER
      WRITE(2,99987) (B(I,J),J=1,ORDER)
   60 CONTINUE
99987 FORMAT(1X,30I4)
      WRITE(2,99990) TOT
99990 FORMAT(1X,14HORDER OF GRAPH,I6)
      WRITE(2,99995)
99995 FORMAT(1X,11HEIGENVALUES)
      WRITE(2,99994) (RR(I),I=1,ORDER)
99994 FORMAT(1X,12F10.6)
      WRITE(2,99989)
99989 FORMAT(1X,14HMULTIPLICITIES)
      WRITE(2,99994) (RM(J),J=1,ORDER)
      WRITE(2,190)
  500 RETURN
      END
```

Function COMP is used in subroutine CODE to order and compact three integers between 0 and 99.

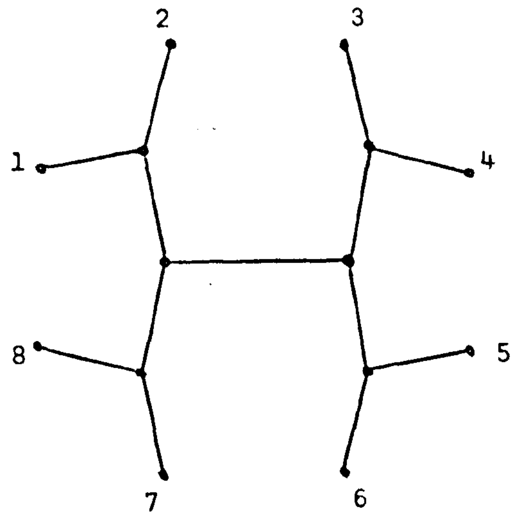
```
      INTEGER FUNCTION COMP(IX,IY,IZ)
      L=IX
      M=IY
      IU=IZ
      IF(IY.GE.L) GO TO 10
      L=IY
      M=IX
   10 IF(IZ.GE.M) GO TO 20
      IU=M
      M=IZ
      IF(L.LE.M) GO TO 20
      K=M
      M=L
      L=K
   20 COMP=L*10000+M*100+IU
      RETURN
      END
```

APPENDIX 2:THE UNIQUENESS OF THE (3,12)-CAGE

Certain trivalent cages of even girth g achieve the minimum possible order,that is $2(2^{g/2}-1)$. It is well known (see for example (4,p159)) that this can only occur if g=4,6,8 or 12. The existence and uniqueness of the (3,4)-,(3,6)- and (3,8)-cages with this property is easily settled (42), and the existence of a (3,12)-cage on 126 vertices was demonstrated by Benson (2). Benson's graph was shown to have the interesting property of being locally 7-arc-transitive but not vertex-transitive by Bouwer and Djokovic (7). However the uniqueness of the (3,12)-cage does not appear to have been established.We set out to find all the (3,12)-cages by an extension of the method normally used for the other cases.

Example.There is a unique (3,6)-cage on 14 vertices.

Proof.If there is a (3,6)-cage on 14 vertices it must consist of two disjoint edge-subgraphs,firstly a spanning tree



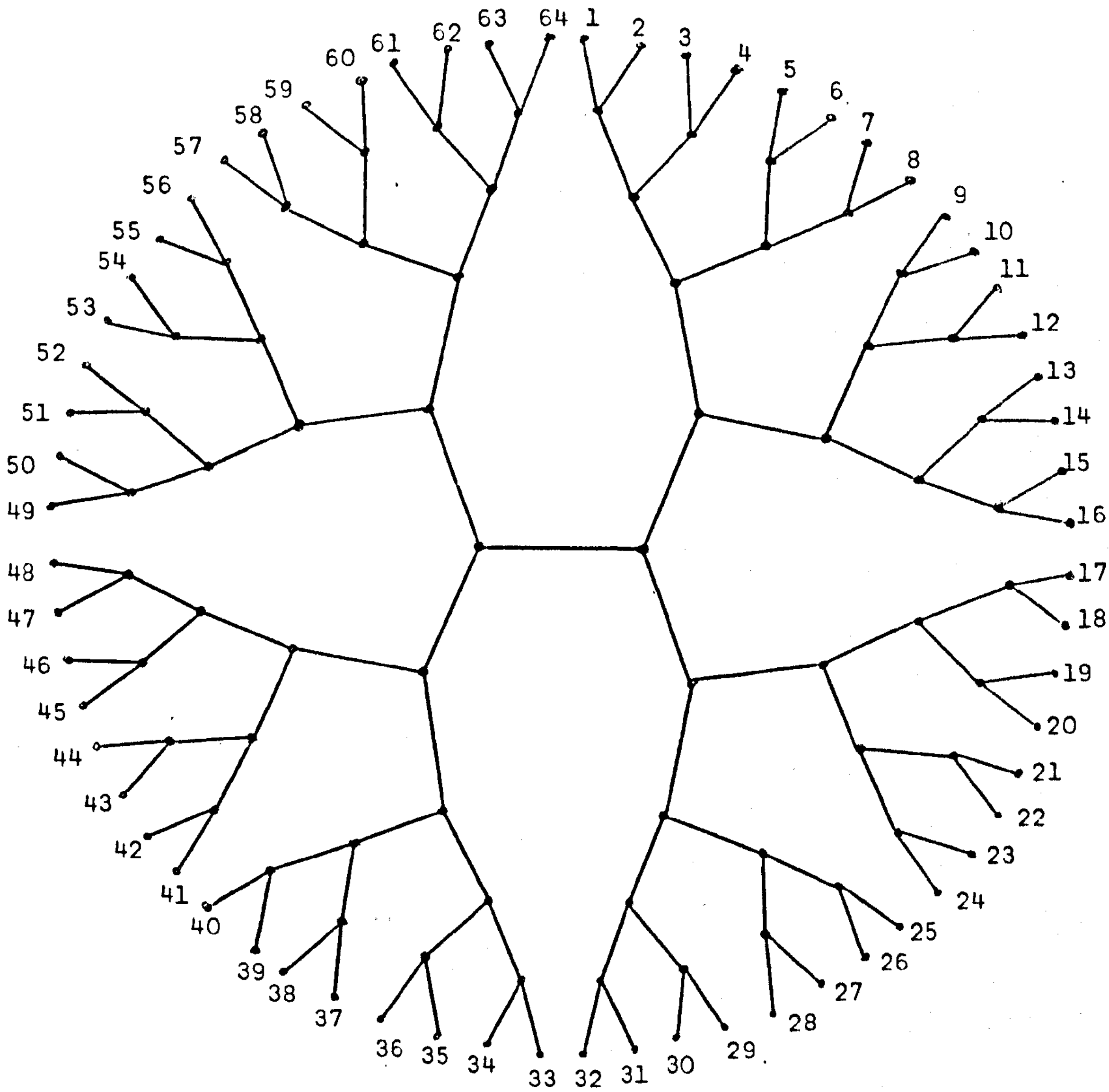
and secondly a divalent graph on vertices 1 to 8.

Vertex 1 is not adjacent to 2,7 or 8 by consideration of the girth,and it is clear that an edge from 1 to 3 would be equivalent to an edge from 1 to 4, from 1 to 5 or from 1 to 6. So without loss of generality 1 adj 3. Now vertex 3 is adjacent to none of 2,4,5,6 by girth so without loss of generality 3 adj 7. Similarly 7 adj 5, 5 adj 2, 2 adj 4, 4 adj 8, 8 adj 6, and 6 adj 1.

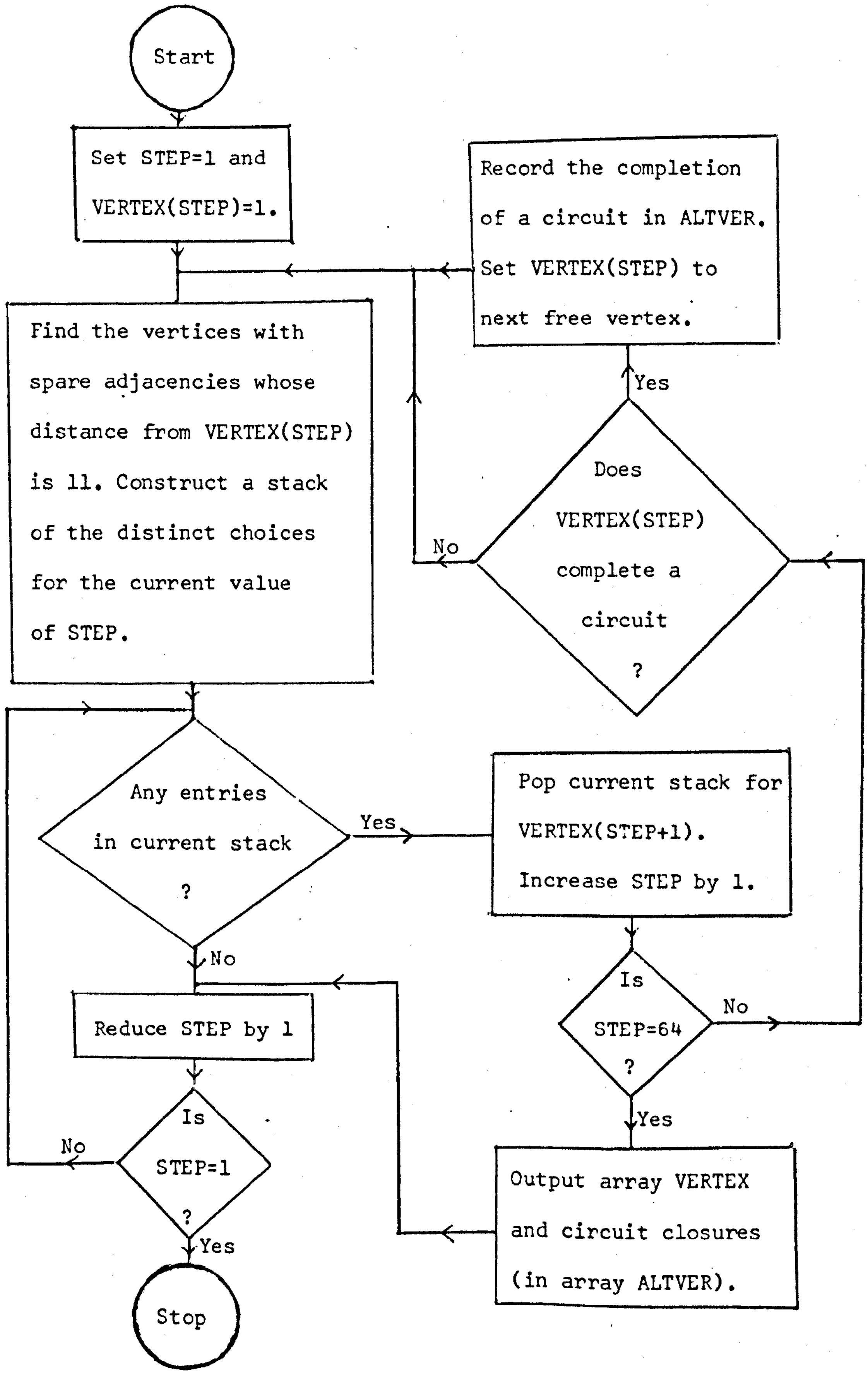
Now any circuit of length less than 6 must involve at least one of the edges of the divalent edge-subgraph,and it is clear that we would have to join two vertices whose distance apart is less than 5 if there is to be such a circuit.Since we have only joined vertices whose distance apart is 5, the graph constructed is a (3,6)-cage,and the construction is clearly unique. //

The construction of the (3,8)-cage is equally straightforward, but for $g>8$ we find that it is not always possible to choose just one vertex without loss of generality.Consider the tree given below for a (3,12)-cage on 126 vertices.Without loss of generality 1 adj 33, 33 adj 17, 17 adj 49, 49 adj 9, 9 adj 41, 41 adj 25, and 25 adj 57. However,at this stage we must make a choice,for it is clear that while 57 may be adjacent to 3, 4,5,6,7, or 8, an edge joining 57 and 3 is not equivalent to an edge joining 57 and 5.

The spanning tree of a (3,12)-cage:



Thus if we wish to find all (3,12)-cages we must construct a back-tracking algorithm to pursue all non-isomorphic choices of vertex. A flow-chart for such an algorithm follows:



In order to locate vertices whose distance from VERTEX(STEP) is 11 we construct a 64×64 matrix DIST giving the distance of every endpoint of the spanning tree from every other after the edges so far of the divalent graph have been added.

We decide that two choices,say x and y, are not distinct if the following occurs:-There is no endpoint z of the spanning tree such that

i) z is incident with an edge of the divalent graph so far constructed

and ii) d(x,z)≤d(x,y), where the distances are measured within the spanning tree only,that is before any edges of the divalent graph are added.

Thus in the case of the (3,12)-cage when we wish to add an edge incident with 57, the distinct choices in the range 1 to 8 may be taken to be 1,2,3 and 5.

The program used was,as in Appendix 1, written in extended FORTRAN for the UMRCC 1906/7600.

```
*     PROGRAM CAGE(INPUT,OUTPUT,TAPE7=INPUT,TAPE2=OUTPUT,DEBUG
     1=OUTPUT)
*     IMPLICIT INTEGER(A-Z)
      DIMENSION ODIST(64,64),DIST(64,64),VERTEX(64)
      DIMENSION ALTVER(64),CHOICE(64,16),TIP(64)
```

We begin by constructing ODIST, the original distances of the endpoints of the tree from one another,and setting the initial values of the other variables.

```
  100 DO 150 I=1,64
      K=I+1
      DO 145 J=K,64
```

```
      N=5
  120 IF((I-1)/2**N.NE.(J-1)/2**N) GO TO 110
      N=N-1
      GO TO 120
  110 ODIST(I,J)=2*N+2
      IF(N.EQ.5) ODIST(I,J)=11
      ODIST(J,I)=ODIST(I,J)
      DIST(I,J)=ODIST(I,J)
  145 DIST(J,I)=ODIST(J,I)
  150 CONTINUE
      DO 130 I=1,64
      ODIST(I,I)=0
  130 DIST(I,I)=0
      STEP=1
      VERTEX(1)=1
      DO 160 I=1,64
      DO 160 J=1,16
  160 CHOICE(I,J)=0
      FLAG=0
```

The next section finds the distinct choices of vertex to which VERTEX(STEP) may be joined.It is necessary

i) to find the vertices whose distance from VERTEX(STEP) is 11,

ii) to check that each such vertex is not already adjacent to two other endpoints of the spanning tree

and iii) to check that each such vertex is not equivalent as a choice to one already chosen.

```
  200 DO 210 J=1,64
*     IF(DIST(VERTEX(STEP),J).NE.11) GO TO 210
```

```
      DO 215 L=1,64
      ONES=0
      IF(DIST(J,L).EQ.1) ONES=ONES+1
      IF(ONES.EQ.2) GO TO 210
  215 CONTINUE
      IF(TIP(STEP).EQ.0) GO TO 260
      N=5
* 240 IDIV=(CHOICE(STEP,TIP(STEP))-1)/2**N
      JDIV=(J-1)/2**N
      IF(IDIV.NE.JDIV) GO TO 230
      N=N-1
      GO TO 240
  230 PAR1=IDIV*2**N+1
      PAR2=(JDIV+1)*2**N
      DO 250 II=1,64
      IF(VERTEX(II).EQ.0) GO TO 210
      IF(VERTEX(II).LT.PAR1) GO TO 250
      IF(VERTEX(II).GT.PAR2) GO TO 250
      GO TO 260
  250 CONTINUE
  260 TIP(STEP)=TIP(STEP)+1
*     CHOICE(STEP,TIP(STEP))=J
  210 CONTINUE
```

If we are currently starting a new circuit at vertex a, say, and there are only two distinct choices, b and c say,available, then time can be saved by eliminating one of the choices.For the second and last of the vertices of a circuit can only be equivalent when STEP=VERTEX(STEP)=1 and here there is only one distinct choice of second vertex.Thus a circuit whose second vertex is b must return to a via a vertex equivalent to c and vice versa.

```
      IF(ALTVER(STEP).EQ.0) GO TO 300
      IF(TIP(STEP).NE.2) GO TO 300
      CHOICE(STEP,2)=0
      TIP(STEP)=1
```

If there are no choices in the current stack we must step backwards.If STEP becomes 1 we have considered all the possibilities and the program stops.Otherwise we must enquire again whether the current stack contains any choices.

```
  300 IF(TIP(STEP).NE.0) GO TO 600
  320 VERTEX(STEP)=0
      ALTVER(STEP)=0
      STEP=STEP-1
      FLAG=1
      IF(STEP.EQ.1) STOP
      GO TO 300
```

When the current stack is not empty we pop the stack for the next vertex and increase STEP by 1. If STEP is 64 we have constructed a (3,12)-cage and we output VERTEX and ALTVER, from which the divalent edge-subgraph is easily constructed,and then step backwards. Otherwise we carry on to the next section.

```
* 600 VERTEX(STEP+1)=CHOICE(STEP,TIP(STEP))
*     CHOICE(STEP,TIP(STEP))=0
      TIP(STEP)=TIP(STEP)-1
      STEP=STEP+1
      IF(STEP.NE.64) GO TO 500
```

```
      WRITE(2,610) (VERTEX(K),K=1,64)
  610 FORMAT(1X,19H12-CAGE VERTEX LIST,/,1X,64I2)
      WRITE(2,620) (ALTVER(K),K=1,64)
  620 FORMAT(1X,17HCIRCUIT CLOSED AT,/,1X,64I2,//)
      GO TO 320
```

In the next stage we must construct DIST for the current value of STEP. If we have not stepped backwards since DIST was last constructed (indicated by FLAG=0) we may construct the new DIST simply from its previous value.If however we have stepped backwards (when FLAG=1) we must construct DIST from scratch,beginning with ODIST and adding the edges indicated by the vertices held in VERTEX and ALTVER, from 1 to STEP.

```
  500 S=STEP-1
      IF(FLAG.EQ.0) GO TO 510
      DO 530 I=1,64
      DO 530 J=1,64
  530 DIST(I,J)=ODIST(I,J)
      DO 540 K=2,S
      U=VERTEX(K-1)
      V=VERTEX(K)
      IF(ALTVER(K).NE.0) V=ALTVER(K)
      DO 540 I=1,63
      II=I+1
      DO 535 J=II,64
      DIST(I,J)=MIN0(DIST(I,J),DIST(I,U)+DIST(V,J)+1,
     1DIST(I,V)+DIST(U,J)+1)
  535 DIST(J,I)=DIST(I,J)
  540 CONTINUE
      FLAG=0
```

```
  510 DO 520 I=1,63
      II=I+1
      DO 525 J=II,64
*     DIST(I,J)=MINO(DIST(I,J),DIST(I,VERTEX(STEP-1))
     1+DIST(VERTEX(STEP),J)+1,DIST(I,VERTEX(STEP))
     2+DIST(VERTEX(STEP-1),J)+1)
  525 DIST(J,I)=DIST(I,J)
  520 CONTINUE
```

Now it is necessary to decide whether VERTEX(STEP) completes a circuit and if so which is the next "free" vertex from which we can start the next.Once this has been done we return to 200.

```
  550 DO 560 C=1,S
      IF(VERTEX(C).EQ.VERTEX(STEP)) GO TO 570
  560 CONTINUE
      GO TO 200
  570 ALTVER(STEP)=VERTEX(STEP)
  590 IF(VERTEX(STEP).EQ.64) VERTEX(STEP)=0
      VERTEX(STEP)=VERTEX(STEP)+1
      DO 580 J=1,S
      IF(VERTEX(J).EQ.VERTEX(STEP)) GO TO 590
  580 CONTINUE
      GO TO 200
      END
```

When the program is run it generates twelve divalent edge-subgraphs,each consisting of four 16-circuits.The twelve corresponding cages are easily seen to be isomorphic so we have the following proposition:

Proposition.There is exactly one (3,12)-cage on 126 vertices. //

For one of the divalent graphs generated the four circuits are:

<1,33,17,49,9,41,25,57,3,37,19,53,11,45,27,61>

<2,43,29,59,12,35,21,51,4,47,31,63,10,39,23,55>

<3,36,26,56,13,48,18,60,7,40,28,52,15,44,20,64>

<6,46,24,58,16,34,32,54,8,42,22,62,14,38,30,50> .

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