Keele University

THE SEPARATION TECHNIQUE FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: GENERAL RESULTS AND ITS CONNECTION WITH OTHER METHODS

by

Inaam A. Malloki

Ph.D. Thesis
Department of Mathematics
University of Keele
April, 1987

ACKNOWLEDGEMENTS

I would like to express my thanks to Dr. A. D. Osborne, my supervisor, for his wonderful guidance and help throughout this thesis, and to High Education Ministry, Iraq whose grant supported this research.

I would also like to express my utmost appreciation to my family for their unfailing encouragement and support during my studies, and my husband for his advices and help.

Finally, I wish to express my thanks to Mrs. O. Brindley for much care and patience in typing of this thesis.

ABSTRACT

This thesis is primarily a study of the separation method for solving nonlinear partial differential equations, which is a generalisation of the classical separation approach as applied to linear equations. The method is of importance since it produces physically useful solutions, such as travelling wave and soliton solutions, to the interesting nonlinear equations of current interest in a simple and economical way.

This work is also concerned with the relationships between this method and the other standard systematic methods for solving nonlinear partial differential equations. As a start in this direction, a preliminary investigation of the correspondence of separable and similarity solutions is carried out. The thesis also uses the separation technique to study the non-solvability of equations by IST assuming that the Ablowitz conjecture is true.

The thesis commences with a general introduction which includes the standard systematic methods for solving nonlinear partial differential equations. Chapters two and three deal with the properties and applications of the separation technique and extend existing results. Chapters four and five use the separation technique in the Painlevé test and the final chapter concerns the connection between similarity solutions and separable solutions.

CONTENTS

continued . . .

continued . . .

CHAPTER ONE

Introduction

1.1 Linear and Nonlinear Partial Differential Equations

The theory of partial differential equations (p.d.e.'s) has become one of the most important fields of study in mathematical analysis, mainly due to the frequent occurence of such equations in many branches of physics, engineering and other sciences. In fact, we can say that every problem that can be formulated in mathematical physics involves the solution of a p.d.e. There are problems which require only the solution of an ordinary differential equation, but these are usually obtained by introducing simplifying assumptions into a more general problem governed by a p.d.e.

A p.d.e. (which is any equation involving a function of several variables and its partial derivatives) is said to be linear if it is linear in the unknown function and all its derivatives, with coefficients depending only on the independent variables.

Linear equations are so much easier to deal with than nonlinear equations. Hence they produced a plentiful harvest of results when they were first investigated. In a linear system, the analyst may formulate the problem of finding an unknown function of a p.d.e. satisfying appropriate initial or boundary conditions before proceeding to the solution development, using the principle of superposition. This was essentially responsible for the great success of constructing effective theories for linearized physical phenomena - any particular solutions can be combined to yield useful, more general solutions. The fact that such a superposition principle holds for linear equations means that Fourier-type analysis can be applied - often, an interesting solution of an equation i.e., one which fits physically reasonable initial/boundary conditions, can be expressed as a series of functions which are simple solutions of the equation.

The problem is especially tractable if the "simple solutions" are the solutions of the boundary value problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda w(x)y = 0, \qquad (1.1\text{-}1)$$

where $a \leqslant x \leqslant b$, and $y(x)$ satisfies the boundary conditions

$$\left.\begin{array}{l} \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{array}\right| \qquad (1.1\text{-}2)$$

which is known as the <u>Sturm-Liouville problem</u> obtained by applying the separation of variables technique to the linear p.d.e.

The set of solutions to (1.1-1) and (1.1-2), $\{\phi_k(x)\}$, $k = 1, 2, \ldots$, for all possible values of λ are <u>orthogonal</u> where the inner product of any two functions $\phi_i(x)$ and

$\phi_j(x)$ over the interval $a \leq x \leq b$, is defined as

$$(\phi_i, \phi_j) = \int_a^b \rho(x)\phi_i(x)\phi_j(x)dx,$$

for some weight function $\rho(x)$ where $\rho(x) > 0$ for $a \leq x \leq b$. Thus any separable solution of the original p.d.e. which satisfies (1.1-2) can be expressed as a Fourier series of the eigenfunctions $\phi_k(x)$, $k = 1, 2, \ldots$, [101].

Examination of the Cauchy problem [83] - the appropriate generalization of the initial value problem for linear p.d.e.'s to higher dimensions - gives rise to a natural classification of second order linear equations which is based upon the possibilities of reducing the equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad (1.1\text{-}3)$$

where the coefficients are functions of x and y, by a coordinate transformation to canonical or standard forms, namely, hyperbolic, parabolic and elliptic according to whether $(B^2 - 4AC)$ is positive, zero or negative in a domain. In the case of two independent variables, a transformation can always be found to reduce the given equation to the canonical form using the (so called), characteristic equation. There is a decisive distinction between the three canonical forms, of which shows an entirely different behaviour regarding properties and construction of solutions [74].

Some "classical" linear p.d.e.'s of the second order are,

The Laplace equation, (elliptic type):

$$u_{xx} + u_{yy} = 0. \tag{1.1-4}$$

The wave equation, (hyperbolic type):

$$u_{xx} - u_{yy} = 0. \tag{1.1-5}$$

The heat equation, (parabolic type):

$$u_x - u_{yy} = 0. \tag{1.1-6}$$

These three equations are so widely applicable that they are often called "the differential equations of physics" [45]. The possibility of solving them depends chiefly on the fact that they are separable in several systems of coordinates.

Example (1.1-1)

Taking, for instance, the two dimensional wave equation in rectangular coordinates,

$$u_{xx} + u_{yy} = u_{tt} \tag{1.1-7}$$

we try a form of solution $\phi = X(x)Y(y)T(t)$. Substitution in eq. (1.1-7) and division by XYT yields,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{T}\frac{d^2T}{dt^2}.$$

Each term is a function of only one of the independent variables. Hence if the equation is to hold for all values x, y, t each term must be constant, and every expression of the form

$$A\exp\{i(\ell x + my - nt)\},$$

where A, ℓ, m, n are constants, and $\ell^2 + m^2 = n^2$, is a solution and so is any sum of expressions of this form. The complex exponentials can evidently be replaced by cosines and sines. Using Fourier analysis to sum up solutions we get that

$$u(x, y, t) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sin\frac{\ell\pi x}{a} \sin\frac{m\pi y}{b}(A_{\ell,m}\cos nt + B_{\ell,m} \sin nt),$$

with $n^2 = \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right)$, is a solution for the rectangular membrane problem whose corners are at (0, 0), (a, 0), (0, b), (a, b) [45].

If the initial values of u and $\frac{\partial u}{\partial t}$, are known, then they are expandable in a double Fourier sine series and comparison of coefficients determines $A_{\ell,m}$ and $B_{\ell,m}$. Thus the solution is found.

Although most of the hardest physical problems are non-linear, there has recently been a revival of interest in these problems, the phenomena being modelled by <u>nonlinear</u> partial differential equations - equations which are not linear. This is due to several reasons. One important reason is that many equations have solutions which posses a number of common properties such as "soliton" behaviour. The term soliton was recently coined to describe a pulselike nonlinear wave (solitary wave) which emerges from a collision with a similar pulse having unchanged shape and speed. The concept of a solitary wave was introduced well over a century ago by J. Scott-Russell [79].

In the physical sciences, the development of precise instruments for measurements, has induced extensive study of nonlinear models which, in turn, requires mathematical methods for nonlinear partial differential equations.

Nonlinear evolution equations are the most important equations in mathematical physics of current interest and these include the Korteweg-de Vries (KdV), the modified KdV, the sine-Gordon (sG) and cubic Schrödinger equations. These equations have been extensively studied for the past 20 years according to many common properties of these equations, in particular, soliton behaviour; and according to the development of many major methods of solution such as, similarity methods, inverse scattering transform, Backlund transformation and other methods.

The Korteweg-de Vries (KdV) Equation

The most extensively studied nonlinear p.d.e., is the KdV equation. In 1895 Kortweg and de Vries provided a simple analytic foundation for the study of solitary waves by developing an equation for shallow water waves. This equation is the KdV equation [79]

$$u_{xxx} + \alpha u u_x + u_t = 0, \tag{1.1-8}$$

where α is a constant. The linearization of the KdV equation was found by Gardner, Green, Kruskal and Miura [35] in 1967. Similarity solutions have been studied by many authors [81, 12].

A generalization of the KdV equation is,

$$\phi_t + \alpha\phi^p\phi_x + \phi_{x^{2r+1}} = 0, \tag{1.1-9}$$

where α is a constant, p and r non negative integers and

$$\phi_{x^{2r+1}} = \frac{\partial^{2r+1}\phi}{\partial x^{2r+1}}.$$

<u>The Modified KdV Equation</u>

The most simple generalization included in eq. (1.1-9) is the modified KdV (mKdV) equation

$$\phi_t - \beta\phi^2\phi_x + \phi_{xxx} = 0, \tag{1.1-10}$$

where β is a constant. This equation has been used to describe acoustic waves in certain anharmonic lattices [79].

Miura (1968) [25] established a transformation which relates solutions of the KdV eq. (1.1-8) and the mKdV eq. (1.1-10) as follows:

$$u = \frac{1}{\alpha}\left(-\beta\phi^2 + \varepsilon(6\beta)^{\frac{1}{2}}\phi_x\right), \quad \varepsilon = \pm 1. \tag{1.1-11}$$

Clearly, if ϕ is a solution of (1.1-10), then u defined by (1.1-11) is a solution to the KdV equation.

The Burgers Equation

The well known equation

$$u_t + uu_x = \delta u_{xx}, \tag{1.1-12}$$

where δ is a constant, which is called the Burgers equation and occurs in viscosity dominated systems [25], is a famous equation for including nonlinearity and dissipation together in the simplest form and because it can be linearized through the Hopf-Cole transformation [25]

$$u = \alpha\frac{\partial}{\partial x}(\log F) \tag{1.1-13}$$

to give the linear equation,

$$F_t - Fc(t) = \delta F_{xx} \tag{1.1-14}$$

where c(t) is "constant" of integration. Eq. (1.1-14) reduces to the heat equation when c = 0.

Transformations similar to (1.1-13) has been applied by many authors to the KdV equation, hoping to linearize it, but this transformation with $\alpha = 1$, gives the homogeneous equation [25],

$$FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^{\ 2} + FF_{xt} - F_xF_t = 0.$$

This means that the Burgers equation has a nonlinear superposition principle whereas it seems as though the KdV has not.

The Sine-Gordon Equation

Much attention has been paid recently to the sine-Gordon (sG) equation,

$$\phi_{xx} - \phi_{tt} = m^2 \sin \phi \qquad (1.1\text{-}15)$$

where m is a constant, because it appears in several important physical problems [9]. In particular it is of use in the theory of plane or cylindrical Josephson junctions which requires the solutions of sG equation in space dimensions and in the theory of solitons where the solutions in one spatial and one time coordinates are of interest.

Solitary wave solutions of the sG equation

$$\phi = 4\tan^{-1}\exp[m\gamma(x - vt) + \delta], \qquad (1.1\text{-}16)$$

where $\gamma^2 = (1 - v^2)^{-1}$, δ, v are constants.

Recently, Ablowitz et al. [3] have solved the initial value problem for the transformed sG equation:

$$\phi_{xt} = \sin \phi. \qquad (1.1\text{-}17)$$

Separable solutions have been developed by many authors [56], [69]. Similarity solutions [55], as well as, Bäcklund transformations [9] have been found for the sG equation.

Comparing theories of linear and nonlinear equations, we see that while linear equations have that special property, which greatly facilitates their treatment, namely, the superposition principle, the treatment is much less comprehensive. Nevertheless, for some special non-linear equation there are nonlinear superposition principles.

Example (1.1-2) [8]

Consider the linear equation,

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u. \tag{1.1-18}$$

Setting $u = f\big(v(x, y)\big)$, eq. (1.1-18) becomes

$$av_x + bv_y = cf/f'. \tag{1.1-19}$$

Since eq. (1.1-18) is linear, $U = \sum_{i=1}^{k} u_i$ is a solution if the u_i, $i = 1, 2, \ldots, k$ are solutions. However if $u_i = f(v_i)$, $i = 1, \ldots, k$, and $V = f^{-1}(U)$, then.

$$V = f^{-1}[f(v_1) + f(v_2) + \ldots + f(v_k)] \tag{1.1-20}$$

is a solution to eq. (1.1-19). This implies that eq. (1.1-19) has a superposition principle.

Thus for some nonlinear equations there are nonlinear superposition principles [8, 12, 13], but in these cases any Fourier series-type analysis would be a lot more difficult than in the linear case due to many reasons. One important reason is the lack of orthogonality of any "simple solutions" in most cases.

The earliest and most obvious classification of linear p.d.e.'s into the three types, hyperbolic, elliptic and parabolic was made on a formal basis. However, there is still more important principles by which linear p.d.e.'s may classified [30]. In contrast, nonlinear equations can't be easily classified according to their forms. The available classifications are related to the solutions or techniques of solution of the equations.

1.2 Nonlinear Equations: Methods of Solution

The theory of linear p.d.e.'s has been studied deeply and extensively for the past 200 years, and is fairly complete. However, very little of a general nature is known about nonlinear equations. We survey some of the central ideas and methods of this subject here.

The "old fashioned" techniques for nonlinear p.d.e.'s which find the general solution seem nowadays, to be not practical in most cases especially in physical sciences. Nevertheless, it would be unfair if we did not at least make a passing mention of some of the work done on nonlinear p.d.e.'s at the turn of this century.

1.2-1 The general method:

We shall begin our discussion by defining the various types of integrals possessed by nonlinear p.d.e.'s.

A nonlinear equation of the first order, involving two independent variables x and y and a dependent variable z will be denoted by,

$$F(x, y, z, p, q) = 0, \tag{1.2-1}$$

where $p = \frac{\partial z}{\partial x}$, and $q = \frac{\partial z}{\partial y}$.

A complete integral of eq. (1.2-1) is any solution containing two arbitrary constants, say α and β, and this will be denoted by,

$$f(x, y, z, \alpha, \beta) = 0. \tag{1.2-2}$$

If the arbitrary constants α and β in (1.2-2) are not independent i.e.,

$$\beta = \phi(\alpha)$$

for some function ϕ, equation (1.2-2) represents a one-parameter family of surfaces,

$$f[x, y, z, \alpha, \phi(\alpha)] = 0. \tag{1.2-3}$$

For each choice of the function ϕ, we get, in general, a distinct family, the envelope of which is found by eliminating α between (1.2-3) and the equation obtained by differentiating (1.2-3) partially with respect to α. The totality of all such envelopes, derived from the equations

$$f(x, y, z, \alpha, \phi(\alpha)) = 0, \quad \frac{\partial f}{\partial \alpha} = 0$$

for all possible choices of ϕ, is known as the general integral of the p.d.e. (1.2-1).

For the nonlinear p.d.e. (1.2-1), there are at least three methods for obtaining the complete solutions. These methods are: Cauchy's method of characteristics, Jacobi's method and Charpit's method.

Charpit's method, illustrates the fundamental idea, which is to introduce a second first order p.d.e.,

$$g(x, y, z, p, q, a) = 0, \tag{1.2-4}$$

involving an arbitrary constant a and such that, eq. (1.2-1) and (1.2-4) can be solved to give,

$$p = p(x, y, z, a), \quad q = q(x, y, z, a)$$

and the resulting equation

$$dz = pdx + qdy,$$

is integrable.

A nonlinear equation of the second order involving two variables x and y and dependent variable z will be denoted by,

$$F(x, y, z, p, q, s, r, t) = 0, \tag{1.2-5}$$

where $s = \frac{\partial^2 z}{\partial x \partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, and $t = \frac{\partial^2 z}{\partial y^2}$.

One way of attacking the problem of solving (1.2-5), is to assume that there exists a solvable first order equation which is such that the original second order relation can be obtained from it and the two relations derived by partial differentiation. Such a first order equation is called an <u>intermediate integral</u>. Assuming that the intermediate integral exists, there are various ways of proceeding to the constructing of that integral.

One of these ways is Monge's method, another method is associated with the name of Boole [32].

The method devised by Ampére deals with equations of the second order with no assumption of the existence of an intermediate integral. The method will be illustrated by the following example:

Example (1.2-1)

Ampére's method is based upon a transformation of the independent variables.

Consider the Borne Infeld (BI) equation,

$$z_{xx}(1 - z_y^2) + 2z_x z_y z_{xy} - (1 + z_x^2)z_{yy} = 0. \qquad (1.2\text{-}6)$$

(1.2-6) can be written as,

$$f(x, y, z, p, q, s, r, t) = (1 - q^2)r + 2pqs - (1 + p^2)t = 0,$$

using the notation above.

Let new independent variables α and β be introduced; they are not determinated until the effect of the transformation is being considered. These variables may be functions of both variables x and y.

The equation for the arguments (see [32]) is

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial r}\left(\frac{\partial y}{\partial x}\right)^2 = 0.$$

This equation has equal roots if and only if,

$$\left(\frac{\partial f}{\partial s}\right)^2 - 4\left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial f}{\partial t}\right) = 0,$$

i.e., $\alpha = \beta$ iff $1 + p^2 - q^2 = 0$.

Suppose that $\alpha \neq \beta$, i.e., $1 + p^2 - q^2 \neq 0$, and let us substitute the values [30],

$$s = \frac{\partial q}{\partial x} - t\frac{\partial y}{\partial x},$$

$$r = \frac{\partial p}{\partial x} - \frac{\partial q}{\partial x}\frac{\partial y}{\partial x} + t\left(\frac{\partial y}{\partial x}\right)^2,$$

in the equation. We then get:

$$(1 - q^2)(p' - y'q' + ty'^2) + 2pq(q' - ty') - (1 + p^2)t = 0,$$

where a dash denotes $\frac{\partial}{\partial x}$.

This equation is a linear polynomial in t,

$$p_0 + p_1 t = 0,$$

where,

$$p_0 = (1 - q^2)(p' - q'y') + 2pqq'$$

$$p_1 = (1 - q^2)y'^2 - 2pqy' - (1 + p^2)$$

Consider $p_0 = p_1 = 0$ together with the Pfaffian relation, $z' = p + qy'$,

$$(1 - q^2)(p' - q'y') + 2pqq' = 0 \qquad (1.2\text{-}7)$$

$$(1 - q^2)y'^2 - 2pqy' - (1 + p^2) = 0 \tag{1.2-8}$$

$$z' = p + qy'. \tag{1.2-9}$$

From (1.2-8) we get,

$$y' = (pq \pm \sqrt{1 + p^2 - q^2})/(1 - q^2) \quad \text{if} \quad q \neq 1.$$

Since $\alpha \neq \beta$, suppose that $1 + p^2 - q^2 = w^2$, say ($w \not\equiv 0$).

Hence,

$$\begin{aligned} y' &= (pq - w)/(1 - q^2), && \text{(a)} \\ p' &= q'(-pq - w)/(1 - q^2), && \text{(b)} \\ z' &= (p - qw)/(1 - q^2), && \text{(c)} \end{aligned} \tag{1.2-10}$$

being the system when α is constant; and

$$\begin{aligned} y' &= (pq + w)/(1 - q^2), && \text{(a)} \\ p' &= q'(-pq + w)/(1 - q^2), && \text{(b)} \\ z' &= (p + qw)/(1 - q^2), && \text{(c)} \end{aligned} \tag{1.2-11}$$

being the system when β is constant. We need an integral equation for each system.

Let us solve (1.2-10(b)), i.e.,

$$p'(1 - q^2) - q'(-pq - \sqrt{1 + p^2 - q^2}) = 0.$$

This equation can be expressed as a Clairaut equation [32], and its integral is

$$(-pq + w)/(1 - q^2) = K,$$

where K is a constant of integration, so that we take,

$(-pq + w)/(1 - q^2) = \alpha$ (or β) and then

$$-\beta = \frac{pq + w}{1 - q^2} \tag{1.2-12(a)}$$

$$-\alpha = \frac{pq - w}{1 - q^2}. \tag{1.2-12(b)}$$

Now we shall write x, y and dz as functions of α and β:

(i) $\frac{\partial y}{\partial \beta} = y'\frac{\partial x}{\partial \beta} = -\beta\frac{\partial x}{\partial \beta}, \quad \frac{\partial y}{\partial \alpha} = y'\frac{\partial x}{\partial \alpha} = -\alpha\frac{\partial x}{\partial \alpha}$

$$\therefore \frac{\partial^2 y}{\partial \alpha \partial \beta} = -\beta\frac{\partial^2 x}{\partial \alpha \partial \beta} = -\alpha\frac{\partial^2 x}{\partial \alpha \partial \beta}$$

$$\Rightarrow (\beta - \alpha)\frac{\partial^2 x}{\partial \alpha \partial \beta} = 0$$

$$\Rightarrow x = \phi'(\alpha) + \psi'(\beta), \quad (\text{as } \alpha \neq \beta)$$

for some functions, ϕ and ψ.

(ii) Similarly as (i) we can take

$$y = \phi(\alpha) - \alpha\phi'(\alpha) + \psi(\beta) - \beta\psi'(\beta).$$

(iii) Now, $dz = \left(p\frac{\partial x}{\partial \alpha} + q\frac{\partial y}{\partial \alpha}\right)d\alpha + \left(p\frac{\partial x}{\partial \beta} + q\frac{\partial y}{\partial \beta}\right)d\beta$

i.e., $dz = p(\alpha)d\alpha + q(\beta)d\beta$,

where,
$$p(\alpha) = (p - \alpha q)\phi''$$
$$q(\beta) = (p - \beta q)\psi''.$$

Writing p and q as functions of α and β we get,

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = (-pq \pm w)/(1 - q^2) \quad \text{(as above)}.$$

Taking the positive sign first, yields $p - \alpha q = -\sqrt{\alpha^2 - 1}$, and similarly for the negative sign, $p - \beta q = -\sqrt{\beta^2 - 1}$. Hence,

$$dz = \left[-\sqrt{\alpha^2 - 1}\right]\phi''(\alpha)\,d\alpha + \left[-\sqrt{\beta^2 - 1}\right]\psi''(\beta)\,d\beta,$$

And so,

$$\left.\begin{aligned} z &= -\int\sqrt{\alpha^2 - 1}\ \phi''(\alpha)\,d\alpha - \int\sqrt{\beta^2 - 1}\ \psi''(\beta)\,d\beta \\ x &= \phi'(\alpha) + \psi'(\beta) \\ &\text{and} \\ y &= \phi(\alpha) - \alpha\phi'(\alpha) + \psi(\beta) - \beta\psi'(\beta) \end{aligned}\right\} \qquad (1.2\text{-}13)$$

is the general solution of BI equation.

Methods of finding general solutions of nonlinear p.d.e.'s can be powerful tools. Some of these methods lead to general solutions of some important nonlinear wave equations of current interest. For instance Liouville's equation,

$$\sigma_{xt} = \exp(\sigma) \qquad (1.2\text{-}14)$$

which is of a great interest, has the general solution [32],

$$\sigma(x, t) = \ln\{-2f'(x)g'(t)/[1 + f(x)g(t)]^2\},$$

where f and g are arbitrary differentiable functions.

If $f(x) = e^{-ax}$, $g(t) = e^{bt+\delta}$, where a, b, δ are constants, then

$$e^{\sigma} = \frac{2abe^{-ax+bt+\delta}}{(1 + e^{-ax+bt+\delta})^2},$$

which is a "solitary" solution [14].

1.2-2 Travelling wave solutions:

Often travelling wave solutions of nonlinear evolution equations are sought. These are found by using the ad-hoc assumption that a given equation for u(x, t) has a solution of the form,

$$u(x, t) = f(x - \mu t),$$

where μ is a constant, which represents the speed of the travelling wave. We consider one specific example of this technique here.

Example (1.2-2)

Solution of one-dimensional TDGL equation [86]:

The one-dimensional time-dependent Ginzburg-Landau equation which can be written as,

$$\frac{\partial}{\partial t}u(x, t) = L\left\{\frac{\partial^2}{\partial x^2}u(x, t) + g\left[u(x, t) - \frac{1}{6}u^3(x, t)\right]\right\}, \tag{1.2-15}$$

where L and g are constants, is investigated by Y. Tagami [86]. This equation is associated with the potential energy function,

$$v(u(x)) = -\frac{1}{2}u^2 + \frac{1}{4!}u^4.$$

Assuming the travelling wave solution,

$$u(x, t) = M(\xi), \quad \xi = x - \nu t,$$

where ν the velocity of the travelling wave, $M(\xi)$ satisfies the following equation,

$$\frac{d^2M}{d\xi^2} + \frac{\nu}{L}\frac{dM}{d\xi} + g\left(1 - \frac{1}{6}M^2\right)M = 0 \qquad (1.2\text{-}16)$$

the "kink" solution of which is known as

$$M_k(\xi) = \pm\sqrt{6}\ \tanh(\sqrt{(g/2)}\,\xi).$$

The author seeks the solution for (1.2-16) in the form of expansion in terms of a small parameter $\nu/L = \varepsilon$ ($\varepsilon > 0$) as

$$M(\xi;\ \varepsilon) = M_0(\xi) + \varepsilon M_1(\xi) + \varepsilon^2 M_2(\xi) + \ldots, \qquad (1.2\text{-}17)$$

substituting (1.2-17) in eq. (1.2-16), yields that M_0 satisfies the equation,

$$\frac{d^2M_0}{d\xi^2} + g\left(1 - \frac{1}{6}{M_0}^2\right)M_0 = 0,$$

$$\Rightarrow\ M_0(\xi) = \pm\sqrt{12/(1+k^2)}\ h\left(\sqrt{g/(1+k^2)}\,\xi\right),$$

where h is given by the following triad:

$$h(y) = \begin{cases} \mathrm{sn}(ky,\ 1/k), & k > 1 \\ \tanh y, & k = 1 \\ k\,\mathrm{sn}(y,\ k), & 1 > k > 0, \end{cases}$$

M_1 satisfies the equation

$$\frac{d^2M_1}{d\xi^2} + g\left(1 - \tfrac{1}{2}{M_0}^2\right)M_1 = -\frac{dM_0}{d\xi}$$

and is found to be,

$$M_1(\xi) = \pm\frac{1}{\sqrt{3g}}\,\frac{1}{kk'^4}\mathrm{dn}\ y\ \mathrm{cn}\ y\{[k^2y - (1+k^2)E(y) + \mathrm{sn}\ y$$

$$(\mathrm{dc}\ y + k^4\mathrm{cd}\ y)]^2 - k'^2(\mathrm{tn}^2y - k^6\mathrm{sd}^2y)\}.$$

Calculations of the functions $M_2(\xi)$, $M_3(\xi)$, ... in eq. (1.2-17) are similar.

1.2-3 Ad-hoc techniques:

As an alternative to methods which find general solutions or travelling wave solutions, various ad-hoc techniques which seem to be needed for some specific physical equations, prove useful (e.g. Navier Stokes equations in fluid dynamics, Maxwell's equations in electrodynamic field theory, Einstein field equations in general relativity [34]).

More recently, systematic methods which lead to solutions of p.d.e.'s of physical interest and significance have been employed. These major methods, namely, similarity and inverse scattering methods will be discussed separately in the following sections.

1.3 The Inverse Scattering Transform

Ever since the Cole-Hopf solution of the Burgers equation, countless people must have tried similar 'tricks' to solve the KdV equation, but the eventual method of solution needs much more than a simple 'trick'.

The soliton was discovered (and named) in 1965 by Zabusky and Kruskal in numerical calculations. It was observed that two distinct solitary waves, i.e., with distinct amplitudes, interact nonlinearly but emerge from the interaction unchanged. This resemblance of these solitary waves to particles led to the name "solitons", yet the real breakthrough occurred in 1967, when the idea of the inverse scattering transform (IST) method, as a tool for solving p.d.e.'s, was first discovered by Gardner, Greene, Kruskal, and Miura (GGKM). They showed that associated with the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1.3\text{-}1)$$

is a linear eigenvalue problem,

$$v_{xx} - \big(u(x, t) - \lambda\big)v = 0, \qquad (1.3\text{-}2)$$

where u(x, t) is a solution of equation (1.3-1), so that $\nu(x, t)$ and $\lambda(t)$ depend parametrically on t. The equation (1.3-2) is the well known Schrödinger eigenvalue problem with u(x, t) playing the role of a potential.

Some major questions still remain, the answers to which are only partially understood. One question is why is the Schrodinger equation the appropriate linear eigenvalue problem for the KdV equation?

Historically the conservation laws have played an important role in the development of IST. Indeed it was the Miura transformation

$$u = -(w^2 + w_x), \tag{1.3-3}$$

where w is a solution for the mkdV equation which led GGKM to the choice of Schrodinger equation. This may be viewed as a Riccati equation for w in terms of u; the well known transformation $w = v_x/v$ linearizes (1.3-3) yielding

$$v_{xx} + uv_x = 0.$$

Since the KdV equation is Galilean-invariant, and in order to be as general as possible, GGKM considered (1.3-2).

It turns out that (1.3-2) provides an implicit linearization of the KdV equation.

Soon afterwards, Lax indicated how this approach could be applied to general class of evolution equations. Then Zakharov and Shabat demonstrated the applicability of this method to the (so called) nonlinear Schrödinger equation,

$$iq_t(x, t) + q_{xx}(x, t) + c|q(x, t)|^2 q(x, t) = 0. \quad (1.3\text{-}4)$$

The way was thereby opened to the search and discovery of many other nonlinear p.d.e.'s solvable by the same technique.

The technique of IST can be regarded as an extension of the Fourier transform method as applied to linear equations. The approach consists of first defining a formal "scattering problem", or eigenvalue problem, in which the solution, u, of the original p.d.e. plays the role of a "potential".

We now outline the conceptual steps (shown in figure (1.3-1)) in order to obtain the solution of the KdV equation (1.3-1) [4], when the initial data $u(x, 0)$ is given and $u(x, t) \to 0$ sufficiently rapidly as $|x| \to \infty$.

Associate with (1.3-1) the linear eigenvalue problem (1.3-2), as before, with $x \in (-\infty, \infty)$. The eigenvalues of (1.3-2) may be computed. The Schrödinger equation (1.3-2) will have a finite number of negative energy bound states ($E = -k_n$, $n = 1, 2, \ldots, N$) and a continuous spectrum for positive E ($E = k^2$, k real). For fixed t, scattering solutions of (1.3-2) are defined by the boundary conditions:

(i) For $\lambda = k^2$, k is real,

$$\psi(k, x, t) \to e^{-ikx} + R(k, t)e^{ikx}, \quad \text{as } x \to +\infty$$

$$\psi(k, x, t) \to T(k, t)e^{-ikx}, \quad \text{as } x \to -\infty.$$

These relations serve to define the reflection coefficient $R(k, t)$ and the transmission coefficient $T(k, t)$, which can be shown to satisfy $|R|^2 + |T|^2 = 1$.
(ii) For $\lambda = ik_n$, the bound states solution is defined by the boundary conditions:

$$\psi_n(k, x, t) \to e^{-k_n x}, \quad \text{as } x \to +\infty$$

$$\psi_n(k, x, t) \to C_n(t)e^{k_n x}, \quad \text{as } x \to -\infty.$$

The spectrum of the Schrödinger equation, together with the coefficients $C_n(t)$, $R(k, t)$, $T(k, t)$ are called the scattering data of a given potential $u(x, t)$. The problem of finding scattering data is called the direct problem.

We now turn to the inverse scattering problem, which consists of determining the potential u from its scattering data. It has been found that the initial scattering data is determined by the potential $u(x, 0)$, evolve according to these formulas: $k_n(t) = k_n(0)$; $C_n = \exp(-4k_n{}^3t)C_n(k_m, 0)$, for $n = 1, \ldots, N$; and $R(k, t) = \exp(8ik^3t)R(k, 0)$. This stage of the technique is called time evolution of the scattering data.

$u(x, t)$ is obtained from the scattering data, through the use of Gel'fand-Levitan integral equation,

$$K(x, Y, t) + B(x + y, t) + \int_x^{\infty} B(z + Y, t)K(x, z, t)dz = 0, \qquad Y > x \qquad (1.3\text{-}5)$$

with,

$$B(\xi, t) = \sum_{n=1}^{N} C_n^{\,2}(t)e^{-k_n\xi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k, t)e^{ik\xi}dk. \qquad (1.3\text{-}6)$$

Then:

$$u(x, t) = 12\frac{d}{dx}K(x, x, t). \qquad (1.3\text{-}7)$$

The IST is the nonlinear analogue of transform methods used for solving linear equations which arise naturally from Fourier analysis. This tends to imply that there exists a method of separation for nonlinear equations which is a generalization of the linear technique. In the IST one maps the initial data into the scattering data, follows the evolution of the set of scattering data at any desired time, and inverts the mapping with (1.3-5), thereby recovering the solution u(x, t) to the partial differential eq. (1.3-1). We may summarize the situation schematically as follows:

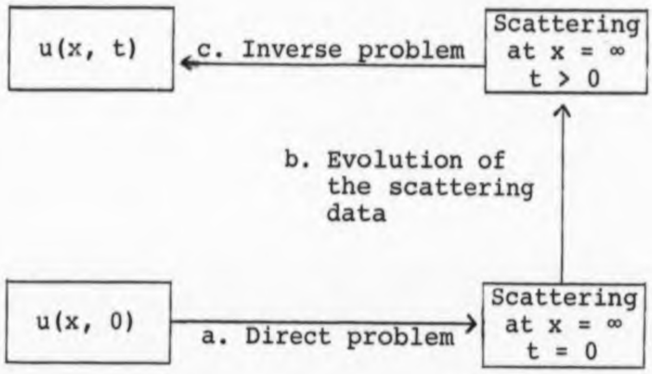

Fig. (1.3-1)

This method of solving the KdV equation was soon expressed in an elegant and general form by Lax [28]. To summarize the general formulation of Lax, consider a general nonlinear wave equation

$$\phi_t = K(\phi) \tag{1.3-8}$$

where K denotes a nonlinear operator on some suitable space of functions. Suppose we can find linear operators L and B which depend upon ϕ, and satisfy the operator equation

$$iL_t = BL - LB. \tag{1.3-9}$$

Eq. (1.3-9) automatically implies that the eigenvalues E of L, which appear in

$$L\psi = E\psi \tag{1.3-10}$$

are independent of time. Furthermore, the eigenfunctions, ψ, may be shown to evolve in time according to

$$i\psi_t = B\psi. \tag{1.3-11}$$

Example (1.3-1)

The modified KdV equation [91]:

Consider the modified Korteweg-de Vries (mKdV) equation in the following form,

$$u_t + u^2 u_x + \mu u_{xxx} = 0, \quad 0 < \mu < \frac{1}{6}. \tag{1.3-12}$$

Eq. (1.3-12) can be written in the form of

$$\frac{\partial L}{\partial t} = i[L, A] \tag{1.3-13}$$

where

$$L = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} D + \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}, \tag{1.3-14}$$

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} D^3 + D\begin{pmatrix} f & g \\ g^* & h \end{pmatrix} + \begin{pmatrix} f & g \\ g^* & h \end{pmatrix} D + \begin{pmatrix} 0 & k \\ k^* & 0 \end{pmatrix}, \tag{1.3-15}$$

and where

$$D = -i\left(\frac{\partial}{\partial x}\right), \quad \alpha = -1 + \sqrt{1 - 6\mu}, \quad \beta = -1 - \sqrt{1 - 6\mu},$$

and $\rho = -2\mu(1 - 6\mu)/(2 - 3\mu)$.

Consider the eigenvalue problem

$$L\psi = \lambda\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{1.3-16}$$

The transformation

$$\left.\begin{aligned}\psi_1 &= \sqrt{-\beta}\ \exp(-i\lambda x/\alpha\beta)\nu_2\\ \psi_2 &= \sqrt{-\alpha}\ \exp(-i\lambda x/\alpha\beta)\nu_1\end{aligned}\right\}\tag{1.3-17}$$

yields

$$\left.\begin{aligned}\nu_1' + i\xi\nu_1 &= q(x)\nu_2\\ \nu_2' - i\xi\nu_2 &= -q^*(x)\nu_1\end{aligned}\right\}\tag{1.3-18}$$

where

$$\xi = -(1 + \alpha)\lambda/\alpha\beta = (1 + \beta)\lambda/\alpha\beta, \tag{1.3-19}$$

and

$$q(x) = iu(x, t)/\sqrt{\alpha\beta} = iu(x, t)/\sqrt{6\mu}. \tag{1.3-20}$$

The potential $q(x)$ is pure imaginary, which leads to the fact that the eigenvalues for bound states, ξ_j, are

$\xi_j = i\eta_j$, η_j; real positive constant.

The latter gives the time dependence of the coefficients as follows:

$$a(\xi, t) = a(\xi, 0),$$

$$b(\xi, t) = b(\xi, 0)\exp(8i\xi^3\mu t),$$

$$c_j(t) = c_j(0)\exp(8\eta_j{}^3\mu t).$$

The N-soliton solution can be obtained as the special case, $b(\xi) = 0$,

$$u(x, t) = \pm\sqrt{6\mu}\left[\frac{\partial^2}{\partial x^2} \log \Delta\right]^{\frac{1}{2}},$$

where,

$$\Delta = \det(f_{jk})$$

$$f_{jk} = \delta_{jk} + \sum_{\ell=1}^{N} \lambda_j \lambda_\ell^{2} \lambda_k / (\eta_j + \eta_\ell)(\eta_\ell + \eta_k)$$

and

$$\lambda_j = \lambda_j(0)\exp(-\eta_j x + 4\eta_j^{3}\mu t).$$

The method is quite powerful as it is used to produce <u>specific solutions</u> (solitons or stationary waves), but is also quite specialized [2]. Certainly, not all nonlinear evolution equations can be solved by this method. In particular, the method deals with <u>evolution equations</u>, by which we mean that the equation describes how a particular quantity evolves in time from a special initial state [4]. Some of the equations solvable by IST, directly or indirectly, are the KdV, sG, mkdV, Boussinesq, and cylindrical KdV equations [1, 6, 63, 55] The IST has to date not been applied to the Zakharov equations, governing the sonic-Langmuir soliton dynamics in a two-component plasma, and indeed there is disagreement among mathematicians about the very possibility of doing this [36].

The equations then that can be solved by IST are known to be very special and one of the major outstanding problems in the field is to characterize these equations [80].

It has been noticed for some time that, when an equation which is solvable by IST - any such equation is called of IST type - (the mKdV equation for instance) is reduced, the resulting o.d.e. has the "Painlevé property". To explain what we mean by the Painlevé property or (P-property) we must look at the fundamental work done, at the turn of the century, by Painlevé and Gambier, who studied ordinary differential equations of the form,

$$\frac{d^2w}{dz^2} = F\left(z,\ w,\ \frac{dw}{dz}\right) \qquad (1.3\text{-}21)$$

where F is analytic in z, algebraic in w, and rational in $\frac{dw}{dz}$.

The problem proposed by Painlevé and Gambier was to establish conditions under which the critical points of any solution of (1.3-21), i.e., branch points and essential singularities, would be fixed points instead of movable points. Thus any function which was the solution of an equation in this class would have only poles as movable singularities.

The investigation resulted in the discovery of 50 canonical types of equations with the desired property. Of these, all but 6 were found to be integrable in terms of elementary or classical functions, or transcendents defined by linear equations. But the remaining 6 equations required the introduction of new transcendental functions for their solutions. The functions are called Painlevé transcendents [44], [24].

The Boussinesq equation [1],

$$u_{tt} - u_{xx} - 6(u^2)_{xx} + u_{xxxx} = 0, \qquad (1.3\text{-}22)$$

was linearized exactly with IST. It can be reduced to the defining equation for the first Painlevé transcendent,

$$\frac{d^2w}{dz^2} = 6w^2.$$

It is an example to demonstrate that there is a close connection between these nonlinear ordinary differential equations without movable critical points and nonlinear partial differential equations that can be linearized exactly by an inverse scattering transform [1, 5, 6, 80] Hence Ablowitz et al. [5] state their conjecture:

"Every nonlinear o.d.e. obtained by direct reduction of a nonlinear p.d.e. solvable by some inverse scattering transform has the Painlevé property."

This relation can be used to investigate either the o.d.e.'s or the p.d.e.'s. If, however, an o.d.e., obtainable by an exact reduction of a p.d.e., fails to possess the P-property then the conjecture (if it is true) states that the p.d.e. is not solvable by IST. On the other hand, passing the Painlevé test does not guarantee that the original p.d.e. is solvable by IST; to our knowledge, there is no systematic way to obtain all the possible o.d.e.'s obtainable from a p.d.e.

Another point about the conjecture, is that it relates to o.d.e.'s obtained from equations solved directly by IST. There are many examples of equations solved only indirectly by IST; the sine-Gordon equation is perhaps the best known example [6].

This conjecture has been verified in various specific cases [49, 50, 66, 87, 88].

One consequence of this conjecture is an explicit algorithm for necessary, but not sufficient, conditions to determine whether an o.d.e. meets the P-property. This algorithm is given by Ablowitz, Ramani, and Segur [6].

Another consequence of the conjecture is the investigation of a "Painlevé property" for partial differential equations. Weiss et al. [93] have, recently, introduced the P-property for p.d.e.'s and its relation with the integrability behaviour. The definition is, briefly, that a partial differential equation has the P-property when the solutions of the p.d.e. are <u>single valued</u> about the movable, singularity manifold and the singularity manifold is <u>noncharacteristic</u>. This definition of the Painlevé property allows the Ablowitz conjecture to be stated directly for the p.d.e. instead of the o.d.e. [94].

Since Weiss et al.'s paper, several papers appearing, recently, have concerned the Painlevé property of a partial differential equation and its integrability

[22, 37, 41, 77, 84, 94, 95]. But Goldstein and Infeld [36] have shown that having Painlevé property for a p.d.e. is not equivalent to complete integrability.

1.4 Lie's Similarity Method

The original similarity method was developed about hundred years ago by the mathematician Lie and his followers. Using the "group properties" of ordinary differential equations, he achieved two important results involving:

(1) how to construct an integrating factor for a first order o.d.e.,

and,

(2) how to reduce a second order differential equation to a first order equation by a change of variables.

These two results are all the more important, because they do not require the equation to be linear.

A group property of a system of differential equations is defined as a property of the system which remains unchanged when the independent and dependent variables are subjected to certain groups of transformations [73].

It has shown [44], that when a first order equation is invariant under a known group, an integrating factor may, at least theoretically, be found and the equation integrated by quadrature. For higher order equations or systems, a reduction to lower order plus a suitable number of quadratures can be carried out for a definite class of problems [11].

The practical application of one-parameter continuous transformation groups to the solution of differential equations is given below.

Consider the family of transformations

$$x_1 = \phi(x, y; \alpha), \quad y_1 = \psi(x, y; \alpha) \tag{1.4-1}$$

where α is a parameter which can vary continuously over a given range. Several simple examples of basic transformations, which must have group properties are of the types:

(a) The group of <u>translations</u>, defined by

$$x_1 = x + \alpha, \quad y_1 = y. \tag{1.4-2}$$

This is of fundamental importance, since any transformation of the type (1.4-1) is equivalent, by a change of variables, to a translation group [21].

(b) The group of <u>rotations</u>, defined by

$$x_1 = x \cos \alpha - y \sin \alpha, \quad y_1 = x \sin \alpha + y \cos \alpha. \tag{1.4-3}$$

(c) The <u>affine</u> group, defined by

$$x_1 = \alpha x, \quad y_1 = y, \quad 0 < \alpha < \infty. \tag{1.4-4}$$

And lastly,

(d) The <u>stretched</u> group, defined by

$$x_1 = (\alpha + 1)x, \quad y_1 = (\alpha + 1)y. \tag{1.4-5}$$

Example (1.4-1)

As an example, consider the o.d.e.

$$\dot{y} = \frac{y(x - y^2)}{x^2} \tag{1.4-6}$$

which can be written in the standard form,

$$M(x, y)dx + N(x, y)dy = 0$$

as follows,

$$y(y^2 - x)dx + x^2 dy = 0.$$

If the one-parameter family of integral curves of (1.4-6) is represented by $\phi(x, y) = c$, where c is a constant, then along any such curves,

$$\phi_x dx + \phi_y dy = 0. \tag{1.4-7}$$

Eqs. (1.4-6,7)yield the integrating factor μ as

$$\frac{\phi_x}{y(y^2 - x)} = \frac{\phi_y}{x^2}.$$

It is shown that [29] eq. (1.4-6) is invariant under the stretching group of transformations,

$$\left.\begin{aligned} y_1 &= \lambda^{\frac{1}{2}} y \\ x_1 &= \lambda x \end{aligned}\right\} \quad 0 < \lambda$$

then individual integral curves transform into other integral curves:

$$\phi(\lambda x, \lambda^{\frac{1}{2}}y) = c(\lambda). \tag{1.4-8}$$

If we differentiate (1.4-8) with respect to λ and set $\lambda = 1$, we get,

$$x\phi_x + \frac{1}{2}y\phi_y = \left(\frac{dc}{d\lambda}\right)_{\lambda=1} = \text{constant (say 1)}.$$

This can be written as

$$x\left(\mu y(y^2 - x)\right) + \frac{1}{2}y\mu x^2 = 1.$$

Hence,

$$\mu = \left(xy(y^2 - x) + \frac{1}{2}x^2y\right)^{-1}. \tag{1.4-9}$$

Lie did not restrict himself to stretching transformations but dealt with more general groups of transformations.

Now let α_0 be that value of the parameter α in (1.4-1) which corresponds to identity transformation, so that

$$\phi(x, y; \alpha_0) = x, \quad \psi(x, y; \alpha_0) = y.$$

Then if ε is small, the transformation

$$x_1 = \phi(x, y; \alpha_0 + \varepsilon), \quad y_1 = \psi(x, y; \alpha_0 + \varepsilon) \tag{1.4-10}$$

will be such that x_1 differs only infinitesimally from x, and y_1 from y. This transformation therefore differs only infinitesimally from the identity transformation, and is said to be an <u>infinitesimal transformation</u>.

It is shown [44], that every one-parameter group (1.4-1) contains an infinitesimal transformation. Using Taylor's theorem, (1.4-10) now reads,

$$x_1 = \phi(x, y; \alpha_0 + \varepsilon) = \phi(x, y; \alpha_0) + \varepsilon\frac{\partial\phi}{\partial\alpha}(x, y; \alpha_0) + \frac{\varepsilon^2}{2!}\frac{\partial^2\phi}{\partial\alpha^2}(x, y; \alpha_0) + \ldots$$

$$(1.4\text{-}11)$$

$$y_1 = \psi(x, y; \alpha_0 + \varepsilon) = \psi(x, y; \alpha_0) + \varepsilon\frac{\partial\psi}{\partial\alpha}(x, y; \alpha_0) + \frac{\varepsilon^2}{2!}\frac{\partial^2\psi}{\partial\alpha^2}(x, y; \alpha_0) + \ldots$$

or, neglecting higher order terms, (1.4-11) will be:

$$\left.\begin{aligned} x_1 &= x + \varepsilon\xi(x, y) \\ y_1 &= y + \varepsilon\eta(x, y) \end{aligned}\right\} \qquad (1.4\text{-}12)$$

where ξ and η are given [44] by

$$\delta f(x, y) = \left\{\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}\right\}\delta t.$$

Hence the infinitesimal transformation (1.4-12) is completely represented by the symbol,

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}. \qquad (1.4\text{-}13)$$

Consider now $p\left(= \frac{dy}{dx}\right)$ as a third variable which under the group (1.4-1) becomes p_1 where,

$$p_1 = \frac{dy_1}{dx_1} = \frac{d\psi}{d\phi} = \chi(x, y; p; \alpha).$$

If (1.4-1) is a group of transformation, then so is the set of transformations,

$$x_1 = \phi(x, y; \alpha), \quad y_1 = \psi(x, y; \alpha), \quad p_1 = \chi(x, y, p; \alpha). \tag{1.4-14}$$

This group is known as the <u>extended group</u> of the given group. Suppose $\zeta(x, y, p)$ is defined as the coefficient of α in the Taylor's series expansion of p_1 in powers of α. Then the infinitesimal transformation of the extended group is, represented by,

$$U'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial p}. \tag{1.4-15}$$

If the first order differential equation

$$F(x, y, p) = 0$$

is invariant under the extended group (1.4-14), then

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dp}{\zeta}.$$

It can be shown that when one solution of the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}$$

is known, the most general differential equation of first order invariant under Uf can be constructed and this equation is integrable by quadratures. In particular, if

$$Pdy - Qdx = 0$$

is invariant under a group of infinitesimal transformation (1.4-14), then an integrating factor is $\frac{1}{P\eta - Q\xi}$.

Example (1.4-2)

The differential equation,

$$\frac{d}{dx}(x^3y') = \nu_0\frac{x^3}{(1 + \alpha x^2)^3} - q_0\frac{x^7}{(1 + \alpha x^2)^5 y^2} \qquad (1.4\text{-}16)$$

is invariant under the group given by [11],

$$\left.\begin{aligned} \xi(x, y) &= x + \alpha x^3 \\ \eta(x, y) &= 2y \end{aligned}\right\} \qquad (1.4\text{-}17)$$

This equation can be reduced to a first order equation by finding two invariants u, v of the group.

These are found from solving the characteristic differential equations

$$\frac{dx}{x + \alpha x^3} = \frac{dy}{2y} = \frac{dP}{[2 - (1 + 3\alpha x^2)]p}. \qquad (1.4\text{-}18)$$

Integration of the first two of (1.4-18) gives

$$u(x, y) = \frac{y(1 + \alpha x^2)}{x^2}$$

while the integration of the first and third gives,

$$v(x, y, p) = \frac{p}{x}(1 + \alpha x^2)^2.$$

A first order differential equation for v(u) can be found directly, by differentiating, as follows,

$$\frac{du}{dx} = \frac{p}{x^2} - 2\frac{y}{x^3} + \alpha p = \frac{1}{x(1 + \alpha x^2)}(v - 2u) \qquad (1.4\text{-}19)$$

$$\frac{dv}{dx} = \frac{dp}{dx}\,\frac{(1 + \alpha x^2)^2}{x} - p\frac{(1 + \alpha x^2)^2}{x^2} + 4\alpha(1 + \alpha x^2)p. \qquad (1.4\text{-}20)$$

(1.4-19) provides a mapping back to the x coordinate along a trajectory v(u):

$$\frac{dx}{x(1 + \alpha x^2)} = \frac{du}{v - 2u}.$$

The expression for $\frac{dp}{dx}$ from (1.4-20) can be substituted in the original equation (1.4-18) written as,

$$\frac{dp}{dx} + \frac{3}{x}p = \frac{1}{(1 + \alpha x^2)^3}\left\{\nu_0 - \frac{q_0}{u^2}\right\} \qquad (1.4\text{-}21)$$

to yield an equation for $\frac{dv}{du}$,

$$(v - 2u)\frac{dv}{du} + 4v = \nu_0 - \frac{q_0}{u^2}. \qquad (1.4\text{-}22)$$

The method of continuous (Lie) transformation groups can be extended to partial differential equations. It can be shown that the fact that a given p.d.e. is invariant under a given group of transformations can be used to reduce the number of independent variables [11]. The resulting solutions are usually termed "similarity solutions".

Attention was called, to the method "similarity" for solving nonlinear p.d.e.'s, by Birkhoff in 1950. Using the algebraic symmetry of the p.d.e., he showed how solutions can be found merely by solving a related o.d.e. Morgan showed that the determination of the similarity solutions for a p.d.e. is equivalent to determination of the invariant solutions of these equations under a group of transformation [66].

After Birkhoff's and Morgan's work, similarity solutions were found for many physical problems [8]. Several attempts have been made to ease or remove some limitation of the method.

Example (1.4-3)

To show the particular features of the method, consider the linear diffusion equation

$$u_t = u_{xx} \tag{1.4-23}$$

which is invariant [29] to the group of transformations,

$$\left.\begin{aligned} u_1 &= \lambda^\alpha u \\ t_1 &= \lambda^2 t \\ x_1 &= \lambda x \end{aligned}\right\} \quad 0 < \lambda < \infty \qquad (1.4\text{-}24)$$

where α can have any value.

The condition for the solution to be invariant, reads,

$$\lambda^\alpha f(x, t) = f(\lambda x, \lambda^2 t). \qquad (1.4\text{-}25)$$

If we differentiate (1.4-25) with respect to λ and set $\lambda = 1$, we get

$$x f_x + 2t f_t = \alpha f. \qquad (1.4\text{-}26)$$

The characteristic equations,

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{df}{\alpha f}$$

have the two integrals $s = xt^{-1/2}$ and $y = ft^{-\alpha/2}$, so the most general form f can be taken if it is invariant to (1.4-24) is

$$f(x, t) = t^{\alpha/2} y(x/t^{1/2}). \qquad (1.4\text{-}27)$$

If f(x, t), given by (1.4-27) is to satisfy (1.4-23), then,

$$\ddot{y} = \frac{\alpha}{2} y - \frac{s}{2}\dot{y}.$$

Recently, infinitesimal transformations have been applied in the analysis of finding similarity solutions.

Consider a second-order p.d.e. with one dependent variable u and two independent variables x and t:

$$H(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{1.4-28}$$

Let a one-parameter (ε) group of transformations of the variables x, t, u be taken as,

$$\left.\begin{aligned} x_1 &= x_1(x, t, u; \varepsilon) \\ t_1 &= t_1(x, t, u; \varepsilon) \\ u_1 &= u_1(x, t, u; \varepsilon) \end{aligned}\right\} \tag{1.4-29}$$

which maps the (u, x, t) space into itself. More specifically, we now consider the following infinitesimal transformations,

$$\left.\begin{aligned} x_1 &= x + \varepsilon\xi(x, t, u) + O(\varepsilon^2) \\ t_1 &= t + \varepsilon\tau(x, t, u) + O(\varepsilon^2) \\ u_1 &= u + \varepsilon\eta(x, t, u) + O(\varepsilon^2) \end{aligned}\right\}. \tag{1.4-30}$$

The infinitesimal form of invariance condition of the solution surface becomes, with eqs. (1.4-30),

$$\theta[x + \varepsilon\xi, t + \varepsilon\tau] = \theta(x, t) + \varepsilon\eta(x, t, u) + O(\varepsilon^2). \tag{1.4-31}$$

Upon expanding the left-hand side of eq. (1.4-31), it is found,

$$\xi(x, t, \theta)\frac{\partial\theta}{\partial x} + \tau(x, t, \theta)\frac{\partial\theta}{\partial t} = \eta(x, t, \theta). \qquad (1.4\text{-}32)$$

The characteristic (Lagrange) equations resulting from eq. (1.4-32) are,

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{d\theta}{\eta}. \qquad (1.4\text{-}33)$$

These are solvable in principle. The general solution of this equation will involve two arbitrary constants of which one constant takes the role of similarity variable, say s, and the other say f(s), which plays the role of a dependent variable.

Thus, finally, the similarity form of solutions is obtained as,

$$u(x, t) = F\big(x, t, s, f(s)\big).$$

By substituting this relation in eq. (1.4-28) we can obtain an o.d.e. for f.

The classical method to determine the infinitesimal transformations ξ, η, τ stems from the invariance fact. This provides, usually a large number of simultaneous p.d.e.'s to solve, which at best is very tedious, and the most general transformations are difficult to deal with as it will be shown in the following example.

<u>Example (1.4-4)</u>

The infinitesimal transformations of the Lie group of transformations leaving invariant the equation,

$$u_{xx} = u_x^2 u_t$$

are [11],

$$\xi(x, t, u) = -\gamma x\left(\frac{u^2}{4} + \frac{t}{2}\right) - \frac{\delta}{2}xu + \lambda x + g(u, t)$$

$$\tau(x, t, u) = \alpha + 2\beta t + \gamma t^2$$

$$\eta(x, t, u) = k + \delta t + \beta u + \gamma tu,$$

where $\{\alpha, \beta, \gamma, \delta, k, \lambda\}$ are arbitrary constants and $g(u, t)$ satisfies,

$$g_{uu} = g_t.$$

Note that, finding the general solution to the Lagrange's equations (1.4-33), in this case, is very difficult, if it is not impossible.

In spite of the limitation of the method, at present, there is a revival of interest in the group theoretic analysis of nonlinear p.d.e.'s. The main reason behind that is the so called "Painlevé conjecture" (see §1.3).

In the early stages in the study of solitons, it was shown [81], that the similarity solution of KdV eq.

$$u_t + uu_x + u_{xxx} = 0 \qquad (1.4\text{-}34)$$

satisfies a third order nonlinear o.d.e. Using special values of infinitesimal, it was found that, the similarity variables are

$$s = \frac{1}{2}a^{-5/3}\left(2a^2x - 3\alpha(at + \delta) + 6(a\beta - \alpha\delta)\right)/(at + \delta)^{1/3}$$ (1.4-35)

and,

$$f(s) = (at + \delta)^{2/3}\left[\alpha - \frac{2}{3}au\right]\frac{2}{3}a^{5/3} \quad (1.4\text{-}36)$$

while, we have found that, the correct forms for s and f(s) are given by replacing $\frac{2}{3}a^{5/3}$ in (1.4-36) by $\frac{3}{2}a^{-5/3}$.

Lakshmanan and Kaliappan [55], present the result of investigation of invariance properties of a large class of nonlinear evolution equations under a one-parameter continuous (Lie) group of transformations. Many examples of similarity solution can be found in the literature (see ch. 6).

1.5 Other Methods for Producing Solutions to Nonlinear P.d.e.'s

In addition to the major methods mentioned in the previous sections, there are many other techniques, which play a very important role in the study of nonlinear p.d.e.'s.

The Bäcklund transformation first appeared in 1875 and was introduced by the Swedish mathematician, Bäcklund, when he was considering a problem in differential geometry involving the theory of surfaces of constant negative curvature [25]. It is a powerful method for constructing new solutions out of known ones. A Bäcklund transformation can be considered as a type of generalization of the superposition principle for linear p.d.e.'s.

There is no generally accepted definition of a Bäcklund transformation [26]. To describe it in some cases, consider a second order p.d.e. The Bäcklund transformation consists of a pair of first order p.d.e.'s relating a solution of the given second order equation to another solution of the same equation - in this case it is called an auto-Bäcklund transformation - or to a solution of another equation.

Indeed, the Miura transformation (1.1-11) constitutes [25], [28] one half of a Bäcklund transformation. The other half of the transformation is obtained by repeatedly substituting (1.1-11) into (1.1-10) until all the x derivatives of ϕ have been eliminated. The complete Bäcklund transformation so obtained consists of the pair of coupled p.d.e.'s,

$$\left.\begin{aligned} \phi_x &= \frac{\varepsilon}{(6\beta)^{\frac{1}{2}}}(\alpha u + \beta\phi^2) \\ \phi_t &= -\frac{\alpha\varepsilon}{(6\beta)^{\frac{1}{2}}}\left[\frac{1}{3}\beta\phi^2 u + u_{xx} + \left(\frac{2\beta}{3}\right)^{\frac{1}{2}}\varepsilon\phi u_x + \frac{\alpha u^2}{3}\right] \end{aligned}\right\} . \qquad (1.5\text{-}1)$$

The reduction of Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0 \quad (\nu > 0) \qquad (1.5\text{-}2)$$

to the heat equation,

$$\phi_t - \nu\phi_{xx} = 0 \quad (\nu > 0) \tag{1.5-3}$$

can be written as a Bäcklund transformation [96]

$$\left.\begin{aligned} \phi_x &= -\frac{u\phi}{2\nu} \\ \phi_t &= -(2\nu u_x - u^2)\frac{\phi}{4\nu} \end{aligned}\right\}. \tag{1.5-4}$$

Eq. (1.5-3) [99] has an auto-Bäcklund transformation,

$$\left.\begin{aligned} \phi' - \sqrt{\nu}\phi'_x &= 0 \\ \sqrt{\nu}\phi_x - \phi_t &= 0 \end{aligned}\right\} \tag{1.5-5}$$

where ϕ and ϕ' are two solutions of eq. (1.5-3).

It is well known [85], [26], that Liouville's equation

$$u_{xt} = \exp(u) \tag{1.5-6}$$

can be reduced with the help of the Bäcklund transformation,

$$\left.\begin{aligned} u_x' &= u_x + \beta\exp\left[\left(\frac{1}{2}\right)(u + u')\right] \\ u_t' &= -u_t - \left(\frac{2}{\beta}\right)\exp\left[\left(\frac{1}{2}\right)(u - u')\right] \end{aligned}\right\} \tag{1.5-7}$$

to the linear equation

$$u_{xt}' = 0. \tag{1.5-8}$$

Thus, insertion of the general solution

$u'(x, t) = f_1(x) + f_2(t)$ of eq. (1.5-8) into the Bäcklund transformation (1.5-7) and subsequent integration produces the general solution of Liouville's equation,

$$u(x, t) = f_2 - f_1 + 2\ln\left\{-\sqrt{2}/\bigl(f(x) + g(t)\bigr)\right\}, \qquad (1.5\text{-}9)$$

where $f(x) = \int^x e^{-f_1(s)}\,ds$, $g(t) = \int^t e^{f_2(t)}\,dt$, which Liouville (1853) first found by a different method.

Eq. (1.5-9) may be simplified, by writing it in the following form

$$u(x, t) = \ln\left\{\frac{2f'g'}{(f + g)^2}\right\}, \qquad (1.5\text{-}10)$$

which is equivalent, to the following form, which has been given earlier:

$$u(x, t) = \ln\left\{\frac{-2h'g'}{(1 + hg)^2}\right\} \qquad (1.5\text{-}11)$$

where $h = 1/f$.

Example (1.5-1)

To illustrate the applicability of a Bäcklund transformation, let us take sine-Gordon equation in the normalized form,

$$\phi_{uv} = \sin\phi, \quad u = \frac{x - t}{2}, \quad v = \frac{x + t}{2}. \qquad (1.5\text{-}12)$$

The appropriate Bäcklund transformation is,

$$\phi_u' = \phi_u + 2\lambda\sin\left(\frac{\phi' + \phi}{2}\right) \tag{1.5-13}$$

$$\phi_v' = -\phi_v + \frac{2}{\lambda}\sin\left(\frac{\phi' - \phi}{2}\right) \tag{1.5-14}$$

where λ is an arbitrary parameter. Differentiating (1.5-13) with respect to v, (1.5-14) with respect to u and using eq. (1.5-13,14) gives,

$$\phi_{uv}' = \sin\phi'. \tag{1.5-15}$$

Thus, (1.5-13, 14) is an auto-Bäcklund transformation for (1.5-12). Consider now the solution $\phi = \phi_0$, where $\phi_0 = 0$ is the trivial solution of (1.5-12). Another solution is found by application of (1.5-13, 14), to be given by,

$$\frac{\partial\phi_1}{\partial u} = 2\lambda\sin\frac{\phi_1}{2}, \quad \frac{\partial\phi_1}{\partial v} = \frac{2}{\lambda}\sin\frac{\phi_1}{2}. \tag{1.5-16}$$

Solving these relations, separately, gives,

$$2\lambda u = 2\ln\left(\tan\tfrac{1}{4}\phi_1\right) + f(v)$$

$$2\lambda^{-1}v = 2\ln\left(\tan\tfrac{1}{4}\phi_1\right) + g(u),$$

where f and g are constants of integration. It can easily be shown that the solution is the solitary wave one, i.e.,

$$\tan\frac{\phi_1}{4} = c\exp\left(\lambda u + \frac{v}{\lambda}\right) = c\exp\left\{\frac{x - Ut}{\sqrt{1 - U^2}}\right\}, \tag{1.5-17}$$

where $U = \frac{\lambda^2 - 1}{\lambda^2 + 1}$.

This method may be repeated, so that ϕ_1 generates a new solution, ϕ_2, which is the Perring-Skyrme result for two interacting solitary waves. The technique may be repeated further to find solutions representing the interaction of many solitons.

During the last decade, much work has been carried out on Bäcklund transformations especially for nonlinear evolution equations. The technique is intimately connected with the applicability of IST [26 , 28], similarity solutions [57 , 13], the existence of soliton solutions [98].

Bäcklund transformations have been applied in different ways by several authors. A chain of Bäcklund transformation for the KdV equation [61], a generalized Bäcklund transformation [62], and a class of parabolic equations that admit Bäcklund transformations [64] have been found.

In trying to solve the KdV equation, (1.1-8) which can be reduced to the equation,

$$F\left(F_t + F_{xxx}\right)_x - F_x\left(F_t + F_{xxx}\right) + 3\left(F_{xx}{}^2 - F_x F_{xxx}\right) = 0 \tag{1.5-18}$$

(see §1.1), we firstly look at a solution of the form,

$$F = 1 + \sum_{n=1}^{N} \varepsilon^n F^{(n)} . \tag{1.5-19}$$

Assuming that the expansion and its derivatives converge only for sufficiently small values of the parameter ε, it can be shown that

$$\left(F_t^{(1)} + F_{xxx}^{(1)}\right)_x = 0.$$

Hence we may take

$$\left.\begin{aligned} F^{(1)} &= \exp\theta_1 + \exp\theta_2 \\ \theta_i &= a_i x - a_i^3 t + \delta_i, \quad i = 1, 2 \end{aligned}\right\}.$$

This exact solution for $F^{(1)}$ can be subsituted in the equation for coefficients of terms of $0(\varepsilon^2)$ in (1.5-19) to give,

$$\left(F_t^{(2)} + F_{xxx}^{(2)}\right)_x = 3a_1a_2(a_1 - a_2)^2\exp(\theta_1 + \theta_2),$$

which integrates to

$$F^{(2)} = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 \exp(\theta_1 + \theta_2).$$

Repeating the same method and choosing $F^{(3)} = 0$, we see that all the subsequent $F^{(n)} = 0$ $(n > 3)$. Hence one solution to the KdV equation is,

$$\left.\begin{aligned} u(x, t) &= \frac{\partial^2}{\partial x^2}\log F(x, t) \\ F(x, t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + A\exp(\theta_1 + \theta_2) \\ A &= \left((a_1 - a_2)/(a_1 + a_2)\right)^2 \end{aligned}\right| . \tag{1.5-20}$$

This method of reducing the KdV equation to one or more bilinear equations has become known as Hirota's direct method. Hirota's technique has been described by Jimbo and Miwa [25] as it has a deep theoretical significance. Hirota describes it as a direct and systematic way of finding exact solutions of certain nonlinear evolution equations [26]. Its principle drawback [20] (apart from the guesswork element) is that it gives only soliton solutions.

With the basic symbol D, which is defined [20] as,

$$D_x^{\,m} D_t^{\,n}(a.b) = \left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x, t)b(x', t')\right]_{\substack{x'=x \\ t'=t}} \tag{1.5-21}$$

and which has certain properties [26], the nonlinear equation will transform into bilinear differential equations of the following special form,

$$F\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}, \frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)a(x, t)b(x', t')\Big|_{\substack{x=x' \\ t=t'}} = 0.$$

Hirota's technique is regarded as a kind of perturbational approach. It has been applied to the sG, double sG, NS, mkdV, Bergers, wave-wave interactions, 2DkdV, and 2DsG equations [20 , 26]. On the other hand there is a connection between Hirota's bilinear method and the inverse scattering transform [75].

Example (1.5-2)

Consider the sine-Gordon (sG) equation,

$$\phi_{xx} - \phi_{tt} = \sin \phi. \tag{1.5-22}$$

By using the transformation

$$\phi = 4\tan^{-1}(g/f),$$

eq. (1.5-22) reduces to [20],

$$\left.\begin{aligned} (D_x^{\,2} - D_t^{\,2})(f.g) &= fg \\ (D_x^{\,2} - D_t^{\,2})(f.f. - g.g) &= 0 \end{aligned}\right\}. \tag{1.5-23}$$

We now put,

$$\left.\begin{aligned} f &= 1 + \varepsilon^2 f^{(1)} + \varepsilon^4 f^{(2)} + \ldots \\ &\text{and} \\ g &= \varepsilon g^{(1)} + \varepsilon^3 g^{(2)} + \varepsilon^5 g^{(3)} + \ldots \end{aligned}\right\} \tag{1.5-24}$$

Equating coefficients of powers of ε gives,

$$g_{xx}^{(1)} - g_{tt}^{(1)} = g^{(1)}. \tag{1.5-25}$$

The simplest nontrivial solution to this equation is,

$$g^{(1)}(x, t) = e^{\theta(x,t)}, \tag{1.5-26}$$

where

$$\theta(x, t) = kx - wt + \delta$$

with

$$k^2 - w^2 = 1.$$

All the other terms can be chosen to be zero so that a solution to (1.5-22) is,

$$\phi(x, t) = 4\tan^{-1} e^{\theta(x,t)}, \tag{1.5-27}$$

which is the well-known one-kink solution.

For the two-kinks solution, we put,

$$g^{(1)} = \exp \theta_1 + \exp \theta_2 \tag{1.5-28}$$

where

$$\theta_i = k_i x - w_i t + \delta_i$$

with

$$k_i^2 - w_i^2 = 1.$$

Then the simplest choice for $f^{(1)}$ is,

$$f^{(1)} = a(1, 2)\exp(\theta_1 + \theta_2) \tag{1.5-29}$$

where

$$a(1, 2) = \frac{1 - k_1k_2 + w_1w_2}{1 + k_1k_2 - w_1w_2} = -\left(\frac{k_1 - k_2}{w_1 + w_2}\right)^2 = -\left(\frac{w_1 - w_2}{k_1 + k_2}\right)^2 . \tag{1.5-30}$$

Again all other terms can be chosen to be zero which gives,

$$\phi(x, t) = -4\tan^{-1}\left(\left|\frac{k_1 + k_2}{w_1 - w_2}\right|\frac{\cosh\frac{1}{2}(\theta_1{}' - \theta_2{}')}{\sinh\frac{1}{2}(\theta_1{}' + \theta_2{}')}\right) \tag{1.5-31}$$

where

$$\theta_i{}' = \theta_i - \tfrac{1}{2}\ln\left|\frac{k_1 + k_2}{w_1 - w_2}\right| .$$

Solutions for higher numbers of kinks can be found in a similar way.

There are many individual techniques which solve nonlinear p.d.e.'s, but which are less general and important than the methods mentioned before. The Toda technique, for instance, has been used in some situations, [28, 96].

1.6 Separation of Variables

Historically, the theory of variables separation of partial differential equations has been developed and proved most useful for linear p.d.e.'s especially when it is used together with Fourier analysis to give useful solutions for boundary value problems.

However, it is quite natural to try this method on nonlinear p.d.e.'s, as some nonlinear problems are solvable, under special conditions, by familiar "linear" methods. One of familiar linear methods is the Fourier transform, which is analogous to the inverse scattering transform for nonlinear p.d.e.'s.

The major advantage of using variables separation for computation of explicit solutions of p.d.e.'s is that the problem is reduced to solving ordinary differential equations (the separation equations).

Basically, a partial differential equation is separable in the independent variables $x_1, \ldots, x_n$ if the equation admits a nontrivial solution of the form $u(x_1, \ldots, x_n) = \prod_{i=1}^{n} S^{(i)}(x_i)$. One can also talk about additive separation $v(x_1, \ldots, x_n) = \sum_{i=1}^{n} T^{(i)}(x_i)$ (which is equivalent to the product separation above by using dependent variable transformation $v = \ln u$).

Several examples of direct separation (additive or multiplication) for nonlinear p.d.e.'s can be found in the literature.

Oplinger's problem [7] of solving the hyperbolic equation

$$u_{tt} - c^2\left(1 + \alpha\int_0^L u_x{}^2 dx\right)u_{xx} = 0, \tag{1.6-1}$$

where c, α, L are constants, can be solved by trying the solution

$$u = F(x)G(t)$$

which separates eq. (1.6-1) into the following o.d.e.'s,

$$F'' + \nu^2 F = 0$$

$$G'' + \nu^2 c^2 (1 + \alpha I G^2) G = 0$$

where ν^2 is the separation constant, and $I = \int_0^L \frac{dF}{dx}^2 dx$.

Solutions of physical problems possessing symmetry with respect to a group can usually be given a simplified mathematical form by the introduction of suitable variables associated with this group. Birkhoff [10] has shown how such substitution lead to "separation of variables" of equations occurring in fluid mechanics.

Recently, Cosgrove [23], has formulated Einstein's equations for the stationary axisymmetric vaccum gravitational field. In this case, γ, is the basic field variable which satisfies a field equation which is fourth-order p.d.e. This equation has been solved by separation of variables in the form $\gamma = \gamma_1(\rho) + \gamma_2(\tau)$, where $\gamma_1(\rho)$ is either zero or a very simple function and $\gamma_2(\tau)$ satisfies an ordinary differential equation of the fourth order.

Johnson and Thompson [46] have shown that the method of separation of variables can be used to solve the appropriate scalar Gel'fand-Levitan equation - which is occur in IST technique. This produces many new solutions (with soliton interactions) and in particular, introduces a new rational exponential soliton.

As might be expected the classical method of simple separation can be generalized, and separable solutions of nonlinear equations can be achieved by dependent or independent variable transformations.

In illustration of the idea, consider the sine-Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin \phi. \tag{1.6-2}$$

Eq. (1.6-2) is not separable in the classical sense. Lamb [56] has shown that sG equation admits a solution of the form

$$\phi = 4 \tan^{-1}\left(\frac{X(x)}{T(t)}\right). \tag{1.6-3}$$

Upon substitution (1.6-3) into the sG equation it can be shown that X and T must satisfy

$$(X')^2 = kX^4 + mX^2 + n,$$

$$(T')^2 = -kT^4 + (m - 1)T^2 - n,$$

where k, m, and n are arbitrary constants. This generalized notation of a separable solution will be fully investigated in the next two chapters.

1.7 Contributions of this Research

As mentioned before, this study is devoted to the separation of variables method applied to nonlinear partial differential equations. It is divided in two parts.

Part I, which is represented by chapters 2 and 3, is devoted to the separation technique itself. We start this part by giving a brief general description of the classical separability method for linear equations. We then list the definitions (as found in the literature) of the concept of "separation of variables" for linear and nonlinear equations. We show that in some cases, the classical linear technique can be applied directly to nonlinear equations. The historical background to the general nonlinear technique represented by the separable solutions of sG equation is then given. We give, for the rest of the two chapters, a full description of the separation of variables technique including notations, definitions, remarks, examples and theorems. We shall distinguish between two types of separability; "simple" and "implicit". Simple separability, defined roughly as the equation is separable as it stands, is studied extensively in chapter 2 which ends with an introductory description about implicit separability, which means that the equation is separable by a transformation. In chapter 3, we carry on describing implicit separability by using different transformations and restrictions.

Part II, which is represented by chapters 4, 5 and 6, is devoted to the connection between the separation of variables method and other known methods.

In chapter 4, we use the relation between the Painlevé property and the separation method to derive sets of second order equations which are not solvable by the inverse scattering transform (IST), according to the Ablowitz conjecture.

In chapter 5, the analysis for finding sets of equations, not solvable by IST is extended to higher order p.d.e.'s.

A comparison of the similarity and separation methods through examples and theorems is shown in chapter 6. We have found, in general that there are close ties between the methods since they both can be considered as methods of reduction a p.d.e. to an o.d.e.

CHAPTER TWO

Separation of Variables I

2.1 Direct Separation of Variables

The classical method of separation of variables is one of the powerful systematic methods, of solving linear partial differential equations (p.d.e.'s). In this method, the p.d.e. is broken down into ordinary differential equations (o.d.e.'s) by direct separation of variables and the final solution is built up from particular solutions of these o.d.e.'s. The basic idea is to assume that a solution, $\phi(x, t)$ of a given p.d.e. in two independent variables x and t, is of the form,

$$\phi(x, t) = X(x)T(x), \tag{2.1-1}$$

where X is a function of x only, and T is a function of t alone. More generally, if the independent variables are $x_1, x_2, \ldots, x_n$ in the p.d.e. then a solution can be taken as,

$$\phi(x_1, \ldots, x_n) = \prod_{i=1}^{n} X_i(x_i). \qquad (2.1\text{-}1)'$$

The purpose of this assumption is to simplify the problem from one of solving a p.d.e. to that of solving an o.d.e. This method has been shown to provide solutions for the standard equations of mathematical physics, in many coordinate systems, and with many given boundary conditions (see §1.1). More specifically, separation of variables may be used to solve initial boundary value problems (IBVP's) and applied to problems where the p.d.e., with constant or variable coefficients, is linear and homogeneous and the boundary conditions are of the form, [31]:

$$\left.\begin{aligned} \alpha\phi_x(0, t) + \beta\phi(0, t) &= 0 \\ \gamma\phi_x(1, t) + \delta\phi(1, t) &= 0 \end{aligned}\right\} \qquad (2.1\text{-}2)$$

where α, β, γ, and δ are constants. The general idea for the method is to find an infinite number of solutions $\phi_n(x, t) = X_n(x)T_n(t)$, and the final solution is the sum of these solutions according to the superposition principle which is applied to linear equations.

The method is most easily explained by considering the following second order p.d.e.:

$$A(x)\phi_{xx} - B(t)\phi_{tt} + C(x)\phi_x - D(t)\phi_t + (E(x) - F(t))\phi = 0. \qquad (2.1\text{-}3)$$

Substituting $\phi(x, t) = X(x)T(x)$, in the equation and divising by XT, yields the following identity:

$$A(x)\frac{X''}{X} + C(x)\frac{X'}{X} + E(x) = B(t)\frac{T''}{T} + D(t)\frac{T'}{T} + F(t).$$

This identity is true if both sides are equated to a constant ($-\lambda$ say) which gives,

$$\left.\begin{array}{l} A(x)X'' + C(x)X' + (E(x) + \lambda)X = 0 \\ B(t)T'' + D(t)T' + (F(t) + \lambda)T = 0 \end{array}\right\} \qquad (2.1\text{-}4)$$

If $C(x)r(x) = (A(x)r(x))'$ and $D(t)s(t) = (B(t)s(t))'$ for some functions, r and s, then each equation is called a Sturm-Liouville equation in the honour of the German mathematician Sturm (1803-1855) and French mathematician Liouville (1809-1887) who, independently, were the first to formulate the equation. The boundary value problem, (2.1-4) with boundary conditions of the type (2.1-2) is called a Sturm-Liouville problem. Each value of λ for which the problem has non-trivial solutions is called an eigenvalue for the problem. Each non-trivial solution is called an eigenfunction for the problem.

Theorem (2.1-1) [15]

Suppose that the Sturm-Liouville problem (2.1-4) and (2.1-2) is given. Then

(i) There exists a countably infinite set of eigenvalues. The eigenvalues may be placed in an increasing sequence λ_n, $n \in Z^+$ (i.e. $\lambda_n < \lambda_{n+1}$).

For each n, let $\phi_n(x)$ denote an eigenfunction corresponding to λ_n, then

(ii) the sequence of eigenfunctions $\{\phi_n(x)\}$ is orthogonal,

(iii) for each n, $\phi_n(x)$ is uniquely determined up to a non-zero factor,

(iv) if $f^2(x)$ is integrable on (a, b), then the set of partial sums of its Fourier series with respect to eigenfunctions of the system converges in the mean to $f(x)$. □

Proof of this theorem can be found, for example in [90]. ■

Example (2.1-2)

Let us start with an equation with variable coefficients,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + w^2u = 0 \qquad (2.1\text{-}5)$$

where w is a constant. Eq. (2.1-5) is derived from Helmoltz equation $u_{xx} + u_{tt} + w^2u = 0$, when using polar coordinates.

Suppose that $u(r, \theta) = f(r)g(\theta)$, not identically zero, then for some λ, f and g satisfy the o.d.e.'s,

$$r^2f''(r) + rf'(r) + (w^2r^2 - \lambda^2)f(r) = 0 \qquad (2.1\text{-}6)$$

$$g''(\theta) + \lambda^2g(\theta) = 0. \qquad (2.1\text{-}7)$$

The general solutions of the o.d.e.'s (2.1-6,7) are,

$$f(r) = \alpha_1J_\lambda(wr) + \alpha_2J_{-\lambda}(wr)$$

$$g(\theta) = \beta_1e^{i\lambda\theta} + \beta_2e^{-i\lambda\theta}$$

where J_λ and $J_{-\lambda}$ are Bessel functions of the first kind, which are orthogonal [10].

Example (2.1-3)

The classical method is easily extended to equations with more than two independent variables. We now apply the method to Laplace's equation in rectangular cartesian co-ordinates,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \qquad (2.1\text{-}8)$$

Substituting $\phi = X(x)Y(y)Z(z)$ in (2.1-8) gives,

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0. \qquad (2.1\text{-}9)$$

Dividing eq. (2.1-9) by XYZ gives,

$$\frac{X''(x)}{X(x)} = -\left(\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right) = a$$

where a is a constant. This is equivalent to,

$$\left.\begin{aligned} &\frac{d^2X}{dx^2} - aX = 0 \\ &\frac{d^2Y}{dy^2} - bY = 0 \\ &\frac{d^2Z}{dz^2} + (a + b)Z = 0 \end{aligned}\right\} \qquad (2.1\text{-}10)$$

where b is another constant. In this case, the solutions to eq.'s (2.1-10) are well known:

$$X = k_1 e^{\pm\sqrt{a}x}, \quad Y = k_2 e^{\pm\sqrt{b}y}, \quad Z = k_3 e^{\pm\sqrt{a+b}z},$$

where k_1, k_2 and k_3 are constants.

Substituting back into equation (2.1-8) yields,

$$\phi(x, y, z) = C\exp(\pm\sqrt{a}x \pm \sqrt{b}y \pm \sqrt{a+b}z),$$

where $C = k_1k_2k_3$.

Since Laplace's equation is linear, a sum of solutions is also a solution. Therefore,

$$\phi = \sum_{a,b} C_{ab}\exp(\pm\sqrt{a}x \pm \sqrt{b}y \pm \sqrt{a+b}z),$$

is a solution.

Some "separability" definitions

Many definitions of "separable equations" appear in the Literature. We present a few of these definitions here.

Definition (2.1-4) [82]

A second order homogeneous linear p.d.e. in two variables,

$$a_{11}u_{xx} + a_{12}u_{xt} + a_{22}u_{tt} + a_{10}u_x + a_{01}u_t + a_{00}u = 0, \tag{2.1-11}$$

where a_{ij} $(i, j = 0, 1, 2)$ are functions of x and t, is called separable if for each solution u of the form $u(x, t) = X(x)T(t)$, it is possible to write eq. (2.1-11) in the form,

$$\frac{1}{X}f(D)X = \frac{1}{T}g(D')T,$$

where $f(D)$, $g(D')$ are quadratic functions of $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial t}$ respectively.

In [78] a necessary condition for the previous definition is stated as follows:

Theorem (2.1-5)

The p.d.e. (2.1-11) can be reduced to two o.d.e.'s of second order both containing an arbitrary constant λ, by separation, if there exists a transformation,

$$\left.\begin{array}{l} \xi = \xi(x, t) \\ \eta = \eta(x, t) \end{array}\right\}, \quad \frac{\partial(\xi, \eta)}{\partial(x, t)} \neq 0$$

so that the resulting equation,

$$A_{11}u_{\xi\xi} + A_{22}u_{\eta\eta} + A_{10}u_{\xi} + A_{01}u_{\eta} + A_{00}u = 0, \qquad (2.1\text{-}12)$$

does not contain a term of the type $u_{\xi\eta}$ and if there exists a function $B(\xi, \eta)$ so that $A_{11}/B = C_{11}$, $A_{10}/B = C_{10}$, are functions of ξ only and $A_{22}/B = C_{22}$, $A_{01}/B = C_{01}$, are functions of η only, and if A_{00}/B can be split up into a function of ξ and a function of t as $A_{00}/B = C_{00}^{(1)}(\xi) + C_{00}^{(2)}(\eta)$. □

A more general definition, applicable to any order p.d.e. is, Definition (2.1-6) [59]

Any homogeneous, linear p.d.e., in two independent variables,

$$L(\phi(x,\ t)) = 0, \tag{2.1-13}$$

is separable if the assumption,

$$\phi = \frac{X(x)T(t)}{R(x,\ t)}$$

permits the separation of the p.d.e. into two o.d.e.'s. The equation is said to be simply separable if R is a constant, and R-separable if R is not.

This definition is equivalent to the following definition.

Definition (2.1-7) [92]

Eq. (2.1-13) is separable, providing a function $f(x,\ t)$ exists, such that

$$L(\phi) = L(X(x)T(t)) = (g(x) + h(t))f(x,\ t)X(x)T(t).$$

Definition (2.1-8) [53]

The p.d.e.,

$$F\left(\frac{\partial^{i+j}u(x,\ t)}{\partial x^i \partial t^j},\ x,\ t;\ \ i,\ j \in \mathbb{Z} \geqslant 0\right) = 0, \tag{2.1-14}$$

where F is analytic, is called separable for (x, t), if there are two analytic o.d.e.'s

$$X^{(k)}(x) + f\left(X^{(k-1)}(x), \ldots, X(x), x, \alpha\right) = 0$$

$$T^{(\ell)}(t) + g\left(T^{(\ell-1)}(t), \ldots, T(t), t, \alpha\right) = 0$$

for some k and ℓ, both equations jointly depending in an analytic way on the complex parameter α, such that the function

$$u(x, t) = X(x)T(t)$$

is a solution of (2.1-14).

The basic idea of separation is the same in all these apparently different definitions. The "definition" often depends on the authors and their aims. For instance, additive separability, may be taken as the basic definition for separability [47], [58]. On the other hand additive separability can be regarded as a special case of multiplicative separability if dependent variable transformations are allowed to play a role in the definitions.

One of the characteristics of separation of variables for linear p.d.e.'s is that the range of practical applications is greatly increased by the introduction of additional coordinate systems. In this connection, however, the Stäckel matrix is the fundamental mechanism for variables separation [39, 47, 48, 59].

In a number of papers, several people have used the ordinary separation of variables technique (additive or multiplicative) to obtain solutions of specified nonlinear p.d.e.'s. A good example of an equation which the separation method provides solutions for, directly, is the nonlinear diffusion equation,

$$\frac{\partial}{\partial x}(u^n u_x) = u_t, \quad n > 0.$$

Also see §1.6.

2.2 Implicit Separation of Variables

It is quite natural to ask whether the classical technique of separation of variables can be modified to apply to a wider class of nonlinear p.d.e.'s rather than the special cases, as mentioned in the previous chapter. Lamb [56] in 1971, in an implicit manner, showed that the sG equation,

$$\phi_{xx} - \phi_{tt} = \sin \phi \tag{2.2-1}$$

has a class of solutions of the form,

$$\phi(x, t) = 4\tan^{-1}(X(x)T(t))$$

where the functions $X(x)$ and $T(t)$ are solutions of the o.d.e.'s,

$$\left.\begin{aligned}(X')^2 &= pX^4 + mX^2 + q \\ (T')^2 &= -qT^4 + (m - 1)T^2 - p\end{aligned}\right\},$$

and where p, q and m are arbitrary constants.

Zagrodzinski [100] extended Lamb's analysis by showing that equations of the form,

$$\phi_{xx} + \varepsilon\phi_{tt} = \sin\phi, \quad (\varepsilon = \pm 1) \tag{2.2-2}$$

have solutions of the form,

$$\phi(x, t) = \pm 4\tan^{-1}(X(x)T(t)) + (1 - \delta)\frac{\pi}{2},$$

provided that $X(x)$, $T(t)$ satisfy the o.d.e.'s,

$$\left.\begin{aligned} (X')^2 &= pX^4 + \delta mX^2 + q \\ \varepsilon(T')^2 &= qT^4 + \delta(1 - m)T^2 + p \end{aligned}\right\},$$

where $\delta = \pm 1$ and p, q, m are arbitrary constants.

For the sG equation (2.2-1) and its elliptic variant (2.2-2) the analysis above shows that they are "separable" not in terms of the original dependent variable ϕ, but in terms of a new dependent variable ψ, where

$$\psi(x, t) = \tan\left(\frac{\phi}{4}\right), \tag{2.2-3}$$

and for which the equation takes the form,

$$(1 + \psi^2)(\psi_{xx} + \varepsilon\psi_{tt}) - 2\psi(\psi_x^{\ 2} + \varepsilon\psi_t^{\ 2}) = \psi(1 - \psi^2). \tag{2.2-4}$$

This raises the question of finding more dependent variable transformations which lead to a separable equation for $\psi(x, t)$, and of finding a mathematical tool for analysing the separability of the various other nonlinear p.d.e.'s, i.e., of modifying the classical method so that it can be applied to nonlinear p.d.e.'s.

In this connection, Osborne and Stuart [69] have found a general class of dependent variable transformations under which the sG equation is separable, and that the most general transformation which does this is,

$$\phi = 2\cos^{-1}\mathrm{sn}\left\{\frac{\ln(\alpha\psi)}{k\beta}, k\right\}, \tag{2.2-5}$$

where ψ is the new dependent variable and $\alpha, \beta, k \in \mathbb{R} - \{0\}$ with $k \leq 1$ and sn is a Jacobian elliptic function with sine amplitude of modulus k.

Putting $k = 1$, and $\beta = -n$, (2.2-5) reduces to,

$$\psi = \frac{1}{\alpha}\tan^n\left(\frac{\phi}{4}\right),$$

which is a two parameter generalization of (2.2-3). It can be shown that the final solution of (2.2-1) is independent of choice of α and n.

The existence of this general transformation means, of course that there are several classes of separable solutions. In this connection, Bryan et al. [17], have classified a set of separable solutions of the sine-Gordon equation in one space and one dimension and of its Laplacian or elliptic variant. They have found three structural groups: a) a one-soliton sector containing the single soliton and antisoliton, b) a two-soliton sector which includes the doublet solutions and c) general sector in which the solutions are products of Jacobian elliptic functions with coupled periods.

Hudak [43] has obtained a vortex solution to the

(2 + 0) dimensional sG equation,

$$\Delta\phi = 1/\lambda_p^{-2}\sin(p\phi), \tag{2.2-6}$$

using the same transformation, $\phi = 4\tan^{-1}(X(x)T(t))$ (in which $p = 1$ and $\lambda_1 = 1$ is taken for simplicity).

It has been shown that one special set of all possible solutions is,

$$\phi = \pm 4\tan^{-1}\left[\left[a/(2 - a)\right]^{\frac{1}{2}}\left\{\sum_n c_n \sinh\left\{\left[(2 - a)/2\right]^{\frac{1}{2}}(x - x_n)\right\}\right\}\right.$$
$$\left.\times \left\{\sinh\left[(a/2)^{\frac{1}{2}}(t - t_0)\right]^{-1}\right\}\right], \tag{2.2-7}$$

where t_0, x_n, c_n are arbitrary real numbers satisfying the conditions,

$$\sum_{n,m} c_n c_m \cosh\left\{\left[(2 - a)/2\right]^{\frac{1}{2}}(x_n - x_m)\right\} = 1. \tag{2.2-8}$$

The sum in (2.2-7) and (2.2-8) run over a set of integers n, m.

The vortex type solution of eq. (2.2-6) with $p = 1$ and $\lambda_1 = \lambda$ has the form (2.2-7) where the only non-zero constant c_n is $c_0 = 1$, i.e.,

$$\phi = \pm 4\tan^{-1}\left\{\left(\frac{a}{2 - a}\right)^{\frac{1}{2}}\sinh\left[\left(\frac{2 - a}{2}\right)^{\frac{1}{2}} \frac{(x - x_0)}{\lambda}\right]\right.$$
$$\left.\sinh\left[\left(\frac{1}{2}a\right)^{\frac{1}{2}} \frac{(t - t_0)}{\lambda}\right]^{-1}\right\}.$$

In [67, 70, 71, 72], the separation technique for obtaining a general transformation for sG equation, is used to obtain solutions of several important wave equations including KdV, Burgers, and Fisher's equations.

2.3 Definitions of the Separability of Nonlinear Equations

In their paper [69], Osborne and Stuart, in attempting to obtain the dependent variable transformation, used by Lamb [56] and Zagrodzinski [100], to separate the sine-Gordon (sG) equation, studied in some details, the existence of separable solutions of the sG equation and similar quasilinear p.d.e.'s. The algorithm they used is to first assume a dependent-variable transformation which reduces the original equation to a separable form, and then expand the derivatives of the separating functions, as power series, in terms of the functions themselves. As a consequence of recurrence relations which occur when equating coefficients, an ordinary differential equations appears, whose solutions are the transformations which separate these equations into two o.d.e.'s. The authors have found a general set of dependent-variable transformations, which lead to separable forms of the sG equation. One particular transformation in this general class is Lamb's transformation.

In the following paper [71] separability of the sG equation was studied using combination of independent and dependent-variable transformation.

Although, the separability of the sG equation and other equations was, successfully, examined by Osborne and Stuart [69, 70, 71, 72], and mathematical algorithms were devised, a definition of a "separable equation" was absent from their work. However, in this section, an attempt is made to overcome this problem. Basic definitions involving separation of variables, which may meet the requirements of separation of a large class of nonlinear p.d.e.'s will be stated. These definitions cover the sG equation and other important equations, studied by Osborne and Stuart. An interesting feature of these definitions is that they are flexible and general, so that they can be applied to nonlinear and linear p.d.e.'s as well as o.d.e.'s. Moreover, ordinary separation of variables for linear equations will be a special case of these definitions (see §2.4).

Let

$$H(\phi(x, t)) = 0 \tag{2.3-1}$$

be a general real partial differential equation of the dependent variable ϕ and two independent variables x and t.

For the following definitions and remarks, we will use the series of functions of x (or t) that has the form,

$$\sum_{n=0}^{\infty} a_n X(x)^{n+\lambda}, \quad \lambda \in \mathbb{R}$$

and call it a "<u>power series in X</u>" if a_n is constant $\forall n$, and a "<u>generalized (g.) power series in X</u>" if $\exists j$ such that a_j

is a function of x.

Definition (2.3-1)

Eq. (2.3-1) is said to be simply separable if $\phi(x, t) = X(x)T(t)$, where X is a function of x alone, T is a function of t alone, and if for some $\lambda, \rho \in \mathbb{R}$,

$$(X')^r = \sum_{n=0}^{\infty} a_n(x)X^{n+\lambda} \qquad (2.3\text{-}2(a))$$

$$(T')^s = \sum_{n=0}^{\infty} b_n(t)T^{n+\rho}, \qquad (2.3\text{-}2(b))$$

where r and s are constants to be chosen such that the derivatives $\left\{\frac{\partial^n \phi}{\partial x^{n_1}\partial t^{n_2}}: n, n_1, n_2 \in \mathbb{Z} \geqslant 0, n_1 + n_2 = n\right\}$ in the equation, can be represented as a product of g. power series in X or T. $a_n \neq 0\ \forall n$ and $b_n \neq 0\ \forall n$.

Definition (2.3-2)

Eq. (2.3-1) is said to be implicitly separable if it is reducible to a simply separable equation.

Definition (2.3-3)

A separable equation is a simply or implicitly separable equation.

Justification of the definitions

Four important properties of infinite series will be needed here:

<u>Lemma (2.3-4)</u> [52], [89]

A series $\sum f_n(x)$ may be differentiated term by term if the differentiated series is a uniformly convergent series of continuous functions. □

<u>Lemma (2.3-5)</u> [16]

If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge absolutely in the interval $(-\ell, \ell)$, their product is given by $\sum_{n=0}^{\infty} c_n x^n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0 \quad \forall n,$$

which converges absolutely in the same interval. □

<u>Lemma (2.3-6)</u> [52]

For every positive integer k,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty} a_n^{(k)} x^n,$$

where the constant coefficients $a_n^{(k)}$ are constructed from the constant coefficients a_n in a perfectly determinate manner $\forall n, k$. And these series are all absolutely convergent, so long as $\sum_{n=0}^{\infty} a_n x^n$ itself is. □

<u>Reversion theorem for power series</u> [52]

Given the expansion

$$y = a_1x + a_2x^2 + \ldots,$$

convergent for $|x| < r$, the function $y = f(x)$ thereby determined is reversible in the neighbourhood of the origin, under the sole hypothesis that $a_1 \neq 0$; there then exists one and only one function $x = \phi(y)$ which is expressible by a power series, convergent in a certain neighbourhood of the origin, of the form

$$x = b_1y + b_2y^2 + \ldots$$

and for which, in that neighbourhood, we have

$$f(\phi(y)) \equiv y.$$

Moreover $b_1 = 1/a_1$. □

Remark (2.3-7)

It is easy to see that a simply separable equation, is an implicitly separable one.

Remark (2.3-8)

The requirement of the first derivative in (2.3-2) is not very stringent, but just for convenience. The apparently more general case of (2.3-2) is,

$$(X^{(p)})^r = \sum_{n=0}^{\infty} a_n(x)X^{n+\lambda} \qquad (2.3\text{-}3(a))$$

$$(T^{(q)})^s = \sum_{n=0}^{\infty} b_n(t)T^{n+\rho} \qquad (2.3\text{-}3(b))$$

where p and q are positive integers. This case, in fact is included in (2.3-2) as a special case, i.e., (2.3-2) implies (2.3-3) by lemma (2.3-4).

<u>Remark (2.3-9)</u>

If $a_n = 0\ \forall n$, or $b_n = 0\ \forall n$ in (2.3-2) then the final solution of the p.d.e. will be a function of one of the independent variables x or t only.

<u>Remark (2.3-10)</u>

In (2.3-2), $a_0(x) \neq 0$, $b_0(t) \neq 0$ without loss of generality, for if, for instance,

$$(X')^r = \sum_{n=k}^{\infty} a_n(x) X^{n+\lambda}$$

where $a_k(x) \neq 0$, then

$$(X')^r = \sum_{m=0}^{\infty} a_{m+k}(x) X^{m+(k+\lambda)}$$

(by letting $m = n - k$).

Let $\alpha_m(x) = a_{m+k}(x)\ \forall m$, and $k + \lambda = \lambda_1$. Then,

$$(X')^r = \sum_{m=0}^{\infty} \alpha_m(x) X^{m+\lambda_1} = \sum_{n=0}^{\infty} \alpha_n(x)^{n+\lambda_1},$$

where $\alpha_0(x) \neq 0$. (Similarly for $(T')^s$.)

<u>NB</u>: With this conversion, if λ is a positive integer then λ is not necessarily zero.

Remark (2.3-11)

The definitions can be easily extended to equations consisting of one dependent variable and several independent variables.

Remark (2.3-12)

In case of equations with no explicit dependence on the independent variables x or t, (2.3-2) might be simplified by taking the functions a_n's and b_n's to be constants $\forall n$. This enables us to solve a system of algebraic equations rather than system of differential equations.

The disadvantage of this choice is that some solutions may be lost. The following result illustrates the fact that, in at least some cases, no solutions are lost. One must impose the following conditions on X and T; (they will be illustrated for X only):

(i) $X = f(x)$ and $F(x) = f(x)/x$ have Taylor series with $F'(0) \neq 0$.

By the reversion theorem for power series [52], the above condition implies that there exists one and only one function $g(X)$, such that $x = g(X)$, with $g(X)/X = G(X)$ expressible as a Taylor series and $G'(0) \neq 0$ $(G'(0) = 1/F'(0))$.

(ii) $\lambda = 0$ in (2.3-2(a)) or λ is a positive integer.

Then the following theorem will hold;

Theorem (2.3-13)

If the coefficients, a_n, in (2.3-2(a)) are analytic in x, and conditions (i) and (ii) hold, then a_n is constant $\forall n$, without loss of generality. □

Proof Lemma (2.3-5) implies that (2.3-2(a)) can be written as,

$$(X')^r = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\alpha_{mn}x^m\right]$$

for some constants α_{mn}, where $\sum_{m=0}^{\infty}\alpha_{mn}x^m = a_n(x)X^{n+\lambda}$ for each n. Therefore

$$(X')^r = \sum_{m=0}^{\infty}\alpha_m x^m \qquad (2.3\text{-}4)$$

where α_m is constructed from the coefficients of x in the Taylor series for a_n and X^n, for each n, so that each α_m is constant.

Applying reversion theorem now to X(x), and substituting the result in (2.3-4) gives, by the above lemmas,

$$(X')^r = \sum_{n=0}^{\infty}\beta_n X^{n+\lambda}, \quad \beta_0 \neq 0,$$

where β_n's are constants, i.e., the coefficients in (2.3-2(a)) are constants without loss of generality. ∎

In general, it seems reasonable to suppose that when applying the definitions to equations with no explicit (x, t)-dependence a_n and b_n in (2.3-2) can be taken to be constants $\forall n$ without loss of generality. Unfortunately, it is difficult to prove any result which is more general than theorem (2.3-13) due to the lack of results concerning reversion of $\mathcal{G}$· series.

If in any case, a_n and b_n are not constants for all n, and at least one of series (2.3-2) is not finite, then there is the added difficulty of showing that the series is uniformly convergent if it is pointwise convergent. Of course any convergent power series is automatically uniformly convergent.

The following result concerns the choice of the real numbers r and s in (2.3-2).

Theorem (2.3-14)

Suppose that the conditions of theorem (2.3-13) are satisfied. If the powers of all the x-derivatives, $(\phi_x n: n \in \mathbb{N})$, in a simply separable equation,

$$H(\phi(x, t)) = 0 \qquad (2.3\text{-}1)$$

are positive, then the reciprocal of the L.c.m. of the denominator of all these powers, is the value of the constant r in (2.3-2) without loss of generality provided that the powers of $\phi_x n$, $n > 1$ are integers. $\square$

Proof Since the p.d.e. (2.3-1) is simply separable $\phi(x, t) = X(x)T(t)$ where X and T satisfy (2.3-2).

Let $p_i/q_i \in R^+$, $i = 1, 2, \ldots, N_1$, where p_i and q_i have no common factors, be the powers of all the (N_1) first x-derivatives, ϕ_x in the p.d.e. (2.3-1).

Let $m_i \in \mathbb{N}$, $i = 1, 2, \ldots, N_2$, be the powers of all the (N_2) higher x-derivatives (ϕ_{x^n}, $n > 1$) in (2.3-1).

To look for a suitable constant r for the identity (2.3-2(a)) which ensures that all the power of derivatives, $(X^{(n)})^a$, are power series, we claim that,

$1/r = \text{L.c.m.}(q_1, q_2, \ldots, q_{N_1})$, without loss of generality.

Now $X' = \left[\sum_0^\infty a_n x^{n+\lambda}\right]^{1/r} = (F(x, X))^{1/r}$ say, so that $F(x, X)$ has a power series expansion.

Thus, each term, $(\phi_x)^{p_i/q_i}$, in (2.3-1) will lead to an expression of the form $(X')^{p_i/q_i}$ where

$$(X')^{p_i/q_i} = F(x, X)^{p_i/(rq_i)} \quad \forall i.$$

If $1/r = \text{L.c.m.}(q_1, \ldots, q_{N_1})$, then

$1/r = \alpha_i q_i \ \forall i \ (\alpha_i \in \mathbb{Z})$, so that

$p_i/(rq_i) = \alpha_i p_i \in \mathbb{Z}$. Hence $(X')^{p_i/q_i}$ has a power series expansion for all i.

If $1/r \neq \alpha_i q_i \ \forall i$, where $\alpha_i \in \mathbb{Z}$, then for at least one value of i, $p_i/(rq_i) \notin \mathbb{Z}$, since p_i and q_i contain no common factors. For this value of i, $(X')^{p_i/q_i}$ has no power series expansion. Thus, $1/r = \alpha_i q_i \ \forall i$, and so $1/r = \text{L.c.m.}(q_1, \ldots, q_{N_1})$, without loss of generality.

Since (X') has a power series expansion, by the differentiation lemma (2.3-4), $X^{(n)}$ has a power series expansion $\forall n \geq 1$.

Thus, by lemma (2.3-6) each term of the form $(X^{(n)})^{m_i}$, which is derived from a term of the form $(\phi_{x^n})^{m_i}$ in (2.3-1), has a power series expansion, as required. ∎

A similar theorem holds for the t-derivatives of ϕ.

Given any p.d.e., as it stands, if we assume that it has separable solutions as in definition (2.3-1) then this assumption will lead to recurrence relations for the coefficients a_n's and b_n's. If these relations are inconsistent or only have trivial solutions then original assumption is false and the equation has no separable solutions of the given form. If recurrence relations are solved then separable solutions are produced.

2.4 Applications of the Definition

In this section we will show that the definition of a simply separable equation can be applied to linear p.d.e.'s as well as o.d.e.'s.

Linear p.d.e.'s

To establish the relationship between the technique for nonlinear equations as in the previous section and the classical linear technique, let us restrict ourselves to general second order linear equation

$$A_{11}(x,\ t)u_{xx} + A_{12}(x,\ t)u_{xt} + A_{22}(x,\ t)u_{tt} + B_1(x,\ t)u_x + B_2(x,\ t)u_t + C(x,\ t)u = 0. \qquad (2.4\text{-}1)$$

The separability of this equation has been studied by many authors. To be precise, let us take Koornwinder's theorem, for the necessary and sufficient conditions for (2.4-1) to be separable by the classical linear technique.

Theorem (2.4-1) [53]

Eq. (2.4-1) must take the following form:

$$A_{11}(u_{xx} + b_1(x)u_x + c_1(x)u) + A_{22}(u_{tt} + b_2(t)u_t + c_2(t)u) = 0, \qquad (2.4\text{-}2)$$

if it is separable. Furthermore, if u has the form $u = X(x)T(t)$ and if u is not identically zero, then u is a solution of eq. (2.4-2) if and only if, the functions X and T are solutions of the o.d.e.'s

$$X''(x) + P_1(x)X' + Q_1(x)X = 0 \qquad (2.4\text{-}3(a))$$

$$T''(t) + P_2(t)T' + Q_2(t)T = 0 \qquad (2.4\text{-}3(b))$$

where P_i and Q_i are analytic functions of b_i and c_i and the separability constant α, for each $i = 1, 2$. □

The following theorem proves that the o.d.e. (2.4-3(a)) can be represented as a g. power series for X' as in definition (2.3-1). (Similarly for (2.4-3(b)).)

Theorem (2.4-2)

Eq. (2.4-1) is separable in the sense of theorem (2.4-1) if and only if it is separable by the series technique, i.e. it is implicitly separable.

Proof (i) Eq. (2.4-1) is separable by the classical linear technique means that it is reducible to eq. (2.4-2), which gives eq.'s (2.4-3) as the separating o.d.e.'s. Without loss of generality, let us prove that eq. (2.4-2) is simply separable, which implies that eq. (2.4-1) is implicitly separable.

Consider the following equation,

$$X''(x) + P(x)X' + Q(x)X = 0, \qquad (2.4\text{-}4)$$

which represents eq. (2.4-3(a)) or (2.4-3(b)).

Let

$$P(x) = \sum_0^\infty p_n x^n, \quad Q(x) = \sum_0^\infty q_n x^n \qquad (2.4\text{-}5)$$

where p_n's and q_n's are constants, since P(x) and Q(x) are analytic by theorem (2.4-1). This equation is solvable by Frobenius method which suggests solution of the form,

$$X(x) = \sum_0^\infty a_n x^{n+\lambda} \qquad (2.5\text{-}6)$$

where λ is a real constant to be determined, and $a_0 \neq 0$, without loss of generality.

Substituting (2.4-5) and (2.4-6) in eq. (2.4-4) gives

$$\sum_0^\infty (n + \lambda)(n + \lambda - 1)a_n x^{n+\lambda-2} + \left(\sum_0^\infty p_n x^n\right)\left(\sum_0^\infty (n + \lambda)a_n x^{n+\lambda-1}\right)$$

$$+ \left(\sum_0^\infty q_n x^n\right)\left(\sum_0^\infty a_n x^{n+\lambda}\right) = 0.$$

We now form the indicial equation by equating coefficients of the lowest power of x (here $x^{\lambda-2}$):

$$\lambda(\lambda - 1)a_0 = 0.$$

Hence λ is a non-negative integer, which implies that the solution X(x) of (2.4-4) is represented by a Taylor series. Then $X(x) = \sum_0^\infty a_n x^n$ without loss of generality, where a_0 could be zero.

We have now two different cases:

<u>Case (a)</u> ($a_0 = 0$)

Since we are interested in nontrivial solutions, ($X \not\equiv 0$), we claim that the assumption $a_0 = 0$ implies that $a_1 \neq 0$:

Suppose that $a_1 = 0$. This means that $X(0) = X'(0) = 0$. It is then clear that $X''(0) = 0$, by eq. (2.4-4). We now prove that $X^{(n)}(0) = 0$ for $n \geqslant 3$. Differentiating eq. (2.4-4) n times using Leibnitz's formula, gives,

$$X^{(n+2)} + \left[PX^{(n+1)} + \binom{n}{1}P'X^{n} + \ldots + P^{(n)}X'\right]$$

$$+ \left[QX^{(n)} + \binom{n}{1}Q'X^{(n-1)} + \ldots + Q^{(n)}X\right] = 0. \qquad (2.4\text{-}7)$$

Suppose $X^{n}(0) = 0$, for $n = 1, 2, \ldots, k + 1$, then by (2.4-7) it follows that $X^{(k+2)}(0) = 0$. Hence $X^{(n)}(0) = 0$ $\forall n$ by induction. Therefore $X \equiv 0$ which contradicts our assumption. Hence $a_1 \neq 0$ if $a_0 = 0$.

Now, it is possible to apply the reversion theorem of power series [52] to X(x) to get that x is a Taylor series in X. Since X' is a Taylor series of x, X' is a Taylor series in X, by lemma (2.3-6), as required.

Case (b) $(a_0 \neq 0)$

Consider the series,

$$X - a_0 = a_1x + a_2x^2 + a_3x^3 + \ldots \qquad (2.4\text{-}8)$$

Suppose that a_k is the first non-zero coefficient on the r.h.s. of (2.4-8). If $k > 1$, multiply (2.4-8) by x^{1-k} to get,

$$(X - a_0)x^{1-k} = a_kx + a_{k+1}x^2 + a_{k+2}x^3 + \ldots.$$

Denoting $(X - a_0)x^{1-k}$ by Y(x), and using the reversion theorem of power series on Y gives,

$$x = b_1Y + b_2Y^2 + b_3Y^3 + \ldots \qquad (2.4\text{-}9)$$

or

$$x = b_1(X - a_0)x^{1-k} + b_2(X - a_0)^2 x^{2(1-k)} +$$

$$b_3(X - a_0)^3 x^{3(1-k)} + \ldots. \qquad (2.4\text{-}9)$$

Using the binomial theorem and rearranging the coefficients of X gives,

$$\begin{aligned} x = &\left(-b_1a_0x^{1-k} + b_2a_0{}^2x^{2(1-k)} + \ldots\right) + \\ &\left(b_1x^{1-k} - 2b_0a_0x^{2(1-k)} + \ldots\right)X + \\ &\left(b_2x^{2(1-k)} - 3b_3a_0x^{3(1-k)} + \ldots\right)X^2 + \ldots \end{aligned}$$

$$\text{i.e. } x = \sum_{n=0}^{\infty} c_n(x)X^n, \qquad (2.4\text{-}10)$$

for some $c_n(x)$.

Substituting (2.4-10) in X', which is a Taylor series in x, gives

$$X' = \sum_{n=0}^{\infty} b_n(x)X^n,$$

for some $b_n(x)$, as required.

(ii) To prove the converse result it is sufficient to notice that, assuming convergent power series for the derivatives X' and T' as in (2.3-2) with sum functions F(x, X) and G(t, T) means that we get the o.d.e.'s

$$X' = F(x, X)$$

$$T' = G(t, T)$$

which means that the equation is separable by the classical linear method (see [53] definition 2.1). ∎

Example (2.4-3)

As an example of the advantage of the nonlinear separation method over the linear separation method, consider the following p.d.e.,

$$\phi_x + \phi_{xt} + \phi_{tt} = 0. \tag{2.4-11}$$

This equation can not be separated into two o.d.e.'s for X and T, if $\phi(x, t) = X(x)T(t)$. However, one particular solution is given by,

$$X' = -\frac{b^2}{b + 1}X, \quad T' = bT$$

where b is a constant. Thus, the equation is separable by the nonlinear technique.

Example (2.4-4)

As another example, consider the linearized KdV equation:

$$\phi_t + \phi_x + \phi_{xxx} = 0. \tag{2.4-12}$$

This equation can be separated to the o.d.e.'s

$$X''' + X' = -\lambda X \tag{2.4-13(a)}$$

$$T' = \lambda T, \tag{2.4-13(b)}$$

where λ is the separation constant, yet a separable solution $\phi = XT$ can not be written down at once.

Let $\quad X' = aX^p, \quad T' = bT^q, \quad p, q \in \mathbb{Q}, a, b \in \mathbb{R}.$

Substitution of X', T' in (2.4-12) gives

$$bXT^q + aX^pT + p(2p - 1)a^3X^{3p-2}T = 0.$$

It is clear that $p = q = 1$, and $b + a + a^3 = 0$.

Hence the separable solution $\phi = XT = Ke^{ax-(a^3+a)t}$, (where K is constant) can be easily achieved.

Ordinary differential equations

The method of "nonlinear separation" can also be used for an o.d.e. which has a first integral, which is difficult to obtain by the usual methods. Let us illustrate this by considering the following example.

Example (2.4-5)

Using the separation technique we will show that the first order o.d.e.,

$$y' = xy^2 + ky - 1$$

is the first integral of the 2nd order o.d.e.,

$$yy'' = y'^2 + (xy^2 + 1)y' + y^3. \qquad (2.4\text{-}14)$$

Applying a degenerate case of definition (2.3-1), where

$y(x) = y(x).1.$

Consider the expansion of y' as a generalized power series of x and y as,

$$y' = \sum_{n=0}^{\infty} a_n(x) y^{n+p} \quad (a_0 \neq 0 \text{ w.l.o.g.})$$

Substituting y' and y'' into eq. (2.4-14) we get,

$$\sum_{n=0}^{\infty} a_n' y^{n+p+1} + \left(\sum_{n=0}^{\infty} (n+p) a_n y^{n+p}\right)\left(\sum_{n=0}^{\infty} a_n y^{n+p}\right)$$

$$= \left(\sum_{n=0}^{\infty} a_n y^{n+p}\right)\left(\sum_{n=0}^{\infty} a_n y^{n+p}\right) + (xy^2 + 1)\left(\sum_{n=0}^{\infty} a_n y^{n+p}\right) + y^3.$$

Equating the coefficient of the lowest power of y^{n+p}'s which is y^p gives that $p = 0$.

Equating the coefficient of y^0 gives that $a_0{}^2 + a_0 = 0$, i.e., $a_0 = -1$.

Equating the coefficient of y gives that a_1 is arbitrary. The obvious choice for a_1 is to be constant, which gives that, upon equating coefficient of y^2 that $a_2 = x$.

It is worth checking the available result after a few stages of equating coefficients. Here in this example there is no necessity to carry on the procedure since the resulting identity

$$y' = -1 + Ky + xy^2,$$

where K is constant, is the first integral of eq. (2.4-14).

The separation technique, applied on an o.d.e., can be used in example (2.4-4) to solve eq. (2.4-13(a)).

2.5 Simple Separability

The simplest version of 'separability' is simple separability which is in fact the trivial case of implicit separability. As might be expected not every equation is simply separable. The sine-Gordon equation is an example of an equation which is not simply separable [71]. On the other hand every separable solution $\phi = X(x)T(t)$ is a solution for the equation

$$\phi\phi_{xt} - \phi_x\phi_t = 0.$$

To see if there are some classes of equations which are non-trivially simply separable, it is sufficient to find equations which have at least one non-trivial separable solution. (By a trivial solution we mean a solution which is a function of one independent variable only.) Dealing with equations with no explicit dependence on the independent variables x, t make the task much easier. Thus the conditions of theorem (2.3-13) will be assumed here for simplicity.

Second order linear equations

Theorem (2.5-1)

Any second order constant coefficient linear equation is simply separable. □

Proof Consider the second order linear equation,

$$A\phi_{xx} + B\phi_{xt} + C\phi_{tt} + D\phi_x + E\phi_t + F\phi = 0, \qquad (2.5\text{-}1)$$

where all the coefficients are constants. Substitution the separable solution $\phi = X(x)T(t)$, where $X' = aX$, $T' = bT$, and a and b are constants, in (2.5-1) gives,

$$Aa^2 + Bab + Cb^2 + Da + Eb + F = 0. \qquad (2.5\text{-}2)$$

This algebraic relation between the constants a and b, means that there are an infinite number of solutions for (2.5-1), of the given form, so that any second order linear equation is simply separable. ■

Second order quasilinear equations

A more general equation than (2.5-1) is the quasilinear p.d.e. of second order with constant coefficients:

$$A\phi_{xx} + B\phi_{xt} + C\phi_{tt} + D\phi_x + E\phi_t = f(\phi). \qquad (2.5\text{-}3)$$

Theorem (2.5-2) [71]

If $f(\phi) = \gamma\phi$ or $A = C = D = E = 0$ and $f(\phi) = \gamma\phi^{\lambda}$, $\lambda \in \mathbb{R}$, where γ is a constant, then eq. (2.5-3) is simply separable. Otherwise it is not. □

Second order general quasilinear equations

Consider the p.d.e.,

$$F(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{xt}, \phi_{tt}) = 0. \tag{2.5-4}$$

Eq. (2.5-4) is called a general quasilinear equation, if it is a linear equation in the dependent variable's derivative, i.e., eq. (2.5-4) may be written as,

$$F_1(\phi)\phi_{xx} + F_2(\phi)\phi_{xt} + F_3(\phi)\phi_{tt} + F_4(\phi)\phi_x + F_5(\phi)\phi_t + F_6(\phi) = 0, \tag{2.5-5}$$

where the coefficients F_i's $(i = 1, \ldots, 6)$ are functions of ϕ only.

Let $F_i(\phi)$ be a polynomial in ϕ for each $i = 1, \ldots, 6$. Let N be the maximum degree of the polynomials F_i. Then:

$$F_i(\phi) = f_{i0} + f_{i1}\phi + \ldots + f_{iN}\phi^N, \quad (i = 1, \ldots, 6), \tag{2.5-6}$$

where f_{ij} $(j = 0, \ldots, N)$ are constants, some of which could be zero. Hence we have the theorem:

Theorem (2.5-3)

If there exists a nontrivial solution $(a \neq 0, b \neq 0)$ for the system

$$FX = 0 \tag{2.5-7}$$

where,

$$X = \begin{bmatrix} a^2 \\ ab \\ b^2 \\ a \\ b \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} f_{10} & f_{20} & f_{30} & f_{40} & f_{50} & f_{61} \\ f_{11} & f_{21} & f_{31} & f_{41} & f_{51} & f_{62} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{1N} & f_{2N} & f_{3N} & f_{4N} & f_{5N} & f_{6,N+1} \end{bmatrix},$$

$$f_{6,N+1} = 0$$

then eq. (2.5-5) is simply separable. □

<u>Proof</u> This result can be easily proved when it is assumed that $\phi = XT$, $X' = aX$, $T' = bT$, which implies that

$$\phi_{xx} = a^2\phi, \quad \phi_{xt} = ab\phi, \quad \phi_{tt} = b^2\phi, \quad \phi_x = a\phi,$$
$$\phi_t = b\phi.$$

Thus if (2.5-7) has a nontrivial solution then eq. (2.5-5) is simply separable. ■

<u>Second order polynomial equations</u>

Eq. (2.5-4) is of <u>polynomial class</u> if the function F is a polynomial in all its arguments. Then it may be written as,

$$\sum_{i=1}^{N} \alpha_i \, \phi^{a_{oi}} \, \phi_x^{a_{1i}} \, \phi_t^{a_{2i}} \, \phi_{xx}^{a_{3i}} \, \phi_{xt}^{a_{4i}} \, \phi_{tt}^{a_{5i}} = 0, \tag{2.5-8}$$

where α_i's are constants, and a_{ji} ($j = 0, \ldots, 5$) are non-negative integers. We investigate the separability of eq. (2.5-8), by assuming that,

$$\phi = XT, \quad X' = aX, \quad T' = bT, \tag{2.5-9}$$

where a, b are constants.

Let

$$\left.\begin{aligned} A_i &= \alpha_i\, a^{a_{1i}+a_{4i}+2a_{3i}}\, b^{a_{2i}+a_{4i}+2a_{5i}} \\ B_i &= \sum_{j=0}^{5} a_{ji} \end{aligned}\right\}, \quad (i = 1, \ldots, N).$$

$$(2.5\text{-}10)$$

Without loss of generality, assume that

$$\left.\begin{aligned} &B_{k_0+1} = B_{k_0+2} = \cdots = B_{k_1} \\ &B_{k_1+1} = B_{k_1+2} = \cdots = B_{k_2} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &B_{k_{n-1}+1} = B_{k_{n-1}+2} = \cdots = B_{k_n} \end{aligned}\right\} \tag{2.5-11}$$

$k_0 = 0$

$k_n = N$

$k_i < N$ if $i < n$.

The following theorem will be hold:

<u>Theorem (2.5-4)</u>

Eq. (2.5-8) is simply separable if there exists a nontrivial solution ($a \neq 0$, $b \neq 0$) for the system,

$$\sum_{i=k_{j-1}+1}^{k_j} A_i = 0, \quad j = 1, \ldots, n. \ \square \qquad (2.5\text{-}12)$$

<u>Proof</u> Substituting (2.5-9) as in theorem (2.5-3), in eq. (2.5-8), the problem is reduced to that of solving the equation

$$\sum_{i=1}^{N} A_i(a, b)\phi^{B_i} = 0$$

where A_i and B_i are as in (2.5-10) which according to (2.5-11) gives the system (2.5-12). ■

Theorems (2.5-1,3,4) and the first part of theorem (2.5-2) can be easily extended to higher order equations or equations involving more than two independent variables.

The second part of theorem (2.5-2) may be extended as follows:

<u>Theorem (2.5-5)</u>

The equation,

$$\phi_{x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}} = \gamma\phi^{\lambda}$$

where γ and λ are constants, $r_i \in \mathbb{N}$, is simply separable. $\square$

In the following example, we demonstrate that (2.3-2) can be true power series, as opposed to finite sums.

Example (2.5-6)

Consider the p.d.e.,

$$\phi_{xt}^{\;2} + \phi_{xxtt} = 0. \tag{2.5-13}$$

The substitution $\phi(x, t) = X(x)T(t)$, such that,

$$X'^2 = 2\sum_{n=0}^{\infty} a_n X^n \tag{2.5-14(a)}$$

$$T'^2 = 2\sum_{n=0}^{\infty} b_n T^n \tag{2.5-14(b)}$$

in eq. (2.5-13) gives,

$$4\left(\sum_{n=0}^{\infty} a_n X^n\right)\left(\sum_{n=0}^{\infty} b_n T^n\right) + \left(\sum_{n=1}^{\infty} na_n X^{n-1}\right)\left(\sum_{n=1}^{\infty} nb_n T^{n-1}\right) = 0.$$

Equating to zero the coefficient of $X^n T^m$ gives the following relation,

$$4a_n b_m + (n + 1)(m + 1)a_{n+1}b_{m+1} = 0 \quad \forall n, m \in \mathbb{Z} \geqslant 0, \tag{2.5-15}$$

which may be written as,

$$\left.\begin{aligned} b_{m+1} &= -4\left(\frac{a_n}{a_{n+1}}\right)\frac{b_m}{(n + 1)(m + 1)} \\ \text{or,} & \\ a_{n+1} &= -4\left(\frac{b_m}{b_{m+1}}\right)\frac{a_n}{(n + 1)(m + 1)} \end{aligned}\right\}, \quad \forall n, m \in \mathbb{Z} \geqslant 0.$$

This is a symmetric relation between the a's and b's and it gives,

$$\frac{a_0}{a_1} = \frac{a_1}{2a_2} = \frac{a_2}{3a_3} = \ldots \qquad (2.5\text{-}16(a))$$

and,

$$-\frac{4a_0}{a_1} = \frac{b_1}{b_0} = \frac{2b_2}{b_1} = \frac{3b_3}{b_2} = \ldots. \qquad (2.5\text{-}16(b))$$

Substitution of (2.5-16) back into (2.5-14) implies the following infinite series for X and T,

$$\left.\begin{aligned} X'^2 &= 2a_0\left[1 + (pX) + \frac{(pX)^2}{2!} + \frac{(pX)^3}{3!} + \ldots\right] \\ T'^2 &= 2b_0\left[1 + \left(\frac{-4}{p}T\right) + \frac{1}{2!}\left(\frac{-4}{p}T\right)^2 + \frac{1}{3!}\left(\frac{-4}{p}T\right)^3 + \ldots\right] \end{aligned}\right|$$

where $p = a_1/a_0$.

It is clear that the first series converges to $2a_0e^{pX}$ $\forall X$ and the second series converges to $2b_0e^{(-4/p)T}$, $\forall T$.

In the general case, (2.3-2) is likely to be a true power series and the investigation of convergence of such a series may be difficult.

The flexibility of the definitions of separable equations enables us to have, in some cases, more than one choice of derivative for the power series. Without loss of generality a power series for X' and T' is one choice. If there is another power of another derivative of X which will ensure that all other derivatives will be power series too, then it may be useful to use it instead of X' (similarly for T(t)). For instance, for the sine-Gordon equation [70], there are two choices of power series, one for X' and one for X'^2 (similarly for T' and T'^2). The first choice gives an infinite series while the second choice gives a finite sum. For theoretical purposes it may be an advantage to have an infinite series, while for calculation purposes it may be better to have finite sum. As another example of such equations besides the sG equation consider the following example.

<u>Example (2.5-7)</u>

In this example we will illustrate the possibility of many choices for the power series involving X and T. Furthermore, we will show that applying the separation technique to some p.d.e.'s reduces the problem to that of solving an o.d.e.

Consider the second order equation,

$$\phi^2\phi_{xx} + \phi_{tt} - 2\phi\phi_x^{\,2} + \phi_t^{\,2} = 2(\phi + \phi^2 - \phi^3). \qquad (2.5\text{-}17)$$

The substitution $\phi = X(x)T(t)$ and division by X gives,

$$XT^3X'' + T'' - 2T^3X'^2 + XT'^2 = 2(T + XT^2 - X^2T^3). \qquad (2.5\text{-}18)$$

First, let us assume that $T' = \sum_{n=0}^{\infty} b_n T^n$ leaving X's derivatives undefined at the moment. Rewriting (2.5-18) in terms of powers of T, we get

$$XT^3X'' + (b_0 + b_1T + b_2T^2 + \ldots)(b_1 + 2b_2T + 3b_3T^2 + \ldots)$$

$$- 2T^3X'^2 + X(b_0 + b_1T + b_2T^2 + \ldots)(b_0 + b_1T + b_2T^2 + \ldots)$$

$$= 2(T + XT^2 - X^2T^3).$$

Comparing coefficients of X, T and T^2 gives that $b_0 = 0$, $b_1 = \sqrt{2}$ and $b_2 = 0$ respectively.

Now it can be easily proved, by induction that $b_n = 0$ $\forall n \geq 2$. Suppose that $b_2 = b_3 = \ldots = b_k = 0$ $k \geq 2$. To prove that $b_{k+1} = 0$, consider the coefficient of XT^{k+2}. This is

$$b_0b_{k+2} + b_1b_{k+1} + b_2b_k + \ldots + b_kb_2 + b_{k+1}b_1 + b_{k+2}b_0 = 0$$

Since $b_0 = b_2 = \ldots = b_k = 0$ then $b_{k+1} = 0$. Thus we have,

$$T' = \sqrt{2}T. \qquad (2.5\text{-}19)$$

This result for T(t) shows that there is no need to have another choice for T. Let us now substitute (2.5-19) back into (2.5-18) which leads to the following o.d.e.

$$XX'' - 2X'^2 = -2X^2. \qquad (2.5\text{-}20)$$

There are three choices here. Either (2.5-20) may be solved by ordinary methods, or it may be solved by the separation technique with two different choices of power series involving X.

Let

$$X'^2 = 2\sum_{n=0}^{\infty} a_n X^n$$

which implies that

$$\left(\sum_{n=0}^{\infty} na_n X^n\right) - 2\left(\sum_{n=0}^{\infty} 2a_n X^n\right) = -2X^2.$$

Equating to zero the coefficient of X^k gives,

$$(k - 4)a_k = \begin{cases} 0 & \text{if} \quad k \neq 2 \\ -2 & \text{if} \quad k = 2. \end{cases}$$

Hence, $a_2 = 1$, a_4 is arbitrary and $a_n = 0 \; \forall n \neq 2, 4$.
Therefore

$$X'^2 = 2(X^2 + aX^4) \tag{2.5-21}$$

where a is arbitrary constant. Note that (2.5-21) is the first integral of (2.5-20).

If, instead, we make the substitution $X' = \sum_{n=0}^{\infty} a_n X^n$ in (2.5-20) then the resulting identity in X leads to recurrence relations with quadratic terms, solving these relations gives the following results: $a_1{}^2 = 2$, a_3 is arbitrary and,

$$a_n = \begin{cases} 0 & \text{if} \quad n = 0, 2, 4, \ldots \\ \\ \begin{pmatrix} \frac{1}{2} \\ (n-1)/2 \end{pmatrix} \dfrac{(2a_3)^{(n-1)/2}}{a_1{}^{(n-3)/2}} & n = 5, 7, \ldots \end{cases}$$

The infinite series for X', now, can be written as a series of odd terms only, i.e.,

$$X' = a_1X + \binom{\frac{1}{2}}{1}(2a_3)X^3 + \binom{\frac{1}{2}}{2}\frac{(2a_3)^2}{a_1}X^5 + \ldots +$$

$$\binom{\frac{1}{2}}{(n-1)/2}\frac{(2a_3)^{(n-1)/2}}{a_1^{(n-3)/2}}X^n + \ldots$$

or,

$$X' = a_1X\left[1 + \binom{\frac{1}{2}}{1}(a_1a_3X^2) + \binom{\frac{1}{2}}{2}(a_1a_3X^2)^2 + \ldots +\right.$$

$$\left.\binom{\frac{1}{2}}{(n-1)/2}(a_1a_3X^2)^{(n-2)/2} + \ldots\right]. \qquad (2.5\text{-}22)$$

The infinite series on the R.H.S. is nothing but the binomial series for $(1 + a_1a_3X^2)^{\frac{1}{2}}$. Hence

$$X' = a_1X(1 + a_1a_3X^2)^{\frac{1}{2}}, \qquad (2.5\text{-}23)$$

where $a_1 = \sqrt{2}$.

Since a_3 is arbitrary, so is a_1a_3. This implies that (2.5-23) is equivalent to (2.5-21).

<u>The choice of λ and ρ in the power series (2.3-2)</u>

The technique of simple separation for p.d.e.'s is similar to the Frobenius method for solving the ordinary differential equation,

$$y'' + p(x)y' + q(x)y = 0. \qquad (2.5\text{-}24)$$

The method of Frobenius seeks a solution of (2.5-24) in the form

$$y(x) = x^s \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+s} \qquad (c_0 \neq 0)$$

where s is left completely undetermined. This power series is capable of describing [19]

(a) analytic functions that do not vanish at the origin ($s = 0$),

(b) analytic functions with a zero of order m at the origin ($s = m$, positive integer),

(c) functions with a pole of order m at the origin ($s = -m$, a negative integer),

(d) functions with certain types of branch points at the origin (s noninteger).

For partial differential equations, when dealing with the assumptions $X' = \sum_{n=0}^{\infty} a_n X^{n+\lambda}$, $T' = \sum_{n=0}^{\infty} b_n T^{n+\rho}$, where a_n and b_n are constants $\forall n$, it is reasonable to investigate whether these power series can be reduced to Taylor or Laurent series in X or T.

The assumption that $a_0 \neq 0$ and $b_0 \neq 0$ will be used since nothing is gained by taking $a_0 = 0$ as this would simply mean that the series $\sum_{n=0}^{\infty} a_n X^{n+\lambda}$ would start with the term $a_1 X^{1+\lambda}$ or, if $a_1 = 0$, the term $a_2 X^{2+\lambda}$ and so on. Suppose the first non-zero term is a $a_m X^{m+\lambda}$; this is equivalent to using the series with λ replaced by $\lambda + m$ and a_m replaced by a_0. (Similarly for the series of T'.)

Quasilinear equations

There are two cases, in which eq. (2.5-3) is simply separable: (see theorem (2.5-2)).

Case (i) $A = C = D = E = 0$ or $f(\phi) = \nu\phi^\gamma$, $\nu, \gamma \in \mathbb{R}$.

Lemma (2.5-8)

For case (i), the constants λ and ρ for the power series for X(x), T(t) satisfy

$\lambda = \rho = \gamma$. $\square$

Proof Substituting $\phi = XT$ and comparing terms in $X^{n+\lambda}$ $T^{m+\rho}$ gives

$$a_n b_m X^{n+\lambda} T^{m+\rho} = \begin{cases} \nu X^\gamma T^\gamma & \text{if} \quad n = m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $a_0 b_0 \neq 0$, hence $\lambda = \rho = \gamma$; if γ is zero of a positive integer then the series are Taylor series; if γ is a negative integer then the series are Laurent series. $\blacksquare$

Case (ii) $f(\phi) = F\phi$; hence we have the linear equation

$$A\phi_{xx} + B\phi_{xt} + C\phi_{tt} + D\phi_x + E\phi_t = F\phi \qquad (2.5\text{-}25)$$

without loss of generality.

Lemma (2.5-9)

For case (ii), λ, $\rho \in \mathbb{Z} \geqslant 0$, so that the power series (2.3-2) are Taylor series. $\square$

Proof Suppose that λ is not a non-negative integer. We will show that this implies that $a_0 = 0$, which is a contradiction.

Substituting $\phi = XT$ in (2.5-25) and expanding each series for X' and T' according to definition (2.3-1) gives,

$$AT\left(\sum_{n=0}^{\infty} (n + \lambda)a_n X^{n+\lambda-1}\right)\left(\sum_{n=0}^{\infty} a_n X^{n+\lambda}\right) +$$

$$B\left(\sum_{n=0}^{\infty} a_n X^{n+\lambda}\right)\left(\sum_{n=0}^{\infty} b_n T^{n+\rho}\right) +$$

$$CX\left(\sum_{n=0}^{\infty} (n + \rho)b_n T^{n+\rho-1}\right)\left(\sum_{n=0}^{\infty} b_n T^{n+\rho}\right) + DT\left(\sum_{n=0}^{\infty} a_n X^{n+\lambda}\right) +$$

$$EX\left(\sum_{n=0}^{\infty} b_n T^{n+\rho}\right) - FXT = 0. \tag{2.5-26}$$

Suppose first of all $B \neq 0$, and that $\lambda \neq 1$, $\rho \neq 1$. Comparing coefficient of $X^{\lambda}T^{\rho}$ gives, $a_0 b_0 = 0$, which is a contradiction; hence λ, $\rho \in \mathbb{Z} \geqslant 0$ as required.

Thus, now suppose $B = 0$. Then for (2.5-25) to be a p.d.e., $A \neq 0$ or $D \neq 0$ and $C \neq 0$ or $E \neq 0$.

Suppose that $A \neq 0$. Comparing the coefficient of $TX^{2\lambda-1}$ gives the "indicial equation":

$$Aa_0{}^2\lambda + Da_{\lambda-1} = 0.$$

Thus if $\lambda \notin \mathbb{Z} \geq 0$, $a_{\lambda-1} = 0$; $a_0 \neq 0 \Rightarrow \lambda = 0$ which is a contradiction.

Thus if $A \neq 0$, $\lambda \in \mathbb{Z} \geq 0$. Similarly if $C \neq 0$, $\rho \in \mathbb{Z} \geq 0$.

The remaining case is when $A = 0$, $B = 0$, $D \neq 0$. If $\lambda \neq 1$, the coefficient of TX^λ gives, $Da_0 = 0 \Rightarrow D = 0$. This is a contradiction. Thus $\lambda = 1$. Similarly, if $A = 0$, $B = 0$, $E \neq 0$ then $\rho = 1$.

In all cases, $\lambda, \rho \in \mathbb{Z} \geq 0$ as required. ■

<u>Theorem (2.5-10)</u>

For the linear equation,

$$\frac{\partial^\ell \phi}{\partial x^\ell} + \frac{\partial^m \phi}{\partial t^m} = \alpha\phi, \qquad (2.5\text{-}27)$$

where ℓ and m are positive integers, and α is a constant, the constants λ and ρ in the power series (2.3-2) are non-negative integers. □

<u>Proof</u> Let $\phi = XT$, $X^{(\ell)} = \sum_{n=0}^{\infty} a_n X^{n+\lambda}$, $T^{(m)} = \sum_{n=0}^{\infty} b_n T^{n+\rho}$. Consider the coefficient of XT; if $\lambda \neq 1$ then $a_0 = 0$ which is a contradiction; if $\rho \neq 1$ then $b_0 = 0$ a contradiction too. Thus $\lambda, \rho \in \mathbb{Z} \geq 0$. ■

<u>Polynomial equations</u>

Consider the polynomial p.d.e.,

$$\sum_{i=1}^{N} \alpha_i \phi^{a_i} \phi_x^{b_i} \phi_t^{c_i} \phi_{xt}^{d_i} \phi_{xx}^{e_i} \phi_{tt}^{f_i} = 0. \qquad (2.5\text{-}8)$$

where $\alpha_i \neq 0$ without loss of generality.

Theorem (2.5-11)

If $\exists i$ such that $\forall j$, $j \neq i$, $j = 1, \ldots, N$, one of the following conditions satisfies, then $\lambda^{e_i} \rho^{f_i} = 0$:

(i) $(a_i + c_i + f_i - e_i) + \lambda(b_i + d_i + 2e_i) <$
$(a_j + c_j + f_j - e_j) + \lambda(b_j + d_j + 2e_j)$

(ii) $(a_i + b_i + e_i - f_i) + \rho(c_i + d_i + 2f_i) <$
$(a_j + b_j + e_j - f_j) + \rho(c_j + d_j + 2f_j)$. □

Proof Expansion of the derivatives as usual gives the following identity:

$$\sum_{i=1}^{N} \alpha_i X^{A_i} T^{B_i} \left(\sum_{n=0}^{\infty} a_n X^n\right)^{C_i} \left(\sum_{n=0}^{\infty} b_n T^n\right)^{D_i} \left(\sum_{n=0}^{\infty} (n + \lambda) a_n X^n\right)^{E_i}$$

$$\left(\sum_{n=0}^{\infty} (n + \rho) b_n T^n\right)^{F_i} = 0,$$

where

$$\left.\begin{aligned}
A_i &= a_i + c_i + F_i - E_i + \lambda(b_i + d_i + 2E_i) \\
B_i &= a_i + b_i + E_i - F_i + \rho(c_i + d_i + 2F_i) \\
C_i &= b_i + d_i + E_i \\
D_i &= c_i + d_i + F_i \\
E_i &= e_i \\
F_i &= f_i
\end{aligned}\right\}$$

The coefficient of $X^{A_k} T^{B_k}$ will be

$\alpha_k a_0^{C_k+E_k} b_0^{D_k+F_k} \lambda^{E_k} \rho^{F_k}$ if $A_k < A_j$ (or $B_k < B_j$)

$\forall j = 1, \ldots, N, j \neq k$. Hence $\lambda^{E_k} \rho^{F_k} = 0$ as required. ∎

It is difficult to determine the explicit character of ρ and λ in the case of the general nonlinear eq. (2.5-4).

2.6 The Effects of a Dependent Variable Transformation on a P.d.e.

To obtain more separable solutions of simply separable equations and separable solutions of non-simply separable equations, a transformation is sought such that the resulting equation is simply separable. In other words, implicitly separable solutions are sought. In this section, we will restrict ourselves to simple dependent-variable transformations. This kind of transformation has been used by Osborne and Stuart (see §2.2) to find separable sets for the sG equation and other equations.

We shall only consider p.d.e.'s of the form,

$$\sum_{n=1}^{N} F_n(x, t, \phi) \prod_{i=1}^{m_n} \phi_{x^{p_i} t^{q_i}} = F_0(x, t, \phi). \qquad (2.6\text{-}1)$$

Such equations will be transformed to the possibly simply separable equation of a dependent variable ψ and the independent variables x and t:

$$L(\psi(x, t)) = 0, \qquad (2.6\text{-}2)$$

using the dependent variable transformation,

$$\phi(x, t) = g(\psi(x, t)). \qquad (2.6\text{-}3)$$

Lemma (2.6-1)

Any travelling wave solution, $\phi = f(x - \mu t)$, where μ is a constant, is an implicitly separable solution.

Conversely, the implicitly separable solution $\phi = g(\psi)$, $\psi = \exp(\alpha x + \beta t + \gamma)$, where α, β and γ are constants includes any travelling wave solution as a special case. $\square$

Proof (First part)

$$f(x - \mu t) = f(\ln(e^{x}.e^{-\mu t})) = g(\psi),$$

where $g = f \circ \ln$ and $\psi = X(x)T(t)$, $X = e^{x}$, $T = e^{-\mu t}$. Hence the travelling wave solutions are implicitly separable.

(Second part) To prove the converse, it is easily seen that

$$\phi = f(\alpha x + \beta t + \gamma)$$

where $f = g \circ \exp$. If $\alpha = 1$, $\beta = -\mu$, $\gamma = 0$ then

$$\phi = f(x - \mu t) \text{ as required. } \blacksquare$$

Theorem (2.6-2)

Eq. (2.6-2) has the following properties if eq. (2.6-1) does:

(i) There is no explicit dependence on the independent variables.

(ii) There are no mixed derivative terms. □

Proof If eq. (2.6-1) possesses the properties (i) and (ii) then it can be written as,

$$\sum_{i=1}^{N} F_{i1}(\phi) \prod_{j=1}^{k_i} \phi_{x^{p_j}} + \sum_{i=1}^{M} F_{i2}(\phi) \prod_{j=1}^{\ell_i} \phi_{t^{q_j}} = F_0(\phi). \qquad (2.6\text{-}4)$$

For the sake of simplicity some indices of variables will be omitted if no confusion arises from this.

The proof is a substitution of $\phi = g(\psi)$ in eq. (2.6-4). It is sufficient to prove that,

$$\sum_{i=1}^{N} F_{i1}(\phi) \prod_{j=1}^{k_i} \phi_{x^{p_j}} \qquad (2.6\text{-}5)$$

is mapped to, the form,

$$\sum_{i=1}^{N_1} G_{i1}(\psi) \prod_{j=1}^{K_i} \psi_{x^{p_j}}. \qquad (2.6\text{-}6)$$

Using the transformation (2.6-3) and the formula for the x-derivatives (see appendix, lemma (2)), (2.6-5) will be

$$\sum_{i=1}^{N} F_{i1}(g(\psi)) \prod_{j=1}^{k_i} \left\{ \sum_{n=1}^{Q_j} \alpha_n \left(\prod_{\ell=1}^{b(n)} \psi_{x^{a_n}} \right) g(\psi)^{(b(n))} \right\}. \qquad (2.6\text{-}7)$$

It is clear that (2.6-7) can be written as,

$$\sum_{i=1}^{N} G_{i1}{}^{*}(\psi) \sum_{n=1}^{R_i} H_n(\psi) \prod_{\ell=1}^{k_n} \psi_{x^{a_n}} \quad \text{for some } H_n(\psi),\ G_{i1}{}^{*}(\psi)$$

for each n, i, where $R_i = \prod_{j=1}^{k_i} Q_j$. More precisely, (2.6-7) is of the form,

$$\sum_{m=1}^{N_1} G_{i1}(\psi) \prod_{\ell=1}^{k_m} \psi_{x^{a_\ell}}$$

where $N_1 = \sum_{i=1}^{N} R_i$, and $G_{i1}(\psi)$ is a function of ψ constructed from $G_{i1}{}^{*}$ and H_n. ■

Theorem (2.6-3)

If eq. (2.6-4) satisfies the conditions of theorem (2.3-14) then eq. (2.6-2) does. □

Proof The conditions of theorem (2.3-14) are:

"All the powers of ϕ's derivatives are positive and the powers of higher derivatives are integers."

Eq. (2.6-2) is obtained by applying the transformation $\phi = g(\psi)$ on (2.6-4). The derivatives of the form ϕ_{x^p} are given in the appendix. Notice that all powers of all the derivatives ψ_{x^q}, $q \in \mathbb{N}$ in the r.h.s. of eq. (1), are positive integers. This implies that if $\phi_x = \psi_x g'$ is raised to any positive power so is ψ_x. (Similarly for ψ_t.)

In the case of higher derivatives, applying the binomial theorem many times to eq. (6) gives that all the integer powers of the r.h.s. of eq. (6) contain only integer powers of $\psi_x a$. (Similarly for $\psi_t a$.) ■

In practice if eq. (2.6-4) is transformed by the use of a dependent-variable transformation $\phi = g(\psi)$, the resulting equation is simply separable if g is a non constant solution of an ordinary differential equation. Letting $\phi(x, t) = g(\psi(x, t))$ in eq. (2.6-4) the following identity for ψ and g is obtained:

$$\sum_{i=1}^{N} F_{i1}(g) \prod_{j=1}^{k_i} \left| \sum_{n=1}^{\Omega_j} \alpha \prod_{\ell=1}^{b(n)} \psi_x a \right| g^{(b(n))}$$

$$+ \sum_{i=1}^{M} F_{i2}(g) \prod_{j=1}^{\ell_i} \left| \sum_{n=1}^{\Omega_j} \alpha \prod_{\ell=1}^{b(n)} \psi_x a \right| g^{(b(n))} = F_0(g) \qquad (2.6\text{-}8)$$

where the constants are as defined in lemma (2) and its simplification.

<u>Theorem (2.6-4)</u>

The identity (2.6-8) is an ordinary differential equation in ψ and g if $\psi = X(x)T(t)$ and $X' = k_1 X^r$, $T' = k_2 T^s$, for some constants r, s, k_1, k_2. □

<u>Proof</u> Identity (2.6-8) can be written in the form,

$$\sum_{i=1}^{N_1} G_{i1}(g) \prod_{n=1}^{k} \psi_x n^{a_{in}} + \sum_{i=1}^{M_1} G_{i2}(g) \prod_{n=1}^{\ell} \psi_t n^{b_{in}} = F_0(g)$$

$$(2.6\text{-}9)$$

where $N_1 = \sum_{i=1}^{N} Q_i$, $M_1 = \sum_{i=1}^{M} Q_i$, G_{ij}, $(j = 1, 2)$ are constructed from F_{ij} $(j = 1, 2)$, derivatives of g and the constants α . k denotes the highest x derivative and ℓ denotes the highest t derivative.

To prove that (2.6-9) is an o.d.e. in $g(\psi)$ it is sufficient to show that

$$\sum_{i=1}^{N_1} \psi_x^{a_{i1}} \psi_{x^2}^{a_{i2}} \dots \psi_{x^k}^{a_{ik}} + \sum_{i=1}^{M_1} \psi_t^{b_{i1}} \psi_{t^2}^{b_{i2}} \dots \psi_{t^\ell}^{b_{i\ell}} \tag{2.6-10}$$

is a function of ψ only.

(2.6-10) can be written as,

$$\sum_{i=1}^{N_1} T^{\sum_{n=1}^{k} a_{in}} (X')^{a_{i1}} (X^{(2)})^{a_{i2}} \dots (X^{(k)})^{a_{ik}} +$$

$$\sum_{i=1}^{M_1} X^{\sum_{n=1}^{\ell} b_{in}} (T')^{b_{i1}} (T^{(2)})^{b_{i2}} \dots (T^{(\ell)})^{b_{i\ell}}$$

or,

$$\sum_{i=1}^{N_1} \alpha_i T^{\sum_{n=1}^{k} a_{in}} X^{a_{i1}r+a_{i2}(2r-1)+\dots+a_{ik}(kr-k+1)} +$$

$$\sum_{i=1}^{M_1} \beta_i X^{\sum_{n=1}^{\beta} b_{in}} T^{b_{i1}s+b_{i2}(2s-1)+\dots+b_{i\ell}(\ell s-\ell+1)} \tag{2.6-11}$$

where $(\alpha_i,\ i = 1, \ldots, N_1), (\beta_i,\ i = 1, \ldots, M_1)$ are the constants constructed from the derivatives $X(x)^{(n)}$ and $T(t)^{(m)}$ respectively, where $n = 1, \ldots, k$, $m = 1, \ldots, \ell$.

Now, if F_0 is not identically zero then, the above expression is a function of ψ if (and only if),

$$\sum_{n=1}^{k} a_{in} = a_{i1}r + a_{i2}(2r - 1) + \ldots + a_{ik}(kr - k + 1),$$
$$i = 1, \ldots, N_1$$

and,

$$\sum_{n=1}^{\ell} b_{in} = b_{i1}s + b_{i2}(2s - 1) + \ldots + b_{i\ell}(\ell s - \ell + 1),$$
$$i = 1, \ldots, M_1$$

which implies that $r = s = 1$.

If F_0 is identically zero, then multiplication by X^{α}, for some α (without loss of generality) is possible to balance the powers of X and T:

Let

$$\alpha = \left(\sum_{n=1}^{k} a_{1n}\right) - a_{11}r - a_{12}(2r - 1) - \ldots -$$
$$a_{1k}(kr - k + 1).$$

then we have two cases:

<u>Case (i)</u> $(N_1 = 1)$

If $N_1 = 1$ then we obtain the following system of M_1

relations if (2.6-10) is to be a function of ψ:

$$b_{i1}s + b_{i2}(2s - 1) + \dots + b_{i\ell}(\ell s - s + 1) = b_{i1} + b_{i2} + \dots + b_{i\ell} + a_{11} + a_{12} + \dots + a_{1k} - a_{11}r - a_{12}(2r - 1) - \dots - a_{1k}(kr - k + 1), \quad (i = 1, \dots, M_1). \tag{2.6-12}$$

Which may be solved to give that $r = s = 1$, or just a relation between r and s. Note however, that the system always has one solution that being $r = s = 1$.

Case (ii) ($N_1 \geq 2$)

If $N_1 \geq 2$, then we have a system of the following (N - 1) relations

$$a_{i1} + a_{i2} + \dots + a_{ik} = a_{i1}r + a_{i2}(2r - 1) + \dots + a_{ik}(kr - k + 1) + a_{11} + a_{12} + \dots + a_{1k} - a_{11}r - a_{12}(2r - 1) - \dots - a_{1k}(kr - k + 1), \quad (i = 2, \dots, N_1) \tag{2.6-13}$$

together with the above M_1 relations (2.6-12). One solution is always $r = 1$ which implies that $\alpha = 0$ and $s = 1$. ∎

The following result follows easily from the previous theorem:

Corollary (2.6-5)

Eq. (2.6-4) is implicitly separable if there exists a non-constant solution to the o.d.e. (2.6-8). □

Corollary (2.6-5) provides us with guaranteed implicitly separable solutions for eq. (2.6-4). One solution of the transformed p.d.e. is always,

$$\psi(x, t) = \exp(\alpha x + \beta t + \gamma),$$

where α, β and γ are arbitrary constants (see proof of theorem (2.6-4).

Other solutions, if they exist, are of the form

$$\psi(x, t) = (k_1 x + c_1)^{A_1}(k_2 t + c_2)^{A_2}$$

where k_i, c_i, $(i = 1, 2)$ are arbitrary constants

$$A_i = \frac{1}{1 - \alpha}, \; \alpha = \begin{cases} r & i = 1 \\ s & i = 2 \end{cases}.$$

Note that from proof of theorem (2.6-4), if $r = 1$ then $s = 1$ and conversely, so that there are no guaranteed solution of the form,

$$\psi(x, t) = (k_1 x + c_1)^{A_1}\exp(\beta t + \gamma)$$

for instance.

The question now arises "Does every p.d.e. have an implicitly separable solution using (2.6-3)?". The answer to this question is no. Consider the p.d.e.,

$$\phi_{xx} + \phi_{tt} = f(\phi), \qquad (2.6\text{-}14)$$

where $f(\phi) = \begin{cases} 1 & \text{if } \phi \in \mathbb{R}/\mathbb{Q} \\ 0 & \text{if } \phi \in \mathbb{Q} \end{cases}.$

Applying (2.6-3) to (2.6-14), where $\psi = XT$, $X' = k_1X^r$, $T' = k_2T^s$ gives the o.d.e.,

$$(k_1{}^2\psi^2 + k_2{}^2\psi^{2s})g'' + (rk_1{}^2\psi + sk_2{}^2\psi^{2s-1})g' = f(g). \tag{2.6-15}$$

Clearly (2.6-15) has no solution, hence (2.6-14) has no implicitly separable solution.

Some p.d.e.'s are implicitly separable using (2.6-3) only if $g(\psi) = \psi$ or g = constant. The following p.d.e. is an example of this case:

$$\phi_x + \phi_t = 0.$$

Now, to generalize theorem (2.6-4), let $\psi(x, t) = X(x)T(t)$ in eq. (2.6-4) and expand $X'(x)$, $T'(t)$ as series of X and T as in definitions, where the constants r and s are taken to be one according to theorems (2.6-3) and (2.3-14). The coefficients $a_n(x)$, $b_n(t)$ are taken to be constants, λ and ρ may be taken zero for simplicity.

Eq. (2.6-4) is separable if the coefficient of $(X(x))^i(T(t))^j$ for all i, j in (2.6-8) as an identity in $X(x)T(t)$ in this identity are consistent. This leads to an ordinary differential equation which we will call it a general o.d.e. to recognize it from the above o.d.e. (in the guaranteed case) which is called a guaranteed o.d.e. Clearly the guaranteed o.d.e. is a special case of the general o.d.e.

In this generalized case, besides the necessity of the existence of a non-constant solution to the o.d.e., the function g (or its derivatives) must be expanded as a power series in ψ, so that the transformed p.d.e. will be an identity in $(X^n T^m)$. For the sG equation,

$$\phi_{xx} + \phi_{tt} = \sin \phi, \tag{2.6-16}$$

when applying (2.6-3) to (2.6-16), (see [69]), the following identity is obtained:

$$(\psi_{xx} + \psi_{tt})g' + (\psi_x{}^2 + \psi_t{}^2)g'' = \sin g. \tag{2.6-17}$$

Assuming that $\psi = X(x)T(t)$ and substituting the series expansion

$$X'^2 = \sum_{n=0}^{\infty} 2a_n X^n, \quad T'^2 = \sum_{n=0}^{\infty} 2b_n T^n \tag{2.6-18}$$

into (2.6-17) gives,

$$\left\{\sum_{n=1}^{\infty} n(a_n X^{n-1} T + b_n T^{n-1} X)\right\} g' + \left\{\sum_{n=0}^{\infty} 2(a_n X^n T^2 + b_n T^n X^2)\right\} g'' =$$
$$\sin g.$$

This identity can be written as:

$$2(a_2 + b_2)(\psi g' + \psi^2 g'') - \sin g = 0 \tag{2.6-19}$$

and

$$\left\{\sum_{\substack{n=1\\n\neq 2}}^{\infty} n(a_n X^{n-1} T + b_n T^{n-1} X)\right\} g' + \left\{\sum_{\substack{n=0\\n\neq 2}}^{\infty} 2(a_n X^n T^2 + b_n T^n X^2)\right\} g''$$

$$= 0. \tag{2.6-20}$$

It follows, therefore that a necessary condition for (2.6-16) to be implicitly separable that there exists a non-constant solution for (2.6-19). These conditions are sufficient to have the guaranteed separable solution ($\psi = K \exp(\sqrt{2a_2}x + \sqrt{2b_2}t)$). To seek other solutions, g' and g" must be expanded as power series in ψ.

Consider the solution for (2.6-19);

$$g = 2\cos^{-1} \mathrm{sn}\left\{\frac{\ln \alpha\psi}{k\beta}, k\right\}, \tag{2.6-21}$$

where α, k non-zero constants with $k \leq 1$, $\beta^2 = 2(a_2 + b_2)$ and sn is a Jacobian elliptic function sine amplitude of modulo k.

To get the subclass of non elliptic transformation [for in the general elliptic case, g' and g" can not be expanded in power series], put k = 1. Then if $\beta = -n$ then (2.6-21) reduces to

$$g = 4\tan^{-1}(\alpha\psi)^{1/n}.$$

Substitution g' and g" in (2.6-20) gives that,

$$\left.\begin{aligned}(X')^2 &= pX^4 + mX^2 + q\\ (T')^2 &= \alpha^{2/n} qT^4 + (1 - m)T^2 + \alpha^{-2/n} p\end{aligned}\right\}.$$

The above example shows the difficulties involved when considering the conditions onto g's (or its derivatives) expansion. However, this condition is not necessary in some cases. Consider the p.d.e.,

$$\phi_x + \phi_t = f(\phi).$$

Let $\phi = g(\psi)$ and $X' = \sum_{n=0}^{\infty} a_n X^n$, $T' = \sum_{n=0}^{\infty} b_n T^n$,

then

$$(a_1 + b_1)\psi g' - f(g) = 0 \qquad (2.6\text{-}22)$$

and

$$T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} a_n X^n\right) + X\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} b_n T^n\right) = 0.$$

Existence of a non constant solution for the o.d.e. (2.6-22) is sufficient for this case.

Before going any further with the separation method, it is better to illustrate it by some useful examples.

2.7 Examples of the Application of the Separation Technique

This section presents examples illustrating the use of the separation technique described above to get useful separation solutions of equations which occur in practice.

Example (2.7-1)

Separable solutions of the Burgers equation

In this example, we apply the dependent variable transformation, $\phi = g(\psi)$, to the Burgers equation,

$$\phi_{xx} + 2\phi\phi_x + \phi_t = 0. \tag{2.7-1}$$

Simple separability

Eq. (2.7-1) is separable as it stands, (i.e., simply separable). Let $\phi = XT$, then (2.7-1) reduces to

$$X''T + 2XTX'T + T'X = 0.$$

Substituting $X' = \sum_{n=0}^{\infty} a_n X^{n+\lambda}$, $T' = \sum_{n=0}^{\infty} b_n T^{n+\rho}$ gives that $\rho = 0$ and $T' = b_1T + b_2T^2$. Then $\lambda = 0$ and $X' = a_0$ and hence $b_1 = 0$.

Thus $X' = a_0$, $T' = -2a_0T^2$ is the simply separable solution.

Implicit separability

Substituting $\phi = g(\psi)$ in (2.7-1) gives the following expression,

$$\psi_x{}^2 g'' + 2\psi_x g g' + (\psi_{xx} + \psi_t)g' = 0 \tag{2.7-2}$$

$g(\psi)$ is chosen so that (2.7-2) is a simply separable p.d.e., and so by definition,

$$\psi(x, t) = X(x)T(t) \tag{2.7-3}$$

where

$$(X')^r = \sum_{n=0}^{\infty} a_n X^{n+\lambda} \tag{2.7-4a}$$

$$(T')^s = \sum_{n=0}^{\infty} b_n T^{n+\rho} \tag{2.7-4b}$$

where r, s, λ, ρ and a_n, b_n $\forall n$ are constants. Then we have two cases:

Case (i) (The Guaranteed solution case.)

In this case (2.7-4) reduces to

$$X' = aX^p \tag{2.7-5a}$$

$$T' = bT^q \qquad (27.-5b)$$

where a and b are arbitrary constants and p and q are constants to be determined. If we, now, substitute (2.7-3) and (2.7-5) in (2.7-2), we obtain,

$$a^2X^{2p}T^2g'' + 2aX^pTgg' + (pa^2X^{2p-1}T + bXT^q)g' = 0. \tag{2.7-6}$$

(2.7-6) will be, then, an o.d.e. of the form

$$a^2\psi^2g'' + 2a\psi gg' + (a^2 + b)\psi g' = 0 \tag{2.7-7}$$

if and only if p = q = 1. i.e., the transformed equations will always have solutions of the type

$\psi(x, t) = K \exp(ax + bt)$, where K is arbitrary constant. However this exponential solution is the only guaranteed solution, as $p = q = 1$ are the only values which make (2.7-6) an o.d.e. in g, where $g(\psi) \neq \psi$ (see simple separability).

Eq. (2.7-7) is an Euler equation, which is upon the independent transformation $\psi = \exp(s)$, reduces to the constant coefficients equation,

$$a^2\ddot{g} + 2ag\dot{g} + b\dot{g} = 0; \quad g = g(s); \quad \dot{g} = \frac{dg}{ds}.$$

This equation can be easily integrated to,

$$a^2\dot{g} + ag^2 + bg = K,$$

where K is an integration constant. Substituting $g(s) = h(s) + \gamma$, where γ satisfies $a\gamma^2 + b\gamma - K = 0$, we obtain that, if $(b^2 + 4Ka)$ is not equal to zero, then the general solution to eq. (2.7-7) is

$$g_1(\psi) = \frac{1}{(c\psi^\alpha - \beta)} + \gamma, \qquad (2.7\text{-}8)$$

where β and c are non zero arbitrary constants, $\alpha = \frac{1}{\beta a}$ and γ can be written as $\gamma = \frac{1}{2\beta} - \frac{b}{2a}$.

In the case $(b^2 + 4Ka = 0)$, the general solution is,

$$g_2(\psi) = \frac{a}{\ln\psi + K_1} - \frac{b}{2a} \qquad (2.7\text{-}9)$$

where K_1 is arbitrary constant.

Hence two solutions to Burgers equation are obtainable using (2.7-8) and (2.7-9).

<u>Case (ii)</u> (The general case.)

In this case, the constants r and s in (2.7-4) can be taken to be unity without loss of generality, by theorem (2.3-14). Substituting (2.7-4) with r = s = 1 and (2.7-3) into (2.7-2) gives,

$$\left(\sum_{n=0}^{\infty} a_n X^{n+\lambda}\right)^2 T^2 g'' + 2\left(\sum_{n=0}^{\infty} a_n X^{n+\lambda}\right) Tgg' +$$

$$\left[\left(\sum_{n=0}^{\infty} (n+\lambda) a_n X^{n+\lambda-1}\right)\left(\sum_{n=0}^{\infty} a_n X^{n+\lambda}\right) T + \left(\sum_{n=0}^{\infty} b_n T^{n+\rho}\right) X\right] g' = 0 \tag{2.7-10}$$

where a_0, $b_0 \neq 0$ without loss of generality.

Division by gg' is possible for nontrivial solutions; Assuming that λ is not a nonnegative integer gives that the coefficient of $X^\lambda T$ equal to zero (i.e., $a_0 = 0$) which is a contradiction. Hence $\lambda \in \mathbb{Z} \geq 0$.

To prove that ρ is a nonnegative integer, divide by g'. If ρ is not a nonnegative integer then $b_0 = 0$ which is a contradiction. Thus $\lambda = \rho = 0$ without loss of generality. Thus, (2.7-10) can be written as,

$$T^2\left(\sum_{n=0}^{\infty} a_n X^n\right)^2 g'' + 2T\left(\sum_{n=0}^{\infty} a_n X^n\right) gg' + \left[T\left(\sum_{n=0}^{\infty} n a_n X^{n-1}\right)\left(\sum_{n=0}^{\infty} a_n X^n\right)\right.$$

$$\left. + X\left(\sum_{n=0}^{\infty} b_n T^n\right)\right] g' = 0. \tag{2.7-11}$$

Let $\left(\sum_{n=0}^{\infty} a_n X^n\right)^2 = \sum_{n=0}^{\infty} A_n X^n$, and $\left(\sum_{n=0}^{\infty} n a_n X^{n-1}\right)\left(\sum_{n=0}^{\infty} a_n X^n\right) =$

$\sum_{n=0}^{\infty} B_n X^n$.

Comparing coefficients of $X^n T^m$ shows that $g(\psi)$ must satisfy the o.d.e.,

$$A_2\psi^2 g'' + 2a_1\psi g g' + (B_1 + b_1)\psi g' = 0,$$

$$A_2 = B_1 = 2a_0 a_2 + a_1^2, \tag{2.7-12}$$

and the identity,

$$T^2\left(\sum_{\substack{n=0\\n\neq 2}}^{\infty} A_n X^n\right) g'' + 2T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} a_n X^n\right) g g' + \left[T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} B_n X^n\right) + X\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} b_n T^n\right)\right] g' = 0, \tag{2.7-13}$$

Eq. (2.7-7) now can be considered as a special case of eq. (2.7-12).

Solution of eq. (2.7-12)

From eq. (2.7-12), it is seen that, the choice of g seems to be arbitrary if the constants A_2, a_1, and b_1 vanish. But if the guaranteed solution ($a_1 \neq 0$, $b_1 \neq 0$) is required, then the form of g is quite specific. Thus we have two cases:

<u>Case (i)</u> $(A_2 = a_1 = b_1 = 0)$

Even if g, as a solution to (2.7-12), seems to be arbitrary in this case, yet it has to satisfy the identity (2.7-13). One special value to g is $g(\psi) = \psi$ which gives that Burgers equation is simply separable. To see if there are other values for g which satisfy (2.7-13), we need to assume that $g(\psi)$ has a power series expansion of the form

$$g(\psi) = \sum_{n=0}^{\infty} c_n \psi^{n+\alpha}, \tag{2.7-14}$$

where $\alpha \in \mathbb{R}$ is a constant, $c_0 \neq 0$.

Substitution of (2.7-14) into (2.7-13) gives,

$$T^2\left(\sum_{\substack{n=0\\n\neq 2}}^{\infty} A_n X^n\right)\left(\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)c_n X^{n+\alpha-2}T^{n+\alpha-2}\right)$$

$$+\ 2T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} a_n X^n\right)\left(\sum_{n=0}^{\infty} c_n X^{n+\alpha}T^{n+\alpha}\right)\left(\sum_{n=0}^{\infty}(n+\alpha)c_n X^{n+\alpha-1}T^{n+\alpha-1}\right)$$

$$+\left[T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} B_n X^n\right) + X\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} b_n T^n\right)\right]\left(\sum_{n=0}^{\infty}(n+\alpha)c_n X^{n+\alpha-1}T^{n+\alpha-1}\right) = 0.$$

$$\tag{2.7-15}$$

Terms in $XT^m X^r T^r$ are terms in $TX^n X^s T^s$, iff $m + r = 1 + s$ and $r + 1 = n + s$ $\forall m, r, n, s \in \mathbb{Z} \geq 0$. Thus $m + n = 2$; as $m, n \in \mathbb{Z} \geq 0$ m must be less than or equal to 2. Hence

$$T' = b_0 + b_2T^2.$$

Now, consider the lowest possible terms in X^mT^m which are,

$$T^2\left(\sum_{\substack{n=0\\n\neq 2}}^{\infty} A_nX^n\right)(\alpha(\alpha - 1)c_0X^{\alpha-2}T^{\alpha-2}) + 2T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} a_nX^n\right)$$

$$(\alpha c_0{}^2X^{2\alpha-1}T^{2\alpha-1}) + \left[T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} B_nX^n\right) + X\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} b_nT^n\right)\right]$$

$$(\alpha c_0X^{\alpha-1}T^{\alpha-1}), \tag{2.7-16}$$

If $\alpha \geq -1$ then $\alpha - 2 \leq 2\alpha - 1$ and we have $\alpha(\alpha - 1)c_0 = 0$ which implies that $\alpha = 0$, 1 since $c_0 \neq 0$.

If $\alpha < -1$ then $2\alpha - 1 \notin \mathbb{Z} \geq 0$; hence $\alpha c_0{}^2 = 0$ or $\alpha = 0$. In all cases,

$$\alpha \in \mathbb{Z} \geq 0,$$

and hence

$$g(\psi) = \sum_{n=0}^{\infty} c_n\psi^n,$$

without loss of generality. Then the identity (2.7-15) reduces to,

$$T^2\left(\sum_{\substack{n=0\\n\neq 2}}^{\infty} A_n X^n\right)\left(\sum_{n=0}^{\infty} n(n-1)c_n X^{n-2}T^{n-2}\right) + 2T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} a_n X^n\right)$$

$$\left(\sum_{n=0}^{\infty} c_n X^n T^n\right)\left(\sum_{n=0}^{\infty} nc_n X^{n-1}T^{n-1}\right)$$

$$+ \left\{T\left(\sum_{\substack{n=0\\n\neq 1}}^{\infty} B_n X^n\right) + X(b_0 + b_2T^2)\right\}\left(\sum_{n=0}^{\infty} nc_n X^{n-1}T^{n-1}\right) = 0. \tag{2.7-17}$$

Assume that $a_0 \neq 0$ (hence $a_2 = 0$), as in simply separable case, then it can be shown, by equating to zero the coefficients of X^nT^m $\forall n, m \in \mathbb{Z} \geqslant 0$, that

$$\left.\begin{aligned} a_n &= 0, \quad n = 1, 2, \ldots \\ b_n &= 0, \quad n = 0, 1, 3, \ldots \\ b_2 &= -2c\ a_0 \\ g(\psi) &= c\psi \end{aligned}\right\} \tag{2.7-18}$$

(2.7-18) leads to the fact that, if $a_0 \neq 0$ then the simply separable solution is the only one in this case.

Now, if $a_0 = 0$, then we have $a_2c_0 = 0$. If $a_2 \neq 0$, $c_0 = 0$ this leads to contradiction. If $a_2 = 0$, $c_0 \neq 0$ then $X' = 0$. This leads to $\psi(x, t)$ is a function of t only, which is excluded from definition (2.3-1).

<u>Case (ii)</u> ($A_2 \neq 0$ or $a_1 \neq 0$ or $b_1 \neq 0$)

In this case, we have to solve (2.7-12) (i.e., to find a nonconstant solution which can be expanded as a power series to satisfy (2.7-13)). However this case is included in case (i) and there is no benefit to solve eq. (2.7-12) since the only expandable g which makes Burgers equation separable is $g(\psi) = \alpha\psi$.

Example (2.7-2)

Separable solutions of the simplified Harry-Dym equation

Among the nonlinear evolution equations, solvable by means of the inverse spectral transform, is the so called Harry-Dym equation

$$\phi_t = \lambda\phi^3\phi_{xxx} \tag{2.7-19}$$

where λ is a constant [57].

Consider, now, the simplified Harry-Dym equation,

$$\phi^2\phi_{xx} = \phi_t. \tag{2.7-20}$$

This equation is separable by the linear separation technique and the separating o.d.e.'s are,

$$XX'' = k, \quad T' = kT^3,$$

where k is the separating constant.

Simple separability

To check if eq. (2.7-20) is simply separable, in the sense of the definition, expand X" and T' as follows:

$$X'' = \sum_{n=0}^{\infty} a_n(x)X^{n+\lambda}, \quad T' = \sum_{n=0}^{\infty} b_n(t)T^{n+\rho}$$

where $\phi = X(x)T(t)$ as usual. It is clear that both series are reduced to one term series, $X'' = a(x)X^p$ and $T' = b(t)T^q$, for some $p, q \in \mathbb{R}$, when X'' and T' are substituted in eq. (2.7-20) with the assumption $\phi = XT$. This gives,

$$a(x)X^{1+p}T^3 = b(t)T^q$$

which means that $a(x)$ and $b(t)$ are constants; $p = -1$ and $q = 3$. In other words, we recover the separable solution obtainable by the linear technique. This is what theorem (2.4-2) predicts.

Implicit separability

Applying the transformation $\phi = g(\psi)$, where $\psi = X(x)T(t)$ to eq. (2.7-20) gives the identity

$$g^2(\psi_x{}^2 g'' + \psi_{xx} g') = \psi_t g'$$

or,

$$X'^2 T^2 g'' + X''Tg' = XT'g^{-2}g'. \tag{2.7-21}$$

Guaranteed solutions

Assuming that $X' = aX^p$, $T' = bT^q$, where $a, b, p, q \in \mathbb{R}$ are constants, in eq. (2.7-21) implies,

$$a^2X^{2p}T^2g'' + pa^2X^{2p-1}Tg' = bXT^qg^{-2}g'.$$

This is equivalent to an o.d.e. for $g(\psi)$ iff

$q = 3 - 2p$. Then it is equivalent to the following o.d.e. if multiplied by X^{2-2p},

$$a^2\psi^2 g'' + pa^2\psi g' = b\psi^{3-2p} g^{-2} g'. \qquad (2.7\text{-}22)$$

If $p = 1$ (which implies $q = 1$) then the transformation $\psi = e^s$ reduces (2.7-22) to,

$$a^2\ddot{g} - bg^{-2}\dot{g} = 0, \quad g = g(s), \quad \dot{g} = \frac{dg}{ds}.$$

Letting $h = g$ gives $a^2\frac{dh}{dg} = \frac{b}{g^2}$ which implies that

$$a^2\dot{g} = a^2 h = K - \frac{b}{g}$$

or,

$$s + L = \int \frac{a^2 dg}{K - b/g} = \int \left(\frac{a^2}{K} + \frac{ba^2}{K(Kg - b)}\right) dg$$

where K and L are integration constants. Let $c = \frac{K}{a^2}$ $L = \ln k$, then

$$\exp(g/c) = \exp\left(\ln\psi + \ln k - \frac{ba^2}{K^2}\ln(Kg - b)\right)$$

$$\Rightarrow \exp(g/c) = k\psi(ca^2 g - b)^{-b/ca^2}$$

is an implicit form for g.

<u>Solutions of eq. (2.7-22) when $p \neq 1$</u>

The transformation

$$g = \psi^{1-p}u(z), \quad z = \ln\psi \quad (p \neq 1)$$

transforms eq. (2.7-22) to,

$$u^2u'' + (1 - p)u'u'' - \frac{b}{a^2}u' - \frac{b}{a^2}(1 - p)u = 0$$

which can be integrated at once using the transformation $u' = h(u)$ to give

$$hh' + \left(1 - p - \frac{b}{a^2}u^{-2}\right)h - \frac{b}{a^2}(1 - p)u^{-1} = 0.$$

This equation is Abel's equation of the 2nd kind [60].

CHAPTER THREE

Separation of Variables II

3.1 Introduction

In the previous chapter, we investigated simple separability of nonlinear p.d.e.'s, and implicit separability of nonlinear p.d.e.'s using dependent-variable transformations. We found that, using dependent variable transformations we can work in two directions: one is to transform the p.d.e. to a separable one (which is the aim of the method) and the second direction is to reduce the p.d.e. to a nonlinear o.d.e. (which is a tool to find the transformation).

Transformations in general are, perhaps the most powerful general analytic tool available in solving non-linear p.d.e.'s. Some of these transformations linearize the equation, or transform it to a more simple p.d.e. (for instance, the Hopf-Cole transformation [42], the inverse scattering method, and Bäcklund transformations), while other transformations reduce the p.d.e. to a nonlinear o.d.e. (for example the similarity transformation (see §1.6) and some other individuals' work [18]).

3.2 Extensions of the Dependent-Variable Transformation

To get possibly more separable solutions to nonlinear p.d.e.'s, it seems reasonable to extend the dependent-variable transformation, used in the previous chapter, so that solutions provided by the separation technique will tie up with solutions provided by other known techniques.

An "obvious" extended transformation to reduce the general p.d.e.,

$$H(\phi(x, t)) = 0, \tag{3.2-1}$$

to a simply seperable one is the affine transformation in g

$$\phi(x, t) = u(x, t)g(\psi) + h(x, t), \tag{3.2-2}$$

where $\psi(x, t) = X(x)T(t)$ is a solution of the transformed p.d.e. Clearly $\phi = g(\psi)$ is a special case of (3.2-2). (The choice of (3.2-2) is justified in a later section.)

In this chapter, a complete description of our new transformation (3.2-2) will be given as theorems and comments for the p.d.e.'s with no explicit dependence on the independent variables, x and t. The method will be sketched out for other p.d.e.'s.

Once again, the main objective of the method is to reduce a given p.d.e. to an o.d.e. for g. Using all the details of the definitions (2.3-1,2), with unknown u and h, the problem turns into a complicated one. Hence some simplifying conditions here seems to be necessary.

We will concentrate here on equations of polynomial class and investigate the properties of the functions u and h, using one term 'series' $X'(x) = aX^n$, $T'(t) = bT^m$ for some $n, m \in \mathbb{R}$, to reduce the original p.d.e. to an o.d.e. For nonpolynomial equations, where a transcendental function term exists, (3.2-2) reduces to simpler form where $u \equiv 1$ or $h \equiv 0$. For sine-Gordon equation, for instance, u must be 1, while for the equation $\phi_{xx} - \phi_t = \ln\phi$, h must be zero at once.

Lemma (3.2-1)

In (3.2-2), if u is a product of the function ψ and its derivatives and $X' = aX^n$, $T' = bT^m$ then $u = T^q$ (or X^q) for some $q \in \mathbb{R}$. □

Proof It can be easily seen that if $X' = aX^n$, $T' = bT^m$ for some n and m, then any derivative of ψ is $\gamma X^\alpha T^\beta$ for some α, β and γ. Therefore, if u is a product of ψ and its derivatives then $u = \gamma X^\alpha T^\beta$ for some α, β, γ which is equivalent to $u = \psi^\alpha T^{\beta-\alpha}$ (or $X^{\alpha-\beta}\psi^\beta$).

Substituting this result into (3.2-2) gives,

$$\phi = T^q g(\psi) + h(x, t),$$

where $q = \beta - \alpha$; or

$$\phi = X^q g(\psi) + h(x, t)$$

where $q = \alpha - \beta$. (And where the notation has been changed slightly.) ■

In what follows, we take $u = T^q$ without loss of generality. (The case $u = X^q$ is implicitly included. In practice, one form of u may have an advantage over the other for purposes of calculating derivatives of ϕ which occur in a particular equation.)

Consider the following p.d.e.

$$\sum_{n=1}^{r} \alpha_n \phi^{M_n} \phi_{x^n} + \sum_{n=r+1}^{r+s} \alpha_n \phi^{M_n} \phi_{t^{n-r}} = \alpha_0 \phi^{M_0} \qquad (3.2\text{-}3)$$

(where r and s denote the highest x and t derivatives respectively, M_i ($i = 0, \ldots, r + s$) are non-negative integers, and α_n are constants $\forall n \in \mathbb{Z} \geq 0$).

Many famous equations are included in (3.2-3) as special cases: The generalized KdV, KdV-Burgers, Fishers, Harry-Dym and other equations.

Assume that $\exists j \in \{x: x = 1, \ldots, r\}$ such that $M_j > M_i$ ($\forall i = 1, \ldots, j - 1, j + 1, \ldots, r + s$). Then we have the following:

<u>Theorem (3.2-2)</u>

Suppose that (3.2-3) reduces to an o.d.e. in $g(\psi)$, using (3.2-2). Then $u = T^q$ for some q if one of the following statements is true:

(i) $r > s$ and $j \neq r$

(ii) $r < s$

(iii) $r = s$ and $M_r \neq M_s$. □

<u>Proof</u> Clearly we can choose $\alpha_i = 1$ $\forall i = 0, \ldots, r + s$ in (3.2-3) without loss of generality.

Substituting (3.2-2) into (3.2-3) and using lemma (3) for the p^{th} derivatives (see appendix) gives the following identity:

$$(ug + h)^{M_1}(u\psi_x g' + F_1 + h_x) + \dots + (ug + h)^{M_j}$$
$$(u\psi_x{}^j g^{(j)} + F_j + h_{x^j}) + \dots + (ug + h)^{M_r}$$
$$(u\psi_x{}^r g^{(r)} + F_r + h_{x^r}) + (ug + h)^{M_{r+1}}(u\psi_t g' + G_1 + h_t)$$
$$+ \dots + (ug + h)^{M_{r+s}}(u\psi_t{}^s g^{(s)} + G_s + h_{t^s}) = (ug + h)^{M_0},$$

(3.2-4)

where F_i ($i = 1, \dots, r$), G_i ($i = 1, \dots, s$) are functions of u, ψ, g defined in lemma (3), (and its analogue for ϕ_{t^p}).

We divide by $u^{M_j+1}\psi_x{}^j$ ($u^{M_j+1}\psi_{x^j} \neq 0$ as $\psi_x \neq 0$ for non-trivial solutions), to guarantee that the coefficient of $g^{M_j}g^{(j)}$ is unity then we have three cases.

Case (a) $r > s$

In this case, if $j \neq r$, then the coefficient of $g^{M_r}g^{(r)}$, that is,

$$(u^{M_r+1}\psi_x{}^r)/(u^{M_j+1}\psi_x{}^j) = u^{M_r-M_j}\psi_x{}^{r-j}$$

must be a function of ψ (say ψ^p for some p). Hence u is a product of ψ and its x-derivatives, i.e., $u = T^q$ for some q, by lemma (3.2-1).

Case (b) $r < s$

The choice of the coefficient of $g^{M_s} g^{(s)}$ will be convenient here to show that u is a product of ψ and its derivatives which means again that $u = T^q$ for some q, as in case (a).

Case (c) $r = s$

In this case, if $M_r \neq M_s$ then we will choose the maximum of M_r and M_s and this case will be reduced to case (a) if $M_r > M_s$, or it can be reduced to case (b) if $M_r < M_s$. ■

Following the proof above, one can easily calculate the value of q in each case.

Lemma (3.2-3)

In (3.2-2), if $h(x, t) = u(x, t)f(\psi)$ then $h = 0$ without loss of generality. □

The proof is obvious. ■

Theorem (3.2-4)

If $\exists j \in \{x: x = 1, \ldots, r\}$ such that $M_j - 1 > M_i$ $(\forall i = 1, \ldots, j - 1, j + 1, \ldots, r + s)$ and if eq. (3.2-3) reduces to an o.d.e., using (3.2-1) then $h = 0$. □

Proof Consider the coefficients in (3.2-4) after a division by $u^{M_j+1} \psi_x^{\,j}$. In particular the coefficient of

$g^{M_j - 1} g^{(j)}$ which is $M_j h/u$ must be a function of ψ (say $f(\psi)$). Thus,

$$h = u(x, t)f(\psi)$$

which gives that $h = 0$ without loss of generality by lemma (3.2-4). ∎

Similar results to theorems (3.2-2) and (3.2-4) can be achieved if $j \in \{x: x = r + 1, \ldots, r + s\}$.

We consider now the KdV-type equation,

$$\phi_{x^r} + \phi^k \phi_x + \phi_t = 0 \qquad (3.2\text{-}5)$$

where $k \in \mathbb{N}$. As it is well known, $r = 2$ and $k = 1$ in eq. (3.2-5) gives the Burgers equation; $r = 3$ and $k = 1$ gives the KdV equation; $r = 3$ and $k = 2$ gives the modified KdV equation and $r = 3$ and $k \geq 2$ gives a generalized KdV equation.

Examining the properties of the functions u and h in (3.2-2) when it is applied to (3.2-5) we see by theorem (3.2-2) that $u = T^q$, where $q = \frac{r - 1}{k}(1 - n)$ if $k \geq 1$, while $h = 0$ if $k > 2$, by theorem (3.2-4). We now examine the case when $k = 1$ separately. The following result shows, once again, that $h = 0$ without loss of generality.

<u>Theorem (3.2-5)</u>

Applying the transformation,

$$\phi = T^q g(\psi) + h(x, t) \qquad (3.2\text{-}6)$$

to eq. (3.2-5), where $k = 1$ yields that $h \equiv 0$ without loss of generality if $n \neq 1$ and $r \neq 2$. □

Proof Substituting (3.2-6) in (3.2-5) gives the following identity:

$$(u\psi_x{}^r g^{(r)} + F_r + u\psi_{x^r} g' + h_{x^r}) + (ug + h)(u\psi_x g' + h_x) +$$

$$(u\psi_t g' + u_t g + h_t) = 0, \qquad (3.2\text{-}7)$$

where $u = T^q$, and F_r, as defined in appendix, lemma (4).

After division by $u^2\psi_x$ (which is non-zero for non-trivial solutions), eq. (3.2-7) becomes an o.d.e. in $g(\psi)$ iff all the coefficients are functions of ψ. In particular the coefficient of g', using lemma (5) is given by,

$$I = \frac{u\psi_{x^r} + hu\psi_x + u\psi_t}{u^2\psi_x} = a^{r-1}n(2n - 1) \ldots$$

$$((r - 1)n - (r - 2))\psi^{-q} + J$$

where $J = \frac{h}{u} + \frac{\psi_t}{u\psi_x}$, is necessary a function of ψ. We have here two possibilities:

Either, $J = 0$ i.e., $h = -\frac{b}{a}X^{1-n}T^{m-1}$,

Or, J is a function of ψ which implies that $h = X^s T^{s+q}$ for some $s \in \mathbb{Z} \geq 0$.

In the second case, $h = T^q\psi^s$ which implies that $h = 0$ without loss of generality by lemma (3.2-3).

To prove that $h = 0$, in the first case, we have to consider the coefficient of g which is

$$G = \frac{h_x}{u\psi_x} + \frac{u_t}{u^2\psi_x}.$$

Substituting $h_x = -b(1 - n)T^{m-1}$ and other terms in G gives

$$G = \frac{b}{a}(q - (1 - n))X^{-n}T^{m-2-q}.$$

Hence either $q = 1 - n$ or $q = n + m - 2$. By theorem (3.2-2) $q = (r - 1)(1 - n)$. Thus $r = 2$ or $n = 1$ if $q = 1 - n$ which are excluded. Thus $q = n + m - 2$ which implies that

$$\phi = T^{n+m-2}g(\psi) - \frac{b}{a}\psi^{1-n}T^{n+m-2}$$

which means that $h \equiv 0$ without loss of generality by lemma (3.2-3). ∎

For the existence of separable solutions of an assumed type, the problem must be reduced to an o.d.e. in $g(\psi)$ where X', T', u and h may be defined.

In the following existence theorem, the p.d.e.,

$$\sum_{k=1}^{N_1} \alpha_k \phi^{a_{k0}} \phi_x^{a_{k1}} \cdots \phi_{x^r}^{a_{kr}} +$$

$$\sum_{k=N_1+1}^{N} \alpha_k \phi^{a_{k0}} \phi_t^{a_{k1}} \cdots \phi_{t^s}^{a_{ks}} = 0 \qquad (3.2\text{-}8)$$

where a_{ki}'s $(i = 0, \ldots, s)$ are non-negative, r and s denote the highest degree of derivatives of ϕ with respect to x and t respectively, and N_1 is the number of x terms, will be reduced to an o.d.e.

Theorem (3.2-6)

Eq. (3.2-8) is implicitly separable if $\phi = T^q g(\psi)$, $\psi = X(x)T(t)$, $X' = aX^n$ and $T' = bT^m$. Moreover n, m and q satisfy the system of linear equations

$$q\left(\sum_{i=0}^{r} a_{ki} - a_{1i}\right) - (n - 1) \sum_{i=1}^{r} i(a_{ki} - a_{1i}) = 0,$$

$$k = 2, \ldots, N_1 \qquad (3.2\text{-}9(a))$$

and

$$q\left(\sum_{i=0}^{s} a_{ki} - \sum_{i=0}^{r} a_{1i}\right) + (m - 1) \sum_{i=1}^{s} ia_{ki} + (n - 1)$$

$$\sum_{i=1}^{r} ia_{1i} = 0, \quad k = N_1 + 1, \ldots, N. \ \square \qquad (3.2\text{-}9(b))$$

Proof It is sufficient to prove that the problem is reduced to that of solving an o.d.e. Substituting $\phi = T^q g(\psi)$, using lemma (6), appendix gives,

$$\sum_{k=1}^{N_1} \alpha_k (T^q F_0)^{a_{k0}} (T^{q+1-n} F_1)^{a_{k1}} \ldots (T^{q+r(1-n)} F_r)^{a_{kr}} +$$

$$\sum_{k=N_1+1}^{N} \alpha_k (T^q G_0)^{a_{k0}} (T^{q+m-1} G_1)^{a_{k1}} + \ldots + (T^{q+s(m-1)} G_s)^{a_{ks}} = 0$$

which can be written as,

$$\sum_{k=1}^{N_1} T^{\sum_{i=0}^{r}[q+i(1-n)]a_{ki}} F_k^* + \sum_{k=N_1+1}^{N} T^{\sum_{i=0}^{s}[q+i(m-1)]a_{ki}} F_k^*$$

$$= 0, \qquad (3.2\text{-}10)$$

where the functions, F_k*'s are constructed from F_k's or G_k's.

In this stage an o.d.e. can be easily achieved, if eq. (3.2-10) is multiplies by $T^{-q\left(\sum_{i=0}^{r} a_{1i}\right)-(1-n)\sum_{i=0}^{r} ia_{1i}}$, as well as the linear homogeneous equations (3.2-9). ∎

(3.2-9) is a system of homogeneous linear equations in the unknowns q, 1 - n, 1 - m if $N_1 \geq 2$. The equations always have the zero solution, q = 1 - n = 1 - m = 0 in which case, the transformation is $\phi = g(\psi)$, where ψ is the travelling wave solution, which we dealt with in the previous chapter. Here we are interested in non-trivial solutions, in particular where q is not zero. If the number of equations is less than 3, the equations have a non-trivial solution (because of that, multiplication of the whole eq. (3.2-10) by $T^{-q\left(\sum a_{1i}\right)-(1-n)\sum ia_{1i}}$ in theorem (3.2-6) is useful). Hence in the case of the Burgers and the KdV equations, non-trivial solutions can be achieved, while in the case of the Boussinesq equation, where the number of equations is 3 they have the trivial solution only.

Example (3.2-7)

The modified KdV (mKdV) equation

Consider the generalized Korteweg-de Vries equation

$$\phi_t + a\phi^n\phi_x + b\phi_{xxx} = 0$$

where a, b ε $\mathbb{R}$ with $a \neq 0$ and n ε $\mathbb{N}$. This is a KdV equation with an $(n + 1)^{th}$ degree of nonlinearity. This equation incorporates the KdV equation (n = 1) and the modified KdV equation (n = 2).

A solitary wave solution for the mKdV equation [25] is

$$\phi = A\left(\frac{a}{6b}\right)^{-\frac{1}{2}}\operatorname{sech}\{Ax - A^3bt + B\},$$

where A and B are arbitrary constants.

Miura [25] in 1968, established a transformation which related solutions of the KdV equation,

$$q_t + \alpha qq_x + q_{xxx} = 0 \qquad (3.2\text{-}11)$$

and the mKdV equation,

$$v_t - \beta v^2v_x + v_{xxx} = 0. \qquad (3.2\text{-}12)$$

This transformation is

$$q = \frac{1}{\alpha}(-\beta v^2 + \varepsilon(6\beta)^{\frac{1}{2}}v_x), \quad \varepsilon = \pm 1.$$

Clearly, if q is a solution of (3.2-11), then v defined by

(3.2-12) is a solution of the mKdV.

In [28], the similarity solution for the equation (3.2-12) with $\beta = 6$ is given as

$$v(x, t) = t^{-\frac{1}{3}} f(x/t^{\frac{1}{3}}), \quad s = x/t^{\frac{1}{3}} \qquad (3.2\text{-}13)$$

and the similarity o.d.e. is

$$f'' - \frac{1}{3}sf - 2f^3 = 0 \qquad (3.2\text{-}14)$$

where f is known as a second Painlevé transcendent [44].

Now, let us apply the transformation (3.2-2) to the mKdV equation. This transformation can be reduced to $\phi = T^q g(\psi)$, $q = 1 - n$ (by theorems (3.2-2) and (3.2-4)) where $\psi = XT$ and $X' = aX^n$, $T' = bT^m$.

Substituting $\phi = T^q g(\psi)$, (where $q = 1 - n$), and its derivatives, into the equation,

$$\phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0 \qquad (3.2\text{-}15)$$

and using theorem (3.2-6) to get that $m = 4 - 3n$ and obtain the identity:

$$(a^3X^{3n}T^{4-n})g''' + (3na^3X^{3n-1}T^{3-n})g'' +$$
$$(na^3(2n - 1)X^{3n-2}T^{2-n} + bXT^{5-4n})g' + b(1 - n)T^{4-4n}g -$$
$$6aX^nT^{4-3n}g^2g' = 0,$$

which, when it is multiplied by $\frac{1}{a}X^{-n}T^{3n-4}$ gives the o.d.e.,

$$a^2\psi^{2n}g''' + 3na^2\psi^{2n-1}g'' + (n(2n - 1)a^2\psi^{2n-2} + \frac{b}{a}\psi^{1-n})g' +$$

$$\frac{b(1 - n)}{a}\psi^{-n}g - 6g^2g' = 0, \qquad (3.2\text{-}16)$$

which can be integrated to,

$$a^2\psi^{2n}g'' + na^2\psi^{2n-1}g' + \frac{b}{a}\psi^{1-n}g - 2g^3 - K = 0, \qquad (3.2\text{-}17)$$

where K is the integration constant.

<u>Case (1)</u> $(n = 1 \Rightarrow q = 0)$

In this case eq. (3.2-17) can be transformed by the independent transformation $s = \ln\psi$ to,

$$a^2\ddot{g} + \frac{b}{a}g - 2g^3 - K = 0$$

which can be integrated to,

$$a^2\dot{g}^2 = g^4 - B\ g^2 + 2Kg + K_1 \qquad (3.2\text{-}18)$$

where K_1 is arbitrary constant, and $B = \frac{b}{a}$.

Eq. (3.2-18) can be solved using elliptic and non-elliptic methods:

<u>Non-elliptic solutions of (3.2-18)</u>

(i) $K = K_1 = 0$ gives that,

$$\int\frac{dg}{g\sqrt{g^2 - \beta^2}} = \frac{1}{\beta}\sec^{-1}\frac{1}{\beta}g = \frac{1}{a}\ln\psi + c, \quad (\beta^2 = B).$$

Hence,

$$\phi = g(\psi) = \beta\sec\left(\frac{\beta}{a}\ln\psi + c\right)$$

where c is another constant of integration. This is a solitary wave solution.

(ii) $K = 0$, $B^2 = 4K_1$ gives,

$$\int \frac{dg}{\sqrt{g^4 - Bg^2 + \frac{B^2}{4}}} = -\frac{1}{\sqrt{2B}}\ln\left|\frac{g + \sqrt{\frac{B}{2}}}{g - \sqrt{\frac{B}{2}}}\right| = \frac{1}{a}\ln\psi + c.$$

Hence,

$$\phi = g(\psi) = \frac{\sqrt{\frac{B}{2}}\left(c\psi^{\sqrt{2B}/a} - 1\right)}{\left(c\psi^{\sqrt{2B}/a} + 1\right)}.$$

(iii) If B, K, and K_1 are chosen such that the equation,

$$g^4 - Bg^2 + 2Kg + K_1 = 0$$

has four equal roots (α) then,

$$g = -\frac{a}{\ln\psi + c} + \alpha.$$

(iv) If B, K and K_1 are chosen such that,
$g^4 - Bg^2 + 2Kg + K_1 = (g^2 + \alpha^2)^2$ for some constant α then,

$$\phi = g = \alpha\tan\left(\frac{\alpha}{a^2}\ln\psi + c\right).$$

Elliptic solutions for (3.2-18)

(i) $K = 0$:

We transform the eq. (3.2-18), by the transformation $g = \alpha h$ (α constant) to,

$$h^{2\prime} = \frac{\alpha^2}{a^2}h^4 - \frac{B}{a^2}h^2 + \frac{K_1}{\alpha^2 a^2}. \qquad (3.2\text{-}19)$$

For some choices of α and K_1, the solution of (3.2-19) can be expressed directly in terms of Jacobian elliptic functions. If $K_1 = a^2(B - a^2)$, $\alpha^2 = B - a^2$, then

$$g = \alpha\,\mathrm{sn}(\ln\psi, k), \quad \text{where } k^2 = \frac{B - a^2}{a^2}.$$

If $K_1 = \alpha^2 a^2\left(1 + \frac{\alpha^2}{a^2}\right)$, $\alpha^2 = \frac{B - a^2}{2}$, then

$$g = \alpha\,\mathrm{cn}(\ln\psi, k), \quad k^2 = \frac{a^2 - B}{2a^2}.$$

(ii) $K \neq 0$:

$$a\int \frac{dg}{\sqrt{g^4 - Bg^2 + 2Kg + K_1}} = \frac{a}{2}\,\wp\left(-\frac{H}{G};\ g_2,\ g_3\right)$$

where g_2, g_3 are the quadrivariant and cubinvariant of the quartic G

$$g_2 = K_1 + \frac{B^2}{12}$$

$$g_3 = -\frac{B}{6}K_1 - \frac{K^2}{4} - \frac{B^3}{216}$$

$$G = g^4 - Bg^2 + 2Kg + K_1$$

and

$$H = -\frac{B}{6}g^4 + Kg^3 + \left(K_1 - \frac{B^2}{12}\right)g^2 + \frac{Bk}{6}g - \frac{B}{6}K_1 - \frac{K^2}{4}.$$

Hence the general solution is [38],

$$\frac{a}{2}\wp^{-1}\left(-\frac{H}{G};\ g_2,\ g_3\right) = \ln\psi + c$$

where $\wp$ is Weierstrass elliptic function.

Case (2) $(n \neq 1 \Rightarrow q \neq 0)$

In this case, we indicate that

(i) If $n = 2$, $g = \pm a\psi$ is a solution where $K = \pm b$,

(ii) If $n = 0$, $g = \pm\frac{a}{\psi}$ is a solution where $K = \pm b$.

In fact, we will show in chapter six, that if $n \neq 1$ then $n = 0$ without loss of generality. This means that the similarity solution (3.2-13) is the only separable solution for $n \neq 1$.

Example (3.2-8)

The heat equation

The heat equation in two dimensions is the p.d.e.,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t}. \tag{3.2-20}$$

It has applications in various branches of science, one of the most important of which is in the theory of heat conduction, as the name implies. One important solution to (3.2-20) is the exponential solution $e^{\alpha x+\alpha^2 t}$ where α is real or complex constant. Other basic solutions can be found in [97].

Let us now apply (3.2-2) on (3.2-20), where u and h can be taken as separable functions of X and T; if $u = X^p T^q$ then $u = T^q$ (see proof of lemma (3.2-1)). Hence,

$$\phi = T^q g(\psi) + X^r T^s \tag{3.2-21}$$

where q, r and s are constants to be determined.

Applying (3.2-21) to (3.2-20) gives,

$$T^q \psi_x{}^2 g'' + T^q(\psi_{xx} - \psi_t)g' - bqT^{q-1+m}g + (h_{xx} - h_t) = 0 \tag{3.2-22}$$

where $\psi = XT$, $X' = aX^n$, $T' = bT^m$.

Dividing by $T^q \psi_x{}^2$ and examining the coefficient of g gives,

$$m + 2n = 3.$$

Consider now the coefficient

$$J = \frac{h_{xx} - h_t}{T^q \psi_x{}^2} = r(r - 1 + n)X^{r-2}T^{s-q-2} - \frac{bs}{a^2}X^{r-2n}T^{s-2n-q}.$$

We have many possibilities for J:

(i) If $r + n - 1 = s = 0$, then $J = 0$. In this case, $h \neq 0$ in contrast to previous equations.

(ii) If $r + n - 1 = 0$, $s \neq 0$ then $r = s - q$, which implies that $h = X^{s-q}T^{s}$ or $h = 0$ without loss of generality, as

$\phi = T^q g(\psi) + X^{s-q}T^s = T^q(g(\psi) + \psi^{s-q}) = T^q g(\psi)$.

(iii) If $r + n - 1 \neq 0$, but $s = 0$, then $r = s - q$ and $h = 0$ without loss of generality as in case (ii).

(iv) If $r + n - 1 \neq 0$, $s \neq 0$, then $r = s - q$ and $h = 0$.

In all these above cases, it can be easily seen that eq. (3.2-22) does not vary since $J = 0$ for all the cases, and it will be written as,

$$g'' + \left(\frac{\psi_{xx}}{\psi_x^2} - \frac{\psi_t}{\psi_x^2}\right)g' - bq\frac{T^{m-1}}{\psi_x^2}g = 0.$$

Substituting $\psi = XT$, $X' = aX^n$, $T' = bT^{3-2n}$ gives

$$g'' + \left(\frac{n}{\psi} - \frac{b}{a^2}\psi^{1-2n}\right)g' - \frac{b}{a^2}q\psi^{-2n}g = 0. \qquad (3.2\text{-}23)$$

<u>Solution of (3.2-23); n = 1</u>

Eq. (3.2-23), if $n = 1$, can be transformed to constant coefficient equation

$$\ddot{g} - \frac{b}{a^2}\dot{g} - \frac{bq}{a^2}g = 0, \qquad (3.2\text{-}24)$$

where $g = g(s)$, $\dot{g} = \frac{dg}{ds}$, $s = \ln\psi$.

(3.2-24) is a linear, constant coefficient equation, where the general solution to (3.2-24) is the exponential one. i.e.,

$$g(s) = Ae^{m_1 s} + Be^{m_2 s},$$

where A and B are constsnts; m_1 and m_2 are the roots of $\left(m^2 - \frac{b}{a^2}m - \frac{bq}{a^2} = 0\right)$, or

$$g(\psi) = A\psi^{m_1} + B\psi^{m_2}.$$

Therefore,

$$\phi(x, t) = e^{bqt}\left[Ae^{m_1(ax+bt)} + Be^{m_2(ax+bt)}\right] + h(x, t) \tag{3.2-25}$$

where h is arbitrary, is a separable solution to (3.2-20).

Since $(am_1)^2 = bq + bm_1$, then letting $h \equiv 0$, $B = 0$ in (3.2-25) gives the exponential solution $Ae^{\alpha x+\alpha^2 t}$, $\alpha = m_1 a$ which is mentioned earlier.

3.3 General Form of the Transformation

As might be expected, if one has to find all implicitly separable solutions to a partial differential equation, then the general form of the transformation has to be sought. This is not an easy task in general, in addition to the fact that generalizing (3.2-2) a little more seems to be not practical in many cases. For instance if (3.2-2) is replaced by

$$\phi(x, t) = u_0(x, t) + u_1(x, t)g(\psi) + u_2(x, t)g^2(\psi)$$

then we need to find some extra relations for $u_i(x, t)$ $(i = 0, 1, 2)$ so that u_i $\forall i$ can be found and such that $g(\psi)$ satisfies an ordinary differential equation. Hence generalizing (3.2-2) means that introducing many unknown functions of x and t or many functions of ψ: $g_1, g_2, \ldots, g_n$, which in fact, make the problem too complicated. However, in what follows an attempt is made to indicate the most generalized form of (3.2-2), under a restricted hypothesis.

Consider, the general dependent and independent variable transformation

$$\phi(x, t) = G(x, t, \psi_1(x, t), \psi_2(x, t), \ldots, \psi_n(x, t)).$$

Clearly it is not practical to choose $n > 1$ for the resulting equation will be more complicated, and one needs n functions for each $i = 1, \ldots, n$. Thus, we consider the transformation,

$$\phi(x, t) = G(x, t, \psi, \psi_\alpha), \tag{3.3-1}$$

where ψ_α denotes any derivative of ψ with respect to x or t, applied on the p.d.e. (3.2-1) such that the transformed p.d.e. in $\psi = X(x)T(t)$ is simply separable. If $X' = aX^n$, $T' = bT^m$ for some constants n, m, a and b, then it can be easily proved that every derivative of ψ with respect to x or t is of the form $\gamma T^\alpha \psi^\beta$ for some constants α, β and γ as follows:

$$\psi_{x^p t^q} = X^{(p)} T^{(q)} \qquad p,\ q \in \mathbb{N}.$$

Then by lemma (5) (appendix),

$$\psi_{x^p t^q} = AX^{pn-p+1} T^{qm-q+1} \tag{3.3-2}$$

where A is a constructive constant.

It then follows easily, that

$$\psi_{x^p t^q} = A\psi^{pn-p+1} T^{qm-pn-q+p}.$$

Hence every derivative of ψ is a function of ψ and T, i.e., the transformation (3.3-1) is implicitly of the form,

$$\phi(x,\ t) = G(X(x),\ T(t),\ \psi) \tag{3.3-3}$$

under the hypothesis that $X' = aX^n$, $T' = bT^m$.

If the assumed transformation (3.3-3), is substituted into the p.d.e. $H(\phi(x,\ t)) = 0$, a p.d.e. for G will result, so that the original problem is <u>not</u> simplified. It can be shown that, in order for o.d.e.'s to result for functions of ψ, (3.3-1) is a sum of separable functions of the form,

$$\phi(x,\ t) = G(X,\ T,\ \psi) = \sum_i u_i^{\,1}(X) u_i^{\,2}(T) g_i(\psi). \tag{3.3-4}$$

(Basically, dealing with p.d.e.'s of polynomial type, so that the transformation can be more general than $\phi = g(\psi)$, imposes a necessary condition for the transformation to have "polynomial properties".)

Although (3.3-4) seems to be a more general transformation of ϕ, it is very difficult to deal with in practice, as it involves many unknown functions $u_i(X, T)$, which have to be identified to construct the o.d.e.'s for the g_i. Two unknown functions in (3.3-4) seems to be a reasonable assumption, as we had shown in the previous section.

3.4 P.d.e.'s with Explicit Dependence on x or t, and the Separation Technique

In applying the separation technique directly to equations with no explicit dependence on the independent variables x or t, we have indicated that, using constants coefficient infinite series for $X'(x)$ and $T'(t)$ is a reasonable approach for finding solutions although some of the solutions for that case might be lost. However for equations with explicit dependence on x or t, clearly the variable coefficients infinite series must not be modified. This severer condition gives rise to many difficulties in solving such equations, by the separation technique. For instance, the following p.d.e. is not directly separable unless it satisfies a special condition:

Consider

$$\phi_{xx} + \gamma\phi\phi_x + A(x, t)\phi_x + B(x, t)\phi + \phi_t = 0. \qquad (3.4\text{-}1)$$

Substituting $\phi = X(x)T(t)$, $X' = \sum_{n=0}^{\infty} a_n(x)X^{n+\lambda}$,

$T' = \sum_{n=0}^{\infty} b_n(t)T^{n+\rho}$ in (3.4-1) gives,

$$T\left[\left(\sum_{n=0}^{\infty}(n + \lambda)a_n(x)X^{n+\lambda-1}\right)\left(\sum_{n=0}^{\infty}a_n(x)X^{n+\lambda}\right) + \right.$$

$$\left.\left(\sum_{n=0}^{\infty}a_n'(x)X^{n+\lambda}\right)\right] + \gamma T^2 X\left(\sum_{n=0}^{\infty}a_n(x)X^{n+\lambda}\right) +$$

$$A(x, t)T\left(\sum_{n=0}^{\infty}a_n(x)X^{n+\lambda}\right) + B(x, t)XT + X\left(\sum_{n=0}^{\infty}b_n(t)T^{n+\rho}\right) = 0.$$

Examining the coefficients of lowest powers of X and T leads to $\lambda = \rho = 0$.

If we consider the coefficients of X^nT^m $\forall n, m$ we get the following relations:

$b_0 = b_3 = \ldots = 0$ (from coefficient of XT^m
$m = 0, 3, 4, \ldots$)

$a_1 = a_2 = \ldots = 0$ (from coefficient of $X^{n+1}T^2$, $n \geq 1$)

$\gamma a_0(x) + b_2(t) = 0$ (from coefficient of XT^2).

The third relation implies that a_0 and b_2 are both constants. Eq. (3.4-1) can be written now, as

$$\gamma a_0 XT^2 + A(x, t)a_0 T + B(x, t)XT + b_1 XT + b_2 XT^2 = 0.$$

If we carry on equating to zero the coefficients of X^nT^m, $(n = 0, 1; m = 1, 2)$, then the following conditions will be imposed on A and B: $A(x, t) \equiv 0$, and $B(x, t)$ is a function of t only and equal to $(-b_1(t))$. On the other hand substituting $X = a_0 x + c$, where c is arbitrary constant, and equating to zero, the coefficient of T^n ($n = 1, 2$) gives,

$$A(x, t)a_0 + (a_0 x + c)(B(x, t) + b_1) = 0.$$

In addition to the equating coefficient technique, used above to find the relations between the coefficients, there are many other possibilities. Consider for example the generalized Laplace's equation,

$$a(x, t)\phi_{xx} + b(x, t)\phi_{tt} = 0. \tag{3.4-2}$$

This equation can not be solved by the classical linear separation technique unless $b(x, t)/a(x, t)$ is a separable function of x and t. The o.d.e.'s obtained by the separation $\phi = XT$ are,

$$A(x)X'' = -\lambda X, \quad B(t)T'' = \lambda T, \tag{3.4-3}$$

where λ is the separation constant, $\frac{A(x)}{B(t)} = \frac{a(x, t)}{b(x, t)}$.

Assume now that,

$$X'' = \sum_{n=0}^{\infty} a_n(x)X^n, \quad T'' = \sum_{n=0}^{\infty} b_n(t)T^n \tag{3.4-4}$$

which when substituted in (3.4-2) (with $\phi = XT$) gives,

$$a(x, t)T\left(\sum_{n=0}^{\infty} a_n(x)X^n\right) + b(x, t)X\left(\sum_{n=0}^{\infty} b_n(t)T^n\right) = 0. \tag{3.4-5}$$

One possibility, here is that $a_0 = a_2 = \ldots = 0$, $b_0 = b_2 = \ldots = 0$, which gives

$$a(x, t)a_1(x) = -b(x, t)b_1(t),$$

and the separable solution for this case is the classical separable solution above.

However, there are other possibilities of solutions obtained by other means (i.e. by <u>not</u> equating coefficients of $X^n T^m$). For instance, suppose that,

$$X'' = a_0(x), \quad T'' = b_0(t) \tag{3.4-6}$$

(as a special case of infinite series (3.4-4)). Then (3.4-5) reduces to,

$$aTa_0(x) + bXb_0(t) = 0.$$

Once again, it is fairly clear that a necessary condition is that $\frac{a(x, t)}{b(x, t)} = \frac{A(x)}{B(t)}$, whence,

$$\frac{A(x)a_0(x)}{X} = \frac{-B(t)b_0(t)}{T} = \frac{1}{\eta}$$

where η is a constant (non-zero). Thus,

$$X(x) = \eta A(x)a_0(x); \quad T(t) = -\eta B(t)b_0(t). \tag{3.4-7}$$

Differentiating the above expressions (3.4-7) and comparing with (3.4-6) give differential equations for $a_0(x)$, $b_0(t)$ in terms of $A(x)$, $B(t)$ respectively and so on.

The basic point above is that, for equations with explicit dependence on the independent variables, because comparing coefficients of X^nT^m is not the only possibility for providing solutions, the problem is a lot more complicated than in the case where no explicit dependence on x or t occurs. Given a large number of different alternatives for finding solutions, it is difficult to find any hard and fast rules for determining which alternative is the best to use etc. This is not only the difficulty in this case; using comparing coefficient to find the solution, the recurrence relations for a_n and b_n are, in general, differential equations, rather than algebraic equations. For eq. (3.4-2) letting $X' = \sum a_n X^n$, $T" = \sum b_n T^n$ gives $b_0 = b_2 = \ldots = 0$ and the following differential equations:

$$\left.\begin{array}{l}
a_0a_1 + a_0' = 0 \\
a(2a_0a_2 + a_1^2 + a_1') + bb_1 = 0 \\
3a_0a_3 + 3a_1a_2 + a_2' = 0 \\
\vdots \\
(n + 1)a_0a_{n+1} + na_1a_n + \ldots + a_na_1 + a_n' = 0 \\
\vdots
\end{array}\right| .$$

Moreover, if a_n (or b_n) are not constant then uniform convergence of infinite power series is not guaranteed.

3.5 The Application of (3.2-1) to a Variable Coefficients Equation

It has been shown [84], that solutions of KdV-type equations with variable coefficients, can be obtained using the transformation

$$\phi(x,\ t) = u(x,\ t)g(\psi).$$

We show here that Burgers-type equations with variable coefficients can be transformed to a simply separable p.d.e. using the above transformation.

Consider the equation,

$$\phi_{xx} + \gamma\phi\phi_x + a(x,\ t)\phi_x + b(x,\ t)\phi + \phi_t = 0 \qquad (3.5\text{-}1)$$

where γ is a non-zero constant, and $a(x, t)$, $b(x, t)$ are defined as follows:

$$\left.\begin{array}{ll}
\text{(i)} & a(x,\ t) = e^{x-t} + 1, \quad b(x,\ t) = e^{t-x} \\
\text{(ii)} & a(x,\ t) = \left(\frac{1}{2}x + 1\right)\frac{1}{t}, \quad b(x,\ t) = \frac{1}{t} \\
\text{(iii)} & a(x,\ t) = 3t^2, \quad b(x,\ t) = h(-x + t^3) \\
\text{(iv)} & a(x,\ t) = \frac{x+1}{t}, \quad b(x,\ t) = \frac{1}{t}.
\end{array}\right\}$$

Applying the transformation $\phi(x, t) = u(x, t)g(\psi)$, with $u_x = 0$, (so that u is a function of t only) for simplicity, on (3.5-1) and dividing by $u^2\psi_x$ (which is non-zero for non-trivial solutions) gives,

$$\frac{\psi_x}{u}g'' + \left(\frac{\psi_{xx}}{u\psi_x} + \frac{a(x,\ t)}{u} + \frac{\psi_t}{u\psi_x}\right)g' + \left(\frac{b(x,\ t)}{u\psi_x} + \frac{u_t}{u^2\psi_x}\right)g$$

$$+ \gamma gg' = 0. \tag{3.5-2}$$

Every coefficient of g and its derivatives in (3.5-2) must be a function of ψ, for (3.5-2) to become an o.d.e. in $g(\psi)$. In particular, the coefficient of g" i.e., $\frac{\psi x}{u}$ must be a function of ψ, $f(\psi)$ say, so that,

$$u = \frac{\psi_x}{f(\psi)}. \tag{3.5-3}$$

Since $u_x = 0$, differentiation of (3.5-3) with respect to x implies,

$$f(\psi)X'' - X^{2\prime}f'(\psi)T = 0,$$

which gives,

$$\frac{X''}{X^{2\prime}T} = h(\psi) = \frac{f'(\psi)}{f(\psi)}$$

where $h(\psi)$ is some function of ψ. Because of T in $\frac{X''}{X^{2\prime}T}$, the only possibility for $h(\psi)$ is $\frac{n}{\psi}$ for some constant n. Then $\frac{X''}{X^{2\prime}T} = \frac{n}{XT} = \frac{f'}{f}$ and hence, $f = c\psi^n$ for some constant c and,

$$X' = k_1X^n, \tag{3.5-4(a)}$$

$$u(x,\ t) = T^{1-n} \tag{3.5-4(b)}$$

where k_1 is a constant.

The correspondong o.d.e. (3.5-2) then becomes,

$$k_1\psi^n g'' + (nk_1\psi^{n-1} + I)g' + (J)g + \gamma gg' = 0 \qquad (3.5\text{-}5)$$

where

$$I = \frac{a(x,\ t)}{u} + \frac{\psi_t}{u\psi_x}, \qquad (3.5\text{-}6)$$

and,

$$J = \frac{b(x,\ t)}{u\psi_x} + \frac{u_t}{u^2\psi_x}. \qquad (3.5\text{-}7)$$

Let u(x, t) and X' be given by (3.5-4). Two cases will appear represented by the following lemmas:

<u>Lemma (3.5-1)</u>

I is a non-zero function of ψ iff,

$$a(x,\ t) = T^{1-n}f(\psi) \qquad (3.5\text{-}8(a))$$

$$T' = k_2T^{3-2n} \qquad (3.5\text{-}8(b))$$

$$b(x,\ t) = k_1X^nT^{2-n}h(\psi) \qquad (3.5\text{-}8(c))$$

where k_2 is a constant, and $f(\psi)$ and $h(\psi)$ are arbitrary functions of ψ. $\square$

<u>Proof</u> Suppose that I is a function of ψ, i.e., a(x, t)/u

and $\psi_t/a\psi_x$ are functions of ψ. This implies that,

$$a(x, t) = T^{1-n}f(\psi) \quad \text{and} \quad T' = k_2T^{3-2n},$$

for some function $f(\psi)$ and a constant k_2.

These conditions together with (3.5-4), when applied to the coefficient of g in (3.5-2) gives,

$$J = \frac{b(x, t)}{k_1T^{2-n}X^n} + \frac{k_2}{k_1}(1 - n)\psi^{-n} \tag{3.5-9}$$

and consequently, as it is necessary for J to be a function of ψ,

$$b(x, t) = k_1X^nT^{2-n}h(\psi)$$

for some function $h(\psi)$.

It can be easily proved now that I is a function of ψ if (3.5-8) are satisfied. I is given by,

$$I = f(\psi) + \frac{k_2}{k_1}\psi^{1-n}. \blacksquare \tag{3.5-10}$$

We calculate, in this case, that (3.5-2) will be the o.d.e.

$$k_1\psi^ng'' + \left[nk_1\psi^{n-1} + f(\psi) + \frac{k_2}{k_1}\psi^{1-n}\right]g' +$$
$$\left[h(\psi) + \frac{k_2}{k_1}(1 - n)\psi^{-n}\right]g + \gamma gg' = 0 \tag{3.5-11}$$

iff the conditions (3.5-4) and (3.5-8) are satisfied.

Eq. (3.5-11) is integrable iff $f'(\psi) = h(\psi)$ and its first integral is,

$$k_1\psi^n g' + \frac{\gamma}{2}g^2 + \frac{k_2}{k_1}\psi^{1-n}g + f(\psi)g = K \qquad (3.5\text{-}12)$$

(K, integration constant).

Example (3.5-2)

If $n = 1$, $X = ce^x$, and $T = \frac{1}{c}e^{-t}$ where c is a non-zero constant, then $\psi(x, t) = e^{x-t}$ and

$$a(x, t) = f(x - t)$$

$$b(x, t) = h(x - t)$$

Thus, eq. (3.5-1)(i) is implicitly separable. One choice of the transformation g (when $K = 0$) is,
$g(\psi) = \left(e^{-\psi}\int\frac{e^{-\psi}}{\psi}d\psi\right)^{-1}$, c is a constant. On the other hand, if $n = 0$, $k_1 = 1/2$, $k_2 = -1/2$ then

$$X(x) = \frac{1}{2}x + 1, \quad T(t) = \frac{1}{\sqrt{t}} \quad \text{and,}$$

$$a(x, t) = \frac{1}{t}\left(\frac{1}{2}x + 1\right) = T\psi$$

$$b(x, t) = \frac{1}{t}$$

satisfy conditions (3.5-8) too. Thus eq. (3.5-1)(ii) is implicitly separable. One choice of the transformation g (when $K = 0$) is $g(\psi) = (\gamma\psi + c)^{-1}$, where c is an arbitrary constant.

Lemma (3.5-3)

I is identically zero in (3.5-5) iff

$$a(x, t) = -\frac{1}{k_1}X^{1-n}f(t) \qquad (3.5\text{-}13(a))$$

and

$$T' = f(t)T \qquad (3.5\text{-}13(b))$$

where f(t) is an integrable function of t. □

Proof Clearly (3.5-13) implies that $I \equiv 0$. Now, if $I \equiv 0$ then

$$a(x, t) = -\frac{1}{k_1}X^{1-n}\,\frac{T'}{T},$$

where $\frac{T'}{T}$ is a function of t (say f(t)), hence (3.5-13) hold. ■

In this case, examining the coefficient of g gives that,

$$n = 1 \Rightarrow b(x, t) = h(\psi)$$

$$n \neq 1 \Rightarrow b(x, t) = \begin{cases} k_1 T^{2-n} X^n \ell(\psi) \\ (n - 1)f(t) \end{cases} \qquad (3.5\text{-}14)$$

As a conclusion for this case, the following o.d.e., exists iff conditions (3.5-4) and (3.5-13) with (3.5-14) are satisfied:

$$k_1\psi^n g'' + nk_1\psi^{n-1}g' + \lambda(\psi)g + \gamma gg' = 0. \qquad (3.5\text{-}15)$$

where

$$\lambda(\psi) = \begin{cases} \dfrac{1}{k_1}\dfrac{h(\psi)}{\psi} & \text{if} \quad n = 1 \quad \text{and} \quad b(x, t) = h(\psi) \\[2ex] 2(\psi) + \dfrac{1 - n}{k_1}\psi^{-n} & \text{if} \quad n \neq 1, \\ & \text{and} \quad b(x, t) = k_1 T^{2-n} X^n \ell(\psi) \\[2ex] 0 & \text{if} \quad n \neq 1, \quad b(x, t) = (n - 1)f(t) \end{cases} \tag{3.5-16}$$

In the second case, $f(t)$ will be T^{2-2n} which implies that $f(t)$ is a constant without loss of generality.

It can be easily shown that eq. (3.5-15) is integrable iff $\lambda(\psi) \equiv 0$. Its first integral is

$$k_1\psi^n g' + \frac{\gamma}{2}g^2 = K, \tag{3.5-17}$$

where K is an integration constant.

Suppose we choose $n = 1$, $k_1 = -1$ in (3.5-4(a)) and $f(t) = 3t^2$. Then $a(x, t) = 3t^2$, $X(x) = ce^{-x}$ and by (3.5-13(b)), $T(t) = de^{t^3}$. Choosing $d = \frac{1}{c}$ gives $\psi = e^{-x+t^3}$ and then (3.5-14) implies that $b(x, t)$ is arbitrary function of $(-x + t^3)$. This shows, in particular, that (3.5-1)(iii) is implicitly separable if $b(x, t)$ is identically zero.

Suppose we now choose $n \neq 1$, $k_1 = -1$, $f(t) = \frac{1}{t}$ and $X^{1-n} = (n - 1)x + 1$ in (3.5-4(a)). Then $a(x, t) = \dfrac{1 + x(n - 1)}{t}$ and by (3.5-14) $b(x, t) = \dfrac{n - 1}{t}$. Thus, in particular, (3.5-1)(iv) is implicitly separable.

(In this case, $T(t) = ct$.)

CHAPTER FOUR

Classes of Equations, not Solvable by the Inverse Scattering Transform

4.1 Introduction

In this chapter, we report on some general classes of nonlinear p.d.e.'s and point out that these equations cannot be solved by the inverse scattering transform (IST). These classes are obtained, on the basis that the Ablowitz, Ramani and Segur (ARS) conjecture is true. Several years ago, Ablowitz, Ramani and Segur conjectured that every nonlinear p.d.e., solvable by IST is directly reducible to a Painlevé type o.d.e., (i.e., if a p.d.e. is directly reducible to an o.d.e. which is not of a Painlevé type then the p.d.e. is not solvable by IST).

An o.d.e. is said to be of the Painlevé type (P-type) if all its solutions possess the Painlevé property. The so called P-property of an o.d.e. (P for Painlevé) which becomes a condition on whether a given p.d.e. is not solvable by IST, is the property that the solutions have no movable critical points (critical points meaning branch points or essential singularities) and this indicates that the only movable singularities of all the solutions are poles [44]. All linear o.d.e.'s have the P-property as all solutions have fixed singularities. The only first order P-type nonlinear o.d.e. is the generalized Ricatti equation [27, 51]

$$\frac{dw}{dz} = p_0(z) + p_1(z)w + p_2(z)w^2.$$

For the second order o.d.e., it is known that there are fifty canonical P-type equations including the defining equations for the six Painlevé transcendents [24]. Recently, Ablowitz, Ramani and Segur presented an explicit algorithm to test whether a given o.d.e., satisfies certain necessary conditions for it to be of a P-type [6]. This algorithm has the advantage of applying to an o.d.e. of any order in contrast to the procedure explained in [44].

In the following sections, the implicit separation technique will be used to obtain classes of second order polynomial type p.d.e.'s with constant coefficients, which are not solvable by IST, from their obtainable o.d.e.'s which are not of P-type. For higher order p.d.e.'s we apply the Ablowitz algorithm to test whether KdV-type p.d.e.'s are solvable by IST in the next chapter.

We first obtain the general form of a second order p.d.e. of polynomial type, which can be analyzed to give the general Painlevé o.d.e.,

$$g'' = L(\psi, g)g'^2 + M(\psi, g)g' + N(\psi, g), \qquad (4.1\text{-}1)$$

which represents the first condition [44] for a second order o.d.e. to have fixed critical points.

To obtain the general p.d.e. for $\phi(x, t)$ which is reduced to eq. (4.1-1), we use the simple separation transformation $\phi = g(\psi)$, where $\psi = X(x)T(t)$, $X' = aX$, $T' = bT$, a and b are constants. Since the Ablowitz conjecture needs one direct transformation to disprove nonsolvability of a p.d.e., there is no benefit to consider a more general transformation than $\phi = g(\psi)$, or to consider a more general case than the 'guaranteed' case above.

We will use the facts:

$$\left.\begin{array}{llll} \phi_x = a\psi g' \quad (a), & \phi_{xx} = a^2(\psi g' + \psi^2 g'') & (c), \\ \phi_t = b\psi g' \quad (b), & \phi_{tt} = b^2(\psi g' + \psi^2 g'') & (d), \\ & \phi_{xt} = ab(\psi g' + \psi^2 g'') & (e), \end{array}\right\} \quad (4.1\text{-}2)$$

which can be easily proved. [It is clear that any derivative of ϕ in (4.1-2) can be mapped to $(\alpha\Phi + \beta\Phi^*)$, where $\Phi = \psi g'$, $\Phi^* = \psi^2 g''$ and α and β are constants.] Therefore the general second order p.d.e., with no explicit (x, t) dependence,

$$H(\phi_{xx}, \phi_{xt}, \phi_{tt}, \phi_x, \phi_t, \phi) = 0 \quad (4.1\text{-}3)$$

is mapped to the o.d.e. in $g(\psi)$,

$$H(\Phi, \Phi^*, g) = 0, \quad (4.1\text{-}4)$$

by the transformation $\phi = g(\psi)$.

Now, if eq. (4.1-4) is of P-type, it is necessarily of the form (4.1-1). Thus in order that (4.1-3) is directly reducible to an equation of P-type, it is necessarily of the form (either explicitly or implicitly),

$$\sum_{i=1}^{N} P_i(\phi)\, \phi_{xx}^{\alpha_i}\, \phi_{xt}^{\beta_i}\, \phi_{tt}^{\gamma_i}\, \phi_x^{\delta_i}\, \phi_t^{\varepsilon_i} = 0, \tag{4.1-5}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i, \varepsilon_i \in \mathbb{Z} \geqslant 0\ \forall i$.

Eq. (4.1-5) can be written as,

$$F(\phi_{xx}, \ldots, \phi) + G(\phi_{xx}, \ldots, \phi) = 0, \tag{4.1-6}$$

where,

$$\left.\begin{aligned} F(\phi_{xx}, \ldots, \phi) &= \sum_{i=1}^{k} P_i(\phi)\, \phi_{xx}^{\alpha_i}\, \phi_{xt}^{\beta_i}\, \phi_{tt}^{\gamma_i}\, \phi_x^{\delta_i}\, \phi_t^{\varepsilon_i} \\ G(\phi_{xx}, \ldots, \phi) &= \sum_{i=k+1}^{N} P_i(\phi)\, \phi_{xx}^{\alpha_i}\, \phi_{xt}^{\beta_i}\, \phi_{tt}^{\gamma_i}\, \phi_x^{\delta_i}\, \phi_t^{\varepsilon_i} \end{aligned}\right\} \tag{4.1-7}$$

and

$$F(\Phi, \Phi^*, g) = 0. \tag{4.1-8}$$

<u>Lemma (4.1-1)</u>

$$F(\Phi, \Phi^*, g) = 0$$

if

$$
\left.
\begin{aligned}
2\alpha_i + \beta_i + \delta_i &= 2\alpha_j + \beta_j + \delta_j && \text{(a)}\\
2\gamma_i + \beta_i + \varepsilon_i &= 2\gamma_j + \beta_j + \varepsilon_j && \text{(b)}\\
\alpha_i + \beta_i + \gamma_i &= \alpha_j + \beta_j + \gamma_j && \text{(c)}\\
\delta_i + \varepsilon_i &= \delta_j + \varepsilon_j && \text{(d)}\\
\sum_{i=1}^{k} P_i(\phi) &= 0. \;\square && \text{(e)}
\end{aligned}
\right\} \qquad (4.1\text{-}9)
$$

Proof Trivial. ∎

So we consider (4.1-5) as F + G = 0, where F satisfies (4.1-9) and G does not. Then any property of the equation G = 0 in the sense of a property of the resulting o.d.e. will be a property of the p.d.e., F + G = 0, (i.e., if G = 0 is not solvable by IST, then neither is F + G = 0).

Now, without loss of generality, we can take P_i, α_i, β_i, γ_i, δ_i and ε_i not to satisfy (4.1-9) for each i for the eq. (4.1-5). Then we have the following lemma:

Let

$$
\left.
\begin{aligned}
A_i &= \alpha_i + \beta_i + \gamma_i\\
B_i &= \delta_i + \varepsilon_i
\end{aligned}
\right\} \qquad (4.1\text{-}10)
$$

where $\min A_i = \min B_i = 0$. Then

Lemma (4.1-2)

Eq. (4.1-5) is not solvable by IST if there exists j, j = 1, ..., n such that either $A_j > 1$ or $A_j = 1, B_j > 0$ or

$A_j = 0, B_j > 2$. □

Proof Transforming (4.1-5) via $\phi = g(\psi)$ and using (4.1-2) gives the identity,

$$\sum_{i=1}^{N} Q_i(a,\ b,\ g)(\psi g' + \psi^2 g")^{A_i}(\psi g')^{B_i} = 0, \qquad (4.1\text{-}11)$$

where $Q_i = a^{2\alpha_i+\beta_i+\delta_i}\, b^{2\gamma_i+\beta_i+\varepsilon_i}\, P_i(g)$; A_i and B_i are as defined above $\forall i$.

Comparing (4.1-11) with (4.1-1) imposes the condition that, $A_i \leqslant 1$. If $A_i = 1$ then $B_i = 0$ or if $A_i = 0$ then $B_i \leqslant 2$, for each $i = 1, \ldots, n$.

Thus if $\exists j = 1, \ldots, n$ for which these conditions do not hold applying Ablowitz conjecture implies that eq. (4.1-5) is not solvable. ∎

NB The following sections are devoted to investigating the remaining p.d.e.'s of the form (4.1-5) where the conditions of lemma (4.1-2) are not satisfied.

4.2 General Quasilinear Equations

We first consider the case in which eq. (4.1-5) is reducible to eq. (4.1-1) where $L \equiv 0$.

Lemma (4.2-1)

Eq. (4.1-5) is reducible to eq. (4.1-1), where $L \equiv 0$ iff it is of the general quasilinear form

$$P_1(\phi)\phi_{xx} + P_2(\phi)\phi_{tt} + P_3(\phi)\phi_{xt} + P_4(\phi)\phi_x + P_5(\phi)\phi_t +$$
$$P_6(\phi) = 0. \quad \square \tag{4.2-1}$$

<u>Proof</u> By the proof of lemma (4.1-2), it is necessary that $\alpha_i + \beta_i + \gamma_i = 1$ with $\delta_i = \varepsilon_i = 0$, or, $\alpha_i + \beta_i + \gamma_i = 0$ and $\delta_i + \varepsilon_i \leq 1$, $\forall i$ in eq. (4.1-5).

Relabelling, this is precisely an equation of the form (4.2-1). ∎

Thus, in this case we consider p.d.e.'s of the form (4.2-1). The method to be adopted for investigating whether (4.2-1) is solvable by IST, breaks up into many distinct stages. The procedure follows from the technique of the Painlevé test for an o.d.e., explained in [44]. At every stage, a set of equations which do not satisfy a necessary condition is derived and then excluded from the whole set (4.2-1). This procedure does not imply that the final remaining set of equations are solvable by IST. In fact, the conditions which have been derived are to identify the sets of equations which fail in any stage of the test, i.e., to find sets of equations which are not solvable by IST according to Ablowitz et al conjecture.

In this case the basic general o.d.e., we consider, is eq. (4.1-1) where $L(\psi, g)$ is identically zero. We will explain in detail how the first condition for the p.d.e. (4.2-1) will be derived according to the necessary conditions of the P-test for the o.d.e. (4.1-1) (with $L \equiv 0$). The investigation has thus to be continued to further stages.

Let

$$\phi(x, t) = g(\psi(x, t)), \tag{4.2-2}$$

be applied to eq. (4.2-1). This gives the following o.d.e., if $\psi = X(x)T(t)$, $X' = aX$ and $T' = bT$ using (4.1-2)

$$a^2P_1(g)(\psi^2g'' + \psi g') + b^2P_2(g)(\psi^2g'' + \psi g') +$$
$$abP_3(g)(\psi^2g'' + \psi g') + aP_4(g)(\psi g') + bP_5(g)(\psi g') + P_6(g) = 0,$$

which can be simplified to

$$g'' = -\frac{a^2P_1 + b^2P_2 + abP_3 + aP_4 + bP_5}{a^2P_1 + b^2P_2 + abP_3}\psi^{-1}g' -$$
$$\frac{P_6}{a^2P_1 + b^2P_2 + abP_3}\psi^{-2}. \tag{4.2-3}$$

The second set of necessary conditions [Ince, p.326] shows that the eq. (4.2-3) is necessarily of the form

$$g'' = (A(\psi)g + B(\psi))g' + C(\psi)g^3 + D(\psi)g^2 + E(\psi)g + F(\psi), \tag{4.2-4}$$

if it is of P-type.

It follows directly that

$$\frac{a^2P_1(g) + b^2P_2(g) + abP_3(g) + aP_4(g) + bP_5(g)}{a^2P_1(g) + b^2P_2(g) + abP_3(g)} =$$
$$(A(\psi)g + B(\psi))\psi, \tag{4.2-5(a)}$$

and

$$\frac{P_6(g)}{a^2P_1(g) + b^2P_2(g) + abP_3(g)} =$$

$$(C(\psi)g^3 + D(\psi)g^2 + E(\psi)g + F(\psi))\psi^2. \qquad (4.2\text{-}5(b))$$

Differentiating both sides of (4.2-5(a)) partially with respect to ψ implies that

$$A(\psi) = \frac{k_1}{\psi} \quad \text{and} \quad B(\psi) = \frac{k_2}{\psi}, \qquad (4.2\text{-}6)$$

where k_1 and k_2 are arbitrary constants.

Similarly, (4.2-5(b)) gives,

$$C(\psi) = \frac{k_3}{\psi^2},\ D(\psi) = \frac{k_4}{\psi^2},\ E(\psi) = \frac{k_5}{\psi^2} \text{ and } F(\psi) = \frac{k_6}{\psi^2}, \qquad (4.2\text{-}7)$$

where the k_i's (i = 3, 4, 5, 6) are arbitrary constants.

At this stage the first set of p.d.e.'s, excluded from the set of all equations of the form (4.2-1), is derived according to (4.2-2), (4.2-5) by the following theorem:

Let

$$\left.\begin{aligned} &R(\phi) = aP_4(\phi) + bP_5(\phi), \\ &\text{and} \\ &S(\phi) = a^2P_1(\phi) + b^2P_2(\phi) + abP_3(\phi) \end{aligned}\right\} \qquad (4.2\text{-}8)$$

Theorem (4.2-2)

Eq. (4.2-1) is not solvable by IST if $\exists$ constants a and b such that, either R/S is not a polynomial of degree $\lesssim 1$;

or P_6/S is not a polynomial of degree $\leqslant 3$. □

Assume thus, in eq. (4.2-1) and $\forall a, b$,

$$\left.\begin{array}{l} R/S \text{ is a polynomial of degree } \leqslant 1 \text{ (say, } k_1\phi + k_2 - 1) \\ P_6/S \text{ is a polynomial of degree } \leqslant 3 \\ \text{(say, } k_3\phi^3 + k_4\phi^2 + k_5\phi + k_6) \end{array}\right\} \quad (4.2\text{-}9)$$

where k_i $(i = 1, \ldots, 6)$ are constants.

This implies that the obtainable o.d.e., is

$$g'' = \frac{1}{\psi}(k_1 g + k_2)g' + \frac{1}{\psi^2}(k_3 g^3 + k_4 g^2 + k_5 g + k_6). \quad (4.2\text{-}10)$$

In order that the general solution of eq. (4.2-10) is free from movable critical points, it is necessary that the equation should be reducible by a substitution of the form,

$$g(\psi) = \lambda(\psi)W(\psi) \quad (4.2\text{-}11)$$

to the equation (4.2-4) where $A(\psi)$ and $C(\psi)$ are constants given in [44] p.328.

<u>Case (a)</u> $(k_1 = k_3 = 0)$

In this case $A(\psi) = C(\psi) = 0$ in (4.2-4). Hence applying the more general transformation,

$$g(\psi) = \lambda(\psi)W(Z) + \mu(\psi), \quad Z = \phi(\psi), \tag{4.2-12}$$

where

$$\lambda = (6/k_4)^{1/5}\,\psi^{2(k_2+1)/5}$$

$$\phi' = (k_4/6)^{2/5}\,\psi^{(k_2-4)/5}$$

$$\mu = (-1/2k_4)(6(k_2+1)^2/25 + k_5)$$

eq. (4.2-10) reduces to

$$W'' = 6W^2 + \overline{S}(\psi), \tag{4.2-13}$$

where

$$\overline{S}(\psi) = (k_4\mu^2 + k_5\mu + k_6)(6/k_4)^{3/5}\,\psi^{-4(k_2+1)/5}.$$

A necessary condition for the o.d.e. (4.2-10) to be Painlevé type is $\overline{S}(\psi)$ is identically zero in (4.2-13), i.e., $k_4\mu^2 + k_5\mu + k_6 = 0$. Then we have the following theorem.

<u>Theorem (4.2-3)</u>

If eq. (4.2-1) satisfies conditions (4.2-9) with $k_1 = k_3 = 0$, then it is not solvable by IST if $(6(k_2 + 1)^2/25)^2 - k_5^2 + 4k_4k_6 \neq 0$. □

<u>Case (b)</u> $(k_1 \neq 0,\ k_3 = 0)$

In this case, we choose $\lambda = \frac{-2}{k_1}\psi$ in (4.2-11). Then

(4.2-10) reduces to,

$$W'' = -2WW' + \left(\frac{k_2 - 2}{\psi}\right)W' - \frac{2}{k_1\psi}(k_1 + k_4)W^2 + \frac{1}{\psi^2}(k_2 + k_5)W$$

$$- \frac{k_1 k_6}{2\psi^3}. \qquad (4.2\text{-}14)$$

The transformation,

$$W(\psi) = w(\psi) + \mu_i, \qquad (4.2\text{-}15)$$

where

$$\mu_i = \begin{cases} \alpha_1/\psi & \text{if} \quad k_1 + 2k_4 \neq 0 \\ \alpha_2 \ln\psi/\psi & \text{if} \quad k_1 + 2k_4 = 0 \end{cases}$$

and where,

$$\left.\begin{aligned} \alpha_1 &= (k_2 + k_5)^{k_1}/2(k_1 + 2k_4) \\ \alpha_2 &= (k_2 + k_5)/2 \end{aligned}\right\}$$

will reduce eq. (4.2-14) to the typical form,

$$w'' = -2ww' + p(\psi)w' + Q(\psi)w^2 + S(\psi) \qquad (4.2\text{-}16)$$

where,

$$\left.\begin{aligned}
p(\psi) &= \left(\frac{k_2 - 2}{\psi} - 2\mu_i\right) \\
Q(\psi) &= -2\left(1 + \frac{k_4}{k_1}\right)\psi^{-1} \\
S(\psi) &= -2\mu\mu_i' + \frac{k_2 - 2}{\psi}\mu_i' - \frac{2}{\psi}\left(1 + \frac{k_4}{k_1}\right)\mu_i^2 + \\
&\quad \frac{1}{\psi^2}(k_2 + k_5)\mu_i - \frac{k_1 k_6}{2\psi^3} - \mu_i''.
\end{aligned}\right\}.$$

Eq. (4.2-16) is free from movable critical points if $p(\psi) = Q(\psi)$. Hence we have the following theorem:

Theorem (4.2-4)

If eq. (4.2-1) satisfies conditions (4.2-9) with $k_3 = 0$, $k_1 \neq 0$ then:

(i) If $k_1 + 2k_4 = 0$ then the equation is not solvable by IST if $k_2 \neq -k_5$ for $k_2 \neq 1$.

(ii) If $k_1 + 2k_4 \neq 0$ then the equation is not solvable by IST if $2k_1k_4(k_2 + 1) - k_1^2k_5 + 4k_4^2 \neq 0$. □

Case (c) ($k_1 \neq 0$, $k_3 \neq 0$)

In this case eq. (4.2-10) is not reducible to an equation of the form (4.2-4), where $A(\psi) = -3$, $C(\psi) = -1$ unless $k_1^2 = -9k_3$; or $A(\psi) = -1$, $C(\psi) = 1$ unless $k_1^2 = k_3$ which gives:

Theorem (4.2-5)

If eq. (4.2-1) satisfies conditions (4.2-9) with $k_1^2 \neq -9k_3$ and $k_1^2 \neq k_3$, then the equation is not solvable by IST. □

Subcase c(i) ($k_1^2 = -9k_3$)

In this subcase we choose $\lambda = -\frac{3}{k_1}\psi$ in (4.2-11). Then (4.2-10) reduces to

$$W'' = \left(-3W + \frac{k_2 - 2}{\psi}\right)W' - W^3 - \frac{3}{\psi}\left(1 + \frac{k_4}{k_1}\right)W^2 + \left(\frac{k_2 + k_5}{\psi^2}\right)W - \frac{k_1 k_6}{3\psi^3}. \tag{4.2-17}$$

The typical equation in this subcase, whose general solution has only fixed critical points is

$$w'' = -3ww' - w^3 + q(\psi)(w' + w^2) \tag{4.2-18}$$

which can be obtained from (4.2-15) by applying the transformation

$$W(\psi) = \lambda(\psi)w(z) + \mu(\psi), \quad z = \phi(\psi) \tag{4.2-19}$$

where λ and μ may be computed as in other cases, only if $k_1(k_2 + 1) + 3k_4 = 0$. Then we have the theorem,

Theorem (4.2-6)

If eq. (4.2-1) satisfies conditions (4.2-9) and if $k_1 \neq 0$, $k_1^2 = -9k_3$ then the equation is not solvable by IST if $k_2 + 1 + 3\frac{k_4}{k_1} \neq 0$. □

<u>Subcase c(ii)</u> $(k_1^2 = k_3)$

We choose, here, $\lambda = -\frac{1}{k_1}\psi$ in (4.2-11). Then (4.2-10) reduces to

$$W'' = \left(-W + \frac{k_2 - 2}{\psi}\right)W' + W^3 - \frac{1}{\psi}\left(\frac{k_4}{k_1} + 1\right)W^2 + \left(\frac{k_2 + k_5}{\psi^2}\right)W - \frac{k_1k_6}{\psi^3}, \tag{4.2-20}$$

which can be reduced to the typical form

$$w'' = -ww' + w^3 + p(\psi)(3w' + w^2) + R(\psi)w + S(\psi), \tag{4.2-21}$$

where

$$p(\psi) = \frac{\alpha_1}{\psi}, \quad R(\psi) = \frac{\alpha_2}{\psi^2}, \quad S(\psi) = \frac{\alpha_3}{\psi^3},$$

and

$$\left.\begin{aligned} \alpha_1 &= (3k_1k_2 - 7k_1 - k_4)/10k_1 \\ \alpha_2 &= 3\gamma^2 - 2\gamma k_4/k_1 - \gamma + k_2 + k_5 \\ \gamma &= (k_1k_2 + k_1 + 3k_4)/10k_1 \\ \alpha_3 &= \gamma^3 - \gamma^2 k_4/k_1 + \gamma k_5 - k_1k_6 \end{aligned}\right\} \tag{4.2-22}$$

by the transformation $W(\psi) = w(\psi) + \frac{\gamma}{\psi}$.

Eq. (4.2-21) is free from movable critical points if one of the following conditions is satisfied:

(i) $R(\psi) = p'(\psi) - 2p^2(\psi)$, $S(\psi) = 0$,

(ii) $\exists q(\psi) \ni p(\psi) = q'(\psi)/2q(\psi)$,

$$R(\psi) = \frac{q''(\psi)}{2q(\psi)} - \frac{q^2(\psi)}{q^2(\psi)} - q(\psi), \quad S(\psi) = 0,$$

(iii) $\exists q(\psi) \ni p(\psi) = \frac{q'(\psi)}{q(\psi)} + q(\psi)$,

$$R(\psi) = p'(\psi) - 2p^2(\psi) - 12q^2(\psi), \quad S(\psi) = -24q^3,$$

(iv) $\exists q(\psi) = 4\psi^3 - \varepsilon\psi^{-k}$, where $\varepsilon = 0$, $k = 1$, or $\varepsilon = 1$

with k arbitrary such that $p(\psi) = -\frac{2q(\psi)}{q'(\psi)}$,

$$R(\psi) = -\frac{24\psi}{q(\psi)}, \quad S(\psi) = 12/q(\psi),$$

and

(v) $\exists q(\psi) \ni q''(\psi) = 6q^2(\psi) + \psi$ and $p(\psi) = 0$,

$$R(\psi) = -12q(\psi), \quad S(\psi) = 12q'(\psi).$$

Hence imposing these conditions on the p.d.e. (4.2-1) gives the following theorem:

Theorem (4.2-7)

If eq. (4.2-1) satisfies conditions (4.2-9) with $k_1 \neq 0$ and $k_1^2 = k_3$, then the equation is not solvable by IST if no one of the following conditions is satisfied.

$$\left.\begin{array}{l}
\text{(i) } \alpha_2 = -\alpha_1 - 2\alpha_1^2, \quad \alpha_3 = 0 \\
\text{(ii) } \alpha_1 = -1,\ \alpha_2 = k \text{ (k arbitrary constant)},\ \alpha_3 = 0 \\
\text{(iii) } \alpha_1 + 1 = (-\alpha_3/24)^{1/3},\ \alpha_3^2 = -1/3(\alpha_1 + 2\alpha_1^2 + \alpha_2^3)
\end{array}\right\}$$

where α_1, α_2 and α_3 are defined in (4.2-22). □

Case (d) ($k_1 = 0$, $k_3 \neq 0$)

We choose $\lambda = \sqrt{\frac{2}{k_3}}\psi$ in (4.2-11). Then the transformed equation will be

$$W'' = \left(\frac{k_2 - 2}{\psi}\right)W' + 2W^3 + \frac{k_4}{\psi}\sqrt{\frac{2}{k_3}}W^2 + \left(\frac{k_2 + k_5}{\psi^2}\right)W + \frac{k_6}{\psi^3}\sqrt{\frac{k_3}{2}} \tag{4.2-23}$$

and we have two subcases:

Subcase d(i) ($k_2 = 2$)

Eq. (4.2-23) may be transformed by $W = w(\psi) + \frac{\alpha}{\psi}$, where $\alpha = -\frac{k_4}{6}\sqrt{\frac{2}{k_3}}$, to the equation,

$$w'' = 2w^3 + R(\psi)w + S(\psi), \tag{4.2-24}$$

where

$$R(\psi) = \frac{\alpha_1}{\psi^2}, \quad S(\psi) = \frac{\alpha_2}{\psi^3}$$

and

$$\left.\begin{aligned} \alpha_1 &= 2 + k_5 - \frac{k_4^{\,2}}{3k_3} \\ \alpha_2 &= \frac{k_4^{\,3}}{54}\left(\sqrt{\frac{2}{k_3}}\right)^3 - \frac{k_4k_5}{6}\sqrt{\frac{2}{k_3}} + k_6\sqrt{\frac{k_3}{2}} \end{aligned}\right\} \tag{4.2-25}$$

A necessary condition for the general solution of eq. (4.2-24) to be free from movable critical points is that $S(\psi)$ is a constant and $R(\psi) = \beta$ or $R(\psi) = \psi + \gamma$, (where β, γ are constants).

Theorem (4.2-8)

If eq. (4.2-1) satisfies conditions (4.2-9) with $k_1 = 0$, $k_3 \neq 0$, $k_2 = 2$, then the equation is not solvable by IST if $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$, where α_1 and α_2 are defined in (4.2-25). □

Subcase d(ii) ($k_2 \neq 2$)

Eq. (4.2-23) is transformed to the equation,

$$w'' = -3q(\psi)w' + 2w^3 - (q' + 2q^2)w, \qquad (4.2\text{-}26)$$

where $q(\psi) = -\frac{1}{3}\left(\frac{k_2 - 2}{\psi}\right)$, by the transformation $W(\psi) = w(\psi) + \frac{\alpha}{\psi}$, $\alpha = -\frac{k_4}{6}\sqrt{\frac{2}{k_3}}$ provided that

$$\left.\begin{aligned} &\beta_1 = 2k_4^{\,3} - 9k_3k_4k_5 + 27k_3^{\,2}k_6 = 0 \\ &\text{and} \\ &\beta_2 = 3k_4^{\,2} - 4k_2k_3 - 9k_3k_5 - 2k_3 - 2k_2^{\,2}k_3 = 0 \end{aligned}\right\} \qquad (4.2\text{-}27)$$

Hence we have the theorem,

Theorem (4.2-9)

If eq. (4.2-1) satisfies conditions (4.2-9) with

$k_1 = 0$, $k_3 \neq 0$, $k_2 \neq 2$, then the equation is not solvable by IST if $\beta_1 \neq 0$ or $\beta_2 \neq 0$ where β_1 and β_2 are defined in (4.2-27). $\square$

4.3 Special Nonlinear Equations

The p.d.e.'s which are now to be dealt with, may be directly reducible to the o.d.e., (4.1-1), where L is not zero. Most of these are of the form,

$$\sum_{i=0}^{5} \phi_i \left(\sum_{j=i}^{5} p_{ij}(\phi)\phi_j \right| = 0, \tag{4.3-1}$$

where $\phi_0 = 1$, $\phi_1 = \phi_x$, $\phi_2 = \phi_t$, $\phi_3 = \phi_{xx}$, $\phi_4 = \phi_{xt}$, $\phi_5 = \phi_{tt}$, and p_{ij} $(i, j = 0, 1, \ldots, 5)$ are polynomials.

Applying the transformation $\phi = g(\psi)$, as before and using (4.1-2) gives the o.d.e.,

$$\psi^4 P_3(g) g''^2 + \left(\psi^3 P_2(g) g' + \psi^2 P_5(g)\right] g'' +$$
$$\left(\psi^2 P_1(g) g'^2 + \psi P_4(g) g' + P_6(g)\right) = 0, \tag{4.3-2}$$

where

$$P_1 = P_2 - P_3 + (a^2 p_{11} + ab p_{12} + b^2 p_{22})$$

$$P_2 = (a^3 p_{13} + a^2 b p_{14} + ab^2 p_{15} + a^2 b p_{23} + ab^2 p_{24} + b^3 p_{25}) + 2P_3$$

$$P_3 = a^4 p_{33} + a^3 b p_{34} + a^2 b^2 p_{35} + a^2 b^2 p_{44} + ab^3 p_{45} + b^4 p_{55}$$

$$P_4 = (a p_{01} + b p_{02}) + P_5$$

$$P_5 = a^2 p_{03} + ab p_{04} + b^2 p_{05}$$

$$P_6 = p_{00}.$$

At this stage we have two cases:

Case one: $P_3 \neq 0$ for some constants a, b

Eq. (4.3-2) has the form (4.1-1) (which represents the first necessary condition for an o.d.e. to be free from movable critical points) iff

$$(P_2P_5 - 2P_3P_4)^2 = (P_2 - 4P_1P_3)(P_5^2 - 4P_3P_6). \qquad (4.3\text{-}3)$$

Thus we have,

Theorem (4.3-1)

Eq. (4.3-1) is not solvable by IST if $P_3 \neq 0$ and (4.3-3) is not satisfied for some constants a and b. □

If (4.3-3) is satisfied ∀a, b then eq. (4.3-1) is reduced to the o.d.e.,

$$g'' = \frac{1}{-2\psi^3 P_3}\left[\psi\left(P_2 \pm \sqrt{P_2 - 4P_1P_3}\right)g' + \left(P_5 \pm \sqrt{P_5^2 - 4P_3P_6}\right)\right]. \qquad (4.3\text{-}4)$$

This equation satisfies the first necessary condition for absence of movable critical points and contains no term in g'^2. This case has been investigated in the previous section and a similar theorem can be stated for eq. (4.3-1).

Case two: $P_3 = 0$, for some constants a, b

In this case, there are many possibilities for the o.d.e., which reduces to,

$$(\psi^3 P_2 g' + \psi^2 P_5) g'' + (\psi^2 P_1 g'^2 + \psi P_4 g' + P_6) = 0. \tag{4.3-5}$$

<u>Subcase two(a)</u>: $(P_2 \neq 0)$

Eq. (4.3-5) is reducible to the equation

$$g'' = -\frac{P_1}{\psi P_2} g' - \frac{P_6}{\psi^2 P_5} \tag{4.3-6}$$

if $P_2^2 P_6 = (P_2 P_4 - P_1 P_5) P_5$.

Once again, this case has been investigated in the previous section.

<u>Theorem (4.3-2)</u>

Eq. (4.3-1) is not solvable by IST if $P_3 = 0$, $P_2 \neq 0$ and $P_2^2 P_6 \neq (P_2 P_4 - P_1 P_5) P_5$ for some constants a and b. □

<u>Subcase two(b)</u>: $(P_2 = 0,\ P_5 \neq 0)$

Eq. (4.3-5) becomes,

$$g'' = -\frac{P_1}{P_5} g'^2 - \frac{P_4}{\psi P_5} g' - \frac{P_6}{\psi^2 P_5} \tag{4.3-7}$$

if $P_2 \equiv 0$. We shall write $-\frac{P_1}{P_5}$, $-\frac{P_4}{P_5}$ and $-\frac{P_6}{P_5}$ as $\ell(g)$, $m(g)$ and $n(g)$ respectively.

We will now state some conditions which ensure that eq. (4.3-7) has movable critical points, according to the procedure outlined in [44]. (Hence in the following theorems, we assume that $P_2 = P_3 = 0$ in eq. (4.3-2) for some constants a and b.)

The first step towards investigating the Painleve property of eq. (4.3-2) is the decomposition of $\ell(g)$ into partial fractions:

<u>Theorem (4.3-3)</u>

Eq. (4.3-1) is not solvable by IST if

$$\ell(\phi) = -\frac{P_1(\phi)}{P_5(\phi)} \neq \sum_{i=1}^{4} \frac{m_i}{\phi - a_i} \tag{4.3-8}$$

for some constants a, b, where the a_i's are constant and the m_i can only have the values listed in [44] p.323. □

Suppose now, that (4.3-8) is not satisfied for each constant a and b. <u>Suppose that $\ell(g)$ has one pole a_1,</u> i.e., $\ell(g) = \frac{\beta}{g - a_1}$, where β is a constant. Then for absence of movable critical points eq. (4.3-7) must transform to the equation

$$w'' = L(w)w'^2 + M(\psi, w)w' + N(\psi, w) \tag{4.3-9}$$

where

$$L(w) = \begin{cases} \frac{1}{w} \\ \frac{m-1}{mw}, \quad m \in \mathbb{Z} \geq 1 \end{cases}$$

using the transformation $g = \frac{1}{w} + a_1$. This yields that

$\beta = 1$ or $\beta = \frac{m+1}{m}$. Therefore, we have

Theorem (4.3-4)

If $\ell(\phi) = -\frac{P_1(\phi)}{P_5(\phi)} = \frac{\beta}{\phi - a_1}$, then eq. (4.3-1) is not solvable by IST if $\beta \neq 1$ and $\beta \neq \frac{m+1}{m}$, $m \in \mathbb{Z} \geq 1$. □

If $\beta = 1$ or $\beta = \frac{m+1}{m}$ then eq. (4.3-7) will be transformed to (4.3-9) where

$$M(\psi, w) = \frac{m(g)}{\psi} = \frac{1}{\psi} m\left(\frac{1}{w} + a_1\right) \qquad (4.3\text{-}10(a))$$

and

$$N(\psi, w) = \frac{w^2}{\psi^2} n(g) = \frac{w^2}{\psi^2} n\left(\frac{1}{w} + a_1\right). \qquad (4.3\text{-}10(b))$$

Comparing with the general canonical form the necessary conditions for absence of movable critical points give

$$m\left(\frac{1}{w} + a_1\right) = k_1 w + k_2 + \frac{k_3}{w} \qquad (4.3\text{-}11(a))$$

and

$$w^2 n\left(\frac{1}{w} + a_1\right) = k_4 w^3 + k_5 w^2 + k_6 w + k_7 + \frac{k_8}{w}, \qquad (4.3\text{-}11(b))$$

where the k_i $(i = 1, \ldots, 8)$ are constants.

Theorem (4.3-5)

If $\ell(\phi) = -\dfrac{P_1(\phi)}{P_5(\phi)} = \dfrac{\beta}{\phi - a_1}$, where a_1 is a constant, $\beta = 1$, or $\beta = \dfrac{m+1}{m}$, $m \in \mathbb{Z} \geqslant 1$ then eq. (4.3-1) is not solvable by IST if,

$$m(\phi) = -\frac{P_4(\phi)}{P_5(\phi)} \neq \frac{k_1}{\phi - a_1} + k_2 + k_3(\phi - a_1)$$

or

$$n(\phi) = -\frac{P_6(\phi)}{P_5(\phi)} \neq \frac{k_4}{\phi - a_1} + k_5 + k_6(\phi - a_1) + k_7(\phi - a_1)^2 + k_8(\phi - a_1)^3$$

for some constants k_i, $i = 1, \ldots, 8$. $\square$

Now suppose that eq. (4.3-9) is of the form

$$w'' = \frac{\alpha w'^2}{w} + \frac{1}{\psi}\left(k_1 w + k_2 + \frac{k_3}{w}\right)w' + \frac{1}{\psi^2}\left(k_4 w + k_5 + \frac{k_6}{w} + \frac{k_7}{w^2} + \frac{k_8}{w^3}\right) \tag{4.3-12}$$

where $\alpha = 1$, or $\alpha = \dfrac{m-1}{m}$, $m \in \mathbb{Z} \geqslant 1$ and the k_i are constants $\forall i = 1, \ldots, 8$.

To transform eq. (4.3-9) possibly to the Painlevé canonical forms we apply $w(\psi) = \lambda(\psi)W(Z) + \mu(\psi)$, $Z = \phi(\psi)$. The term $\dfrac{w'^2}{w}$ in eq. (4.3-9) will be invariant under the transformation iff $\mu = 0$. Hence we get,

$$W'' = \frac{\alpha}{W}W'^2 + \frac{1}{\lambda\phi'^2}\left[\frac{k_1\lambda^2}{\psi}W + \left(\frac{k_2\lambda\phi'}{\psi} - \lambda\phi'' + 2(\alpha - 1)\lambda'\phi'\right.\right.$$

$$\left.\left. + \frac{k_3\phi'}{\psi W}\right]W' + \frac{k_4\lambda^2 W^3}{\psi^2\phi'^2} + \frac{1}{\phi'^2}\left[\frac{k_1\lambda'}{\psi} + \frac{k_5\lambda}{\psi^2}\right]W^2$$

$$+ \frac{1}{\lambda\phi'^2}\left[-\lambda'' + \frac{\alpha\lambda'^2}{\lambda} + \frac{k_2\lambda'}{\psi} + \frac{k_6\lambda}{\psi^2}\right]W + \frac{1}{\lambda\phi'^2}\left[\frac{k_3\lambda'}{\psi\lambda} + \frac{k_7}{\psi^2}\right]$$

$$+ \frac{k_8}{\lambda^2\psi^2\phi'^2 W'} \tag{4.3-13}$$

(where $\alpha = 1$ or $\alpha = \frac{m-1}{m}$, $m \in \mathbb{Z} \geqslant 1$).

<u>Case (i)</u> ($\alpha = 1$)

Comparing eq. (4.3-13) with the canonical equations of type II, we find that:

1°) When $k_1 = k_3 = 0$, then eq. (4.3-13) reduces to canonical types:

(XI) if $k_4 = k_5 = k_7 = k_8 = 0$, $\phi' = C\psi^{k_2}$, where C is arbitrary constant; and λ satisfies the equation

$$-\lambda'' + \frac{\lambda'^2}{\lambda} + k_2\frac{\lambda'}{\psi} + \frac{k_6\lambda}{\psi^2} = 0.$$

(XII) if $k_2 = -1$, $k_6 = 0$, $\phi' = C\psi^{-1}$ and $\lambda = C_1$, where C and C_1 are arbitrary non-zero constants.

2°) When $k_1 \neq 0$, $k_3 \neq 0$, then eq. (4.3-13) reduces to canonical type:

(XIV) if $k_4 = k_5 = k_6 = k_7 = k_8 = 0$, $k_2 = -1$, $\lambda = K$, and $\phi' = C\psi^{-1}$, where C and K are arbitrary non-zero constants.

3°) When $k_1 = 0$, $k_3 \neq 0$, then eq. (4.3-13) reduces to canonical type:

(XV) if $k_4 = k_6 = k_7 = k_8 = 0$, $k_2 = -1$, $\lambda = 1$ and $\phi' = C\psi^{-1}$, where C is an arbitrary non-zero constant.

<u>Theorem (4.3-6)</u>

Eq. (4.3-1) is not solvable by IST if eq. (4.3-13) is not reduced to one of the canonical equations XI, XII, XIV or XV. □

<u>Case (ii)</u> $\left(\alpha = \frac{m-1}{m},\ m \in \mathbb{Z} \geqslant 1\right)$

Comparing eq. (4.3-13) with the canonical equations of type III, we find that:

1°) When $k_1 = k_3 = k_4 = k_8 = 0$, then eq. (4.3-13) reduces to canonical types:

(XVII) if $k_5 = k_7 = 0$, and λ and ϕ' satisfy the equations

$$k_2\frac{\lambda\phi'}{\psi} - \lambda\phi'' - \frac{2}{m}\lambda'\phi' = 0,$$

and

$$-\lambda'' + \frac{m-1}{m}\frac{\lambda'^2}{\lambda} + \frac{k_2\lambda'}{\psi} + \frac{k_6\lambda}{\psi^2} = 0.$$

(XVIII) if $m = 2$, $k_7 = 0$, $\frac{4}{9}(k_2 + 1)^2 + k_6 = 0$,

$\lambda = \frac{4K^2}{k_5}\psi^{\frac{2}{3}(k_2+1)}$ and $\phi' = K\psi^{\frac{1}{3}(k_2-2)}$, where K is arbitrary non-zero constant.

(XIX) if $m = 2$, $k_2 = 2$, $k_7 = 0$, $\frac{2K^2}{k_5}\left(\frac{4}{3} + k_6\right) = 1$,

$\lambda = \frac{4K^2}{k_5}\psi^2$, and $\phi' = K$, where K is arbitrary non-zero constant.

(XX) if $m = 2$, $k_2 = 3$, $k_7 = 0$, $\frac{2K}{k_5}\left(\frac{64}{9} + k_6\right) = \frac{3}{4}$,

$\lambda = \frac{4K^2}{k_5}\psi^2$, and $\phi' = K\psi^{\frac{1}{3}}$, where K is a non-zero arbitrary constant.

(XXI) if $m = 4$, $k_7 = 0$, $3(k_2 + 1)^2 + 4k_6 = 0$,

$\lambda = \frac{3K^2}{k_5}\psi^{k_2+1}$, and $\phi' = K\psi^{\frac{1}{2}(k_2 - 1)}$, where K is arbitrary non-zero constant.

(XXII) if $m = 4$, $k_5 = 0$, λ and ϕ' satisfy the equations

$$\lambda = -\frac{k_7}{\psi^2\phi'^2},$$

$$\frac{k_2\lambda\phi'}{\psi} - \lambda\phi'' - \frac{1}{2}\lambda'\phi' = 0,$$

and

$$-\lambda'' - \frac{3}{4}\frac{\lambda'^2}{\lambda} + k_2\frac{\lambda'}{\psi} + k_6\frac{\lambda}{\psi^2} = 0.$$

(XXIII) if $m = 4$, $\left[k_2 = -1 \text{ or } k_7 = \frac{3}{4}(k_2 + 1)^2 + k_6 = 0\right]$

$\lambda = K\psi^{\frac{k_2+1}{2}}$ and $\phi'^2 = \frac{Kk_5}{3}\psi^{\frac{k_2-1}{2}}$, where K is a non-zero arbitrary constant.

2°) When $k_3 = k_8 = 0$ and $(m + 2)^2k_4 + mk_1^2 = 0$, then eq. (4.3-13) reduces to canonical types:

(XXIV) if $k_7 = 0$, λ and ϕ' satisfy the equations,

$$k_2\frac{\lambda\phi'}{\psi} - \lambda\phi'' - \frac{2}{m}\lambda'\phi' = 0,$$

$$-\lambda'' - \frac{(m - 1)\lambda'^2}{m\lambda} + \frac{k_2\lambda'}{\psi} + \frac{k_6\lambda}{\psi^2} = 0,$$

and

$$\lambda'\left\{1 - \frac{m}{m + 2}\phi'\right\}k_1\psi + \lambda\left\{k_5 + \frac{mk_1}{m + 2}(\psi\phi'' + \phi')\right\} = 0.$$

(XXV) if $m = 4$, $k_7 \neq 0$, λ and ϕ' satisfy the equations,

$$-\lambda\phi'' + \left(\frac{k_2\lambda}{\psi} - \frac{1}{2}\lambda'\right)\phi' - \left(\frac{k_1\lambda\lambda'}{\psi} + \frac{k_5\lambda^2}{\psi^2}\right) = 0$$

and

$$\frac{1}{\phi'}\left[k_2\lambda\phi' - \lambda\phi''\psi - \frac{1}{2}\lambda'\phi'\psi\right] = -\frac{1}{2}(2\lambda\psi\phi'' + 2\lambda\phi' + \lambda'\psi\phi').$$

3°) When $k_1 = k_4 = 0$, $k_8 + k_3^2 = 0$, then eq. (4.3-13) reduces to canonical type:

(XXVI) if $\lambda = \frac{3K^2}{k_5}\psi^{k_2+1}$, $\phi' = K\psi^{\frac{k_2-1}{2}}$ and ($k_2 = -1$, $k_3 = 0$, $k_6{}^2 = -12$) or ($k_2 = 1$, $3 + k_6 = 12K^2$, $k_3k_5 = -36K^3$, $k_5(2k_3 + k_7) = -216K^4$), where K is a non-zero arbitrary constant.

4°) When m is unrestricted, and $(m - 2)^2k_8 + mk_3{}^2 = 0$ then eq. (4.3-13) reduces to canonical type:

(XXVII) if $mk_1{}^2 + k_4(m + 2)^2 = 0$, λ and ϕ' satisfy the equations,

$$\frac{k_2\lambda\phi'}{\psi} - \lambda\phi'' - \frac{2}{m}\lambda'\phi' = -\left(\frac{k_3\lambda'}{\lambda\psi} + \frac{k_7}{\psi 2}\right)$$

$$\frac{k_8}{(\lambda\psi\phi')^2} = -\frac{1}{m}$$

$$k_1\lambda'\psi + k_5\lambda - \frac{mk_1}{m + 2}\left[\lambda'\psi\phi' - \lambda\psi\phi'' - \lambda\phi' + \frac{k_3\lambda'}{\lambda\phi'} + \frac{k_7}{\psi\phi'}\right].$$

5°) When $m = 2$, $k_1 = k_3 = 0$ and $k_4 \neq 0$, then eq. (4.3-13) reduces to canonical type,

(XXIX) if $k_5 = k_7 = k_8 = 0$, $\frac{3}{8}(k_2 + 1)^2 + k_6 = 0$,

$\lambda = \sqrt{\frac{3}{2k_4}}K\psi^{\frac{k_2+1}{2}}$, and $\phi' = K\psi^{\frac{k_2-1}{2}}$, K is a non-zero arbitrary constant.

6°) When $m = 2$, $k_1 = k_3 = k_4 = 0$, $k_8 \neq 0$, then eq.

(4.3-13) reduces to canonical types,

(XXXII) if $k_5 = k_7 = 0$, $k_2 = -1$, ϕ' is arbitrary and λ satisfies

$$-\lambda'' + \frac{1}{2}\frac{\lambda'^2}{\lambda} - \frac{\lambda'}{\psi} + k_6\frac{\lambda}{\psi^2} = 0,$$

$$-2k_8 = (\lambda\psi\phi')^2.$$

(XXXIII) if $k_7 = 0$, $k_2 = -1$, $\lambda = \dfrac{\sqrt{-2k_8}}{\alpha}$, $\phi' = \dfrac{\alpha}{\psi}$ and

$$\alpha = \left(\frac{k_5\sqrt{-2k_8}}{4}\right)^{\frac{1}{3}}.$$

7°) When $m = 3$, $k_4 = \frac{3}{2}k_1{}^2$, $k_8 = -3k_3{}^2$, eq. (4.3-13) reduces to canonical type,

(XXXV) if k_i $(i = 1, \ldots, 8)$, λ and ϕ' satisfy the following relations .

$$\frac{k_1\lambda}{\psi\phi'} = -\frac{2}{3}, \quad \frac{k_3}{\lambda\psi\phi'} = r,$$

$$q = \frac{3}{2\lambda\phi'^2}\left(\frac{k_2\lambda\phi'}{\psi} - \lambda\phi'' - \frac{2}{3}\lambda'\phi'\right) = -\frac{1}{5\phi'^2}\left(\frac{k_1\lambda'}{\psi} + \frac{k_5\lambda}{\psi^2}\right)$$

$$\frac{1}{\lambda\phi'^2}\left(-\lambda'' + \frac{2}{3}\frac{\lambda'^2}{\lambda} + \frac{k_2\lambda'}{\psi} + \frac{k_6\lambda}{\psi^2}\right) = 4q' + r + \frac{8}{3}q^2$$

$$\frac{1}{\lambda\phi'^2}\left(\frac{k_3\lambda'}{\psi\lambda} + \frac{k_7}{\psi^2}\right) = 2qr - 3r'$$

where $q'' = 2q^3 + Sq + T$, $r = -\frac{1}{3}S - \frac{2}{3}(q' + q^2)$, S and T take one of the pairs (0, 0), (α, β), (ϕ, α).

8°) When $m = 5$, $k_4 = 5k_1{}^2$, $k_8 = -\frac{5}{9}k_1{}^2$, eq. (4.3-13) reduces to canonical type,

(XXXVI) if k_i ($i = 1, \ldots, 8$), λ and ϕ' satisfy the equations,

$$\frac{k_1\lambda}{\psi\phi'} = -\frac{2}{5}; \qquad \frac{k_3}{\lambda\psi\phi'} = r$$

$$\frac{q}{5} = -\frac{1}{4\lambda\phi'^2}\left(\frac{k_2\lambda\phi'}{\psi} - \lambda\phi'' - \frac{2}{5}\lambda'\phi'\right) = \frac{1}{14\phi'^2}\left(\frac{k_1\lambda'}{\psi} + \frac{k_5\lambda}{\psi^2}\right)$$

$$\frac{1}{\lambda\phi'^2}\left(-\lambda'' + \frac{4}{5}\,\frac{\lambda'^2}{\lambda} + \frac{k_2\lambda'}{\psi} + \frac{k_6\lambda}{\psi^2}\right) = r - 3q' + \frac{6}{5}q^2$$

$$\frac{1}{\lambda\phi'^2}\left(\frac{k_3\lambda'}{\lambda\psi} + \frac{k_7}{\psi^2}\right) = -\frac{1}{3}(qr + 5r'),$$

where

$$q = \frac{v_2' - v_1'}{v_2 - v_1}, \qquad r = \frac{72}{5}v_1 + \frac{36}{5}v_2 - \frac{9}{5}\left(\frac{v_2' - v_1'}{v_2 - v_1}\right)^2,$$

v_1 and v_2 being solutions of

$$v'' = 6v + s \qquad \left(s = 0,\ \frac{1}{2},\ \text{or } \phi\right).$$

<u>Theorem (4.3-7)</u>

Eq. (4.3-1) is not solvable by IST if eq. (4.3-13) is not reducible to one of the canonical equations mentioned above. □

Let us now investigate the case, where $\ell(g)$ has two poles a_1, a_2:

Applying the transformation $g = \dfrac{a_2 w - a_1}{w - 1}$ to eq. (4.3-7) gives:

$$w'' = \left(\frac{\alpha}{w} + \frac{2 - \beta - \alpha}{w - 1}\right)w'^2 + \frac{M(w)}{\psi}w' + \frac{N(w)}{\psi^2}. \qquad (4.3\text{-}14)$$

For eq. (4.3-14) to be of Painlevé type, α and β are necessarily constants with the following values:

$$\text{(i)}\ \alpha = \tfrac{1}{2},\ \beta = \tfrac{1}{2}, \qquad \text{(ii)}\ \alpha = \tfrac{2}{3},\ \beta = \tfrac{2}{3}$$

$$\text{(iii)}\ \alpha = \tfrac{3}{4},\ \beta = \tfrac{1}{2}, \qquad \text{(iv)}\ \alpha = \tfrac{2}{3},\ \beta = \tfrac{5}{6}.$$

Also,

$$\left.\begin{aligned} M(w) &= m(g) \\ N(w) &= \frac{(w - 1)^2}{a_1 - a_2}n(g) \end{aligned}\right\}. \qquad (4.3\text{-}15)$$

It follows that $\ell(g) = \dfrac{\alpha}{g - a_1} + \dfrac{\beta}{g - a_2}$ for each case.

Thus we have,

Theorem (4.3-8)

If $\ell(\phi) = -\dfrac{P_1(\phi)}{P_5(\phi)} = \dfrac{\alpha}{\phi - a_1} + \dfrac{\beta}{\phi - a_2}$, then eq. (4.3-1) is not solvable by IST if α and β take no values listed (i), (ii), (iii) or (iv). $\square$

If $\ell(g)$ has the required form, then eq. (4.3-14) is possibly reducible to one of the canonical forms given in [44].

Lemma (4.3-9)

The most general transformation possible which reduces eq. (4.3-14) to a possible canonical form is,

$$w(\psi) = W(z), \quad z = v(\psi), \tag{4.3-16}$$

without loss of generality. □

Proof Assume that

$$w(\psi) = \lambda(\psi)W(z) + \mu(\psi), \quad z = v(\psi).$$

Applying this transformation to (4.3-14), the coefficient of W is the transformed equation is,

$$\left(\frac{\alpha}{W + \frac{\mu}{\lambda}} + \frac{2 - \beta - \alpha}{W + \frac{\mu - 1}{\lambda}}\right).$$

Since it is necessary that this coefficient is of the same form as in eq. (4.3-14),

$$\frac{\mu}{\lambda} = 0 \quad \text{and} \quad \frac{\mu - 1}{\lambda} = -1 \quad \Rightarrow \quad \mu = 0 \quad \text{and} \quad \lambda = 1,$$

or

$$\frac{\mu}{\lambda} = -1 \quad \text{and} \quad \frac{\mu - 1}{\lambda} = 0 \quad \Rightarrow \quad \mu = 1 \quad \text{and} \quad \lambda = -1.$$

The latter case implies that $w(\psi) = -W(z) + 1$, which means that $w(\psi) = W(z)$ without loss of generality.

Hence $\mu = 0$ and $\lambda = 1$ for all the cases. ■

Having applied (4.3-16) to eq. (4.3-14) we get,

$$\ddot{W} = \left(\frac{\alpha}{W} + \frac{2 - \beta - \alpha}{W - 1}\right)\dot{W}^2 + \left(-\frac{v''}{v'^2} - \frac{M(W)}{\psi v'}\right)\dot{W} + \frac{N(W)}{\psi^2 v'^2},$$

(4.3-17)

$\left(\text{where } \cdot = \frac{d}{dz}\right)$.

Let us now specify the constants α and β and look for necessary conditions for eq. (4.3-17) to be one of the canonical forms. For the general canonical forms, we will write A_i or B_i to denote functions of Z.

<u>Case (i)</u> $\left(\alpha = \frac{1}{2}, \beta = \frac{1}{2}\right)$

The general canonical form is,

$$w'' = \left(\frac{1}{2w} + \frac{1}{w - 1}\right)w'^2 + \left(A_0 + \frac{A_1}{w - 1}\right)w'$$
$$+ (w - 1)^2\left(B_0 w + \frac{B_1}{w} + B_2 w + \frac{B_3 w}{(w - 1)^2} + \frac{B_4 w}{(w - 1)^3}\right.$$
$$\left. + \frac{B_5 w(w + 1)}{w - 1}\right), \quad w = w(z). \tag{4.3-18}$$

<u>Theorem (4.3-10)</u>

Eq. (4.3-1) is not solvable by IST if eq. (4.3-17) is not reducible to eq. (4.3-18), where A_i and B_i represent the coefficients for the canonical types XXXVII, XXXVIII, XXXIX and XL. □

Case (ii) $\left(\alpha = \frac{2}{3},\ \beta = \frac{2}{3}\right)$

The general canonical form is,

$$w'' = \frac{2}{3}\left(\frac{1}{w} + \frac{1}{w-1}\right)w'^2 + \left(A_0 + A_1 w + \frac{A_2}{w} + \frac{A_3}{w-1}\right)w'$$
$$+ \left(B_0\frac{w}{w-1} + B_1(w-1) + B_2 w + B_3 w(w-1)\right.$$
$$\left. + B_4 w^2(w-1) + B_5\frac{w-1}{w}\right), \quad w = w(z). \qquad (4.3\text{-}19)$$

Theorem (4.3-11)

Eq. (4.3-1) is not solvable by IST if eq. (4.3-17) is not reducible to eq. (4.3-19), where A_i and B_i represent the coefficients of the canonical types XLI and XLII. □

Case (iii) $\left(\alpha = \frac{3}{4},\ \beta = \frac{1}{2}\right)$

The general canonical form is

$$w'' = \frac{3}{4}\left(\frac{1}{w} + \frac{1}{w-1}\right)w'^2 + \left(\frac{A_0}{w-1} + A_1 + \frac{A_2}{w}\right)w'$$
$$+ w(w-1)\left(\frac{B_0}{w} + \frac{B_1}{w-1} + B_2(w-1)\right.$$
$$\left. + B_3(2w-1) + \frac{B_4}{w^2} + \frac{B_5}{(w-1)^2}\right), \quad w = w(z).$$
$$(4.3\text{-}20)$$

Theorem (4.3-12)

Eq. (4.3-1) is not solvable by IST if eq. (4.3-17) is not reducible to eq. (4.3-20), where A_i and B_i represent the coefficients of the canonical types XLIII, XLIV, XLV, XLVI, XLVII, and IX. □

Case (iv) $\left(\alpha = \frac{2}{3},\ \beta = \frac{5}{6}\right)$

The general canonical form is

$$w'' = \left(\frac{2}{3w} + \frac{1}{2(w-1)}\right)w'^2 + \left(A_0 w + A_1 + \frac{A_2}{w}\right)w'$$
$$+ w(w-1)\left(B_0 w + B_1 + \frac{B_2}{w^2} + \frac{B_3}{(w-1)^2} + \frac{B_4}{w} + \frac{B_5}{w-1}\right),$$
$$w = w(z). \qquad (4.3\text{-}21)$$

Theorem (4.3-13)

Eq. (4.3-1) is not solvable by IST if eq. (4.3-17) is not reducible to eq. (4.3-21), where A_i and B_i represent the coefficients of canonical types XLVIII, and X. □

Suppose now that $\ell(g)$ has three or four poles a_1, a_2, a_3, a_4. Then $\ell(g)$ must have the following form

$$\ell(g) = \frac{1}{2}\left(\frac{1}{g-a_1} + \frac{1}{g-a_2} + \frac{1}{g-a_3} + \frac{1}{g-a_4}\right).$$

In this case, eq. (4.3-7) will be transformed to the equation,

$$w'' = \frac{1}{2}\left\{\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-\eta}\right\}w'^2 + M(\psi,\ w)w' + N(\psi,\ w), \qquad (4.3\text{-}22)$$

where

$$\left.\begin{aligned} \eta &= \frac{a_1 - a_3}{a_2 - a_3}\,\frac{a_2 - a_4}{a_1 - a_4} \\ M(\psi,\ w) &= \frac{m(g)}{\psi} \\ N(\psi,\ w) &= \frac{n(g)}{2}\,\frac{(a_1 - a_4)(w - \gamma)^2}{(a_2 - a_4)(a_1 - a_2)} \end{aligned}\right\}$$

using the transformation $w = \dfrac{a_1 - a_3}{a_2 - a_3}\,\dfrac{g - a_2}{g - a_1}$.

The most general transformation for this case, which preserves the essential character of the equation as before is, $w(\psi) = W(z)$, $z = v(\psi)$ and the general canonical form for $W(z)$ is

$$W'' = \frac{1}{2}\left\{\frac{1}{W} + \frac{1}{W - 1} + \frac{1}{W - \eta}\right\}W'^2 + W(W - 1)(W - \eta)$$
$$\left\{\beta + \frac{\gamma}{W^2} + \frac{\delta}{(W - 1)^2} + \frac{\varepsilon}{(W - \eta)^2}\right\}, \qquad (4.3\text{-}23)$$

where η is a constant or $\eta = z$ and γ, δ and ε are constants.

Theorem (4.3-14)

Eq. (4.3-1) is not solvable by IST if eq. (4.3-22) is not reducible to eq. (4.3-23). □

Subcase two (c):

In this case, eq. (4.3-5) is reducible to the first order equation

$$\psi^2 P_1(g)g'^2 + \psi P_4(g)g' + P_6(g) = 0 \qquad (4.3\text{-}24)$$

if $P_2 = P_5 = 0$.

If P_1, P_4, $P_6 \neq 0$ then $g' = \psi^n G(g)$, where $n = 1, -1$, and

$$G(g) = 0, -\frac{P_6(g)}{P_4(g)}, \left(-\frac{P_6(g)}{P_1(g)}\right)^{\frac{1}{2}}, -\frac{P_4(g)}{P_1(g)}, \text{ or}$$

$$\frac{1}{2}\left(-P_4 \pm \sqrt{P_4^{\,2} - 4P_1P_6}\right).$$

A necessary condition for absence of movable critical point is $G(g) = \alpha_0 + \alpha_1 g + \alpha_2 g^2$, for some constants α_i, $i = 0, 1, 2$. Thus,

<u>Theorem (4.3-15)</u>

If $P_2 = P_3 = P_5 = 0$ and $P_1P_4P_6 \neq 0$ for some constants a and b then eq. (4.3-1) is not solvable by IST if

$$G(\phi) = 0, -\frac{P_6(\phi)}{P_4(\phi)}, \left(-\frac{P_6(\phi)}{P_1(\phi)}\right)^{\frac{1}{2}}, -\frac{P_4(\phi)}{P_1(\phi)} \text{ or}$$

$\frac{1}{2}\left(-P_4 \pm \sqrt{P_4^{\,2} - 4P_1P_6}\right)$ is not of the form $\alpha_0 + \alpha_1\phi + \alpha_2\phi^2$ where α_i are constant $i = 0, 1, 2$. □

CHAPTER FIVE

The Painlevé Property for Higher Order Equations

5.1 Introduction

The analysis for finding sets of equations, not solvable by IST can be extended to higher order p.d.e.'s. The technique divides into two stages. Firstly, it is to reduce the p.d.e. into an o.d.e., and secondly, it is to apply the Painlevé test to the o.d.e. In this chapter, we investigate the Painlevé property (i.e., pure poles being the only movable singularities) for the third order, variable coefficient equation

$$\phi_{xxx} + \beta\phi_{xx} + \gamma\phi^N\phi_x + a(x, t)\phi_x + b(x, t)\phi + \phi_t = 0, \tag{5.1-1}$$

where β is a constant, γ is non-zero constant and $N \in \mathbb{Z}^+$. This general equation includes many known equations as special cases. These equations are:

A. Equations with N = 1:

(i) The Korteweg-de Vries (KdV) equation,

$$\phi_{xxx} + 6\phi\phi_x + \phi_t = 0. \tag{5.1-2}$$

(ii) The cylindrical KdV (cKdV) equation, [18]

$$\phi_{xxx} + \phi\phi_x + \frac{1}{2t}\phi + \phi_t = 0. \qquad (5.1\text{-}3)$$

(iii) The KdV-Burgers (KdVB) equation, [54]

$$\phi_{xxx} + \beta\phi_{xx} + \gamma\phi\phi_x + \phi_t = 0. \qquad (5.1\text{-}4)$$

(iv) The equation studied by Roy, [76]

$$\phi_{xxx} + 6\phi\phi_x + \alpha x\phi_x + \gamma\phi + \phi_t = 0. \qquad (5.1\text{-}5)$$

B. Equations with N = 2:

(v) The modified KdV (mKdV) equation,

$$\phi_{xxx} - 6\phi^2\phi_x + \phi_t = 0. \qquad (5.1\text{-}6)$$

(vi) The equation studied by Fung, [33]

$$\phi_{xxx} - 6\phi^2\phi_x + 6\lambda\phi_x + \phi_t = 0. \qquad (5.1\text{-}7)$$

C. Equations with N > 2:

(vii) The generalized KdV (GKdV) equation:

$$\phi_{xxx} + \phi^N\phi_x + \phi_t = 0, \quad N > 2. \qquad (5.1\text{-}8)$$

Recently, Osborne [68] has studied the equation

$$\phi_{xxx} + 6\phi\phi_x + a(x)b(t)\phi_x + c(x)d(t)\phi + \phi_t = 0. \qquad (5.1\text{-}9)$$

He investigated separable solutions by applying the transformation $\phi(x, t) = u(x, t)g(\psi(x, t))$, where $u(x, t)$ is a separable function to be chosen so that the resulting equation in $\psi(x, t)$ is separable, and reducing (5.1-9) to an o.d.e. by letting $\psi(x, t) = X(x)T(t)$, $X' = k_1(x)X^n$, $T' = k_2(t)T^m$, $k_1, k_2 \neq 0$, for some constants n and m.

Following the same techniques used in [68] and section 3.5, it is found that eq. (5.1-1) reduces to the o.d.e.,

$$\beta_1\psi^{2-qN}g'' + \beta_2\psi^{1-qN}g'' + \left(\beta_3\psi^{-qN} + A(\psi)\right)g' + B(\psi)g$$
$$+ \gamma g^N g' = 0 \qquad (5.1\text{-}10)$$

if $u(x, t) = T^q$ for some constant q.

In (5.1-10), $\beta_1 = k_1{}^2$, $\beta_2 = \frac{3k_1{}^2}{2}(2 - qN) + \beta k_1$, and $\beta_3 = \frac{1}{2}(2 - qN)(1 - qN)k_1{}^2 + \beta k_1(1 - qN)$. (Note that $q = 0$ if $\beta \neq 0$.)

The functions $A(\psi)$ and $B(\psi)$ are constructed from the functions $a(x, t)$ and $b(x, t)$ according to the value of q and the term of ψ.

We illustrate this relation in table (5.1-1) below.

For the integrable cases of eq. (5.1-10) we have adopted the Ince procedure to investigate the Painlevé property (see ch. 4). We have analyzed eq. (5.1-10) if it is not integrable by following the method, given by Ablowitz, Ramani and Segur [6] (see §5.3).

From table (5.1-1), we can state the following theorems, which will be useful for the next section.

<u>Theorem (5.1-1)</u>

If $a(x, t) = b(x, t) = 0$ and $N = 2$ in eq. (5.1-1)

<u>Table (5.1-1)</u> The functions a(x, t), b(x, t) for eq. (5.1-1) and $A(\psi)$, $B(\psi)$ in eq. (5.1-10) and the choice of X, T.

Case	a(x, t) and b(x, t)	$A(\psi)$ and $B(\psi)$	X' and T'
IA	$a(x,t) = T^{qN}\xi^*(\psi)$ $b(x,t) = k_1 X^{1-\frac{qN}{2}} T^{1+qN}\eta^*(\psi)$	$A(\psi) = \xi^*(\psi) + \frac{k_2}{k_1}\psi^{\frac{qN}{2}}$ $B(\psi) = \eta^*(\psi) + \frac{qk_2}{k_1}\psi^{\frac{qN}{2}-1}$	$X' = k_1 X^{1-\frac{qN}{2}}$ $T' = k_2 T^{1+\frac{3qN}{2}}$
IB	$a(x,t) = \frac{1}{k_1} X^{\frac{qN}{2}} f(t)$ $b(x,t) = \begin{cases} k_1 X^{\frac{qN}{2}}\tau^*(\psi) & \text{if } q = 0 \\ qf(t) & \text{if } q \neq 0 \end{cases}$	$A(\psi) = 0$ $B(\psi) = \begin{cases} \tau^*(\psi) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$	$X' = k_1 X^{1-\frac{qN}{2}}$ $T' = -f(t)T$

* The functions ξ, η, τ are functions of ψ.

then $A'(\psi) = B(\psi)$ in eq. (5.1-10). □

Proof From table (5.1-1), $a(x, t) = b(x, t) = 0$ implies that $\xi(\psi) = \eta(\psi) = 0$.

$$\text{i.e., } A(\psi) = \frac{k_2}{k_1}\psi^q, \; B(\psi) = \frac{qk_2}{k_1}\psi^{q-1}.$$

Clearly $A'(\psi) = B(\psi)$. ∎

Theorem (5.1-2)

If $a(x, t) = b(x, t) = 0$ and $\beta \neq 0$ in eq. (5. 1-1) then $A'(\psi) = B(\psi)$ in eq. (5.1-10). □

The proof follows from the fact that $\beta \neq 0$ implies $q = 0$, which yields $A(\psi) = \frac{k_2}{k_1}$, while $B(\psi) = 0$, as $a(x, t) = b(x, t) = 0 \Rightarrow \xi(\psi) = \eta(\psi) = 0$ as above. ∎

5.2 The integrability of Eq. (5.1-10)

In this section, we will investigate the cases, where eq. (5.1-10) is directly integrable. It turns out that, a necessary and sufficient condition for eq. (5.1-10) to be integrable is

$$A'(\psi) = B(\psi). \tag{5.2-1}$$

This implies that in the case of the KdV equation, eq. (5.1-10) is integrable only if $q = 0$, which is the value of q giving the travelling wave solutions, while for the mKdV equation and KdVB equation for instance, eq. (5.1-10) is integrable for all q, by theorems (5.1-1) and (5.1-2).

The first integral of eq. (5.1-10), if (5.2-1) is satisfied is

$$k_1^2\psi^{2-qN}g'' + \left(\frac{k_1^2}{2}(2 - qN) + \beta k_1\right)\psi^{1-qN}g' + A(\psi)g + \frac{\gamma}{N + 1}g^{N+1} = K. \tag{5.2-2}$$

where K is a constant of integration.

Eq. (5.1-10) is not of a Painlevé type if eq. (5.2-2) is not, for some K. Clearly the converse is not true (i.e., if eq. (5.2-2) is of a Painlevé type for some K, that does not imply that eq. (5.1-10) is of a Painlevé type). However, following the procedure outlined in Ince [44] as in previous chapter gives the following results:

<u>Case (i)</u> ($N > 2$)

It can be easily seen that eq. (5.2-2) is not of Painlevé type if $N > 2$. (This result will also be achieved when we apply Ablowitz algorithm for eq. (5.1-10) in the next section.)

The close connection between Painlevé property and IST according to the Ablowitz conjecture gives that the GKdV equation is not solvable by IST.

<u>Case (ii)</u> ($N = 2$)

For $q = 0$, eq. (5.2-2) will be

$$k_1^2\psi^2g'' + (k_1^2 + \beta k_1)\psi g' + A(\psi)g + \frac{\gamma}{3}g^3 = K.$$

We make the independent transformation $\psi = e^{s}$ to give

$$\ddot{g} = -\frac{\beta}{k_1}\dot{g} - \frac{\gamma}{3k_1^2}g^3 - \frac{A(\psi)}{k_1^2}g + \frac{K}{k_1^2}, \qquad (5.2\text{-}3)$$

(where a dot denotes differentiation with respect to s).
Making a scale transformation $g = \sqrt{-\frac{6}{\gamma}}k_1 h$ we get, the defining equation for the second Painlevé transcendent if $\beta = 0$ and $A(\psi) = -k_1^2 \ln\psi$. If $A(\psi)$ is a constant then eq. (5.2-3) has elliptic function solutions, $(\beta = 0)$.

For $q \neq 0$, it can be shown that $q = 1$ without loss of generality. (If $q \neq 0$, then the transformation $z = \psi^q$ gives the same result as in the case $q = 1$.)

For example, the analysis above shows that the Ablowitz conjecture seems to be true for the mKdV equation since, in this case, $A(\psi)$ is constant in eq. (5.2-3). For $q \neq 0$, a second Painlevé transcendent equation can be obtained (see previous result).

Case (iii) $(N = 1)$

In this case, the Ince procedure shows that, eq. (5.2-2) is free from movable critical points if $A(\psi)$ is a solution to the following equation (5.2-3), for each possible q and K.

$$\psi^M\left[R_1\psi^{-(q+2)} + R_2\psi^{-1}A' + R_3\psi^{q-2}A^2 + R_4\psi^{q-2} + A''\right]$$
$$= P\ln\psi + Q, \quad \text{if } 3q = \frac{\beta}{k_1} \qquad (5.2\text{-}3(a))$$

or,

$$\psi^{M}\left[R_1\psi^{-(q+2)} + R_2\psi^{-1}A' + R_3\psi^{q-2}A^2 + R_4\psi^{q-2} + A''\right]$$

$$= P\psi^{\frac{3q}{5} - \frac{\beta}{5k_1}} + Q, \quad \text{if } 3q \neq \frac{\beta}{k_1}, \qquad (5.2\text{-}3(b))$$

where P and Q are arbitrary constants.

The constants R_i $(i = 1, 2, 3, 4)$, M are defined as follows:

$$M = 2 - q + \frac{4\beta}{5k_1},$$

$$R_1 = \frac{2\beta k_1}{5}\left[-\frac{43}{30}\,\frac{q^2\beta}{k_1} + \frac{3}{4}q^3 + 17q\left(\frac{\beta}{5k_1}\right)^2 - 6\left(\frac{\beta}{5k_1}\right)^3\right],$$

$$R_2 = 1 - \frac{q}{2} + \frac{\beta}{k_1},$$

$$R_3 = \frac{2}{3k_1^{\,2}},$$

$$R_4 = \frac{\gamma K}{k_1^{\,2}}.$$

In the case of the (KdVB) equation, where $\beta \neq 0$, $A(\psi) = \frac{k_2}{k_1}$ and q must be zero so the LHS of (5.2-3(b)) is $\psi^{\frac{4\beta}{5k_1}}\left(R_1 + R_3\left(\frac{k_2}{k_1}\right)^2 + R_4\right)$ which is not equivalent to $P\psi^{-\frac{\beta}{5k_1}}$ unless $R_1 + R_3\left(\frac{k_2}{k_1}\right)^2 + R_4$ is identically zero. But the former constant is a function of the arbitrary constant K. Hence eq. (5.2-2) has movable critical points in general, i.e., the KdVB equation is not solvable by IST. For the KdV equation (if $q = 0$) $A(\psi) = \frac{k_2}{k_1}$ and the LHS of (5.2-3(a)) is a constant. Thus eq. (5.2-2) is of Painlevé type. This supports the Ablowitz conjecture.

5.3 The Ablowitz-Ramani-Segur Algorithm

We present here the method as developed in [6], and we will illustrate it on the variable coefficient eq. (5.1-10).

At this point, one has to notice that this algorithm does not identify essential singularities and provides only necessary conditions for an equation to be of the Painlevé type. There are three steps in the algorithm:

Step 1: Find the dominant behaviour.

We assume that the solution becomes infinite at the singularity and look for a solution of the form,

$$g = \alpha\xi^{k} \quad \text{with} \quad \mathrm{Re}\,k < 0, \ \xi = \psi - \psi_0 \qquad (5.3\text{-}1)$$

where α, k, and ψ_0 are constants. Substituting (5.3-1) into (5.1-10) we get,

$$\beta_1 \alpha k(k-1)(k-2)\psi_0^{\,2-qN}\left[1 + (2-qN)\frac{\xi}{\psi_0} + \ldots\right]\xi^{k-3}$$

$$+ \beta_2 \alpha k(k-1)\psi_0^{\,1-qN}\left[1 + (1-qN)\frac{\xi}{\psi_0} + \ldots\right]\xi^{k-2}$$

$$+ \beta_3 \alpha k \psi_0^{\,-qN}\left[1 + (-qN)\frac{\xi}{\psi_0} + \ldots\right]\xi^{k-1} + \alpha k A(\psi)\xi^{k-1}$$

$$+ \alpha B(\psi)\xi^{k} + \gamma\alpha^{N+1}k\xi^{Nk+k-1} = 0. \qquad (5.3\text{-}2)$$

For some values of k, some terms of this equation balance when $\psi \to \psi_0$, while the others can be ignored. These are called the leading terms of the equation. If any of the possible k's is not an integer, the equation is not of Painlevé type. Otherwise one has to go to the second step.

If $A(\psi)$ and $B(\psi)$ can be represented as Taylor series in ξ, say

$$A(\psi) = \sum_{i=0}^{\infty} \lambda_i \xi^i, \quad B(\psi) = \sum_{i=0}^{\infty} \gamma_i \xi^i, \qquad (5.3\text{-}3)$$

then there is one possibility for the term in ξ^{k-3} (the lowest power of ξ being $k-3$) to balance with the term in ξ^{Nk+k-1} which implies that,

$$k = -\frac{2}{N} \quad \text{and} \quad \gamma\alpha^N = -\beta_1(k-1)(k-2)\psi_0^{\,2-qN}. \qquad (5.3\text{-}4)$$

Clearly k is an integer only if N = 1, 2 which yields:

Theorem (5.3-1)

Eq. (5.1-10) is not solvable by IST if $N \geqslant 3$, and (5.3-3) is satisfied. □

(Note here that this agrees with result found in §5.2 if eq. (5.1-10) is integrable.)

Now we have two cases:

Case (i) $\left[N = 2 \Rightarrow k = -1,\ \alpha^2 = -\frac{6\beta_1}{\gamma}\psi_0^{2(1-q)}\ \text{by (5.3-4)}\right]$

Step 2: Find the resonances:

For every negative integer value of k, the solution of eq. (5.1-10) has an expression in the form of a Laurent series. The resonances are the powers of $(\psi - \psi_0)$ at which the different arbitrary constants enter in this expansion. To find them, one substitutes the following form of g:

$$g = \alpha(\psi - \psi_0)^k + \beta(\psi - \psi_0)^{k+r} \qquad (5.3\text{-}5)$$

into the equation composed of the leading terms, which is,

$$\beta_1\psi_0^{2-qN}g''' + \gamma g^2 g' = 0. \qquad (5.3\text{-}6)$$

To leading order in β, this equation reduces to,

$$Q(r)\beta(\psi - \psi_0)^p = 0, \quad p \geq k + r - 1. \qquad (5.3\text{-}7)$$

The roots of Q(r) (a polynomial in r) determine the resonances, and one can note that:

(i) One root is always -1.

(ii) If α is arbitrary, then 0 is always a root.

(iii) A root with Rer < 0 is ignorable.

(iv) A root with Re$r > 0$, r non-integer, indicates a (movable) branch point.

If there are no branch points, one has to go further to step 3, dealing only with positive integer resonances.

In our example

$$Q(r) = (r - 3)(r + 1)(r - 4)$$

which implies that the positive integer resonances are $r = 3$, and $r = 4$.

Step 3

Let (k, α) be given as in step 1, and $r_1 < r_2 < \ldots < r_s$, denote the resonances in general, i.e., the positive integer roots of $Q(r)$. (In our case $r_1 = 3$, $r_2 = 4$.) We substitute

$$g = \alpha\xi^k + \sum_{j=0}^{r_s} a_j \xi^{k+j} = \alpha\xi^{-1} + \sum_{j=1}^{4} \xi^{j-1} \qquad (5.3\text{-}8)$$

into the full equation, and calculate by recurrence the coefficients a_j, $\forall_j$, from the equation,

$$Q(j)a_j - R_j(\psi_0, \alpha, a_1, \ldots, a_{j-1}) = 0 \qquad (5.3\text{-}9)$$

which is obtained from the recurrence relations.

(i) For $j < r_1 = 3$, i.e., $j = 1$ and 2, a_j is to be given by

$$a_1 = \frac{-3\beta_1(2 - qN) + \beta_2}{\gamma\alpha}\psi_0^{1-qN}$$

and

$$a_2 = \frac{1}{\alpha\gamma}\left(\left(-3(2 - qN)(1 - qN)\beta_1 + 2\beta_2(1 - qN) - \beta_3\right)\psi_0^{-qN} - \lambda_0 - \gamma a_1^2\right).$$

(ii) For j = 3, we have the identity

$$0.a_3 + R_3(\psi_0, \alpha, a_1, a_2) = 0$$

where $R_3 = \gamma_0 - \lambda_1$.

If $R_3(\psi_0, \alpha, a_1, a_2) \neq 0$, then one has to introduce log terms into the expansion and the equation is not of Painlevé type. If $R_3 = 0$, then a_3 is an arbitrary constant of integration and one has to proceed to the next coefficients as we will see later.

Theorem (5.3-2)

If $A'(\psi) = B(\psi)$ in eq. (5.1-10) with the conditions (5.3-3), then $R_3 = 0$. $\square$

Proof If $A(\psi)$ can be represented as the Taylor series $\sum_{i=0}^{\infty} \lambda_i \xi^i$, and $B(\psi)$ as the Taylor series $\sum_{i=0}^{\infty} \gamma_i \xi^i$ then $A' = B$ yields

$$\sum_{i=0}^{\infty} \gamma_i \xi^i = \sum_{i=0}^{\infty} i\lambda_i \xi^{i-1}$$

which implies that $\gamma_i = (i + 1)\lambda_{i+1}$ $\forall i$.

In particular $\gamma_0 = \lambda_1$ and hence $R_3 = 0$. $\blacksquare$

(iii) For $j = 4$, we have

$$0.a_4 + R_4(\psi_0, \alpha, a_1, a_2, a_3) = 0$$

where $R_4 = Pa_3 + S$.

If $P \neq 0$, then a_4 is a function of a_3 (i.e., a_4 is arbitrary constant of integration) and no logarithms are introduced at any resonances, which means that eq. (5.1-10) has met the necessary conditions for Painlevé property under the assumption that $k < 0$.

We calculate P and S as follows:

$$P = 2\beta_2\psi_0^{1-qN} + 2\alpha\gamma a_1 = 4\beta k_1\psi_0^{1-qN}$$

$$\begin{aligned}
S = {} & \alpha\psi_0^{-2-qN}(-qN)(-qN-1)\left(-\frac{\beta_1}{4}(2-qN)(1-qN)\right. \\
& \left. + \frac{\beta_2}{3}(1-qN) - \frac{\beta_3}{2}\right) + a_2(\beta_3\psi_0^{-qN} + \lambda_0) + a_1\gamma_0 - \alpha\lambda_2 \\
& + \alpha\gamma_1 + \gamma\alpha a_2^2 + \gamma a_1^2 a_2 \\
= {} & \beta\left[\alpha\psi_0^{-(2+qN)}(-qN)(-qN-1)\left(-\frac{k_1}{6}\right)(1-qN)\right. \\
& + \frac{2k_1}{\alpha\gamma}(1-qN)\psi_0^{-mn}\left\{\psi_0^{-qN}\left(-\frac{\alpha k_1^2}{2}(2-qN)(1-qN)\right.\right. \\
& \left. + \beta k_1(1-qN)\right) - \lambda_0 - \frac{1}{\alpha^2\gamma}\psi_0^{2(1-qN)}\left(-\frac{3}{2}(2-qN)\alpha k_1^2\right. \\
& \left.\left.\left. + \beta k_1\right)^2\right\}\right] + \frac{\gamma_0}{\alpha\gamma}\left(-\frac{3}{2}(2-qN)\alpha k_1^2 + \beta k_1\right)\psi_0^{1-qN} - \alpha\lambda_2 \\
& + \alpha\gamma_1.
\end{aligned}$$

Clearly, if $\beta = 0$ then $P = 0$,

$$S = \frac{\gamma_0}{\gamma}\left(-\frac{3}{2}\right)(2 - qN)k_1{}^2\psi_0{}^{1-qN} - \alpha\lambda_2 + \alpha\gamma_1.$$

Case (ii) $\left[N = 1,\ k = -2,\ \alpha = \frac{-12\beta_1}{\gamma}\psi_0{}^{2-q},\ \text{by (5.3-4)}.\right]$

We illustrate this case by applying step 2 and 3 of the Ablowitz algorithm to the o.d.e. which is obtainable from the KdV equation

$$\phi_{xxx} + 6\phi\phi_x + \phi_t = 0. \tag{5.1-2}$$

The positive resonances in this case are $r_1 = 4$ and $r_2 = 6$, and the coefficients a_j's for $j = 1, \ldots, 6$ are as follows:

$$a_1 = (q - 2)\psi_0{}^{1-q}$$

$$a_2 = -\frac{1}{24}(q - 2)(5q - 2)\psi_0{}^{-q} + \frac{1}{6}\psi_0{}^{q/2}$$

$$a_3 = \frac{1}{48}q(q^2 - 4)\psi_0{}^{-q-1}$$

a_4 is arbitrary

$$a_5 = \frac{1}{6}\left\{3a_4(q - 2)\psi_0{}^{-1} - \frac{q}{6}\psi_0{}^{2q-3} + \frac{q}{960}(q - 2)(q + 2)(q^2 + 30q - 16)\psi_0{}^{-q-3}\right\}$$

a_6 is arbitrary.

In this case, no logarithms are introduced at any of the resonances for (k, α) from step 1, which means that eq. (5.1-10) has met the necessary condition for Painlevé property under the assumption that $k < 0$. This supports the Ablowitz conjecture. (This application of the Ablowitz algorithm is well known.)

CHAPTER SIX

The Connection between Similarity and Separation Methods

6.1 Introduction

In this chapter, we consider the problem of investigating the connection between the similarity and separation methods for solving p.d.e.'s with no explicit dependence on the independent variables.

It is well known that the mathematical interpretation of the "general similarity" is a transformation of independent and dependent variables occurring in the equation such that a reduction in the number of independent variables is achieved. This similarity transformation will reduce a problem in two independent variables from a p.d.e. to an o.d.e. The interesting relation between the Painlevé type o.d.e.'s having no movable critical points and the solvability of evolution equations by IST seems to be shown through similarity solutions.

Using independent and dependent transformations, a p.d.e. may be transformed to a simply separable equation (see the previous chapters). The procedure can be considered to transform the p.d.e. into an o.d.e. for the transformation, as we explained previously. Hence the separation transformation is similar to the similarity transform for both of them involve the reduction of a p.d.e. to an o.d.e.

For this reason and other reasons (given later) we study the relationship between the two methods.

The plan of this chapter is as follows: In order that the chapter is self contained, we give a brief outline of the precise versions, we have adopted for comparison of the similarity and separation methods in §6.2. In §6.3-5 we look for possible connections between the two methods through some lemmas, illustrations and examples involving certain nonlinear p.d.e.'s which are being intensively studied at present in theoretical physics and applied mathematics, through the use of the similarity method. In §6.6, we present our comparison results in tabular form for the equations that we have analyzed and many other equations. In §6.7, we give a brief discussion of our results and their implications.

6.2 Lie's Transformation and the Separation Transformation

In recent years, modern algebraic similarity methods have been developed with the aid of group theory. The invariance conditions enable one to find the infinitesimal transformations, associated with a given differential equation. In practice, to use infinitesimal transformations to obtain similarity solutions of a given p.d.e. is to first to seek the largest set of infinitesimal transformations leaving invariant the governing p.d.e. The infinitesimal transformations satisfy a set of "determining equations" which are of such a number that they seem to be solvable in closed form [11].

Having found the infinitesimal transformations

(1.4-22) and solving Lagrange's equation (1.4-25) (see §1.4) the similarity solutions of the equation are given by

$$u = F(x, t, s, f(s)) \tag{6.2-1}$$

where s is called the similarity variable and f(s) becomes the new dependent variable. The dependence of F on (x, t, s, f(s)) is known explicitly and by substituting (6.2-1) into the given p.d.e. we obtain an o.d.e. for f(s).

What we mean by the precise version of the separation method is to use the dependent variable transformation explained and applied in previous chapters as,

$$\phi(x, t) = u(x, t)g(\psi(x, t)) + h(x, t), \quad \psi = X(x)T(t) \tag{6.2-2}$$

where ϕ is the dependent variable for the p.d.e., X' and T' are one term series, u and h are initially unknown functions of x and t, and ψ and g are the independent and dependent variables respectively of the obtainable o.d.e.

6.3 Travelling Wave Solutions

In looking for possible connections between similarity and separable solutions, let SM, SP denote the sets of all possible similarity and separable solutions respectively. Thus the possible connections between the two methods can be illustrated by the following figures:

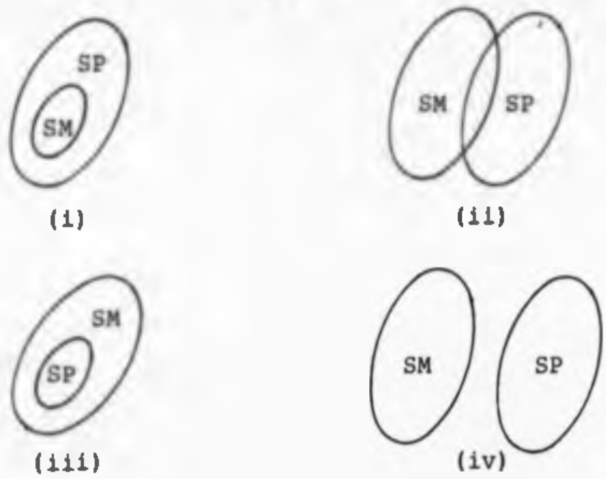

Fig. (6.3-1)

We claim that, the situation (iv) in fig. (6.3-1) is not possible for constant coefficient equations since the two methods both provide travelling wave solutions. This is supported by the following results.

It is rather difficult to prove either situations (i) or (iii). This is due to the nonexistence of the general similarity or separation transformations.

To compare the two methods according to the available solutions, one can see that, for instance, in case of sG equation SP is more general than SM, while for the Burgers equation the opposite is true.

Most material in the rest of this chapter, therefore will concentrate on situation (ii).

Lemma (6.3-1)

The p.d.e.

$$F(\phi,\ \phi_x,\ \phi_t,\ \phi_{xx},\ \phi_{xt},\ \phi_{tt}) = 0 \qquad (6.3\text{-}1)$$

is invariant under the following transformations,

$$\left.\begin{aligned} x^* &= x + \varepsilon\xi(x,\ t,\ \phi) + O(\varepsilon^2) \\ t^* &= t + \varepsilon\tau(x,\ t,\ \phi) + O(\varepsilon^2) \\ \phi^* &= \phi + \varepsilon\eta(x,\ t,\ \phi) + O(\varepsilon^2) \end{aligned}\right\} \tag{6.3-2}$$

where $\xi = \alpha$, $\tau = \beta$, α and β are constants, and $\eta = 0$. □

<u>Proof</u> It can easily be proved that any derivative, $\dfrac{\partial^{p+q}\phi^*}{\partial^p x^* \partial^q t^*}$ where p, q are non negative integers, is equal to the derivative $\dfrac{\partial^{p+q}\phi}{\partial^p x \partial^q t}$, under the hypothesis. Hence

$$F(\phi^*,\ \phi^*_{x^*},\ \phi^*_{t^*},\ \phi^*_{x^*x^*},\ \phi^*_{x^*t^*},\ \phi^*_{t^*t^*})$$

$$= F(\phi,\ \phi_x,\ \phi_t,\ \phi_{xx},\ \phi_{xt},\ \phi_{tt}).\ \blacksquare$$

<u>Theorem (6.3-2)</u>

Eq. (6.3-1) has the similarity solution

$$\phi(x,\ t) = f(s), \quad s = ax + bt \tag{6.3-3}$$

where a, b are constants. □

<u>Proof</u> By lemma (6.3-1), eq. (6.3-1) is invariant under (6.3-2). Hence, solving Lagrange's equation (1.4-33) implies (6.3-3). ■

Thus, travelling wave solutions of eq. (6.3-1) (i.e., the solutions that depend on the space and time variables x and t only in the combination $(k_1x + k_2t)$ where k_1 and k_2 are constants) are similarity solutions.

Theorem (6.3-3)

The travelling wave solution (6.3-3) is a separable solution of eq. (6.3-1). □

Proof It can be easily seen that the solution (6.3-3) is one of the guaranteed separable solutions,

$$\phi(x, t) = u(x, t)g(\psi), \quad \psi = X(x)T(t), \quad X' = aX^n,$$
$$T' = bT^m,$$

of eq. (6.3-1) if $u \equiv 1$ and $n = m = 1$. ■

6.4 Explicit Analysis of Typical Cases

In this section, we will demonstrate the connection between the separation and similarity transformations using a couple of specific nonlinear evolution equations, before representing the results for a full set of equations we wish to explore in the following section. We illustrate, in this section as well, an interesting feature of the KdV and the mKdV equations.

A. The KdV equation

We consider the KdV equation

$$\phi_t + 6\phi\phi_x + \phi_{xxx} = 0. \tag{6.4-1}$$

Applying the transformation $\phi = T^q g(\psi)$ and following the discussion of §5.1 the equation becomes,

$$k_1{}^2\psi^{2-q}g''' + \tfrac{3}{2}k_1{}^2(2 - q)\psi^{1-q}g'' + \left[\frac{k_1{}^2}{2}(2 - q)(1 - q)\psi^{-q} + \frac{k_2}{k_1}\psi^{q/2}\right]g' + q\frac{k_2}{k_1}\psi^{q/2-1}g + 6gg' = 0 \qquad (6.4\text{-}2)$$

which is directly integrable if and only if $q = 0$. In this case, the travelling wave solution is obtained.

For $q \neq 0$, the guaranteed separable solution is

$$\phi = \left(K_2 - \tfrac{3q}{2}k_2t\right)^{-2/3} g\left[\left(K_1 + \tfrac{q}{2}k_1x\right)^{2/q}\left(K_2 - \tfrac{3q}{2}k_2t\right)^{-2/3q}\right] \qquad (6.4\text{-}3)$$

where K_1 and K_2 are constants of integration.

The similarity solution for the KdV equation is obtained by substitution $q = 2$ in (6.4-3) and the obtainable o.d.e. in this case can be reduced to the defining equation for the second Painlevé transcendent [55].

Now, let $z = \psi^{q/2}$, $h(z) = g(\psi)$ and $R(x) = \left(K_1 + \tfrac{q}{2}k_1x\right)$, $S(t) = \left(K_2 - \tfrac{3q}{2}k_2t\right)^{-1/3}$. Thus $\phi = S^2(t)h(z)$, $z = R(x)S(t)$. This implies that $\phi = T^2g(\psi)$ without loss of generality. Therefore $q = 2$ (which provides the similarity solution) represents all the nonzero values of q for the guaranteed solutions, i.e., $q \neq 0 \Rightarrow q = 2$ without loss of generality. Fig. (6.4-1) shows the connection between the sets SM and SP for the KdV equation:

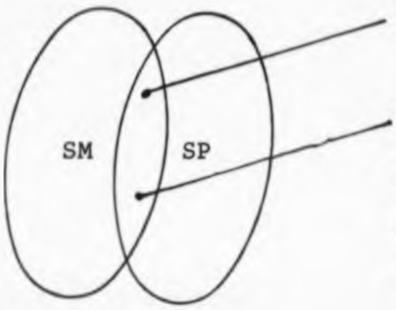

Fig. (6.4-1)

B. The mKdV equation

The guaranteed separable solution for the mKdV equation

$$\phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0 \tag{6.4-4}$$

when $q \neq 0$ is

$$\phi = (K_2 - 3qk_2t)^{1/3}g[(qk_1x + K_1)^{1/q}(K_2 - 3qk_2t)^{1/3q}] \tag{6.4-5}$$

and the similarity solution is obtained by substituting $q = 1$ in (6.4-5) where the o.d.e. for g is

$$g'' = \frac{2}{k_1^2}g^3 - \frac{k_2}{k_1^3}\psi g + \frac{K}{k_1^2} \tag{6.4-6}$$

with K as the constant of integration. Clearly (6.4-6) is the defining equation for the second Painlevé transcendent after a scale transformation.

Applying the independent transformation $z = \psi^q$ to

(6.4-5) gives that $\phi = Tg(\psi)$, i.e., $q = 1$ represents all the non-zero values of q for the mKdV equation. Fig. (6.4-1) again shows the connection between SM and SP in the case of the mKdV equation.

For the above equations, the analysis shows:

Property (6.4-1)

The guaranteed separable solutions are only given by,

$$\phi_1 = g_1(\psi) \quad \text{and} \quad \phi_2 = T^{q^*} g_2(\psi)$$

where q^* is any value of $q \neq 0$ (say $q^* = 1$, without loss of generality). This property can be proved as above for any equation of KdV type (3.2-5).

Property (6.4-2)

The above guaranteed separable solutions are similarity solutions.

In the following section, we will discuss the question of whether the properties (6.4-1,2) are satisfied by any other equations.

6.5 The Class of Equations With No Mixed Derivatives

It is naturally reasonable to see if the properties (6.4-1,2) are applicable to wider class of equations.

In this section, therefore, we study, firstly the properties that the KdV type equations possess such that property (6.4-1) is satisfied and we derive a wider class of equations with this property; secondly, an attempt will be made to prove property (6.4-2) for that class of equations or even a subclass of it.

For equations with no dependence on (x, t), we note (see a previous section) that the travelling wave solution is given by $\phi = g(\psi)$, $\psi = Ke^{k_1x+k_2t}$, where K, k_1 and k_2 are arbitrary constants.

Now, if the guaranteed separable solution is assumed to be $\phi = T^q g(\psi)$ for some constant q and $\psi = T(x)T(t)$, $X' = k_1X^n$ and $T' = k_2T^m$, then we have the following result:

<u>Lemma (6.5-1)</u>

The constants n and m are necessarily of the forms,

$$n = 1 + \mu(q), \quad m = 1 + \nu(q) \qquad (6.5\text{-}1)$$

where q = 0 implies $\mu \equiv \nu \equiv 0$. □

<u>Proof</u> To provide a travelling wave solution, it is necessary that q = 0 implies that n = m = 1. Hence n and m can be considered as functions of q, say $n = \mu_1(q)$, $m = \nu_1(q)$ where $\mu_1 = \nu_1 = 1$ if q = 0. Rewriting $\mu_1(q)$ and $\nu_1(q)$ as

$$\mu_1(q) = 1 + \mu(q), \quad \nu_1(q) = 1 + \nu(q)$$

proves the result. ∎

Let us now study the properties of μ and ν if property (6.4-1) is satisfied; in more precisely, to see when $q \neq 0$ is represented by $q = 1$ without loss of generality.

Lemma (6.5-2)

$q = 1$ represents all the non-zero values of q if and only if

$$\mu(q) = -\alpha q \quad \text{and} \quad \nu(q) = -\beta q \tag{6.5-2}$$

where α and β are constants. □

Proof It is clear that if (6.5-2) is satisfied then

$$X = (k_1 x + c_1)^{1/\alpha q}, \quad T = (k_2 t + c_2)^{1/\beta q}$$

where k_i and c_i $(i = 1, 2)$ are arbitrary constants. Thus

$$\phi = T^q g(\psi) + h(x, t) = (k_2 t + c_2)^{1/\beta} g\left\{\left[(k_1 x + c_1)^{1/\alpha} (k_2 t + c_2)^{1/\beta}\right]^{1/q}\right\} + h$$

which is by the independent transformation $z = \psi^q$, reduced to

$$\phi = (k_2 t + c_2)^{1/\beta} g\left[(k_1 x + c_1)^{1/\alpha} (k_2 t + c_2)^{1/\beta}\right] + h$$

i.e., $q = 1$ without loss of generality.

To prove that the condition (6.5-2) is sufficient, we have to consider

$$\phi = T^q g(\psi) + h(x, t) = (k_2 t + c_2)^{q/\nu(q)}$$

$$g\left[(k_1 x + c_1)^{1/\mu(q)} (k_2 t + c_2)^{1/\nu(q)}\right] + h$$

(where k_i, c_i ($i = 1, 2$) are arbitrary constants), as independent of q. This can happen only when $\mu(q) = \alpha q$ and $\nu(q) = \beta q$ for some constants α and β. ∎

To check if there are some equations, for which (6.5-2) satisfied, let us recall theorem (3.2-6) and use it for the following result:

<u>Corollary (6.5-3)</u>

The class of equations (3.2-8) possess property (6.4-1) if $\exists j \ni \sum_{i=1}^{r} i(a_{ji} - a_{1i}) \neq 0, \quad j = 2, \ldots, N1.$ □

<u>Proof</u> By theorem (3.2-6), applying the transformation $\phi = T^q g(\psi)$ to eq. (3.2-8) gives the system of equations,

$$\left.\begin{aligned} &P_k q + (n-1)Q_k = 0, \quad k = 2, \ldots, N1 \\ &R_k q + (m-1)S_k + (n-1)T_k = 0, \quad k = N1+1, \ldots, N \end{aligned}\right\} \tag{6.5-3}$$

where,

$$\left.\begin{aligned} P_k &= \sum_{i=0}^{r} (a_{ki} - a_{1i}) \\ Q_k &= \sum_{i=1}^{r} i(a_{1i} - a_{ki}) \end{aligned}\right\} \quad k = 2, \ldots, N1$$

$$\left.\begin{aligned} R_k &= \sum_{i=1}^{s} a_{ki} - \sum_{i=0}^{r} a_{1i} \\ S_k &= \sum_{i=1}^{s} ia_{ki} \\ T_k &= \sum_{i=1}^{r} ia_{1i} \end{aligned}\right\} \quad k = N1 + 1, \ldots, N.$$

By the hypothesis, $\exists j \ni Q_j \neq 0$; hence

$$n = 1 + \alpha q$$

where $\alpha = -P_j/Q_j$ which can be zero. Therefore

$$(R_k + T_k\alpha)q + (m - 1)S_k = 0 \quad \forall k = N1 + 1, \ldots, N$$

which implies that

$$m = 1 + \beta q$$

where $\beta = 0$ if $R_k + \alpha T_k = 0$ $\forall k = N1 + 1, \ldots, N$; otherwise

$$\beta = -\frac{S_\ell}{R_\ell + \alpha T_\ell}, \quad R_\ell + \alpha T_\ell \neq 0.$$

Now we have to see whether property (6.4-2) may be satisfied by class of equations (3.2-8). We have to look for similarity solution of the form $\phi = T^{q^*}g(\psi)$, $\psi = XT$ and q^* any non-zero specified constant.

Consider (3.2-8) as a second order equation:

$$H = \sum_{k=1}^{N1} \alpha_k \phi^{a_{k0}} \phi_x^{a_{k1}} \phi_{xx}^{a_{k2}} + \sum_{k=N1+1}^{N} \alpha_k \phi^{a_{k0}} \phi_t^{a_{k1}} \phi_{tt}^{a_{k2}}$$

$$= 0. \tag{6.5-4}$$

Lemma (6.5-4)

Eq. (6.5-4) is invariant under the transformation (6.3-2), where

$$\left.\begin{aligned} \xi &= \lambda_1 x + \nu_1 \\ \tau &= \lambda_2 t + \nu_2 \\ \eta &= \lambda_3 \phi \end{aligned}\right\}. \tag{6.5-5}$$

The constants ν_i $(i = 1, 2)$ are arbitrary, while λ_i $(i = 1, 2, 3)$ satisfy the system of equations,

$$\left.\begin{aligned} &(a_{k0} + a_{k1} + a_{k2})\lambda_3 - (2a_{k2} + a_{k1})\lambda_1 = 0, \\ &\qquad\qquad\qquad\qquad k = 1, \ldots, N1 \\ &(a_{k0} + a_{k1} + a_{k2})\lambda_3 - (2a_{k2} + a_{k1})\lambda_2 = 0, \\ &\qquad\qquad\qquad\qquad k = N1 + 1, \ldots, N \end{aligned}\right\}. \;\square \tag{6.5-6}$$

Proof Following the technique described in [11], a necessary condition for the equation to be invariant is,

$$[\eta_{xx}]\frac{\partial H}{\partial \phi_{xx}} + [\eta_{tt}]\frac{\partial H}{\partial \phi_{tt}} + [\eta_x]\frac{\partial H}{\partial \phi_x} + [\eta_t]\frac{\partial H}{\partial \phi_t} + \eta\frac{\partial H}{\partial \phi} = 0$$

where

$$[\eta_{xx}] = \eta_{xx} + (2\eta_{x\phi} - \xi_{xx})\phi_x + \tau_{xx}\phi_t + (\eta_{\phi\phi} - 2\xi_{x\phi})\phi_x^{\,2}$$

$$- 2\tau_{x\phi}\phi_x\phi_t - \xi_{\phi\phi}\phi_x^{\,3} - \tau_{\phi\phi}\phi_x^{\,2}\phi_t + (\eta_\phi - 2\xi_x)\phi_{xx}$$

$$- 2\tau_x\phi_{xt} - 3\xi_\phi\phi_{xx}\phi_x - \tau_\phi\phi_{xx}\phi_t - 2\tau_\phi\phi_{xt}\phi_x$$

$$= (\lambda_x - 2\lambda_1)\phi_{xx}$$

$$[\eta_{tt}] = \eta_{tt} + (2\eta_{t\phi} - \tau_{tt})\phi_t - \xi_{tt}\phi_x + (\eta_{\phi\phi} - 2\tau_{t\phi})\phi_t^{\,2}$$

$$- 2\xi_{t\phi}\phi_x\phi_t - \tau_{\phi\phi}\phi_t^{\,3} - \xi_{\phi\phi}\phi_t^{\,2}\phi_x + (\eta_\phi - 2\tau_t)\phi_{tt}$$

$$- 2\xi_t\phi_{xt} - 3\tau_\phi\phi_{tt}\phi_t - \xi_\phi\phi_{tt}\phi_x - 2\xi_\phi\phi_{xt}\phi_t$$

$$= (\lambda_3 - 2\lambda_2)\phi_{tt}$$

$$[\eta_x] = \eta_x + (\eta_\phi - \xi_x)\phi_x - \tau_x\phi_t - \xi_\phi\phi_x^{\,2} - \tau_\phi\phi_x\phi_t$$

$$= (\lambda_3 - \lambda_1)\phi_x$$

$$[\eta_t] = \eta_t + (\eta_\phi - \tau_t)\phi_t - \xi_t\phi_x - \tau_\phi\phi_t^{\,2} - \xi_\phi\phi_x\phi_t$$

$$= (\lambda_3 - \lambda_2)\phi_t$$

Therefore,

$$(\lambda_3 - 2\lambda_1)\phi_{xx}\left[\sum_{k=1}^{N1} a_{k2}\alpha_k\phi^{a_{k0}}\,\phi_x^{\,a_{k1}}\,\phi_{xx}^{\,a_{k2}-1}\right]$$

$$+ (\lambda_3 - 2\lambda_2)\phi_{tt}\left[\sum_{k=N1+1}^{N} a_{k2}\alpha_k\phi^{a_{k0}}\,\phi_t^{\,a_{k1}}\,\phi_{tt}^{\,a_{k2}-1}\right]$$

$$+ (\lambda_3 - \lambda_1)\phi_x\left[\sum_{k=1}^{N1} a_{k1}\alpha_k\phi^{a_{k0}}\,\phi_x^{\,a_{k1}-1}\,\phi_{xx}^{\,a_{k2}}\right]$$

$$+ (\lambda_3 - \lambda_2)\phi_t\left[\sum_{k=N1+1}^{N} a_{k1}\alpha_k\phi^{a_{k0}}\,\phi_t^{\,a_{k1}-1}\,\phi_{tt}^{\,a_{k2}}\right]$$

$$+ \lambda_3\phi\left[\sum_{k=1}^{N1} a_{k0}\alpha_k\phi^{a_{k0}-1}\,\phi_x^{\,a_{k1}}\,\phi_{xx}^{\,a_{k2}}\right]$$

$$+ \lambda_3\phi\left[\sum_{k=N1+1}^{N} a_{k0}\alpha_k\phi^{a_{k0}-1}\,\phi_t^{\,a_{k1}}\,\phi_{tt}^{\,a_{k2}}\right] = 0. \qquad (6.5\text{-}7)$$

We now collect together the coefficients of like derivative terms in ϕ and set all of them equal to zero. The resulting equations are the system (6.5-6). ∎

<u>Corollary (6.5-5)</u>

Property (6.4-2) is satisfied by (6.5-4) if there is a nontrivial solution to (6.5-6). □

<u>Proof</u> To satisfy the property, we have to solve the Lagrange's equation

$$\frac{dx}{\lambda_1 x + \nu_1} = \frac{dt}{\lambda_2 t + \nu_2} = \frac{d\phi}{\lambda_3\phi}.$$

Clearly, if $\lambda_1 = \lambda_2 = \lambda_3 = 0$ then the travelling wave solution $\phi = g(\psi)$, $\psi = Ke^{k_1x+k_2t}$ can be obtained, by theorem (6.3-2).

If there is a nontrivial solution to (6.5-6) then

$$s = (\lambda_1 x + \nu_1)^{1/\lambda_1}(\lambda_2 t + \nu_2)^{-1/\lambda_2}$$

and

$$\phi = (\lambda_2 t + \nu_2)^{-\lambda_3/\lambda_2} f(s).$$

Thus, if we denote $(\lambda_1 x + \nu_1)^{1/\lambda_1}$, $(\lambda_2 t + \nu_2)^{-1/\lambda_2}$, s and f(s) by X, T, ψ and $g(\psi)$ respectively then

$$\phi = T^{-\lambda_3} g(\psi). \blacksquare$$

6.6 Various Equations and their Guaranteed Separable and Similarity Solutions

The equations that we consider fall into these categories:

A. IST-solvable evolution equations

We consider the following:

(1) The Korteweg-de Vries (KdV) equation:

$$u_t + uu_x + u_{xxx} = 0. \qquad (6.6\text{-}1)$$

(2) The cylindrical KdV (cKdV) equation:

$$u_t + \frac{1}{2t}u + 6uu_x + u_{xxx} = 0. \qquad (6.6\text{-}2)$$

(3) The modified KdV (mKdV) equation:

$$u_t + u^2 u_x + u_{xxx} = 0. \qquad (6.6\text{-}3)$$

(4) The one-dimensional sine-Gordon (sG) equation:

$$u_{tt} - u_{xx} + \sin u = 0. \qquad (6.6\text{-}4)$$

(5) The Boussinesq (B) equation:

$$u_{tt} + u_{xx} + (u^2)_{xx} + u_{xxxx} = 0. \qquad (6.6\text{-}5)$$

B. Other equations

We consider the following:

(6) The generalized KdV (GKdV) equation:

$$u_t + u^N u_x + u_{xxx} = 0, \quad N > 2. \qquad (6.6\text{-}6)$$

(7) The KdV-Burgers (KdVB) equation:

$$u_t - \mu u u_x + \nu u_{xxx} = \gamma u_{xx}. \qquad (6.6\text{-}7)$$

(8) The Benjamin-Bona-Mahony (BBM) equation:

$$u_t + u_x + u u_x - u_{xxt} = 0. \qquad (6.6\text{-}8)$$

(9) Fisher's equation:

$$u_{xx} - u_t + u - u^n = 0. \qquad (6.6\text{-}9)$$

(10) The phi-four (ϕ^4) equation:

$$u_{tt} - u_{xx} + u - u^3 = 0. \qquad (6.6\text{-}10)$$

(11) The two-dimensional transonic (T) equation:

$$u_x u_{xx} = u_{tt} \qquad (6.6\text{-}11)$$

(12) The Burgers equation:

$$u_t + \mu u u_x - \nu u_{xx} = 0. \qquad (6.6\text{-}12)$$

(13) The nonlinear wave equation:

$$u_{tt} - u^2 u_{xx} = 0. \qquad (6.6\text{-}13)$$

(14) The basic cylindrical transonic (BT) equation:

$$(\gamma + 1) u_x u_{xx} = u_{tt} + \frac{1}{t} u_t \qquad (6.6\text{-}14)$$

The available similarity solutions and guaranteed separable solutions for the above equations will be listed in the following table. Note that all the constants in the table are arbitrary unless specified.

6.7 Discussion

For our results in table (6.6-1), the separable solutions are achieved by using the transformation

$$u = T^q g(\psi) + h(x, t), \quad \psi(x, t) = X(x)T(t), \quad X' = k_1 X^n,$$
$$T' = k_2 T^m$$

where q, k_1, k_2, n and m are constants, h(x, t) is an

Table (6.6-1) Similarity solutions and guaranteed separable solutions for the equations (6.6.1) - (6.6-14) and references for the similarity solutions

Equations	Similarity solutions	Guaranteed separable solutions	References
KdV	$s = \frac{1}{2}\alpha^{-5/3}(\alpha t + \delta)^{-1/3}[2\alpha^2 x - 3\beta(\alpha t + \delta) + 6(\alpha\gamma - \beta\delta)]$ $u = \frac{3\alpha\beta}{2} - \left(\frac{3}{2}\right)^{-2/3}\alpha^{2/3}(\alpha t + \delta)^{-2/3}f(s)$	$\left.\begin{aligned} X' &= k_1 X^{1-q/2} \\ T' &= k_2 T^{1+3q/2} \\ u &= T^q g(\psi) \end{aligned}\right\}$ $q = 0,\ 2$	[81]
cKdV	$s = \left(x + \frac{\delta}{\alpha}\right)t^{-1/3}$ $u = t^{-2/3}f(s)$	$X' = k_1$ $T' = k_2 T^4$ $u = T^q g(\psi)$	[55]
mKdV	$s = 3^{-1/3}\left(x + \frac{3\beta}{\alpha}\right)\left(t + \frac{\delta}{\alpha}\right)^{-1/3}$ $u = 3^{-1/3}\left(t + \frac{\delta}{\alpha}\right)^{-1/3}f(s)$	$\left.\begin{aligned} X' &= k_1 X^{1-q} \\ T' &= k_2 T^{1+3q} \\ u &= T^q g(\psi) \end{aligned}\right\}$ $q = 0,\ 1$	[55]

Table (6.6-1) (continued)

sG	$s = -\left(\frac{1}{2\alpha}\right)\left[\frac{1}{2}\alpha(x^2 - t^2) - \beta x + \delta t - \frac{1}{2\alpha}(\delta^2 - \beta^2)\right]$ $u = f(s)$	Travelling wave solution	[55]
B	$s_1 = x^2/t, \quad u_1 = \frac{1}{t}f_1(s_1) + \frac{1}{2}$ or $s_2 = x - \frac{\delta}{2\theta}t^2, \; u_2 = \frac{1}{2}f_2(s_2) + \frac{\delta^2}{2\theta^2}t^2$	$\left.\begin{array}{l} X' = k_1X^{1-q/2} \\ T' = k_2T^{1+q} \end{array}\right\} q = 0,\ 1$ $u = T^q g(\psi) + \frac{1}{2} \quad \text{if } q \neq 0$	[65, 55]
GKdV	$s = \left(x + \frac{3\beta}{\alpha}\right)(\alpha t + \beta)^{-1/3}$ $u = (\alpha t + \delta)^{-2/3N} f(s)$	$\left.\begin{array}{l} X' = k_1X^{1-Nq/2} \\ T' = k_2T^{1+3Nq/2} \end{array}\right\} q = 0,\ \frac{2}{N}$ $u = T^q g(\psi)$	[55]
KdVB	$s = x + \frac{\mu\beta}{2\alpha}t^2 - \frac{\delta}{\alpha}t$ $u = \frac{\beta}{\alpha}t + f(s)$	Travelling wave solution	[55, 54]

Table (6.6-1) (continued)

BBM	$s = \alpha x + \delta \ln(\alpha t + \beta)$ $u = (\alpha t + \beta)^{-1} f(s) - 1$	$X' = k_1 X$ $T' = k_2 T^{1+q}$ $u = T^q g(\psi) - 1$	[55]
Fisher	$s = \beta x - \alpha t$ $u = f(s)$	Travelling wave solution	[55]
ϕ^4	$s = -\left(\frac{1}{2\alpha}\right)\left[\frac{1}{2}\alpha(x^2 - t^2) + \beta x - \delta t - \frac{1}{2\alpha}(\delta^2 - \beta^2)\right]$ $u = f(s)$	Travelling wave solution	[55]
T	$s = (\alpha x + \beta)^{1/\alpha}(\sigma t + \mu)^{-1/\sigma}$ $u = (\sigma t + \mu)^{(3\alpha-2\sigma)/\sigma} f(s) + \delta t + \lambda$	$X' = k_1 X^n$ $T' = k_2 T^{\frac{1}{2}(q-3n+5)}$ $u = T^q g(\psi) + \delta t + \lambda$	[11]

Table (6.6-1) (continued)

Burgers	$s = \dfrac{b_1\xi - b_2\tau}{(\tau^2 + b_1)^{-1/2}}$, $\tau = a_3 t + \frac{1}{2}a_2$ $\xi = a_3 x + a_5$ $b_1 = a_1 a_3 - \frac{1}{4}a_2^2$ $b_2 = a_3 a_4 - \frac{1}{2}a_2 a_5$ $u = \left(\frac{b_1}{\mu}\right)^{-1/2}(\tau^2 + b_1)^{-1} f(s) + \dfrac{b_2 + \xi\tau}{\mu(\tau^2 + b_1)^{1/2}}$	$\left.\begin{array}{l} X' = k_1 X^{1-q} \\ T' = k_2 T^{1+2q} \end{array}\right\} q = 0, 1$ $u = T^q g(\psi)$	[21]
NW	$s = (\alpha_5 x + \alpha_2)^{1/\alpha_5}(\alpha_1 t + \alpha_3)^{-1/\alpha_1}$, $\alpha_5 = \alpha_1 + \alpha_4$ $u = (\alpha_1 t + \alpha_3)^{\alpha_4/\alpha_1} f(s)$	$X' = k_1 X^n$ $T' = k_2 T^{q-n+2}$ $u = T^q g(\psi)$	[11]
BT	$s = xt^{-\alpha}$ $u = t^{3\alpha-2}\dfrac{f(s)}{\gamma + 1}$	$X' = k_1 X^n$ $T' = k_2 T^m$, $m \neq 1$ $u = T^{3n+2m-5} g(\psi)$	[11]

unknown function of x and t which turns to be zero in many cases (KdV, mKdV, GKdV).

On the other hand, we list the similarity solutions which are available in the literature.

The guaranteed separable solution turns out to be the travelling wave solution in some cases. However, this does not indicate that the similarity solutions are more general than the separable ones. For instance, the sG equation has a travelling wave solution as the guaranteed separable one, yet the general separable solution is more complicated (see previous chapters). As another example is the KdVB equation which has similarity solutions obtainable from the separation technique by assuming that:

$$X' = k_1 X, \quad T' = k_2(t)T \quad \text{and} \quad u = g(\psi) + h(t). \quad (6.7\text{-}1)$$

One type of similarity solution for the Boussinesq equation is tied up with guaranteed separable solution, while the other can be achieved by changing the assumption to (6.7-1).

For the similarity solution of the Burgers equation, the author has made the use of the assumption $a_3 \neq 0$. For $a_3 = 0$, we found that the similarity variable s will be

$$s = (a_1 + a_2 t)^{-1/2}\left[x + \frac{2}{a_2}(a_4 + a_5 t) - \frac{2a_5}{a_2}(a_1 + a_2 t)\right]$$

which is a separable function of x and t only if $a_4 = a_5 = 0$. This leads to the similarity solution

$$u = (a_1 + a_2t)^{-1/2}f(s).$$

Finally, we want to mention that in spite of the differences between the two solutions, one can notice, from the table, in general that both solutions relate to each other in one way or another. To our knowledge, this is due to simplifying the infinitesimal transformations (to obtain similarity solutions in practice) which leads in most cases to separation assumptions. Consider for example the following lemma, written as a problem in [11] p. 152.

Lemma (6.7-1)

If ξ and τ are independent of u and η depends linearly on u then the general solution of (1.4-33) is of the form

$$u = F(x, t)f(s),$$

where f is an arbitrary function of s, s and F are known functions x and t. $\square$

The following lemma shows precisely, which situations lead to separable solutions:

Lemma (6.7-2)

If $\frac{\xi}{\tau}$ is a separable function of x and t and η depends linearly on u then the general solution of (1.4-33) is of the form

$$u = F(x, t)f(s) + h(x, t)$$

where f is arbitrary function of s; s, F and h are known functions of x and t. □

CHAPTER SEVEN

Conclusions and Speculations

In this thesis, we have tried to give a rigorous treatment of the separation method as applied to nonlinear partial differential equations. We have also made a preliminary investigation into the connections between this approach and the other major systematic methods of solution.

As far as the method itself is concerned there are, naturally, a large number of unanswered questions and unexplored avenues. The problem of convergence of the generalized power series in the definition of simple separability has not been fully studied. Also, as stated in chapter two, the lack of theorems available on reversion of generalized power series makes the proof of general theorems concerning the definition difficult.

There is also a lot of work still to be done concerning implicitly separable solutions. As in the case of the sine-Gordon equation, problems arise in finding such solutions when the dependent variable transformation has non-isolated singular points and cannot be expanded in a generalized power series about such points. The relationship between the general and the guaranteed implicitly separable solutions also needs more investigation.

Refering now to the content of chapter three, there is still work to be done on the most general transformation possible which produces separable solutions. There are also many problems involved with equations with mixed derivatives or explicit dependence on the independent variables.

Although a thorough comparison of the separation and similarity techniques has been given in chapter six, a lot more work in the form of general theorems needs to be completed. There does appear to be a very close connection between these techniques and in a number of cases, known similarity solutions turn out to be guaranteed implicitly separable solutions which are simpler to obtain. It may be that a large number of similarity solutions which are expressible in closed form in some sense are obtainable via the separation procedure.

Of course the major drawback in applying the separation technique to nonlinear equations is that there is no general superposition principle for such equations in contrast to linear equations. However, by using the separation procedure together with particular nonlinear superposition principles (such as Backlund transformations), for classes of equations, useful solutions may be obtained.

APPENDIX

In this appendix, we give several formulaes for the p^{th} x-derivative of ϕ, when ϕ is specified. All these lemmas can be easily applied to the p^{th} t-derivative for ϕ.

<u>Lemma (1)</u> [40]

The number of positive integer solutions of the equation,

$$x_1 + x_2 + \ldots + x_n = p$$

where p is positive integer, is $\binom{p-1}{p-n}$. □

<u>Lemma (2)</u>

The p^{th} x-derivative of the function $\phi = g(\psi)$ is

$$\phi_{x^p} = \sum_{i=1}^{p} \left\{ \sum_{m=1}^{q_i} \alpha_{p,i,m} \left\{ \prod_{n=1}^{i} \psi_{x^{a_{p,i,m,n}}} \right\} \right\} g^{(i)} \tag{1}$$

where $\{a_{p,i,m,n}: n = 1, i\}$ is the m^{th} solution to $\sum_{n=1}^{i} a_{p,i,m,n} = p, \forall i, m$; m denotes the solution number so that m takes $\binom{p-1}{p-i} = q_i$ values, and $\alpha_{i,p,m}$ are constants $\forall i, p, m$. □

<u>Proof</u> The proof of this lemma will be by induction using lemma (1).

For $p = 1$, it is clear that the formula is true.

Suppose that the result is true for $p = 1, \ldots, k$; We prove that it is true for $p = k + 1$:

Differentiating the k^{th} derivative of ϕ gives,

$$\phi_{x^{k+1}} = \sum_{i=1}^{k} \left\{ \sum_{m=1}^{\binom{k-1}{k-i}} \alpha_{k,i,m} \left[\psi_x^{1+a_{k,i,m,1}} \left(\prod_{n=2}^{i} \psi_x^{a_{k,i,m,n}} \right) \right. \right.$$

$$\left. \left. + \ldots + \psi_x^{1+a_{k,i,m,i}} \left(\prod_{n=1}^{i-1} \psi_x^{a_{k,i,m,n}} \right) \right] \right\} g^{(i)}$$

$$+ \sum_{i=1}^{k} \left\{ \sum_{m=1}^{\binom{k-1}{k-i}} \alpha_{k,i,m} \psi_x \left(\prod_{n=1}^{i} \psi_x^{a_{k,i,m,n}} \right) \right\} g^{(i+1)} . \tag{2}$$

Now,

$$\sum_{i=1}^{k} \left\{ \sum_{m=1}^{\binom{k-1}{k-i}} \alpha_{k,i,m} \psi_x \left(\prod_{n=1}^{i} \psi_x^{a_{k,i,m,n}} \right) \right\} g^{(i+1)}$$

$$= \sum_{i=1}^{k-1} \left\{ \sum_{m=1}^{\binom{k-1}{k-i}} \alpha_{k,i,m} \psi_x \left(\prod_{n=1}^{i} \psi_x^{a_{k,i,m,n}} \right) \right\} g^{(i+1)}$$

$$+ \alpha_{k,k,1} \psi_x \left(\prod_{n=1}^{k} \psi_x^{a_{k,k,1,n}} \right) g^{(k+1)} .$$

Changing indicies by $i \to i + 1$, the last result can be written as,

$$\sum_{i=2}^{k}\left\{\sum_{m=1}^{\binom{k-1}{k-i+1}} \alpha_{k,i-1,m}\psi_x\left(\prod_{n=1}^{i-1}\psi_x^{a_{k,i-1,m,n}}\right)\right\}g^{(i)}$$

$$+\ \alpha_{k,k,1}\psi_x\left(\prod_{n=1}^{k}\psi_x^{a_{k,k,1,n}}\right)g^{(k+1)}. \qquad (3)$$

Substituting (3) in (2) gives

$$\phi_x^{k+1} = \alpha_{k,1,1}\,\psi_x^{1+a_{k,1,1,i}}g' + \sum_{i=2}^{k}\left\{\sum_{m=1}^{\binom{k-1}{k-i}} \alpha_{k,i,m}\right.$$

$$\left[\psi_x^{1+a_{k,i,m,1}}\prod_{n=2}^{i}\psi_x^{a_{k,i,m,n}} + \dots + \psi_x^{1+a_{k,i,m,i}}\prod_{n=1}^{i-1}\right.$$

$$\left.\left.\psi_x^{a_{k,i,m,n}}\right]\right\}g^{(i)} + \sum_{i=2}^{k}\left\{\sum_{m=1}^{\binom{k-1}{k-i+1}}\alpha_{k,i-1,m}\psi_x\left(\prod_{n=1}^{i-1}\right.\right.$$

$$\left.\left.\psi_x^{a_{k,i-1,m,n}}\right)\right\}g^{(i)} + \alpha_{k,k,1}\psi_x\left(\prod_{n=1}^{k}\psi_x^{a_{k,k,1,n}}\right)g^{(k+1)}, \qquad (4)$$

where,

$$\sum_{n=1}^{i}a_{k,i,m,n} = k \quad \forall i,\ m. \qquad (5)$$

We, at first, want to show that the i^{th} term in (4) for i = 1, ..., k + 1 can be written as:

$$\sum_{m=1}^{\binom{k}{k+1-i}} \alpha_{k+1,i,m} \prod_{n=1}^{i} \psi_x^{a_{k+1,i,m,n}} g^{(i)},$$

for some constants $\alpha_{k+1,i,m}$. And we have to prove that

$$\sum_{n=1}^{i} a_{k+1,i,m,n} = k + 1 \quad \forall m = 1, \ldots, \binom{k}{k+1-i}.$$

Let i = 1:

It is clear that,

$$\alpha_{k,1,1}\psi_x^{1+a_{k,1,1,1}} g' = \sum_{m=1}^{\binom{k}{k+1-1}} \alpha_{k+1,1,m} \prod_{n=1}^{1} \psi_x^{a_{k+1,1,m,n}} g',$$

by choosing, $\alpha_{k+1,1,1} = \alpha_{k,1,1}$ and
$a_{k+1,1,1,1} = 1 + a_{k,1,1,1}$; By (4) it can be easily seen that,

$$\sum_{n=1}^{1} a_{k+1,1,1,n} = k + 1.$$

Now, let i = k + 1:

Choosing

$$\alpha_{k+1,k+1,1} = \alpha_{k,k,1}$$

$$a_{k+1,k+1,1,n} = \begin{cases} a_{k,k,1,n} & n = 1, \ldots, k \\ 1 & n = k + 1 \end{cases}$$

gives,

$$\alpha_{k,k,1}\psi_x \prod_{n=1}^{k} \psi_x a_{k,k,1,n} g^{(k+1)} = \sum_{m=1}^{\binom{k}{k+1-(k+1)}} \alpha_{k+1,k+1,m} \prod_{n=1}^{k+1}$$

$$\psi_x a_{k+1,k+1,m,n} g^{(k+1)}.$$

Clearly,

$$\sum_{m=1}^{k+1} a_{k+1,k+1,1,n} = \sum_{n=1}^{k} a_{k,k,1,n} + a_{k+1,k+1,1,k+1} = k + 1.$$

To investigate the case, where $i = 2, \ldots, k$, we notice that,

$$\alpha_{k,i,m}\psi_x 1+a_{k,i,m,r} \prod_{\substack{n=1 \\ n\neq r}}^{i} \psi_x a_{k,i,m,n} = K_{r,m} \prod_{n=1}^{i} \psi_x a_{k+1,i,m,n}$$

for $i = 2, \ldots, k$

$m = 1, \ldots, \binom{k-1}{k-i}$

$r = 1, \ldots, i$

by choosing

$$K_{r,m} = \alpha_{k,i,m}$$

$$a_{k+1,i,m,n} = \begin{cases} a_{k,i,m,n} & n \neq r \\ 1 + a_{k,i,m,n} & n = r. \end{cases}$$

Moreover

$$\sum_{n=1}^{i} a_{k+1,i,m,n} = \sum_{n=1}^{i} a_{k,i,m,n} + 1 = k + 1.$$

Also,

$$\alpha_{k,i-1,m}\psi_x \prod_{n=1}^{i-1} \psi_x^{a_{k,i-1,m,n}} = L_m \prod_{n=1}^{i} \psi_x^{a_{k+1,i,m,n}}$$

for $i = 2, \ldots, k$

$\quad m = 1, \ldots, \binom{k-1}{k-i+1}$

by choosing,

$$L_m = \alpha_{k,i-1,m}$$

$$a_{k+1,i,m,n} = \begin{cases} a_{k,i-1,m,n} & n = 1, \ldots, i-1 \\ 1 & n = i. \end{cases}$$

Clearly,

$$\sum_{n=1}^{i} a_{k+1,i,m,n} = \sum_{n=1}^{i-1} a_{k,i-1,m,n} + a_{k+1,i,m,i} = k + 1.$$

Since $a_{k+1,i,m,n}$ satisfy $\sum_{n=1}^{i} a_{k+1,i,m,n} = k + 1$ for $i = 2, \ldots, k$ and $m = 1, \ldots, \max\left\{\binom{k-1}{k-i}, \binom{k-1}{k-i+1}\right\}$ then by lemma (1), the number of possible values of m is $\binom{k}{k+1-i}$.

Hence rewriting all the terms together we have

$$\phi_x^{k+1} = \sum_{i=1}^{k+1} \left\{ \sum_{m=1}^{\binom{k}{k+1-i}} \alpha_{k+1,i,m} \prod_{n=1}^{i} \psi_x^{a_{k+1,i,m,n}} \right\} g^{(i)},$$

and $\sum_{n=1}^{i} a_{k+1,i,m,n} = k + 1 \ \forall i, m$, where $\alpha_{k+1,i,m}$ can be constructed from $K_{r,m}$ and L_m $\forall i = 2, \ldots, k$. ■

Formula (1) can be simplified as follows, for application purposes:

Let $Q = \sum_{i=1}^{p} q_i$. Then (1) can be written as

$$\phi_{x^p} = \sum_{n=1}^{Q} \alpha_{p,n} \prod_{i=1}^{b(n)} \psi_{x^{a_{p,n,i}}} g^{(b(n))} \tag{6}$$

$b(n)$ and $\alpha_{p,n}$ can be found by writing n as $n = \sum_{i=1}^{k-1} q_i + m$ for a positive integer k and $0 < m \leq q_k$. Thus $b(n) = k$, $\alpha_{p,n} = \alpha_{p,i,m}$ and $\{a_{p,n,i}: i = 1, \ldots, b(n)\}$ is the m^{th} solution to $\sum_{i=1}^{b(n)} a_{p,n,i} = p$.

<u>Lemma (3)</u>

The p^{th} x-derivative of the function $\phi = u(x, t)g(\psi) + h(x, t)$ is,

$$\phi_{x^p} = u\psi_x{}^p g^{(p)} + F_p + h_{x^p} \tag{7}$$

where,

$$F_p = g^{(p-1)}\left(u_x\psi_x{}^{p-1} + (p - 1)u\psi_x{}^{p-2}\psi_{xx}\right) + \frac{\partial}{\partial x}F_{p-1}$$

$$F_1 = u_x g. \quad \square$$

<u>Proof</u> (By induction)

Let $p = 1$, then

$$\phi_x = u\psi_x g' + u_x g + h_x.$$

Since $F_1 = u_x g$; hence $\phi_x = u\psi_x g' + F_1 + h_x$.

Suppose that (7) is true for $p = 1, \ldots, k$. We will prove it true for $p = k + 1$, by differentiating ϕ_{x^k} with respect to x which gives,

$$\frac{\partial}{\partial x}(\phi_{x^k}) = \phi_{x^{k+1}} = u\psi_x{}^k\psi_x g^{(k+1)} + u_x\psi_x{}^k g^{(k)} +$$

$$k u\psi_x{}^{k-1}\psi_{xx} g^{(k)} + \frac{\partial}{\partial x}F_k\Big) + h_{x^{k+1}}.$$

Denote $\left\{g^{(k)}\left(u_x\psi_x{}^k + ku\psi_x{}^{k-1}\psi_{xx}\right) + \frac{\partial}{\partial x}F_k\right\}$ by F_{k+1}. Thus

$$\phi_{x^{k+1}} = u\psi_x{}^{k+1}g^{(k+1)} + F_{k+1} + h_{x^{k+1}}. \blacksquare$$

<u>Lemma (4)</u>

The p^{th} x-derivative of the function $\phi = u(t)g(\psi) + h(x, t)$ is,

$$\phi_{x^p} = u\psi_x{}^p g^{(p)} + F_p\left[u, \psi, g'', \ldots, g^{(p-1)}\right] + u\psi_{x^p}g' + h_{x^p} \tag{8}$$

where

$$F_p = (p - 1)u\psi_x^{p-2}\psi_{xx}g^{(p-1)} + \frac{\partial}{\partial x}F_{p-1} + u\psi_x\psi_x{}_{p-1}g''$$

$$F_1 = 0. \ \square$$

Proof The proof is similar to the proof of lemma (3). ∎

Lemma (5)

If $X' = aX^n$ then the r^{th} derivative for X^p is,

$$(X^p)^{(r)} = p\big(p + (n - 1)\big) \ldots \big(p + (r - 1)(n - 1)\big)a^r X^{p+r(n-1)}. \ \square \qquad (9)$$

Proof (By induction)

Let $r = 1$ then differentiating X^p gives,

$$(X^p)' = pX^{p-1}X' = paX^{p+n-1}.$$

Suppose that (9) is true for $p = 1, \ldots, k$. We will prove it true for $p = k + 1$:

Differentiating $(X^p)^{(k)}$ gives,

$$(X^p)^{(k+1)} = p\big(p + (n - 1)\big) \ldots \big(p + (k - 1)(n - 1)\big)a^k \big(p + k(n - 1)\big)X^{p+k(n-1)-1}X'.$$

Hence,

$$(X^p)^{(k+1)} = p\big(p + (n - 1)\big) \ldots \big(p + k(n - 1)\big)a^{k+1} X^{p+(k+1)(n-1)}. \ \blacksquare$$

If $p = 1$ in (9) then,

$$X^{(r)} = n(2n - 1) \ldots \big((r - 1)n - (r - 2)\big)a^r X^{rn-(r-1)}.$$

<u>Lemma (6)</u>

If $\phi = T^q g(\psi)$ for some q, then the p^{th} derivatives of ϕ with respect to x and t are

$$\phi_{x^p} = T^{q+p(1-n)} F_p\big(\psi, g', \ldots, g^{(p)}\big) \qquad (10)$$

$$\phi_{t^p} = T^{q+p(m-1)} G_p\big(\psi, g, g', \ldots, g^{(p)}\big) \quad p \geq 1 \qquad (11)$$

respectively, where F_p and G_p are

$$\left.\begin{aligned} F_p &= a\psi^n\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g'}g'' + \ldots + \frac{\partial}{\partial g^{(p-1)}}g^{(p)}\right)F_{p-1}, \\ &\qquad\qquad p \geq 1,\ F_0 = g \\ G_p &= b\big(q + (p-1)(m-1)\big)G_{p-1} + \\ &\quad b\psi\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g}g' + \ldots + \frac{\partial}{\partial g^{(p-1)}}g^{(p)}\right)G_{p-1}, \\ &\qquad\qquad p \geq 2,\ G_1 = bqg + b\psi g' \end{aligned}\right\}. \ \Box \qquad (12)$$

<u>Proof</u> Using the facts that $\psi_x = a\psi^n T^{1-n}$ and $\psi_t = b\psi T^{m-1}$, then by induction, this lemma can be easily proved. First let us prove the x-derivative formula:

Let p = 1, then $\phi_x = T^q \psi_x g'$ by differentiating ϕ with respect to x, or $\phi_x = aT^q\psi^n T^{1-n} g' = T^{q+(1-n)} F_1$ where $F_1 = a\psi^n\left(\frac{\partial}{\partial\psi}\right)g$.

Suppose that the formula is true for p = 1, ..., k;

we will prove it true for $p = k + 1$:

Differentiating ϕ_{x^k} with respect to x gives,

$$\phi_{x^{k+1}} = T^{q+k(1-n)}\frac{dF_k}{dx}.$$

But

$$\frac{dF}{dx} = \psi_x\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g'}g'' + \dots + \frac{\partial}{\partial g(k)}g^{(k+1)}\right).$$

Hence

$$\begin{aligned}\phi_{x^{k+1}} &= aT^{q+k(1-n)}\psi^n T^{1-n}\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g'}g'' + \dots + \frac{\partial}{\partial g(k)}g^{(k+1)}\right) \\ &= T^{q+(k+1)(1-n)}F_{k+1},\end{aligned}$$

where

$$F_{k+1} = a\psi^n\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g'}g'' + \dots + \frac{\partial}{\partial g(k)}g^{(k+1)}\right).$$

To prove the second formula, differentiate ϕ with respect to t and get

$$\begin{aligned}\phi_t &= qT^{q-1}T'g + T^q\psi_t g' \\ &= bqT^{q+(m-1)}g + b\psi T^{q+m-1}g'\end{aligned}$$

$$\Rightarrow \phi_t = T^{q+(m-1)}G_1,$$

where

$$G_1 = bqg + b\psi g'.$$

Second differentiation of ϕ with respect to t gives

$$\phi_{tt} = T^{q+(m-1)}\frac{dG_1}{dt} + \left(q + (m-1)\right)T^{q+m-2}T'G_1$$

$$= T^{q+(m-1)}\psi_t\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g}g' + \frac{\partial}{\partial g'}g''\right)G_1 +$$

$$b\left(q + (m-1)\right)T^{q+2(m-1)}G_1$$

$$\Rightarrow \phi_{tt} = T^{q+2(m-1)}\left[b\psi\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g}g' + \frac{\partial}{\partial g'}g''\right)G_1 +\right.$$

$$\left. b(q + m - 1)G_1\right].$$

Hence the formula is true for p = 2. Suppose that the formula is true for p = 1, 2, ..., k; we will prove it is true for p = k + 1:

$$\phi_{t^{k+1}} = b\left(q + k(m-1)\right)T^{q+k(m-1)-1}T^mG_k + T^{q+k(m-1)}\frac{dG_k}{dt}$$

$$= b\left(q + k(m-1)\right)T^{q+(k+1)(m-1)}G_k + T^{q+k(m-1)}$$

$$\psi_t\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g}g' + \ldots + \frac{\partial}{\partial g(k)}g^{(k+1)}\right)G_k$$

$$= T^{q+(k+1)(m-1)}\left[b\left(q + k(m-1)\right)G_k +\right.$$

$$\left. b\psi\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g}g' + \ldots + \frac{\partial}{\partial g(k)}g^{(k+1)}\right)G_k\right].$$

Thus,

$$\phi_{t^{k+1}} = T^{q+(k+1)(m-1)}G_{k+1},$$

where

$$G_{k+1} = b\big(q + k(m - 1)\big)G_k +$$

$$b\psi\left(\frac{\partial}{\partial\psi} + \frac{\partial}{\partial g}g' + \ldots + \frac{\partial}{\partial g(k)}g^{(k+1)}\right)G_k. \blacksquare$$

REFERENCES

[1] ABLOWITZ, M.J., Phys. Rev. Lett., 38(20), 1977, pp. 1103-1106.

[2] ABLOWITZ, M.J., in the proceedings on mathematical methods and application of scattering theory, Washington, D.C., 1979, Lecture notes in physics 130, Springer-Verlag.

[3] ABLOWITZ, M.J., KAUP, D.J., NEWELL, A.C. and SEGUR, H., Phys. Rev. Lett., 30(25), 1973, pp. 1262-1264.

[4] ABLOWITZ, M.J., KAUP, D.J., NEWELL, A.C. and SEGUR, H., Studies in Appl. Math., 53, 1974, pp. 249-315.

[5] ABLOWITZ, M.J., RAMANI, A. and SEGUR, H., Lettere Al Nuovo Cimento, 23(9), 1978, pp. 333-338.

[6] ABLOWITZ, M.J., RAMANI, A. and SEGUR, H., J. Math. Phys., 21(4), 1980, pp. 715-721.

[7] AMES, W.F., Nonlinear partial differential equations in engineering, Academic Press, New York, Vol. I, 1965.

[8] AMES, W.F., Nonlinear partial differential equations, Academic Press, New York, Vol. II, 1972.

[9] BARONE, A., ESPOSITO, F., MAGEE, C.J. and SCOTT, A.C., Rivista Del Nuovo Cimento, 1(2), 1971, pp. 227-267.

[10] BIRKHOFF, G., Hydrodynamics, Princeton University Press, Princeton, 1960.

[11] BLUMAN, G.W. and COLE, J.D., Similarity methods for differential equations, Springer-Verlag, Berlin, 1974.

[12] BOITI, M. and PEMPINELLI, F., Il Nuovo Cimento, 51B(1), 1979, pp. 70-78.

[13] BOITI, M. and PEMPINELLI, F., Il Nuovo Cimento, 56B(1), 1980, pp. 148-156.

[14] BOS, L. and TORRENCE, R.J., Phys. Lett., 111A(3), 1985, pp. 95-96.

[15] BROMAN, A., Introduction to partial differential equations from Fourier series to boundary-value problems, Addison-Wesley Publishing Company, Reading, 1968.

[16] BROMWICH, T.J.I'A., An introduction to the theory of infinite series, Macmillan and Co. Limited, London, 1947.

[17] BRYAN, A.C., HAINES, C.R. and STUART, A.E.G., Il Nuovo Ciemento, 58B(1), 1980, pp. 1-32.

[18] BUREAU, F.J., Acad. Roy. Belg. Bull. Cl. Sci., (5)68, 1982, pp. 878-897.

[19] BUTKOV, E., Mathematical physics, Addison-Wesley Publishing Company, Reading, 1973.

[20] CAUDREY, P.J., Lecture given at NATO Advanced Study Institute on Nonlinear Equations in Physics and Mathematics, Istanbul, 1977.

[21] CHESTER, W., J. Inst. Maths. Applics., 19, 1977, pp. 343-376.

[22] CLARKSON, P.A., Phys. Lett. 109A(5), 1985, pp. 205-208.

[23] COSGROVE, C.M., J. Phys. A: Math. Gen., 11(12) pp. 2405-2430.

[24] DAVIS, H.T., Introduction to nonlinear differential and integral equations, Dover Publ., New York, 1962.

[25] DODD, R.K., EILBECK, J.C., GIBBON, J.D. and MORRIS, H.C., Solitons and nonlinear wave equations, Academic Press, London, 1982.

[26] DOLD, A. and ECKMANN, B., Backlund transformations, Springer-Verlag, Berlin, 1974.

[27] DORIZZI, B., Lecture Notes in Physics 189: Nonlinear Phenomena, Proceeding of the CIFMO school and workshop held at Oaxtepec, Mexico, Nov. 29 - Dec. 17, 1982, Springer-Verlag, Berlin.

[28] DRAZIN, P.G., Solitons, Cambridge University Press, Cambridge, London, 1983.

[29] DRESNER, L., Similarity solutions of nonlinear partial differential equations, Pitman Advanced Publishing Program, 1983.

[30] DUFF, G.F., Partial differential equations, University of Toronto Press, Toronto, 1956.

[31] FARLOW, S.J., Partial differential equations for scientists and engineers, John Wiley & Sons, Inc., 1982.

[32] FORSYTH, A.R., Theory of differential equations, Part IV, Cambridge University Press, Cambridge, 1906.

[33] FUNG, P.C.W. and AU, C., J. Math. Phys. 25(5), 1984, pp. 1370-1371.

[34] GARABEDIAN, P.R., Partial differential equations, John Wiley & Sons, London, 1964.

[35] GARDNER, C.S., GREENE, J.M., KRUSKAL, M.D. and MIURA, R.M., Phys. Rev. Lett., 19(19), 1967, pp. 1095-1097.

[36] GOLDSTEIN, P. and INFELD, E., Phys. Lett., 103A(1,2), 1984, pp. 8-10.

[37] GRAUEL, A., Lett. Al Nuovo Cimento, 42(8), 1985, pp. 397-402.

[38] GREENHILL, A.G., The application of elliptic functions, Dover Publications, Inc., 1959.

[39] HAINZL, J., SIAM J. Math. Anal., 13(2), 1982, pp. 208-225.

[40] HALL, M., Jr., Combinatorial theory, Blaisdell Publishing Company, Waltham, 1967.

[41] HARADA, H. and OISHI, S., J. Phys. Soc., Japan, 54(1), 1985, pp. 51-56.

[42] HOPF, E., Comm. Pure Appl. Math., 3, 1950, pp. 201-230.

[43] HUDAK, O., Phys. Lett., 89A(5), 1982, pp. 245-248.

[44] INCE, E.L., Ordinary differential equations, Dover Publ., New York, 1956.

[45] JEFFREYS, H. and JEFFREYS, B.S., Methods of mathematical physics, Cambridge University Press, Cambridge, 1956.

[46] JOHNSON, R.S. and THOMPSON, S., Phys. Lett., 66A(4), 1978, pp. 279-281.

[47] KALNINS, E.G. and MILLER, W., Jr., J. Math. Phys., 26(7), 1985, pp. 1560-1565.

[48] KALNINS, E.G. and MILLER, W., Jr., J. Math. Phys., 26(9), 1985, pp. 2168-2171.

[49] KAWAMOTO, S., J. Phys. Soc., Japan, 52(8), 1983, pp. 2613-2616.

[50] KAWAMOTO, S., J. Phys. Soc., Japan, 52(12), 1983, pp. 4059-4065.

[51] KAWAMOTO, S., J. Phys. Soc., Japan, 54(5), 1985, pp. 1701-1709.

[52] KNOPP, K., Theory and application of infinite series, Blackie & Son Limited, London, 1949.

[53] KOORNWINDER, T.H., Preprint from Mathematisch Centrum TW 194/79, Amsterdam, 1979.

[54] LAKSHMANAN, M. and KALIAPPAN, P., Phys. Lett., 71A(2,3), 1979, pp. 166-168.

[55] LAKSHMANAN, M. and KALIAPPAN, P., J. Math. Phys., 24(4), 1983, pp. 795-803.

[56] LAMB, G.L., Rev. Modern Phys., 43(2) Part I, 1971, pp. 99-124.

[57] LEO, M., LEO, R.A., SOLIANI, G., SOLOMBRINO, L. and MARTINA, L., Phys. Rev. D, 27(6), 1983, pp. 1406-1408.

[58] MILLER, W., Jr., in the proceeding of the CIFMO school and workshop, Mexico, 1982, Lecture notes in physics, 108, Springer-Verlag.

[59] MOON, P. and SPENCER, D., Field theory for engineers, D. Van Nostrand Company, Inc., 1961.

[60] MURPHY, G.M., Ordinary differential equations and their solutions, D. Van Nostrand Company, Inc., New York, 1960.

[61] NAKAMURA, A., J. Math. Phys., 22(8), 1981, pp. 1608-1613.

[62] NAKAMURA, A., J. Math. Phys., 22(11), 1981, pp. 2456-2463.

[63] NAKAMURA, A., J. Phys. Soc., Japan, 54(6), 1985, pp. 2105-2109.

[64] NIMMO, J.J.C. and CRIGHTON, D.G., Proc. R. Soc. Lond. A384, 1982, pp. 381-401.

[65] NISHITANI, T. and TAJIRI, M., Phys. Lett., 89A(8), 1982, pp. 379-380.

[66] NISHITANI, T. and TAJIRI, M., J. Phys. Soc., Japan, 53(1), 1984, pp. 79-84.

[67] OSBORNE, A.D., Phys. Lett., 83A(7), 1981, pp. 311-313.

[68] OSBORNE, A.D., (To appear).

[69] OSBORNE, A.D. and STUART, A.E.G., J. Math. Phys., 19(7), 1978, pp. 1573-1579.

[70] OSBORNE, A.D. and STUART, A.E.G., Phys. Lett., 67A(5,6), 1978, pp. 328-330.

[71] OSBORNE, A.D. and STUART, A.E.G., J. Math. Phys., 21(4), 1980, pp. 726-734.

[72] OSBORNE, A.D. and STUART, A.E.G., Phys. Lett., 76A(1), 1980, pp. 5-7.

[73] OVSJANNIKOV, L.V., Group properties of differential equations, Novosibirsk.

[74] PETROVSKY, I.G., Lectures on partial differential equations, Interscience Publishers, Inc., New York, 1954.

[75] RAMANI, A., In a thesis defended at Université Paris-Sud (Orsay), 1981.

[76] ROY CHOWDHURY, A., J. Phys. A.; Math. Gen. 15, 1982, pp. 343-346.

[77] ROY CHOWDHURY, A. and CHANDA, P.K., Phys. Lett., 106A(8), 1984, pp. 353-355.

[78] SAGAN, H., Boundary and eigenvalue problems in mathematical physics, John Wiley & Sons, Inc., New York, 1961.

[79] SCOTT, A.C., CHU, F.Y.F. and McLAUGHLIN, D.W., Proceedings of the IEEE, 61(10), 1973, pp. 1443-1483.

[80] SEGUR, H., in the proceedings on mathematical methods and application of scattering theory, Washington, D.C., 1979, Lecture notes in physics 130, Springer-Verlag.

[81] SHEN, H. and AMES, W.F., Phys. Lett., 49A(4), 1974, pp. 313-314.

[82] SNEDDON, I.N., Elements of Partial differential equations, McGraw-Hill Book Company, Inc., New York, 1957.

[83] STAKGOLD, I., Green's functions and boundary value problems, John Wiley & Sons, New York, 1979.

[84] STEEB, W.H., KLOKE, M., SPIEKER, B.M. and OEVEL, W., J. Phys. A.: Math. Gen., 16, 1983, pp. L447-L450.

[85] STEEB, W.H., KLOKE, M. and SPIEKER, B.M., Z. Naturforsch. 38a, 1983, pp. 1054-1055.

[86] TAGAMI,Y., J. Math. Phys., 25(5), 1984, pp. 1372-1377.

[87] TAJIRI, M. and KAWAMOTO, S., J. Phys. Soc., Japan, 51(5), 1982, pp. 1678-1681.

[88] TAJIRI, M. and KAWAMOTO, S., J. Phys. Soc., Japan, 52(7), 1983, pp. 2315-2319.

[89] TITCHMARSH, E.C., The theory of functions, Oxford University Press, 1939.

[90] TITCHMARCH, E.C., Eigenfunction expansions, Part I, Clarendon Press, Oxford, 1962.

[91] WADATI, M., J. Phys. Soc., Japan, 32, 1972, pp. 1681.

[92] WEINBERGER, H.F., A first course in partial differential equations, Xerox, Massachusetts, 1965.

[93] WEISS, J., TABOR, M. and CARNEVALE, G., J. Math. Phys., 24(3), 1983, pp. 522-526.

[94] WEISS, J., J. Math. Phys., 25(1), 1984, pp. 13-24.

[95] WEISS, J., J. Math. Phys., 26(2), 1985, pp. 258-269.

[96] WHITHAM, G.B., Linear and nonlinear waves, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1973.

[97] WIDDER, D.V., The heat equation, Academic Press, New York, 1975.

[98] XUN-CHENG, H., J. Phys. A.: Math. Gen., 15, 1982, pp. 3367-3372.

[99] XUN-CHENG, H., J. Phys. A.: Math. Gen., 16, 1983, pp. 891-892.

[100] ZAGRODZINSKI, J., Phys. Lett., 57A(3), 1976, pp. 213-214.

[101] ZAUDERER, E., Partial differential equations of applied mathematics, John Wiley & Sons, New York, 1983.