Mathematics for Non-Specialists:

A Study of Two Undergraduate Courses

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Appendix 1

A Summary of the Research Programme and Procedures

Two studies were undertaken: (a) a subsidiary course in mathematics for arts-based undergraduates at a University and (b) a service course in mathematics for first-year students reading for a degree in Economics at a Polytechnic.

(a) The Subsidiary Course in Mathematics for Arts-Based Students

The objectives of the research were:

(i) to examine the purpose and content of a non-vocational or cultural course,

(ii) to create a profile of a typical student on this particular course,

(iii) to identify the salient features as seen by the students,

(iv) to identify the salient features as seen by the staff involved,

(v) to obtain some knowledge of the way students responded to the experience of the course,

(vi) to consider fruitful lines of future development,

(vii) to prepare study booklets and investigate their effectiveness as part of the overall learning experience. (See Ch.5, p.40 et seq.)

The required information from staff and students was obtained using a programme of student interviews, individual discussions with members of staff and a limited number of student questionnaires. A small, but valuable, programme of observations was also included. (p.53 et seq.) Thus, the main sources of data were the sequence of interviews with students (10 occasions spread over a three-year period - see below) and discussions with staff (10 sessions); the role of questionnaires (3 distributions) and observations (7 sessions) being to provide cross-reference on the validity of the information obtained in this way. (p.55)

Student interviews were arranged with volunteers whom it was planned to see in small groups involving about four or five people. (p.56) Except for the first year, (77/78) the same students were asked to attend on four separate occasions throughout the year, (Nov, Feb, Mar, May in 78/79 and 79/80) making it possible to seek for changes in attitude. The implications of the decision (which was unavoidable) to use volunteer interviewees are discussed on pages 57 and 335.

The interviews were semi-structured; a list of questions was prepared but the opportunity to ask supplementary questions about any issues of interest was not overlooked. (p.58) The sets of prepared questions may be seen in Appendix 3.

Questionnaires were distributed to a randomly chosen sample of tutorial groups towards the end of February in 1979 and 1980, (4 groups). There were twenty-eight replies from students who were not interviewees. A summary of the results and the questionnaire are given in Appendix 4. The tutorial groups represented heterogeneous collections of students which were made up at the time of registration. (Some further information was made available when a short discussion was held with a tutorial group (nine students) in November 78. (p.58))

The short programme of observations was carried out between Nov 79 and Feb 80, and involved three lectures and four tutorials. Individual discussions with lecturers were conducted mainly in the period from June 79 to July 80. Seven lecturers were involved, some being interviewed more than once. The aims were to provide general information about attitudes, (all) to discuss the teaching of specific topics (two) and to discuss the design and use of study booklets (two). (Ch. 9, p.147)

Two study booklets were developed and used in 78/79 and 79/80. (p.147, Ch.10 also Appendix 5) Evaluative information was collected from students during the planned programme of interviews in Feb, May and Nov 79 and May 80 (p.153 et seq.). Further information came from a questionnaire distributed to a random sample of tutorial groups in May 80. (Sample size 19, p.159) Lecturers' opinions concerning the booklets were gathered from the discussions with staff in 1979 and 1980.

Information on a subsequent re-design of the course was gathered by interviewing volunteer students in Feb (three) and May (one) of 1981. Discussions with staff (four) on this topic took place in June and July of that year. (Ch.12, p.195)

Some data on the attitudes of former students on the course were gathered by distributing a questionnaire to a randomly chosen sample of graduates in July 80. (Sample size 36, replies 20, p.123)

The Calendar of Events -

Subsidiary Course for Arts-Based Students

1977/78

Student Interviews (a total of nine volunteers) (+++)

Nov 77	n = 6[+]		(see below)
May 78	n = 3	Students: (i), (ii), (iii)	

\+ These six interviews were exploratory and no individual references to them appear in the text. Other interviewees are identified by a roman numeral, see Appendix 2.

1978/79

Student Interviews (a total of ten volunteers) * = present

Students:	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	(xii)	(xiii)	
Nov 78	*	*	*	*	*	*	*	*			n = 8
Feb 79	*		*	*			*	*			n = 5
Mar 79	*		*		*	*	*	*	*	*	n = 8
May 79	*		*	*	*	*	*	*			n = 7

Informal Discussion	Nov 78	Single tutorial group	n = 9
Questionnaire	Feb 79	Sample of tutorial groups (100% response)	n = 21
Staff Discussions	June 79		n = 1 (p.126)

1979/80

Student Interviews (a total of eight volunteers)

Students:	(xiv)	(xv)	(xvi)	(xvii)	(xviii)	(xix)	(xx)	(xxi)	
Nov 79	*	*	*	*	*				n = 5
Feb 80	*	*	*	*	*	*	*		n = 7
Mar 80	*	*	*		*	*		*	n = 6
May 80	*	*	*		*	*	*		n = 5

Questionnaires	Feb 80 Sample of tutorial groups (50% response)	n = 7[++]	
	May 80 On the use of study booklet (100% response)	n = 19	(p.159)
	July 80 On attitudes of former students (56% response)	n = 20	(p.122)
Observations	Nov 79 Lectures (2), tutorials (3)		(p.112)
	Feb 80 Lecture (1), tutorial (1)		
Staff Discussions	Oct 79 - July 80	n = 10	(p.126)

++ The difference in response rates in 79 and 80 occurred because in the former, replies were completed and returned during a class period, in the latter students were asked to return them in their own time.

<u>1980/81</u> (Concerning a new course structure)

Student Interviews (a total of four volunteers)			
	Feb 81	n = 3	(p.200)
	May 81	n = 1	
Staff Discussions June - July 81		n = 4	(p.198)

+++ The detailed analysis of the information derived from students' interviews and questionnaires is the subject of Chapters 6 to 8 (pp.63-125) and a summary of the conclusions is in Chapter 11. This contains a profile of the typical student, (p.166) a staff profile (p.170) and a list of seven significant factors that were considered influential, in shaping the course (p.175). There is a discussion of the extent to which the course aims were seen to be achieved (p.179) and conclusions about the choice and development of course content (p.185). A reconsideration of the research programme is contained in Chapter 20, (p.335) where possible alternative approaches are suggested.

(b) The Service Course for First-Year Economists

In this investigation, the objectives of the research were:

(i) to establish the background to the course, including the attitudes of staff and students to mathematics in relation to the study of economics,

(ii) to design an instructional scheme which was suitable for the students involved and the course they undertook,

(iii) to carry out a formative evaluation of the instructional scheme,

(iv) to assess the effectiveness of the teaching in promoting the skills and confidence necessary to apply mathematics to the study of economics. (Ch.13, p.213)

The programme to collect the relevant data was designed to include student interviews, the distribution of questionnaires to the student body, discussions with specialist economics lecturers and interviews with students who had completed the course. (p.215 et seq.) In this study, the major source of data on attitudes would be the sequence of interviews and supportive evidence would be provided by the questionnaires and cross-reference to staff comments. However the questionnaires, by providing information from the whole student body, would make a significant contribution to the formative evaluation of the course material. In connection with the design and construction of instructional material, a pilot scheme involving self-paced study was introduced in 76/77. (12 students, p.225)

Because the questionnaires had a dual role; (i) as an alternative source of data to interviews which might reveal some influence from the interviewer (who was also the lecturer involved) and (ii) to provide wider evidence for use in formative evaluation, it was necessary to provide a more substantial programme of questionnaires than was the case in the parallel investigation of the subsidiary course for arts students (a). The interviews and questionnaires thus provided complementary evidence more comparable in amount if rather different in nature.

Three interviews (Nov, Feb, May) were planned in each of three successive academic years (77/78 to 79/80). Interviewees were chosen by random selections from the class lists so that on each occasion, a group of about four students would be involved. Students were interviewed once only and no attempt was made to monitor changes in attitude; consequently a total of about one-third of the whole class of between 24 and 35 were interviewed. (p.215) The interviews were, as in (a), semi-structured, a list of questions being prepared beforehand but it was envisaged that interesting points would arise and be followed up. The list of prepared questions appears in Appendix 10.

Questionnaires were distributed to all members of the class who were present at the chosen tutorial and they were completed and returned during the class session. Absentees prevented 100% coverage; the average rate of return was 77% of the class total, apart from one occasion (May 79) when the closeness of the final examination reduced attendance to 39%. Two distributions (Nov, May) were made in 77/78 and three (Nov, Feb, May) in the two following years[+]. (pp.240, 259, 280)

\+ The change from two to three questionnaires was made principally to provide more information for the formative evaluation of the teaching scheme.

Questionnaires and summaries of the returns appear in Appendix 9.

Representative staff views were obtained by individual discussions with three specialist lecturers. These took place during the 79/80 session. (p.296)

Because the investigator was also the lecturer involved, it was possible to observe student activities during tutorial sessions. No formal recordings were attempted of specific behaviour. Notes were made of the activities and general atmosphere during the tutorials and these were used as a cross-check against information gained from interviews and questionnaires. (p.216)

Interviews with students in their second year, which was after their mathematical studies were completed, took place in April 80 (4 students) and March 81 (5 students). In each case, a random sample of five students was chosen from class registers (one absentee, p.306).

Calendar of Events -

Mathematics Service Course for Economists

1976/77				
Pilot Study	Nov 76 - April 77	n = 12		(p.225)
1977/78	No. in class 28			
First-Year Students	Nov 77	n = 4		(p.233)
Interviews	Feb 78	n = 5		
	May 78	n = 4	Total 13 (random sample)	
Questionnaires	Nov 77	n = 24		(p.240)
	May 78	n = 11		

<u>1978/79</u>	No. in class 35		
First-Year Students	Nov 78	n = 6	(p.246)
Interviews	Feb 79	n = 4	
	May 79	n = 2	Total 12
			(random sample)
Questionnaires	Nov 78	n = 28	(p.259)
	Feb 79	n = 26	
	May 79	n = 27	
<u>1979/80</u>	No. in class 24		
First-Year Students	Nov 79	n = 4	(p.268)
Interviews	Feb 80	n = 3	
	May 80	n = 4	Total 11
			(random sample)
Questionnaires	Nov 79	n = 18	(p.280)
	Feb 80	n = 19	
	May 80	n = 17	
Staff Discussions	May - June 80	n = 3	(p.296)
2nd-Year Students			
Interview	Apr 80	n = 4	(p.306)
<u>1980/81</u>			
2nd-Year Students			
Interview	Mar 81	n = 5	(p.306)

Examples of self-paced study notes are in Appendices 7 and 8. The questionnaires and returns are in Appendix 9 and the prepared questions and a verbatim transcript of a typical student interview appear in Appendix 10.

Aspects of the formative evaluation of the study scheme are presented year-by-year. (pp.242, 262, 285) The summary and conclusions in Chapter 19 (p.315 et seq.) present views of the course as seen by the student, the specialist economics lecturer and the student in his second year of studies. The appropriateness of three initial conjectures is considered (p.230) and also the extent to which the course aims, (listed on p.212) are achieved. (p.332) The salient features of the instructional programme are noted, together with some problems worthy of further investigation. (p.323 also p.337) A retrospective view of the research programme is in 20.4. (p.335 et seq.)

Developments in the Research Programme

Developments were implemented mainly in response to the analysis of early data but also, in some cases, to changing circumstances. Some changes were minor, but more significant ones with reference to study (a) were:

(i) students' responses in the exploratory interviews highlighted the inadequacies of some of the writer's preconceived ideas about student attitude and were a necessary pre-requisite for the compilation of the later programme of interviews and questionnaires, (p.63)

(ii) because a major re-structuring of the course was undertaken in 80/81 by the Department concerned, it was possible to carry out a small study of the effects of the changes, (p.195 et seq.)

(iii) a proposal to look in some detail at the attitudes of former students to assess long-term changes was not pursued because the necessary data on individual former students could not be provided, (p.124)

(iv) a distribution problem with one of the study booklets made it necessary to supplement the evaluative data by distributing a questionnaire on this topic in the following year. (p.159)

With reference to study (b), it was noted that:

(i) information gained from the pilot study, combined with ideas expounded in certain published papers, particularly by Davies (1976 and 1977), led to the final decision to design a study scheme which was a blend of lectures, self-paced study and problem-solving activity, (p.221)

(ii) after the first year, the programme of questionnaires was augmented in order to provide information which was more useful in the formative evaluation of the study scheme, (pp.240, 259, 280)

(iii) although strictly speaking part of the formative development, introducing group-work by the students to foster their problem-solving capabilities was, in part, a response to early indications that this aspect was worthy of closer study, (p.245)

(iv) in conjunction with (iii), this change in emphasis was also manifest in the re-design of the self-study notes to promote a more "conjectural" approach. (p.244)

Appendix 2(i)

Subsidiary Mathematics Lecture Programmes

The programmes for 1977/78 and 1979/80 are presented to show the development over the period covered by the investigation.

1977/78 Autumn Term	Approximate Number of Lectures
Introduction to the course.	1
Number notations, duodecimal, binary. Natural numbers, divisors, primes. Counting, number operations.	5
Measurement, rational/irrational numbers. Series, Zeno's paradoxes. Incommensurables ($\sqrt{2}$), towards real numbers.	4
Flow diagrams and calculating procedures. Approximations to $\sqrt{2}, \pi$ etc. Idea of iteration, flowchart construction.	4
Number systems, N, Z, Q, R and their properties.	4
Spring Term	
Euclidean geometry.	3
Cartesian and non-Euclidean geometries.	3
Graphs and derivatives.	4

Theories of motion.	4
Probability.	4

<u>Summer Term</u>

Sets and Groups.	4
Boolean algebra.	3
Examination review.	1
Mathematics and the real world.	2

1979/80	Approximate Number of Lectures
Autumn Term	
Introduction to the course.	1
Number notations, duodecimal and binary. Natural numbers, divisors, primes. Counting, number operations.	5
Flow diagrams and calculating procedures. Numerical approximations to $\sqrt{2}, \pi$ etc.	4
Measurement, rational numbers. Series, Zeno's paradoxes. Incommensurables, towards real numbers.	4
Number systems. N, Z, Q, R and their properties.	4
Spring Term	
Euclidean geometry, finite geometries.	4
Cartesian and non-Euclidean geometries.	4
Graphs, differentiation and integration.	6
Theories of motion.	4

<u>Summer Term</u>

Sets and Groups.	4
Examination review.	1
Boolean algebra.	3
Models in Mathematics.	2

Typical Tutorial Example Sheet

Subsidiary Mathematics Course

For any natural number n the number of divisors of n is written d(n). Similarly, the sum of the divisors of n is written $\sigma(n)$. Look at the given list of values of d(n) for n between 1 and 500, try to answer the following questions for about 4 hours (no more) and then write an account of what you have discovered about d(n).

(i) Look for values of n for which d(n) = 2. State and prove a general theorem about this case.

(ii) Look for values of n for which d(n) is odd. Guess a general theorem but don't try a proof.

(iii) Look for values of n for which d(n) = 3. Again guess (but don't try to prove) a general theorem.

(iv) Find $d(2^3)$, $d(3^3)$, $d(5^3)$. What would you expect $d(7^3)$ to be? How about $d(9^3)$?

(v) List the divisors of p^3 where p is a prime number. Have another look at (iv). List the divisors of 9^3.

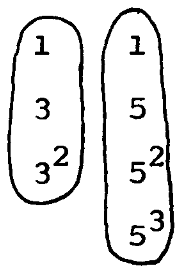
(vi) For what values of n is d(n) = 4? You have met some such numbers already but what do the <u>other</u> ones have in common?

(vii) Stare at $\dfrac{2 \times 2 \times 3 \times 7 \times 7}{2 \times 3 \times 7}$ and $\dfrac{2 \times 2 \times 3 \times 7 \times 7}{3 \times 5}$ and $\dfrac{2 \times 2 \times 3 \times 5}{2 \times 2 \times 2 \times 5}$.

(viii) Don't work out 13 x 17 but list the divisors of 13 x 17.

(ix) Similarly list the divisors of $13^2 \times 17$.

(x) Try to list the divisors of $3^2 \times 5^3$ in an orderly fashion. Can you see the relevance of the picture



?

Can you see 'why' $3^2 \times 5^3$ has 12 divisors?

(xi) Let p, q and r be different prime numbers and let ℓ, m and n be natural numbers. What are $d(p^\ell q^m)$ and $d(p^\ell q^m r^n)$? Check a few cases in the table.

(xii) Look back at the previous questions and try to explain what you have found. In particular try to prove, firstly, that if n is a perfect square then d(n) is odd and secondly, that if d(n) is odd then n is a perfect square.

(xiii) What is the smallest natural number n such that $d(n) = 30$?

Appendix 2(ii)

Background Information on Students who Volunteered for Interview

	Experience	Performance in Subsidiary Mathematics
(i)	Modern O-level. Did not enjoy it very much, could never see the point of it all.	Average
(ii)	Traditional O-level. Enjoyed mathematics, very methodical approach, obtained reasonable grade.	Good
(iii)	Traditional O-level. Did not like mathematics, did not see it as particularly relevant.	Good
(iv)	Traditional O-level. Enjoyed some parts at school. Reading Sociology and Social Studies.	Good
(v)	Traditional O-level. Had a good teacher and enjoyed mathematics. Reading English and Psychology.	Withdrew
(vi)	Traditional O-level. Not too unhappy at school, but found some parts difficult. Reading Sociology and Social Studies.	Good

(vii)	Modern O-level. Not too keen on mathematics, mostly found it boring. Reading English and Sociology.	Average
(viii)	Traditional O-level. Liked some of it, enjoyed those parts that could be "applied". Reading Law and Psychology.	Average
(ix)	Traditional O-level. Could never see the point of it all at school. Reading Law and Psychology.	Poor
(x)	Traditional O-level. Hated most of it, but it was not quite so bad when you got used to it. Reading American Studies and History.	Poor
(xi)	Traditional O-level. Hated it, would go limp at the sight of an x, Reading American Studies and History.	Very Good
(xii)	Took O-level at 15 and got a poor result. Did not enjoy it during the last year at school. Reading French and English.	Average
(xiii)	Modern O-level. Was concerned that they covered rather a lot of "new" mathematics. Reading French and English.	Average

(xiv)	Modern O-level. Obtained a good grade and was reasonably happy without much feeling of enjoyment. Reading History and Politics.	Very Good
(xv)	Traditional O-level. Obtained a good grade. Did not find it easy, had to work quite hard. Reading International Relations.	Fair
(xvi)	Obtained a good CSE but failed the O-level. Did not enjoy it, had problems with the teacher. Reading Psychology and Sociology.	Average
(xvii)	Modern O-level. Did not like it much and gained a reasonable grade only because of hard work. Reading French and Law.	Poor
(xviii)	Traditional O-level. Liked mathematics but was not very good at it. Reading American and Russian Studies.	Good
(xix)	Modern O-level. Obtained an average grade, enjoyed parts but did not get on with the teacher. Reading International Relations.	Very Good
(xx)	Traditional O-level. Enjoyed it at O-level but had problems with the teacher in later studies. Reading Law and American Studies.	Very Good

(xxi) Traditional O-level. Did not like it very much and did not do very well at it. Reading Law and Politics. Poor

Appendix 3

(i) The Prepared Questions for Student Interviews: Subsidiary Course in Mathematics

(a) Interviews in November

What mathematics did you do pre-University? What were your feelings about the subject? Did you experience any enjoyment? What did you find was the hardest part? If you did not particularly like mathematics, can you say why?

Why did you choose to do the mathematics option? Some people find mathematics a particular kind of challenge, what do you think? What do you hope to get out of this particular course?

Can you tell me what are the objectives of the lecturers in this case? Is it possible for you to tell me which topics you will study? What do you think the objectives should be?

What sort of progress are you making? Is it hard or easy? Can you say in what way your present studies may be different from your previous experience of mathematics? Can you give me an example of something you have studied which gave you some level of satisfaction?

How frequently do you find yourself getting lost in the lectures? Can you give me an instance where you appeared to lack some pre-requisite knowledge? How does the amount of material covered compare with your experiences in your other subjects?

How did you rate your progress in studying number theory?

Can you name any point in the lectures when things became obscure? How readily were you able to see what was required in a mathematical proof? How good were you at making conjectures?

What importance would you give to the practical aspects of mathematics on a course like this? Can you name some important practical aspects?

The booklet on flow-charts, how useful did you find it? Was it clear? Did you work through it systematically? (Did you try all the exercises?) Did you read any parts before the lecture? How worthwhile is the optional material? Is there any way in which the booklet is unsatisfactory?

How do you rate the relative merits of lectures and tutorials? Do you think the tutorials complement the lectures? Is there sufficient means of communication in lectures and tutorials?

How do you rate your progress so far? How do you think your progress might be improved? Can you give me a reason why you feel particularly happy or unhappy with this course at the moment?

(b) <u>Prepared Questions for February Interviews</u>

(For those not interviewed previously, ask about mathematical experience at school, feelings about the subject, any experience of enjoyment, the hardest part, reasons, if any, for not particularly liking mathematics, reasons for choosing a mathematics option.)

How difficult do you rate the course at present? How does your progress compare with your other studies? Can you give a reason why the study of mathematics should be considered important? Is the concept of studying a science a sound one? Can mathematics be understood by everyone? Would you still choose mathematics as your science subsidiary?

Can you name any topics you felt were difficult? Can you name any you found interesting or enjoyable? If some degree of challenge is necessary, why do people worry about being lost? Are you aware of what the lecturer's objectives are? Are you always aware of the topics being considered?

Do you do any reading in this subject? Can you tell me how often you have looked at a text-book?

Many students find the tutorials useful, what do you think? Do you see any major problems in the lectures? Would students get lost less often if they made greater effort? To what degree are problems due to students' lack of interest? What effect on your progress has there been because you now have had quite a bit of practice in maths? Do you feel rather "exposed" when trying to do mathematics?

How is your attitude affected by the fact that maths is viewed as a cold subject? Does this course show mathematics as something other than a box of tricks?

What would you teach on this course? Tell me where you found the present topic hard to follow. How could this be improved? Do you feel the whole concept to be too difficult at this stage?

(c) Prepared Questions for March Interviews

How do you rate your mathematical progress since the last interview? How important is it for students like yourself to study mathematics? Would you choose the mathematical option given the choice now? Are only those with a special aptitude likely to progress in mathematics?

To what extent do you see the subject as one in which things are either right or wrong? Having chosen to do this course, what do you now feel is your strongest motivation? Is there any way in which your school view of mathematics has changed? Is there any sense of apathy or frustration developing in relation to this course?

What would you like to see as major objectives? What topic in recent weeks have you found difficult? (rewarding?) How do your ideas about mathematical proof compare with those at the beginning of the course? Here are some aims which might be thought suitable for this course; how worthwhile do you rate each one?

To be shown how mathematicians do things.

To experience for oneself some measure of mathematical activity.

To get the right answer to a set of examples in a variety of topics.

To obtain enough credit to gain a pass grade.

To be told a variety of formulae that could be applied to given problems.

To develop one's own ideas and avoid the spoon-feeding methods of school.

How much reading have you managed recently concerning mathematics? Are the end of session examinations posing any kind of threat? With growing experience, to what extent does the course become easier? Many students seem unclear about the aims of the course, can you say why? Is it possible they have some idea of the aims but feel they should be different?

Can you say how the topics you have been studying may or may not be linked to each other? How do you rate the recent lectures? Tell me what you saw as the objectives of the lecturer when you studied calculus. How useful are the handouts? Why do students sit so far from the lecturer where it must be difficult to hear?

(d) Prepared Questions for May Interviews

How do you rate your mathematical progress since the last interview? What were the particularly good parts? (What were the particularly bad parts?) To what extent has the study of mathematics become more understandable after a year's experience? What are your feelings about the approaching examination? What, in your opinion, is the biggest effect of this course? Does it concern an attitude change rather than skills?

Can you suggest any reason why the idea of a subsidiary science course should be given up? Are you any nearer to answering the question "What is it all about?" What would you say if I put it that topics are deliberately separated so that progress in one is not dependent on others?

How do you rate the lectures since I last saw you? How much idea have you of the lecturer's objectives in each topic?

Have the lecturers assumed you already know the topic? How do you rate the value of the tutorials recently? Have you spent more, less or about the same time on mathematics in recent weeks? How much reading have you managed?

How did you like the booklet on Boolean Algebra? Does it help you to study the subject? Are the examples helpful? Did you work through them systematically? How helpful was the layout with hints and answers at the back? Were there any difficult sections? Were you able to understand the idea of isomorphism?

Can you say to what extent the course should be about the use of mathematics in day-to-day business? Is the course about presenting maths as a worthwhile subject for anyone to study? Has anything happened to the image of maths as dark and gloomy? Are you amongst the maths-haters? Is this a change in attitude?

To what extent do you feel all your hard work has been rewarded by a worthwhile gain in experience? When you finally graduate and look back on this course, what do you think you will remember?

Appendix 3

(ii) Verbatim Transcript of Student Interviews of May 1979

Each comment is attributed to students by a roman numeral, the key to which is given in Appendix 2. The prepared questions are underlined, supplementary questions appear in parenthesis. The students were seen in three groups, (ix), (viii), (vi) and (xi) in one group, (vii) and (x) in another and (iv) individually.

How do you rate your mathematical progress since the last interview?

(iv) I do not know. Difficult to say because of the holiday and everything. What we have been doing this term has been a bit easier, I think everybody has found that.

(ix) I understand a few more topics. Things have got a little bit easier.

(viii) I think they have got better as well, enjoyed them more.

(xi) Same as before really, I did not mind Boolean algebra too much.

(vi) Just the same as before, I am getting through it.

(vii) I do not know if you can say you really make any progress or not because they keep changing topics. If you grasp one you are not necessarily going to grasp the other one. I do not know really. I cannot say I am beginning to think any more mathematically.

(x) Very, very slow, if at all. More recently things have become less and less comprehensible to me, although the recent lectures on motion, that seems fairly straightforward to understand.

Are there any particularly good (or bad) parts?

(iv) I think it has got gradually better as the year has gone by. We are doing motion at the moment and that is really good because that is more historical. The lecturer is approaching it historically, it is less mathematical and it is a lot more interesting.

(ix) Boolean algebra (was good).

(vi) I like motion we are doing motion at the moment. I like the lectures for that.

(ix) It is more historical.

(vii) Not bad at Boolean algebra, quite easy really. I find motion more interesting than the rest of it, it seems to apply more to everyday things. Down to earth. No bad patches, the big problem I do find with the Boolean algebra is when it goes on to switches, that is when I get a bit confused. It is the plans of the switches. I have got a blank where that is concerned.

(x) I was completely foxed by Boolean algebra.

Do you find studying mathematics more understandable after your year's experience?

(iv) Yes, I think I am getting into a mathematical way of thinking. In a way, I suppose every section becomes a little bit easier. I do not know whether they are actually easier, most of them seem to be less difficult things and less of a surprise than before.

(vii) I do not think so. In fact, I think at first I was a bit more enthusiastic about it. Now I just want to get it over and done with, quite honestly.

(ix) I think you get into a way of thinking with maths. Not significantly easier, I find the topics a bit easier.

(xi) No, I have not got better at thinking mathematically, I do not do enough of it.

(vi) No, I think that is true.

(x) No, not at all.

<u>What are your feelings about the approaching examination?</u>

(iv) I have not really thought about it. I have seen past papers and they are quite frightening. I will probably get more worked up about it when it gets nearer. I will not enjoy it at all. Sometimes I quite enjoy exams, but I won't this one.

(ix) Panic, I do not know, I do not think about it that much.

(viii) I had a quick skip through my notes and found I have got a few gaps but I think I am worrying least about that.

(xi) I am panicking, basically. I do not worry about it, but when I do I panic.

(vi) It seems quite easy to do topics as you are going along but whether you can do them when you are faced with it (is different).

(ix) Everybody seems to say "Well you are going to pass it, so many people do." That comes as a comfort.

(vii) I looked at one of the past papers. When they went through it and said what sort of areas the questions will be on, I thought, that will be alright. I can revise that. But when I actually saw the questions I got a bit worried. They are phrased in the sort of way. I do not know.

(x) Dreading it, becoming a real threat. Whether they look leniently or not on subsids I do not know. You hear from one year that it is easy, any old fool can pass it. Others, the worst thing they have ever done. You do not know really. I do not know if it is anything like my Sessional (last year). In that I thought I had failed outright but they passed me.

What, in your opinion, is the biggest effect of the course? Does it concern a change in attitude rather than in mathematical skill?

(iv) I can understand what my mathematician friends are talking about sometimes, even if it is only the terms I understand. I do not think it has really changed my attitude. I suppose I have done things which have proved to be quite interesting. That might make me more happy about mathematics. I did not really get on with mathematics at school, I was not really good at it.

(ix) I think I will just forget about it. I did it because I had to do a science. I do not think it has given me an awful lot.

(viii) Well, I think I will forget most of it but I might sometimes come up with some obscure fact that might impress people. I think I feel better inclined towards mathematics. Been a bit better than O-level anyway.

(xi) I will forget the great majority of it but it might leave some sort of impression. Changed my attitude a bit.

(vi) It has not been continuous enough in the subjects, just sort of jumped about.

(xi) I have not noticed any continuity.

(vii) More likely a change in my attitude because I do not think it is the sort of thing I am going to come across again really, the type of maths that we have done anyway. But I think it helps you to understand a lot of other things. If you happen to be reading and you come across some term you have a bit more understanding of what it is about.

Should the idea of a subsidiary science course be given up. Why (not)?

(iv) No, I do not think so. It is meant to be a broad education here and I think that is (why) most people should do something like that really because they would never do it on their own.

(ix) I do not think so. It is a nice idea but I think they have got to adapt the course.

(viii) Well I think they might be (unsatisfactory). It might be better if they let us take two "subsids" from any subjects, even if they were arts. At least you would be more interested. I do not particularly feel it is worthwhile.

(xi) It would be a pity to drop sciences in a way but I tend to agree you would get more out of something you were more interested in. Possibly if they used continuous assessment on these courses. I am in favour but to the extent that it has not meant an awful lot to me.

(vi) Yes, I would like to see science subsidiary courses, but as (xi) says, use continuous assessment.

(vii) Yes I do, because I think by the time you come to university, people have decided what they want to do and if they are arts inclined it is just a big trauma to a lot of people going through this sort of thing. Just a worry. I think if they want to do a subsid, maybe it should not be an arts but definitely biased that way.

(x) Yes, I think in a way trying too much. A lot of people can appreciate both, I would not deny that, but others are very, very arts based and I am one of them. I never seem to be able to understand anything. I was doing the Boolean algebra, it seemed simple and I thought, this seems easy, why can't I understand it and I just could not.

Are you any nearer an answer to the question "What is it all about?"

(iv) Not any clearer really. The sections of the course still seem a bit unrelated to each other. They do not seem to be moving towards any particular end.

(ix) Not really, just different forms of mathematics.

(viii) No, not really.

(xi) I agree. The most it has done is shatter the odd illusion from school, like geometry, there is only one kind of geometry and things like that.

(vi) I tend to see it as unrelated. It has clarified some points that one had from the O-level course. Apart from that.

(vii) No, obviously I think it is to teach you more about basic concepts behind mathematics and there is quite a few historical things put into it. But no, I really cannot see that it is heading in a certain direction.

What would you say if I put it that topics are deliberately separated so that progress in one is not dependent on others?

(iv) I could understand that but they do not seem to have any relation to each other at all sometimes. Maybe that is doing social sciences where everything tends to fit in. Maybe it is difficult to see it another way, I do not know.

(vi) I can see the point, but then again if you do not master the first one then you try and relate the second one to it, well I do. Unless it is a totally different topic, but say, with Boolean algebra and sets, that confused me because I was trying to relate some things that were done in sets to some aspects of Boolean algebra. (Did you subsequently see that there is a relationship?) That was what I was trying to do.

(viii) I would prefer something more continuous. I might make you work harder if the work was more progressive.

(ix) I think you would feel you had achieved something as well if you can build on something you had first learned.

(xi) Perhaps there could be one or two loose themes so you do not sink all in not understanding the first bit, not that bad but something to link in, a few starting points instead of one a week.

(ix) It seems so futile to write off some topics, I cannot do these but it does not matter because we shall be doing something completely different.

(vi) Yes, because if you have to answer five questions; let us face it, that is what you are aiming for, to get through the exam really. You tend to pick out five areas which you think you can do. But if it led on, you would start getting completely lost by the end of the course. I think you would give up.

How do you rate the lectures you have attended since I last saw you?

(iv) The motion ones are really good. Boolean algebra was easy to understand, quite straightforward really. They have been quite good.

(ix) Much better.

(viii) Yes, I have been able to follow them much better.

(xi) Yes, I think so. Boolean algebra was easier and motion is very good. He does seem to do what the others should have done all through, he makes you laugh.

(ix) He is always telling you what it is used for, not just abstract.

(viii) He does not just turn round and write things on the board.

(ix) He talks to you and asks you questions.

(xi) People actually answer.

(vii) Well, the ones on motion have been good. (Any occasions when you were completely lost?) Yes, this has happened recently. I think if you sat there and really concentrated you would be alright but I tend to switch off after a little while. It does not hold my attention enough.

(x) They are the usual mix, some are quite good and others are (pause). But recently the ones on motion are quite good because he has been trying to generate an interest in the subject again, which is important. He is quite funny too. He seems to try and put across his interest in the subject.

Do you have an idea of what the lecturer sees as the objectives of studying that particular topic?

(iv) No, I think that is half the problem of the course that there is no brief introduction in words about what the topic is going to be and why it is going to be that, that sort of background.

(ix) No, not really.

(xi) You just seem to launch into a topic with no explanation or anything and that is it.

(viii) The lecturers themselves seem so enthusiastic about their subject that they do not seem to realise we are not.

(xi) No, they do not.

(x) No.

<u>Have the lectures assumed you already know the subject?</u>

(iv) They tend to assume that you know somehow what the subject is about, even if the subject is as obscure as Boolean algebra. They never make any attempt to define what the subject is. I find that really difficult because it is hard to take notes and concentrate when you do not know what you are meant to be moving towards.

(ix) I think they do. They seem to in the lectures, not so much in the tutorials.

(xi) In lectures they tend to start off very simply, aware that you are dense, then all of a sudden get carted off on some flight of fancy. When they get into their subject they lose you again, once they have got through the introduction.

(ix) I feel sorry for them really, they are so involved with it themselves it must be a bit crushing for them.

(vii) Sometimes yes, definitely in some of the earlier ones, geometry especially. I think they forget that the last time you probably did any geometry was when you were fifteen or sixteen at school. Deep down somewhere you know it but suddenly you have to go into a lecture and be faced with all these things you have not seen for five years. It is a bit of a shock and I think maybe they ought to have some sort of refresher course.

(x) I think they start off trying to (pause), they think that once they have said it once that is it. You all understand it, we can go straight on. But the majority of people just sit there. They say "Do you understand that?" and you all shake your heads whether you do or not. (What is the problem?) I think sometimes with a subject it is awkward for them, you cannot spend too much time on it. You have got to bomb through it. You cannot stop at every point, you would only do one subject in the whole year.

How do you rate the value of the tutorials over the past few weeks?

(iv) They have been good so far. The Boolean algebra ones cleared up everything that was a bit misty from the lectures.

(Why do the tutorials seem so much more useful?)

(ix) It is because it is a much smaller group. You can say exactly what your problem is. You cannot do that in a lecture. They are very understanding in the tutorials, but once you get them in a lecture hall, off they go.

(viii) I really almost enjoy tutorials but lectures are different.

(xi) Well, just about. They mean more to me than the lectures. Often all I learn about a subject is in a tutorial.

(vi) Yes, I find this is true.

(vii) They are okay. You can usually manage to sort out your difficulties in the tutorials. It is much better to have a small group. The only problem is you ask a question and say you do not understand that point and what usually happens is you get it explained in exactly the same way again and you still do not understand it. You do not feel like asking again. Sometimes it takes a while for it to click. You cannot suddenly get it like that, you have to sort of go round the problem.

(x) I usually seem to understand it on the day. Sometimes

I think, this is good I can understand it. I will go home and do the work. I get home and remember I have got another essay to do. By the end of the week I get round to doing my maths to hand in and find I have forgotten it all. I then try working it out and it just will not go in. Yes, they usually answer my questions in tutorials.

Have you spent more, less or about the same time on mathematics over these past few weeks?

(iv) Less. I do not know if that is because I have found the topics easier and I have just been able to do it without too much bother. That is probably the reason. (Any time for reading?) No I have not. I would like to do some, especially after looking at last year's exam paper. I think some reading would be useful but it is hard to fit it in. I suppose I could if I tried, I would like to.

(ix) Much less because I have got exams in principals coming up so what I decided to do for maths is to do the essay question.

(viii) About the same really. I do not do any reading.

(xi) The same, I just do the tutorial work.

(vi) About the same.

(vii) Less I think, definitely it has gone down since the first term. The first term I was quite interested in some of the things we were doing and I spent a bit more time but as it has got nearer the exams and I have got more work to do I tend to spend less time on maths. No reading of mathematics.

(x) About the same, but in the run up to the examination I will spend more time. No background reading. My sister did it all for me. She is a maths teacher and she was trying to din it in to me.

<u>How did you like the booklet on Boolean algebra?</u>

(iv) It is difficult to say because we had the lecture notes as well. From what I can remember it seemed alright. (<u>Was it clear</u>?) Yes, I think so. (<u>Is a booklet helpful in studying</u>?) Yes, when it gives exercises and that sort of thing. We did a couple in the lectures but I did not do any on my own. (<u>What about the layout</u>?) I think if there are hint-type things they should be incorporated into the text as it went along. Whether this would defeat the object of the exercise I do not know. I suppose it could be difficult having it all together but it is a good idea that it is there anyway.

((vi), (viii), (ix) and (xi) had not received copies of the booklet.)

(vii) Much better in the booklet, they are much easier for reference. Typed out better. (<u>Have you read all the booklet</u>?) I cannot say that I have yet. (<u>What about the layout</u>?) Useful because there is no actual text book that you can work from so if you had a set of these booklets at least there would be something to go back and revise from. Otherwise you have just got bits of paper and they tend to get lost.

(x) I only got (the booklet) the day before yesterday because I overslept and missed a lecture. (<u>Have you looked at it</u>?) I was having a look at it last night. It was not too bad, reasonably clear. It is better when they have got the answers as well so that you can check up. That is a help. Otherwise you do the working, get it back next week and you think, oh well, here are the solutions. They do not seem to bear much relevance to anything.

<u>Were there any difficult sections</u>?

(iv) No, the only thing is, I think, it might have used slightly different symbols or terminology from what we had in the lectures. One or two instances I can remember, but that soon sorts itself out.

<u>Did you notice the isomorphism</u>?

(iv) Not very well, not at first. It took a while to see that that is what was happening. When they started talking about sets and circuits I was getting a bit confused but after the tutorial and after doing some work it was okay.

(vii) It was not obvious, (I) cannot say that it dawned on me.

<u>How useful are the notes that are given out</u>?

(ix) I do not tend to look at them during lectures but I read through them afterwards. They are not too bad. They are better than nothing. In lectures, all I write down is what is on the board. (<u>Do you require more detailed notes</u>?) Yes, possibly, doing it in simple steps.

(viii) I find notes quite useful. I think if they had them in booklet form more people would be inclined to skip the lectures.

(xi) I think instead of one of the lectures, another tutorial if anything.

(vi) The lectures on Wednesday morning seem much better. The two split groups are smaller. (<u>Why is this so</u>?)

(xi) It is the lecture theatre, you feel as if you are sitting at the end of the earth.

(ix) It is so big, so much space. The lecturers are not even looking at you.

(vi) Their voices just seem to fade away.

(ix) Five o'clock on a Monday is not a good time, it is too late.

Can you say if the course should (not) be about the use of mathematics in day to day business?

(iv) I think it could be pointed out that it could be useful, but personally, I am not too bothered about things like how to work out mortgages and interest rates. I suppose it could be an objective but it would not interest me.

(ix) Yes, I think I said that last time, although I wanted it to be historical because that is what I thought it was. But since it is mathematical it might as well bear some relation to business. (Would this replace things like Boolean algebra?) Well, I think I enjoyed that because it was fun.

(viii) Yes, I think it would be more useful but I suspect I would find it deadly boring. (What is interesting in mathematics?)

(ix) I like talking about the philosophy.

(xi) I would prefer it to be applied to everyday life. It has more connection, at the moment it is very set apart, it is not relevant at all.

(vi) I think I would like that (day to day business mathematics) rather than (pause). I do not know though, I would like some of the history as well, to start off with.

(xi) There seems to be a limit on the amount of history you can give in a mathematics course. You cannot tell of a progression without telling what the thing is.

(vii) I think it is more useful; you are more likely to use that later on. As I said before, I do not really think that

I am going to use anything that I have learned. It will just fade into the background like languages when you do not use them, so you do not remember them. Most people are going to have to go out into some sort of commerce or be involved with some sort of mathematics and maybe it would have been better to do that sort of thing.

(x) Yes, I think it should be based more with real life than anything. Things like calculations and mortgages, how it is all done. Theoretical maths at the moment, we cannot seem to relate it. I mean, you can relate it vaguely, like when they say this is the concept of computers. You hardly want to know how a computer works if you are arts-based and you are only doing mathematics as the lesser of all evils.

Is the course about presenting mathematics as a worthwhile subject for anyone to study?

(iv) I think it is saying maths is not as horrific as you thought it was at school and it can be different and interesting and it is not all numbers. (Has this emerged from the course?) It had to be hard, I see maths as a hard subject. It has to be difficult. I think we were introduced to things I had not heard of before so I suppose it achieved that sort of objective. I suppose it was okay in that respect.

(ix) I suppose it must be one of its aims if arts people are to do it. I think this course might be useful to people doing a Certificate in Education and are going to be teachers.

(viii) I think it should be one of its aims but I do not think it is. I do not know why it does not work. There is so much apathy I think.

(ix) No interest for arts students. (They) cannot be bothered with science.

(xi) I cannot see it doing much for people doing a Certificate of Education really, because it is not the sort of maths you do in school. Certainly not the sort of maths a non-specialist maths teacher is going to teach. You are learning how to teach simple maths rather than teaching school children Boolean algebra. The sort of maths we are doing would have to be taught by a specialist.

(vi) I think it is a nice idea, the sort of thing they are trying to do here. I do not think they are succeeding in it.

(ix) I think they would have to change the work load. Two subsids is just too much.

(xi) I agree there. This do or die thing, if you fail it, then bad luck. I do not like that pressure. I say if I could do it for interest but then I know I would not do it for interest if there were no exams and things.

(vii) I do not know really if there is much point in doing that because by the time you get to university you are doing it as a subsidiary. You are not likely to suddenly think, well maths is the thing for me. (What about the ways of thought in mathematics?) I do not know. Arts students tend to have a funny way of looking at things anyway. They are not going to look at it in the same way as (indistinct), and I do not know whether a year is long enough anyway, especially with the time given to it. I cannot say I developed a new scientific way of looking at things.

(x) Some people have a grasp of mathematics and I do not. That is where the problem lies. I am not sure whether the extension might go into the more abstruse and there are quite a few people like me I know, and that sort of thing would be even worse. Fewer people would probably understand what was going on.

<u>Has anything happened to the dark and gloomy image of mathematics</u>?

(iv) Well, it has lost a bit of its mystique really. I now know what people are talking about. I have just come across terms and subjects I might be able to remember in the future.

(ix) Lectures (on motion) have been amusing. They have brought life back into mathematics.

(viii) Well, I have enjoyed the Boolean algebra and I enjoy the motion. I do not really see it as dark and gloomy but it is just not my chosen subject.

(ix) So much depends on the lecturer though.

(viii) I can understand people wanting to study it if you are that way inclined.

(xi) To some extent the old myth has exploded, like geometry. I still see it as basically beyond me. Whether I coped with it well or not, it has made me think slightly differently about that topic.

(vi) No, not really. (<u>Is this a change in feeling</u>?) No, it has been a bit dull but not dark and gloomy. When I did up to O-level I did not really think about it, then I forgot all about it until I came here. Experiences at school were not really that bad.

(vii) I never particularly liked it at school but I do not think that bears much resemblance to what we are doing here. No, I do not see it as a dark and gloomy subject, I see it as a subject that ought to be kept for people who want to do maths, not forced on people who do not want to do it. (<u>Should maths be left for the specialist</u>?) Yes, especially the type we are doing here. It just does not bear any resemblance to everyday life. I will just forget it after this year.

(x) It is not so much the subject. I seem to feel, well, some people can understand maths and I cannot and I accept that. It is in me and not in the subject. Some bits can be absolutely crushingly boring, but others are okay. Some of the work you go through is okay. Sometimes of course, you have understood it and then they start going on for about four weeks about it. It drives you nuts, going on about the same thing. You think, I have understood that so what is the point of carrying on. (Are there any aspects that ordinary people ought to be able to understand?) I think some of the topics we have covered. I think, I should be able to do this but I cannot. I do not understand why I cannot do it. I think there are some things people should be able to understand, I suppose really the thing to do is insurance premiums, mortgages, things like that. Something that is going to be relevant to you later on. How they work out bills and interest and so on. That would be a practical help I think.

Are you amongst the maths-haters? Is this a change in attitude?

(iv) Well, I used to at school, I do not really hate it now. I quite enjoy it. It is better than school. Sometimes it is very frustrating when you cannot do it. I hate it then, but on the whole I quite like it now. Yes, looking back, I think I have gained more than I have lost.

(ix) I was, but I am not anymore I do not think. I think this course probably has produced a change because I have mixed with the lecturers and other people here who are involved and I can see how some things would really interest them. I just wish they could drop down to my level a bit.

(viii) No, I do not hate maths. I would not go out of my way to involve myself in it. My attitude has not changed. I did sessional maths last year and it has been pretty well the same.

(xi) No, just mild dislike. I would not go out of my way to (indistinct), slightly better attitude now.

(vi) No, I do not think I am. I have come to terms with it more as the year has gone on. It started off I did not like it at all because I was made to do it. It was one of the best sciences and having to do economics with my course I felt that I should be able to choose another arts subject instead. Having to do a science I did not like very much but it has not been too bad. (Is there much numeracy in economics?) Well, he started off doing that, diagrams and things, but the second session is more social economics, more geared to my course and it is not too bad.

(x) There are certain subjects I like doing, others I detest. I detest doing a lot of science subjects but maths I just try to avoid rather than dislike because I can never, all the concepts never seem to go in. I can never seem to understand them. Some of the things, real basic simple ones I can, but others! When I have been doing this course I have been looking at them and I think to myself, well, it seems easy but I do not really seem to understand it. Perhaps I am asking too much of it or something, going into it too deeply, I do not know. I am no mathematician I suppose.

Do you feel all your hard work is rewarded by gaining any worthwhile experience?

(iv) Yes, I think it has been worthwhile because of what

I have learned from it. It has not made me a mathematician or anything like that but just the fact that on the whole I have enjoyed it has made it worthwhile.

(ix) I do not think I have gained enough for a whole year's work.

(viii) I do not think I have worked very hard. I could have worked harder, but I did not really think it worth it. No worthwhile experience really.

(xi) No, not really, because when it comes down to it, the crunch of getting it in on time, so in a tutorial you may think, next week I shall be a reformed character and spend a lot of time on it. When it comes to it you put your principals (subjects) first.

(viii) I often feel like that at the end of a tutorial, I really could do this well, but you get put off.

(vi) Sometimes I feel this is quite good, but then.

(vii) I think the reward is if you understand one topic. You feel, ah, I have grasped it. I know something about this, feel quite pleased when this happens. It does not happen that often. Overall I cannot see it is going to leave me with any lasting (impression).

When you finally graduate and look back on this course, what do you think you will remember?

(iv) I do not know. That is difficult. I hope I will not have forgotten everything. The chances are I will. I do not know, I really could not say.

(ix) Having to sit down and do maths homework instead of something towards my principals. Something you had to do.

(viii) Just something you have to do.

(xi) Sitting yawning in the lecture theatre with the overhead projector working away.

(vi) Just something that had to be done. I do not suppose I will remember why I actually did not like it or what bits I liked. I do not feel I actually know anything for having done it.

(xi) Somebody said at the beginning of the year, all you have to do is get it, go to all the tutorials and you are all right.

(vi) It is a comfort in a way but it is a sort of negative.

(vii) Yes, I think I shall remember a few things. Maybe if I came across them or read something it might spark off a memory.

(x) I suppose my impression will be that it has changed my view of maths in that there is more to maths than just what one might call very boring subjects. There are some interesting things, like probability, and so if anyone asked me what did I think of maths, I will be able to say, well, it could be interesting if you are that way inclined. But if in the distant future I become a parent I will be absolutely dreading anyone coming up to me and asking me to do their maths.

Have you become very apathetic?

(iv) I will be glad that it has finished but then that applies to subsids as a whole. No, I do not feel as negative as that, I feel I will have gained something from it. If I look back it will not be just a wasted year.

(ix) I regarded it as a chore until just recently but I was not enjoying it much at all. The recent lectures were better.

(viii) He (the lecturer) has changed people's attitude. They are actually smiling. I found I was a bit (apathetic) last term but it is getting better this term.

(xi) I feel better disposed at the beginning of each new term.

(ix) With the exams coming up I have not got time to be apathetic.

Appendix 4

Subsidiary Mathematics Questionnaire

Questions and summary of replies in 1979 and 1980. The 1979 figures are given first.

1. Write down what you think should be a major objective of the mathematics subsidiary course.

(Typical replies)

To provide an enjoyable contrast to the principal subjects.

To ensure a basic knowledge of mathematics that will be of use in daily life.

To make the course interesting and appealing to those who are not mathematically inclined and merely see the necessity for passing the exam.

To reintroduce maths as a worthwhile subject for anyone to study (as opposed to the approach taken at school).

To get students to pass the exam.

To stimulate an interest in mathematics amongst students who would not normally study the subject.

2. How do you rate mathematics as a subject of study?

very difficult	difficult	average	easy	very easy
2,1	14,4	4,2	0,0	0,0

3. Do you think mathematics can be enjoyable?

never	sometimes	often
0,1	19,5	2,1

4. Name, if possible, two topics that you found particularly difficult.

(Most frequently mentioned:)

Differentiation	12,1	Cartesian geometry	4,2
Integration	6,0	Groups	3,0
Euclidean geometry	5,2	Non-Euclidean geometry	2,2

5. Do you find it easier or harder to understand topics now than at the beginning of the course?

easier	about the same	harder
12,2	8,4	1,1

6. Are the lectures or the tutorials more helpful to your studies?

lectures more helpful	equally helpful	tutorials more helpful
1,0	1,1	19,6

7. What effect do you think a change in the amount of material in the course might have?

better with less material	about the same	worse with less material
12,3	2,2	7,2

8. If you feel some improvement could be made, how would you increase the effectiveness of the (a) lectures (b) tutorials?

(Typical replies)

(a) Lecture theatre not suitable.

Take complex topics slowly, always provide printed notes.

Slower procedure, explanation of structure of lecture, practical use of material taught.

(b) Tutorials are okay.

Deal with specific problems rather than a continuation of lecture course.

Collection and return of work from tutorials would be useful by next week.

9. How often have you referred to a mathematics text book in the last three weeks?

not at all	once	twice	three times	more than three times
18,5	3,2	0,0	0,0	0,0

10. Do you think an understanding of mathematics to be important?

no importance	little importance	some importance	important	very important
0,0	1,0	6,2	9,5	5,0

11. Do you become frustrated when studying mathematics?

very often	often	sometimes	rarely	very rarely
4,1	8,2	6,4	2,0	1,0

12. Do you think mathematics can be satisfying?

rarely	sometimes	often
0,0	17,5	4,2

13. Name, if possible, two topics that you have found to be satisfying.

(Most frequently mentioned were:)

Sets	12,0	Graphs and functions	3,2
Groups	6,0	Number Systems	2,2
Probability	4,0	Differentiation	2,2

(These figures are not directly comparable for the two years since probability was not studied and sets and groups left until the Summer term in 1980. See also question 4.)

Appendix 5

The study booklets developed for use in the subsidiary mathematics course for arts-based students.

(i) Flowcharts

(ii) Boolean Algebra

FLOWCHARTS

In this booklet, sections marked ! indicate where some action is expected of the reader. The section marked (O) is interesting but optional material that can be omitted without losing the main argument.

Flow Charts

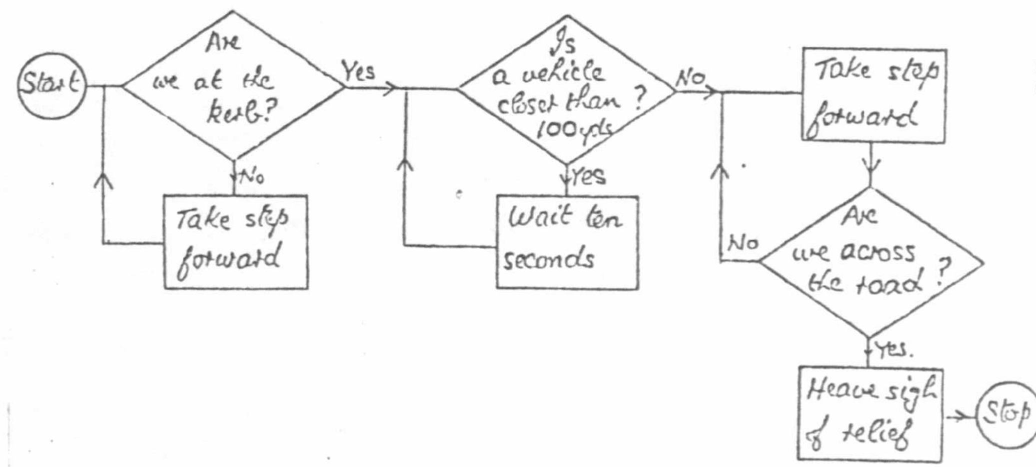
Flow-charts are used in computer programming to describe the sequence of operations which the machine must perform to achieve the given task.

Designing the structure of the flow-chart, the correct sequences, decisions, branches etc., is the most important part of programming. A correct procedure is called an algorithm and a useful way of writing the algorithm is by means of a flow-chart.

What is an algorithm?

The formulation of the logical steps in an order and a manner performable by some device (human or otherwise) for the determination of the solution to a given problem.

Here is an algorithm for crossing a road safely:



Notice the different features that make up the flow-chart:

1) Rectangular boxes used to display some operation.

2) Diamond-shaped boxes used to display a decision.

3) Connectors used to show the logical flow.

4) A loop forming a closed logical circuit.

5) Branches that occur as the result of logical decisions.

6) Exit points, that is points where exits from the loops occur.

Most flow-charts will contain all or most of these features.

In designing a flow-chart we must be careful to account for all possible cases. Notice that in the example given above the system would break down if there is a parked car within 100 yds of the crossing point. Pay special attention when analysing a problem to the following:

(a) Any loops that are necessary

(b) All loops must have some definite way of ending (termination)

(c) Look carefully at the output. There is always a tendency to output too little information. In particular look to see if it is worth printing out intermediate results to indicate how a process is going.

It is important to be clear about the assumptions and possible modes of action that we assume in the device we are trying to illustrate. This is much easier to do for simple machines than for complex organisms like humans.

would, for example, the flow-chart for crossing a road have been different if we knew that it was to apply to a five-year-old child or someone confined to a wheel-chair ?

Although flow-charts are associated with computers and numerical calculations, our definition says nothing about either and indeed there are several interesting non-numerical applications. This booklet contains examples of both numerical and non-numerical applications.

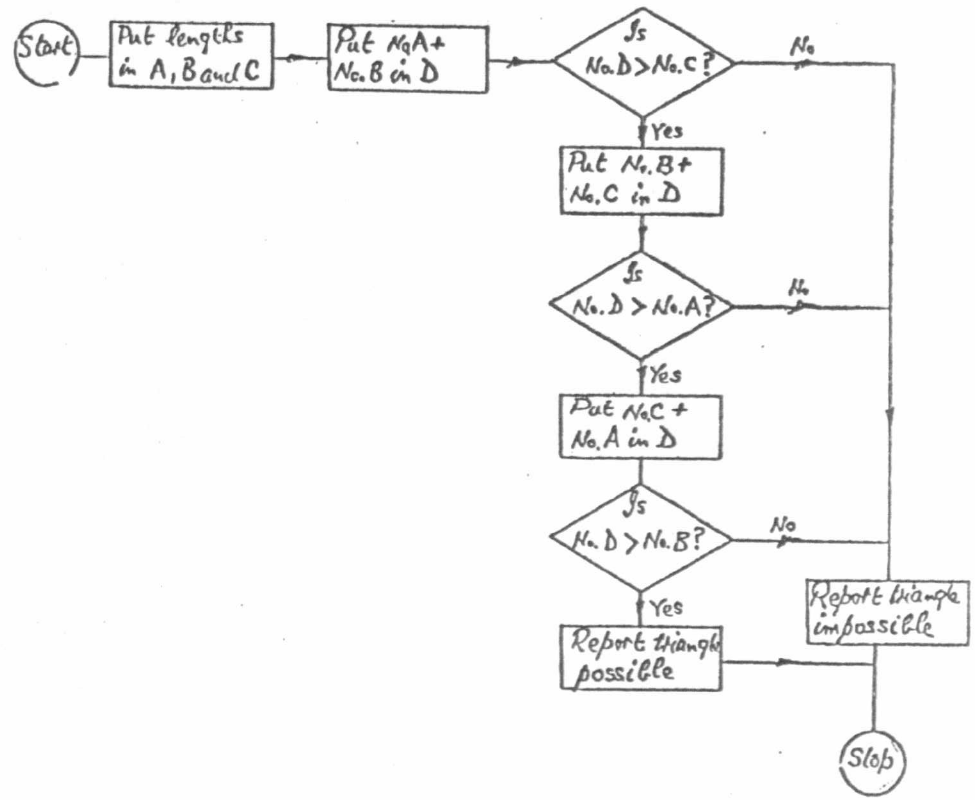
Here are some worked examples for you to study. Have a pen and some paper handy as you may be asked derive some results.

Example 1

A device can add two numbers, compare two numbers and store numbers. Construct a flow-chart to decide whether three rods of given length can form a triangle.

Question: What property are we looking for? Think how the process can fail. For example a = 3, b = 10, c = 4 cannot form a triangle.

Solution: Any two sides of the triangle must together exceed the length of the third. Thus we shall test in turn that a + b > c, b + c > a and c + a > b, otherwise the triangle is impossible.



Example 2

A furniture designer can use four colours, three sorts of wood and two standard sizes in manufacturing kitchen units. How many different units are possible?

The answer is 24 = 4 x 3 x 2 x 1. (Sometimes written as 4! for short and called "factorial 4").

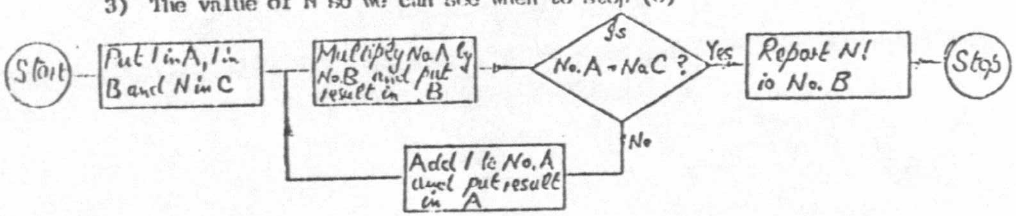
so 2! = 2 x 1 = 2

3! = 3 x 2 x 1 = 6 and so on.

We wish to write a flow-chart to compute N! for any natural number N. We shall assume that a device exists which, as well as storing numbers, can multiply two numbers, compare two numbers and add 1 to a given number.

In this example, we shall start with the number 1 and multiply by 2, then by 3 and so on. Clearly we shall need some check when we have multiplied by N because then it is time to stop. We shall need to store three things.

1) The multiplier which increases by 1 each time (A)
2) The product we have obtained up till then (B)
3) The value of N so we can see when to stop (C)



Work through this flow-chart for N=3 and N=4 and make sure it does what it is supposed to do.

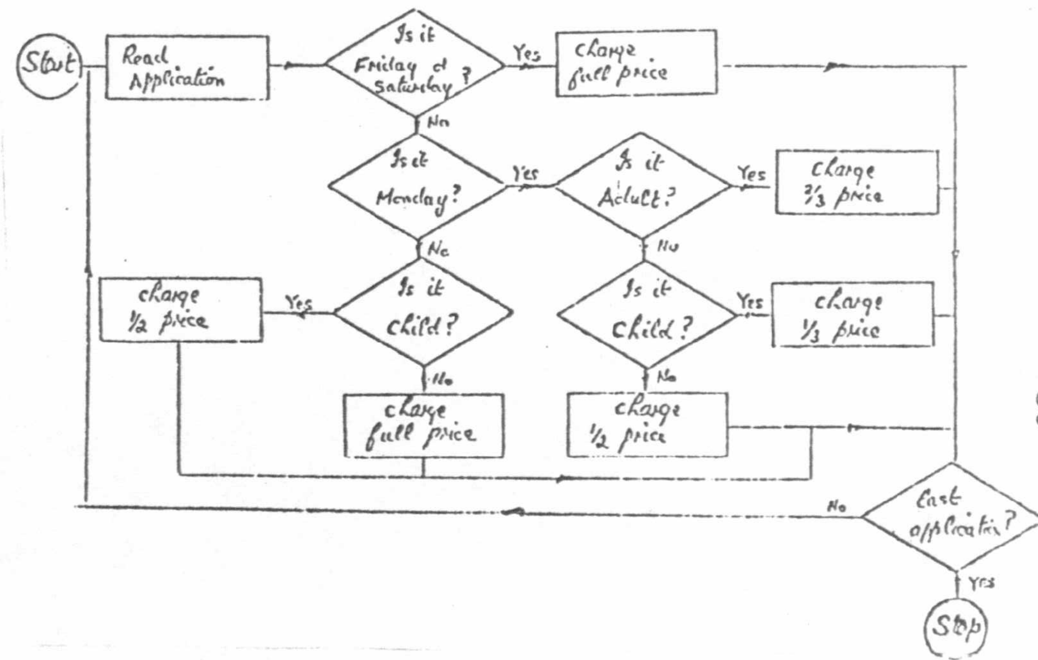
Example 3

This is an illustration of how a flow-chart can be used to display a logical structure far more clearly than written words.

Here is an algorithm for the issue of a theatre ticket:

For Monday performances, children under 14 pay one-third full price, senior citizens pay half-price and everyone else two-thirds full-price. At other performances everyone pays full price except children who pay half-price on Tuesday, Wednesday or Thursday.

Not the easiest of things to read is it. Can you think of other examples of this nature ? A useful way of displaying such information is by means of a flow-chart.



Notice that the flowchart above is not the only flow-chart one could construct in this instance (You may like to try to design one for yourself which does the same job but is different). There is an analogy here with the way in which logically equivalent statements may be constructed in propositional calculus.

Example 4

When a computer requires the value of $\sqrt{2}$ we might imagine it can find the value stored away in its vast memory, just as we use a set of mathematical tables. Strangely however, it is much more efficient to calculate the value by means of a so called iterative process. This means it starts with a guess at the value of $\sqrt{2}$ and then by repeating a set calculation it produces a sequence of results which are closer and closer approximations to $\sqrt{2}$.

Thus, starting with a value (call it x_{old}), generate

$x_{new} = \frac{1}{2}(x_{old} + \frac{2}{x_{old}})$. If this new value is now used as the starting point (swap x_{new} and x_{old}) an even closer approximation is obtained. This process can clearly be repeated.

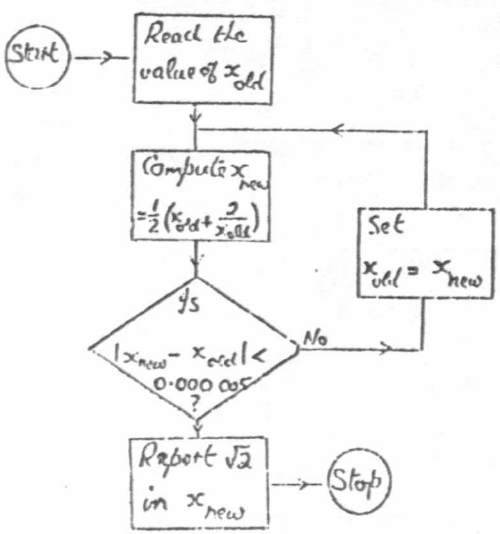
starting with $x_{old} = 1$, we generate the successive approximations:

1.0, 1.5, 1.4167, 1.4142, ...

and starting with $x_{old} = 3$, we get

3.0, 1.8333, 1.4621, 1.4150, 1.4142, ...

check that you can derive the first few terms in each case. Do you think there are any starting values of x_{old} which do not produce a convergent sequence of approximations?



What is the significance of the decision box containing

$|x_{new} - x_{old}| < 0.000005$?

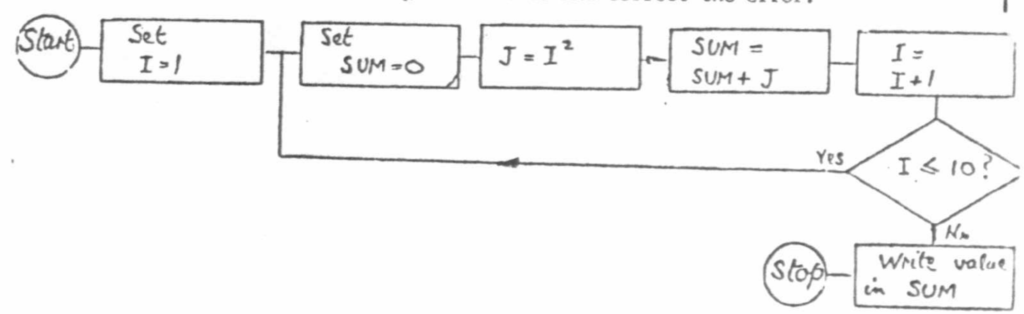
What would happen if this test were excluded Can $\sqrt{2}$ be found with no error at all?

Here are some examples to try yourself. Do not look at the answers in the box (indicated by the large arrow) until you have completed your attempts. Pay special attention to testing your procedure with representative test data.

1. An Electricity Board charges £0.072 per kilowatt-hour for the first 300 kilowatt-hours used in a quarter and £0.027 each for all kilowatt-hours used thereafter. Draw a flow-chart showing how to find the amount due given the number of kilowatt-hours used.

2. Many supermarkets now have automatic change dispensing check-outs. Prepare a flow chart to determine how many one pound notes, 50p, 10p....etc. are to be returned in change when £5 is handed over for a given value of purchase.

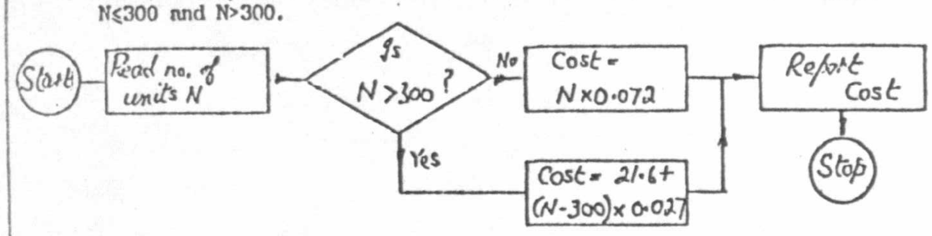
3. The flow-chart given below was designed to find the sum of the squares of the first ten integers. Locate and correct the error.



* There must be an exit point from the loop, otherwise the computation will never terminate.
No, $\sqrt{2}$ remember is irrational and is represented by an infinite decimal.

8

1. You probably have something like the following. Your diagram <u>must</u> work for N≤300 and N>300.



2. If your chart works for all the following inputs, it is probably correct £4.36, £2.82, £3.05 and £1.98.

3. The box [SUM = 0] should <u>not</u> be included in the loop.

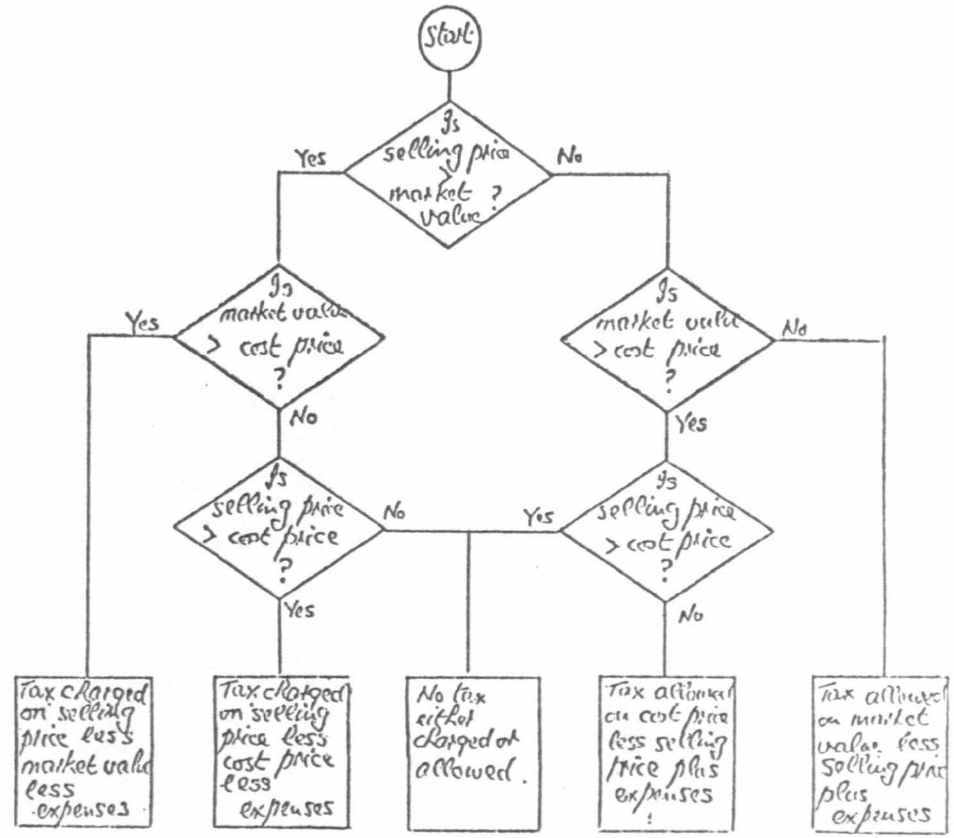
(O) As an illustration that applications of flowcharting can be non-numeric yet far from trivial, consider the following example taken from Lewis B.N. et al. ("Flow charts, logical trees and algorithms for rules and regulations" HMSO (1967).

Extract from Regulations for Capital Gains Tax, subsection F(i) (1966)

"If the assets consist of stocks or shares which have values quoted on a stock exchange (see para.G below) or unit trusts whose values are regularly quoted, the gain or loss (subject to expenses) accruing after 6th April 1965, is the difference between the amount you received on disposal and the market value on 6th April 1965, except that in the case of a gain where the actual cost of the asset was higher than the value at 6th April 1965, the chargeable gain is the excess of the amount you received on disposal over the original cost or acquisition price and in the case of a loss where the actual cost of the asset was lower than the value at 6th April 1965 the allowable loss is the excess of the original cost on acquistion over the amount received on disposal. If the substitution of the original cost for the value at 6th April 1965 turns a gain into a loss or a loss into a gain, there is for the purpose of tax, no chargeable gain or allowable loss".

9

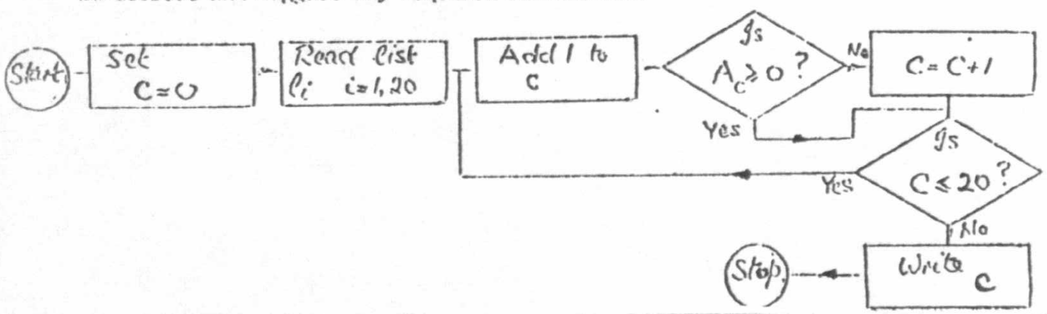
Here is the same information incorporated in the form of a flow-chart. You may judge for yourself which is the easier to follow. Consider some reasons why many people find the flow-chart to be very much easier to follow.



! Here are some simple exercises to help you judge if you have mastered what has been said. Work through them without assistance and check your answers against the given solutions.

1. Design a flow-chart to show how, given a device which can store and compare numbers, the largest of three given values can be found.

2. The flow-chart below is designed to count the number of negative entries there are in a list of 20 numbers. Determine if the flow-chart is correct and suggest any required corrections.



3. A company classifies its products by weight as follows: Class A is 12 pounds or over; Class B is 5 to 11.9 pounds, Class C is less than 5 pounds.

Draw a flow-chart to determine into which class to place an item whose weight is given.

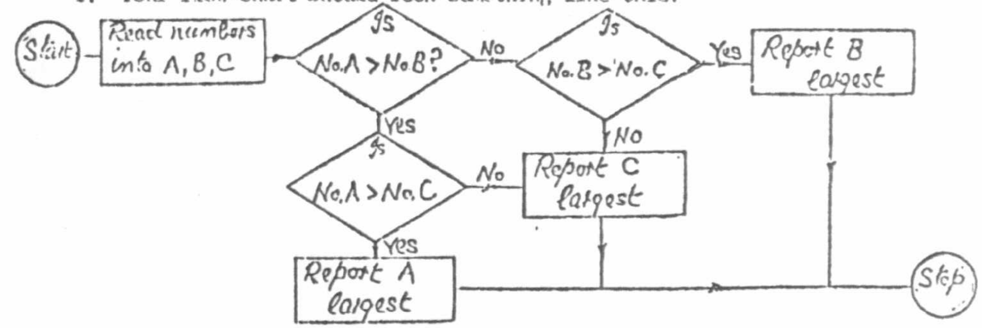
4. The Highway Code says the following about use of the horn:

"When your vehicle is moving use your horn when it is necessary as a warning of your presence to other road users - but never use it as a rebuke. You must not use your horn when your vehicle is moving between the hours of 23.30 and 07.00a.m. in a built-up area. When your vehicle is stopped on the road you may only use your horn at times of danger due to another vehicle moving."

Use this information to design a flow-chart to decide whether or not to use the horn in a given situation.

Solutions to Exercises

1. Your flow-chart should look something like this:

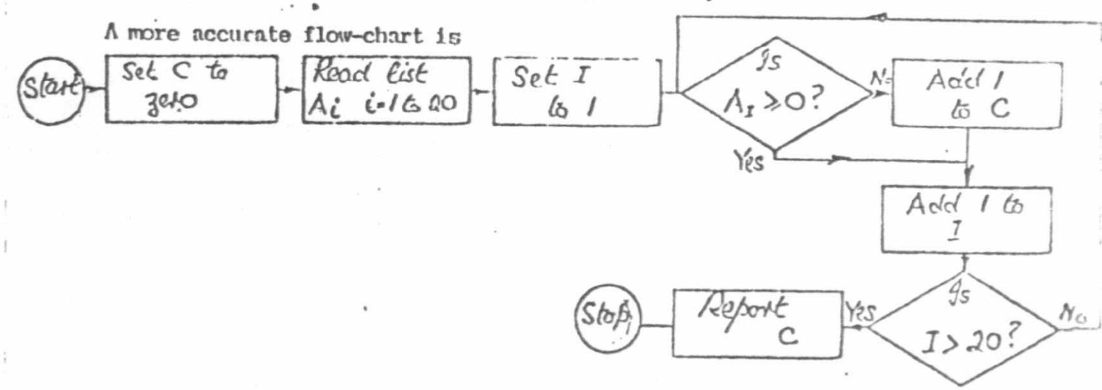


However, if your chart works with all of the following sets of data, then it is correct.

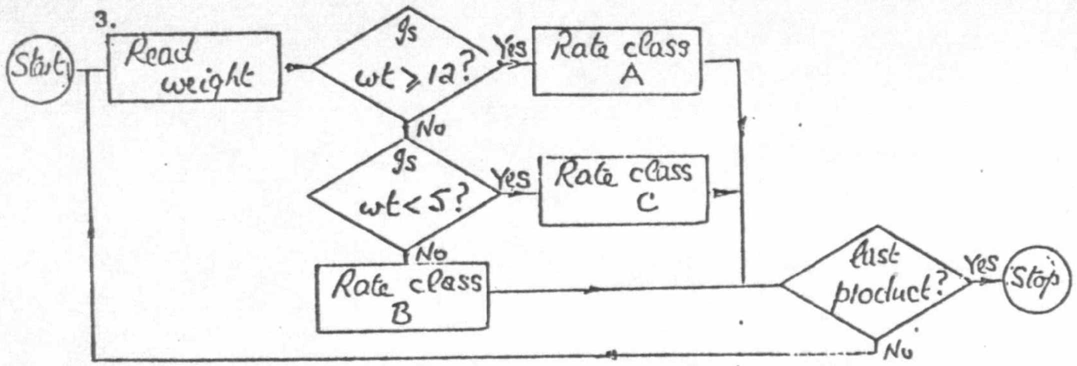
5 10 7	10 5 7	5 7 10
7 10 5	10 7 5	7 5 10

2. The store C is being used for two conflicting purposes. It is used to store the count of the number of negative numbers encountered and as the count for how far the test has gone along the array.

A more accurate flow-chart is

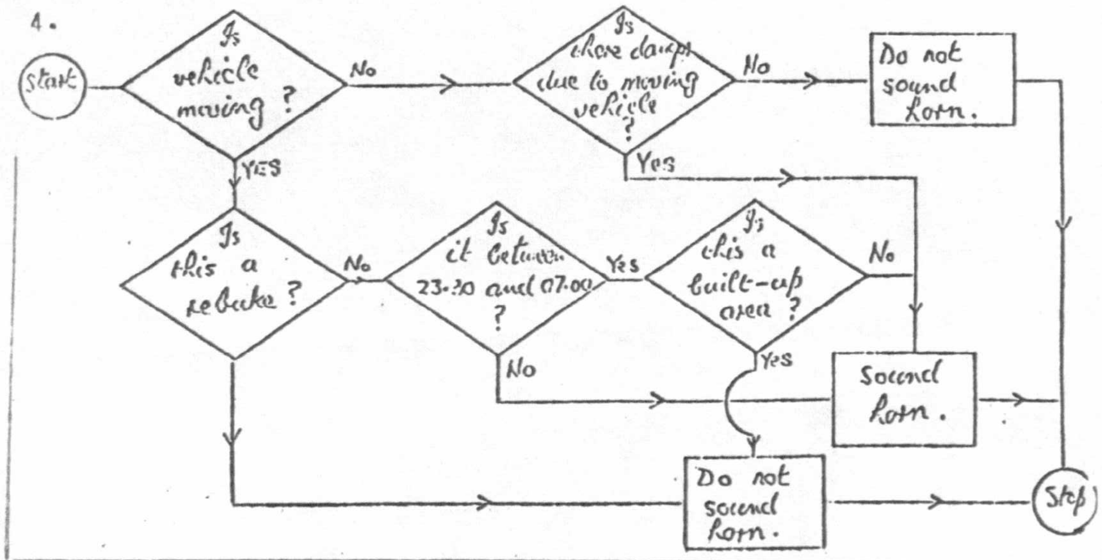


3.



Your flow-chart need not be identical to this to be logically correct. However, there should be only two decisions necessary to differentiate between the three classes. Test your chart with suitable data e.g. three products of weight 14, 3 and 7 pounds respectively.

4.



Subsidiary II — Tutorial 3

For Class discussion

(a) Trace through the flow-chart of example 1 with rod lengths 6,3,1 and then with rod lengths 2,3,4. Draw up a table to show what numbers are in the pigeon holes at each stage.

(b) Follow through the flow-chart of example 2 with N = 4, keeping a record of what numbers are in the pigeon holes at each stage. What happens if you start with N = 0 ?

(c) Redesign the flow-chart of example 2 for a similar device which cannot add 1 to a number but can subtract one from a number.

(d) Design a flow-chart to receive a finite collection of numbers and report the largest one. Assume the device has the basic skills of those you have already considered.

(e) The problem of example 4 is the evaluation of $\sqrt{2}$. Modify the procedure to evaluate $\sqrt{N}$, where both the values of N and x_0 are provided by the user.

(How do you think the relation $x_{new} = \frac{1}{2}(x_{old} + \frac{2}{x_{old}})$ should be modified ?

Test your conjecture with N=25 and x_0 = 4, which should give x_n = 5.125

For homework — Answer either question 1 or question 2

1(i) Design a flow-chart for finding out whether a number is prime or not by looking for divisors less than it. Assume that the device involved, in addition to the usual skills, can also tell whether one number is divisible by another or not.

(Hint let N be the number. Test all numbers less than N from 2 upwards to see if each is a divisor of N. If any divisors are found, report "not prime".)

(ii) Adapt the flow-chart of (i) for finding how many divisors there are of a given number N.

2. Read some of the following and write about "Mathematical Machines".

S. M. Dowly — Mathematics: Art and Science — Chapter 9
Pergamini & Co-Authors — pp 20-30 (look at pictures)
Dubbey — The Development of Pure Mathematics — pp 123-131
Scientific American — p 53 Morrison & Morrison, Charles Babbage
p 325 Davis, Mathematical Machines
p 337-3431 only Ulam, Computers

BOOLEAN ALGEBRA

This booklet covers three systems which have the same basic structure, known as Boolean algebra. This essential sameness is called isomprphism and is shown to apply to the algebra of finite subsets, the algebra of propositions and the algebra of switching networks.

You will find it helpful to go through the notes and try to answer the questions and exercises as you come to them. A ! in the left-hand margin indicates where these occur.

At the back are hints and answers. References are made to these in brackets e.g. [H6]. Try not to look at the answers until you have completed your efforts to answer the questions or find yourself completely stuck.

(0) in the margin indicates interesting optional material that may be omitted without losing the main argument.

An Algebra of Sets

Here are two examples involving the subsets of a finite set of elements and the operations of union and intersection.

Consider a set S with one element; call it {a}. What are the possible subsets of S? How many are there? Call them ϕ and S (why?).

Revision: remember the set operations of union ($\cup$), intersection ($\cap$) and complement ($'$). The symbol ϕ is used to represent the empty set.

Here is the table for the union of the subsets of S:

$\cup$	ϕ	S
ϕ	ϕ	S
S	S	S

i.e. $S \cup \phi = S$ etc.

! Construct tables for $\cap$ and $'$ [H1]

Now consider the next possible finite set, having just two elements, call it X = {a,b}. List the possible subsets. How many are there?*

! Exercise A

Try constructing a table for the operation $\cup$ of the subsets of X. For example $\{a\} \cup \phi = \{a\}$, $\{a\} \cup \{b\} = \{a,b\}$ etc.

What table can you construct for $\cap$? What is the table for the complement $'$? [H2]

An Algebra of Propositions

Propositional calculus concerns statements or propositional formulae which can be either true or false. (Denoted by 1 or 0 respectively).

Thus "8 is less than 5", "The Prime Minister was born in Grantham" are examples of propositions, "Goodbye" is not a proposition.

* There are four subsets, call them ϕ, A, B and X

Consider the proposition "They are poor and they are happy"
(a) (b)

It consists of two statements (marked a and b respectively) linked by the word 'and' (called a connective, and for which the symbol

$\wedge$ is used).

The propositions can be written symbolically as $a \wedge b$.

Notice that $a \wedge b$ is true when both a and b are true. What is its truth value if a and b are (i) both false (ii) one false, the other true?

! Try completing this table

	$\wedge$ (b →)	0	1
a	0		
	1		1

Now consider the proposition "He can swim or he can run".
(a) (b)

There is a different connective here (the word 'or') and the symbol used is $\vee$. Thus, the proposition can be written as $a \vee b$.
! Under what conditions is the proposition false? Now complete the table for $\vee$.

	$\vee$ (b →)	0	1
a	0	0	
	1		

[H3]

Two connectives have been introduced $\wedge$ and $\vee$. There are two more; $'$ meaning 'not' so that if q is true then q' is false and vice versa. $\rightarrow$ is the 'implies' operator and is dealt with later.

Example

Here is an illustration of how a truth table can be used to analyse a complex propositional statement.

Consider the sentence "It is not true that either he does not help or he gets in the way". Someone suggests that this is much better written as "He helps and he does not get in the way". Can we be sure the two sentences say the same thing?

Let p = he helps and q = he gets in the way.

The two sentences correspond to $(p' \vee q)'$ and $p \wedge q'$. We need to show that they have the same truth value for all combinations of p and q.

p	q	p'	q'	$p \wedge q'$	$p' \vee q$	$(p' \vee q)'$
0	0	1	1	0	1	0
1	0	0	1	1	0	1
0	1	1	0	0	1	0
1	1	0	0	0	1	0
				*		*

The equivalence of the two columns marked * shows that the two sentences say the same thing. (Truth tables are so called because they illustrate under what conditions the proposition is true.)

A Word about Truth Tables

The table for $a \vee b$ can be written down in two ways

either

	$\vee$ (b →)	0	1
a	0	0	1
	1	1	1

or

a	b	$a \vee b$
0	0	0
1	0	1
0	1	1
1	1	1

Both display the same information, the first is more compact, but cannot be used to illustrate more than one connective or involve more than two variables. It is sometimes convenient, particularly with many variables to compress the form of the table as much as possible. For example, $x \wedge (y \vee x)$ can be written as either

x	y	z	$y' \vee z'$	$x \wedge (y \vee z)$
0	0	0	0	0
1	0	0	0	0
0	1	0	1	0
1	1	0	1	1
0	0	1	1	0
1	0	1	1	1
0	1	1	1	0
1	1	1	1	1

or

x	$\wedge$	(y	$\vee$	z)
0	0	0	0	0
1	0	0	0	0
0	0	1	1	0
1	1	1	1	0
0	0	0	1	1
1	1	0	1	1
0	0	1	1	1
1	1	1	1	1

Step 2 Step 1

In the second case, form the column under $\vee$ first then combine this with x to produce the column under $\wedge$.

! Exercise B

1. Are the following pairs of propositional formulae equivalent?

   (a) $(p \wedge q)'$ and $p' \vee q'$

   (b) $(p \vee q)'$ and $p' \wedge q'$

   (Check their truth tables.) Can you formulate a theorem here?

(0) 2. Do the following sentences contain the same information?

   (a) It is not true that he is a philosopher and a man of action.

   (b) He is not a philosopher and he is not a man of action. [H4]

Implication

We need a connective for the conjunction "if .... then ....". Unfortunately, when used in ordinary speech, this structure may be ambiguous for certain conditions.

Consider "If it goes on bleeding, I will find the sticking plaster". If it does go on bleeding (p = 1 in the table below), the implication is false if the plaster is not searched for, true if it is. However, if it stops bleeding, (p = 0), the original statement is not shown to be false, whatever action is

taken, so the table is completed with 1's. The easiest way to remember this is that $p \to q$ is 1 unless p = 1 and q = 0. Here is the truth table:

$\to$ (p \ q)	0	1
0	1	1
1	0	1

or

p	q	$p \to q$
0	0	1
1	0	0
0	1	1
1	1	1

Example

What is the truth table for $(p \to q) \wedge (q \to p)$?

p	q	$(p \to q)$	$\wedge$	$(q \to p)$
0	0	1	1	1
1	0	0	0	1
0	1	1	0	0
1	1	1	1	1

! Exercise C

Work out that the following propositional formulae are equivalent:

$p \to q$ and $p' \vee q$ and $(p \wedge q')'$

Use this information to restate the following in an equivalent form:

"If Sam got wet then he sneezed." [H5]

Tautology

Consider this truth table:

p	q	$p \to q$	$(p \to q) \vee p$
0	0	1	1
1	0	0	1
0	1	1	1
1	1	1	1

What do you notice about the value of $(p \to q) \vee p$? This property is called a tautology. An absurdity (or contradiction) is the complement of a tautology, that is, a proposition which is always false.

e.g.

p	p'	$p \vee p'$	$p \wedge p'$
0	1	1	0
1	0	1	0

$p \vee p'$ is a tautology, $p \wedge p'$ is an absurdity.

! Exercise D

1. Which of the following are either a tautology or an absurdity?

   (a) $p \wedge (p \vee q)$ (b) $(p \vee q)' \wedge p$ (c) $p \rightarrow (p \vee q)$ [H6]

2. Compare the tables for the connectives $\vee$ and $\wedge$ with the tables for $\cup$ and $\cap$ of the subsets of S in the section on sets. Are there any similarities? Could we change from one system to the other by changing the symbols? Which symbols correspond? [H7]

(0) This example is an illustration that a more general appreciation of propositional calculus might be an advantage. It is taken from Schools Examination Regulations issued by one of the English Boards.

"Candidates offering History at A-level must take Paper I and two of Papers II,III,IV and V with the following restrictions:

If IV is taken, the third paper must be II;

If V is taken, the third paper must be II or III.

Can this be simplified? Look at the statements in propositional form. Let p be the proposition "take Paper I", q be "take Paper II" and so on. There will be five variables p,q,r,s and t. Now p = 1 so there are now six possibilities (why?) The restrictions reduce the possibilities to four, the propositional formula being:

$$(p \wedge q \wedge r \wedge s' \wedge t') \vee (p \wedge q \wedge r' \wedge s \wedge t') \vee (p \wedge q \wedge r' \wedge s' \wedge t) \vee (p \wedge q' \wedge r \wedge s' \wedge t) .$$

Notice that $p \wedge q$ is common to each of the first three expressions, so the rubric could be rewritten to read, say:

"Candidates offering History at A level must take three papers. Paper I is compulsory. Either Paper II is taken with one from III, IV or V or Paper III is taken with V.

However, the best simplification comes by noticing that some part of the information is redundant.

! List the combinations excluded by the first restriction (involving Paper IV).

List the combinations excluded by the second restriction (involving Paper V)

! Could one of these statements be simplified or excluded? [H8]

Electrical Networks

Two on-off switches can be wired either in series or parallel, like this:

What switches must be on for current to flow in (A) and in (B)? Under what conditions will current not flow? Are these the same for (A) and (B)? (A) is called the "and" condition, (B) the "or" condition (why?). We shall use $\wedge$ for "and" and $\vee$ for "or". A switch which is on will be represented by 1, one which is off by 0. The complement is denoted by $'$, so if $x = 1$, $x' = 0$ and if $y = 0$, $y' = 1$.

$x \wedge y$ represents series connection, $x \vee y$ represents parallel connection. The table for $x \vee y$ is therefore:

either

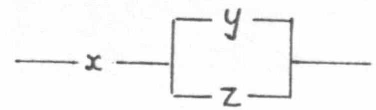
$\vee$	$y=0$	$y=1$
$x=0$	0	1
$x=1$	1	1

or

x	y	$x \vee y$
0	0	0
1	0	1
0	1	1
1	1	1

! Construct a table for the series "and" connection $x \wedge y$ [H9]

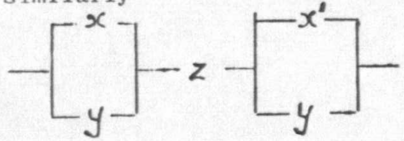
More than two switches may be used in a network. For example



is represented by $x \wedge (y \vee z)$

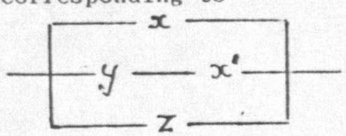
(x is in series with the parallel system $y \vee z$)

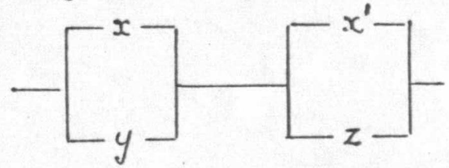
similarly



is the network for
$(x \vee y) \wedge z \wedge (x' \vee y)$

! Note the use of brackets. Try writing down the expressions corresponding to



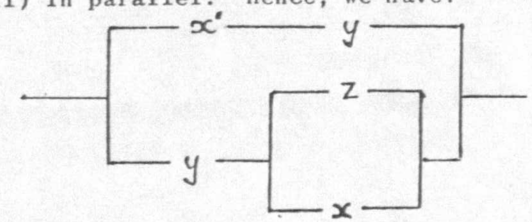


Can you differentiate between the two expressions without using brackets?*

If an expression can be written down to represent a given network, then a network should be possible for any given expression.

Consider $(x' \wedge y) \vee (y \wedge (x \vee z))$

It's best to start right inside and work your way out! So, in this case (i) draw a parallel network $x \vee z$ (ii) place this in series with y (iii) draw a series network $x' \wedge y$ (iv) couple (ii) and (iii) in parallel. Hence, we have:



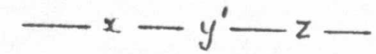
! Exercise E

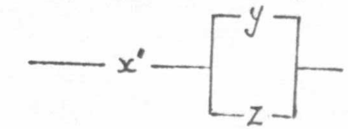
1. Draw networks to express the following conditions:

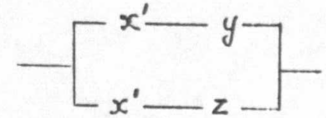
(a) $(x \vee y) \wedge (x \vee z)$ (b) $x' \vee z$ (c) $(x \wedge z) \vee (x' \wedge z) \vee (x' \wedge z')$

(d) $(x \vee y \vee z) \wedge (x' \vee y \vee z')$

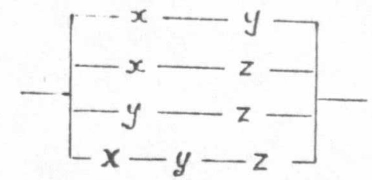
2. What expressions can be written to represent the following networks?

(a) 

(b) 



(a)



(b)

[H10]

Tables can be constructed to discover the conditions under which current will flow for <u>any</u> given expression. These are precisely the same as the truth tables for propositions.

Examples

Find the truth table for $(x \wedge y)' \wedge y$

x	y	$x \wedge y$	$(x \wedge y)'$	$(x \wedge y)' \wedge y$
0	0	0	1	0
1	0	0	1	0
0	1	0	1	1
1	1	1	0	0

Find the truth table for $(x \wedge y) \vee (x' \wedge z)$

* The expressions are best written as $x \vee (y \wedge x') \vee z$ and $(x' \vee y) \wedge (x' \vee z)$

10

x	y	z	x∧y	x′∧z	(x∧y)∨(x′∧z)
0	0	0	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
1	1	0	1	0	1
0	0	1	0	1	1
1	0	1	0	0	0
0	1	1	0	1	1
1	1	1	1	0	1

Networks with Specified Properties

What is required here is a method of working from the truth table to a network which will illustrate the given logic.

How can a network be constructed apart from trial and error?

Consider this table:

x	y	p(x,y)	
0	0	1	*
1	0	0	
0	1	1	*
1	1	1	*

p(x,y) represents an unknown function of x and y.

The steps are:

1. Mark each row in the table in which p(x,y) is 1.

2. For each marked row, write down (x if x=1 in that row and similarly (x′ if x=0 for other switches.

3. Now 'and' the results for each row.

   e.g. x′∧y′ for row 1, x′∧y for row 3, x∧y for row 4.

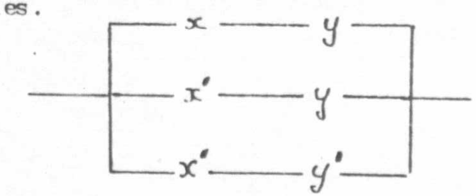
4. Finally 'or' these results.

11

So p(x,y) = (x′∧y′)∨(x′∧y)∨(x∧y)

This may not be the simplest form of p(x,y) but it will have the desired properties.

The network is:



Can you simplify the expression for p(x,y)? [H11]

(0) Example

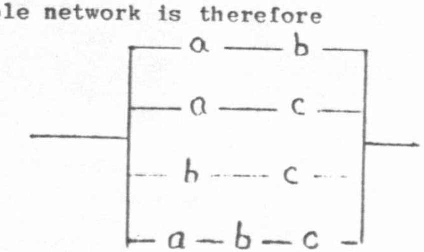
A committee of three people pass resolutions by simple majority. A has two votes, B has two votes and C has 1 vote. Design a voting machine.

In this case, the resolution is passed with three or more votes. Construct the table:

a	b	c	p(a,b,c)	
0	0	0	0	
1	0	0	0	
0	1	0	0	
1	1	0	1	*
0	0	1	0	
1	0	1	1	*
0	1	1	1	*
1	1	1	1	*

And p(a,b,c) = (a∧b)∨(a∧c)∨(b∧c)∨(a∧b∧c)

One possible network is therefore

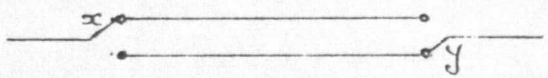


This is not the only wiring. For example since $p=1$ when $a \wedge b=1$ and also when $a \wedge b \wedge c=1$, the switch c is redundant and the last bracketed expression can be removed. [H12]

Two-Way Switches

An example of a network with desired properties is a two-way light switch system for a staircase. In this system, the light may be switched on or off at one end irrespective of the switch state at the other end.

The network is wired like this



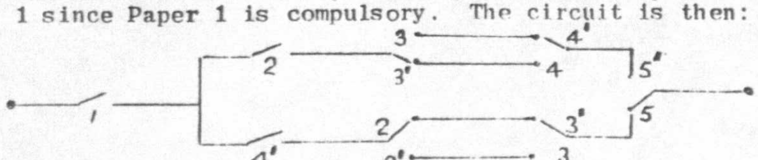
At each end is a switch which is really a combination of two switches (hence the name). A two-way switch has the property that when $x=1$, $x'=0$ and when $x=0$, $x'=1$.

so current can flow when the switch is logically 'off'.

In the example, the network is designed so that current will flow when one of the switches is 'on'. This feature can be generalised to any number of switches in the system by current flowing if an odd number of switches are 'on'. If this is true the state of the network will always be changed by operating any one switch.

(0) Return for a moment to the example of the Examination regulations. Can a network be designed to indicate proper choices? Here, switches must be used to indicate the 'not' conditions, since it must be shown that only three papers are taken.

The network can be simplified by noting that p must always be 1 since Paper 1 is compulsory. The circuit is then:



Notice the use of two, two-way systems in this arrangement.

Isomorphism

We have seen that the algebra of sets, the logic of propositions and switching networks are essentially the same. This essential sameness is called isomorphism. The underlying structure is called a Boolean algebra, after its discoverer, George Boole.

Boolean Algebra

Definition A Boolean algebra is any collection of objects (set), with binary operations $\vee, \wedge$ and unary operation $'$ defined on them such that the following rules hold for all x,y,z.

1. Associative

   $x \vee (y \vee z) = (x \vee y) \vee z$ $\qquad$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

2. Commutative

   $x \vee y = y \vee x$ $\qquad$ $x \wedge y = y \wedge x$

3. Distributive

   $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ $\qquad$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

4. $x \vee x = x$ $\qquad$ $x \wedge x = x$

5. The conditions $x \wedge y = x$ and $x \vee y = y$ are equivalent

6. $(x \vee y)' = x' \wedge y'$ $\quad$ $(x \wedge y)' = x' \vee y'$ $\quad$ $(x')' = x$

   The set contains elements represented by 0 and 1 such that

7. $0 \wedge x = 0$ $\qquad$ $0 \vee x = x$

   $1 \wedge x = x$ $\qquad$ $1 \vee x = 1$

8. $x \wedge x' = 0$ $\qquad$ $x \vee x' = 1$

   each variable can have the value 0 or 1, any Boolean expression will also have the value 0 or 1.

Using tables, we can demonstrate that certain Boolean expressions are equivalent.

For example, 3 above states $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

x	y	z	$y \wedge z$	$x \vee (y \wedge z)$	$(x \vee y)$	$(x \vee z)$	$(x \vee y) \wedge (x \vee z)$
0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	1
0	1	0	0	0	1	0	0
1	1	0	0	1	1	1	1
0	0	1	0	0	0	1	0
1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
				*			*

The equivalence of the two columns marked * proves the identity.

Exercise F

! 1. Try and prove the identities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$(x \vee y)' = x' \wedge y'$

$(x \wedge y)' = x' \vee y'$

2. Construct tables of values for the following Boolean polynomials:

(a) $(x \wedge y') \vee y$ (b) $(x \wedge z) \vee (x' \wedge z) \vee (x' \wedge z')$

(0) 3. Write down the tables for $x \vee y$ and $x \wedge y$.

Now construct tables in which

0 is interpreted as "even positive integer"

1 is interpreted as "odd positive integer"

$\vee$ is ordinary addition

$\wedge$ is ordinary multiplication.

e.g. $0 \wedge 0$ = even integer times even integer = even integer = 0.

$1 \vee 0$ = odd integer plus even integer = odd integer = 1.

Do these tables correspond to those for Boolean algebra? [H13]

Boolean Algebras with more than Two Elements

This is an example of how, in mathematics, the opportunity to generalise often arises.

On page 1, the tables are developed for the algebra of a set with two subsets. This has been shown to be a model of a Boolean algebra with two elements, which were called 0 and 1. Also on page 1, the algebra of a set with four subsets was discussed.

If Boolean algebra is modelled by the algebra of sets, can we learn anything about Boolean algebra by looking at the example with four subsets? So far we have looked at algebras where the elements are one of two values. Look back at set X on page 1, note the number of subsets involved. If this is a model of a Boolean algebra, how many elements does it contain?

! Exercise G

1. List the number of subsets that can be generated from sets containing 1, 2 and 3 elements respectively. How many subsets are there for a set with n elements?

What does this suggest about Boolean algebras with more than two elements?

2. Is it possible to find an example of a Boolean algebra with exactly three elements? [H14]

Look at the first question of the tutorial examples for the tables describing a Boolean algebra with four elements called 0,1,2 and 3.

Tutorial Examples

Sets

1. The following tables are those for a four-element Boolean algebra. Look at the example you have of an algebra of sets with exactly four subsets. Identify which subsets correspond to the elements 0,1,2,3 of the algebra.

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	2	1
3	3	1	1	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

1	′
0	1
1	0
2	3
3	2

Propositions

2. Consider the two propositions p and q,

p: The examination is not both easy and short.

q: The examination is easy.

(a) Show that the value of $p \vee q$ is always true.

(b) Show that the proposition "The examination is either easy or not short" is equivalent to "If the examination is short then it is easy".

(c) Find a statement which is equivalent to "If the examination is not short then it is not easy".

3. Show that the following are tautologies:

(a) $p' \to (p \to q)$ (b) $(p \wedge (p \to q)) \to q$ (c) $p \to (q \to (p \to q))$

4. Show that each of the following is an absurdity.

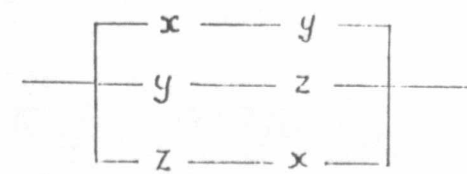
(a) $p \wedge p'$ (b) $q' \wedge p \wedge (p \to q)$

Is $p \to p'$ an absurdity?

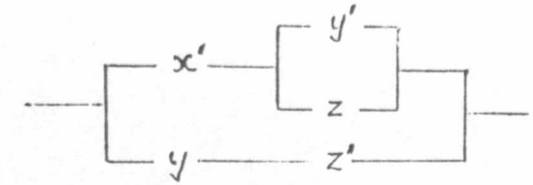
Electrical Networks

5. What are the Boolean polynomials that describe these networks?

(a)



(b)



(c) Draw the networks described by $(x \wedge z) \vee (y \wedge z')$ and $(x \vee y) \wedge ((x' \wedge z) \vee z')$

6. Establish the Boolean identities

$(x \wedge y') \vee y = x \vee y$

$(x \vee y)' \vee (y' \vee z')' \vee (y' \vee z)' = x' \vee y$

Illustrate the first of these identities by drawing two equivalent switching circuits.

7. For each of the networks in question 5 calculate a table of values showing for each setting of the switches whether current will flow.

8. For the staircase lighting problem, draw up a truth table for which $p(x, y) = 1$ when <u>one</u> switch is 'on' and $p(x,y) = 0$ if both switches are 'off' <u>or</u> 'on'.

Use the method explained on page 9 to find the propositional expression for $p(x,y)$. From the expression, construct the equivalent circuit. Compare your answer with the diagram in the text involving the two-way switches.

Try the same problem if there is a third switch z on an intermediate landing.

Hints and Solutions to Exercises

H1 The two subsets are the null set and S itself, i.e. ϕ and {a}

$\cap$	ϕ	S
ϕ	ϕ	ϕ
S	ϕ	S

$'$	
ϕ	S
S	ϕ

H2 There are four subsets of X, ϕ,{a}, {b} and {a,b} and it is convenient to call these ϕ, A, B and X respectively.

$'$	
ϕ	X
A	B
B	A
X	ϕ

$\cup$	ϕ	A	B	X
ϕ	ϕ	A	B	X
A	A	A	X	X
B	B	X	B	X
X	X	X	X	X

$\cap$	ϕ	A	B	X
ϕ	ϕ	ϕ	ϕ	ϕ
A	ϕ	A	ϕ	A
B	ϕ	ϕ	B	B
X	ϕ	A	B	X

H3

$\wedge$ (a \ b)	0	1
0	0	0
1	0	1

$\vee$ (a \ b)	0	1
0	0	1
1	1	1

q	q'
0	1
1	0

H4 1.

p	q	$(p\wedge q)'$	$p'\vee q'$	$(p\vee q)'$	$p'\wedge q'$
0	0	1	1	1	1
1	0	1	1	0	0
0	1	1	1	0	0
1	1	0	0	0	0

The theorem is attributed to De Morgan and states $(p\wedge q)' = p'\vee q'$ and $(p\vee q)' = p'\wedge q'$.

2. Let p = he is a philosopher, q = he is a man of action.

Then (a) = $(p\wedge q)'$ (b) = $p'\wedge q'$

p	q	$(p\wedge q)'$	$p'\wedge q'$
0	0	1	1
1	0	1	0
0	1	1	0
1	1	0	0

(a) and (b) do not contain the same information.

H5

p	q	$p\rightarrow q$	$p'\vee q$	$(p\wedge q')$	$(p\wedge q')'$
0	0	1	1	0	1
1	0	0	0	1	0
0	1	1	1	0	1
1	1	1	1	0	1
		*	*		*

The equivalence of the columns marked * establishes the property. The sentence can be rewritten as either "Either Sam did not get wet or he sneezed" or "It is not true that Sam got wet and did not sneeze".

H6

p	q	$p\wedge(p\vee q)$	$(p\vee q')\wedge p$	$p\rightarrow(p\vee q)$
0	0	0	0	1
1	0	1	0	1
0	1	0	0	1
1	1	1	0	1

(b) is an absurdity (c) is a tautology

H7 By comparison $\phi \equiv 0$, $S \equiv 1$, $\cap \equiv \wedge$ $\cup \equiv \vee$.

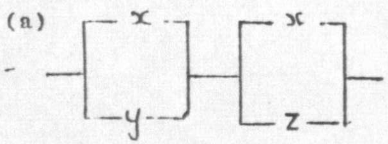
H8 The rubric restrictions merely have to state "Paper IV may only be taken together with paper II"! Combinations III and IV and IV and V are excluded and so the second restriction is redundant.

Another approach would be to list illegal combinations, since there are only two of these. However, instructions involving negatives are often misinterpreted.

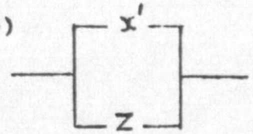
H9

$\wedge$	y = 0	y = 1
x = 0	0	0
x = 1	0	1

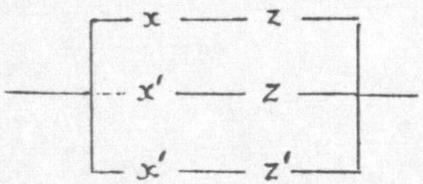
H10 1. (a)



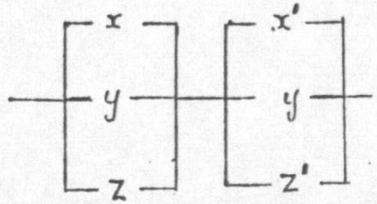
(b)



(c)



(d)



2. (a) $x \wedge y' \wedge z$ (b) $x' \wedge (y \vee z)$ (c) $(x' \wedge y) \vee (x' \wedge z)$

(d) $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \vee (x \wedge y \wedge z)$

H11 Look at the last two brackets, one contains x, the other x', so the condition of x is irrelevant and the expression can be replaced by y alone.

So $p(x,y) = (x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$

becomes $= (x' \wedge y') \vee y$

A similar argument will now show that y' is redundant in the first bracket and hence.

$p = x' \vee y.$

Drawing the truth table:

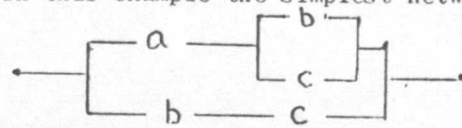
x	y	$x' \vee y$
0	0	1
1	0	0
0	1	1
1	1	1

Clearly $x' \vee y$ has the same truth table as the original p(x,y).

A more formal method is to use the rules of Boolean algebra, which switching networks obey.

$$p(x,y) = (x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$$
$$= (x' \wedge y') \vee (y \wedge (x' \vee x)) \quad \text{distributive law}$$
$$= (x' \wedge y') \vee y \quad \text{since } x' \vee x = 1$$
$$= (x' \vee y) \wedge (y' \vee y) \quad \text{distributive law}$$
$$= (x' \vee y) \quad \text{since } y' \vee y = 1$$

H12 Circuit designers go to great trouble to simplify their networks as much as possible. They often use formal methods of argument, as we did in H11. However, common sense can go a long way in many examples. In this example the simplest network is:-



H13 1.

x	y	z	$y \vee z$	$x \wedge (y \vee z)$	$x \wedge y$	$x \wedge z$	$(x \wedge y) \vee (x \wedge z)$
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0
1	1	0	1	1	1	0	1
0	0	1	1	0	0	0	0
1	0	1	1	1	0	1	1
0	1	1	1	0	0	0	0
1	1	1	1	1	1	1	1

The other identities were encountered as De Morgan's theorem, see H4.

2. (a)

x	y	$x \wedge y'$	$(x \wedge y') \vee y$
0	0	0	0
1	0	1	1
0	1	0	1
1	1	0	1

(b)

x	y	z	$(x \wedge z) \vee (x' \wedge z) \vee (x' \wedge z')$
0	0	0	1
1	0	0	0
0	1	0	1
1	1	0	0
0	0	1	1
1	0	1	1
0	1	1	1
1	1	1	1

3. No, look carefully at the table for v.

H14 If $S = \{a\}$, $n = 1$ and there are two subsets

$X = \{a,b\}$, $n = 2$ and there are four subsets

If $n = p$ how many subsets are there? Verify this for $p = 3$.

There is no way of constructing a set with exactly three subsets, hence there is no Boolean algebra comprising just three elements.

Appendix 6

Notes on Observation of a Lecture during February 1980

The topic was integral calculus; students were familiar with the technique of differentiation and its relation to rate of change.

5.00 First three minutes or more are taken up with students arriving and settling down and the lecturer setting up the overhead projector. A quick count reveals at least ninety students present. They are tending to sit in the rear half of the lecture theatre.

5.05 approx. Lecturer announces topic. Illustrates with example of stone being thrown vertically. Given s = f(t) we can find velocity and acceleration. Hence equation gives not only distance but velocity and acceleration. Integral calculus is reverse process.

Another example:- attempt to shoot down airplane. "If we know a and v can we calculate s?" Assume only force is gravitational; asks "Does anyone know acceleration due to gravity?"
Someone says 32. "Can it be 32?" No - establishes it must be -32: Poses problem:- "What velocity do we need to reach a height of 1000 ft?"

Students are very quiet, busy taking notes. General atmosphere of attentiveness.

The integral is developed as the antiderivative. "Shout out what you think is the value." "What other possibilities are there?" Introduces the idea of a constant of integration.

At this point the image from the OHP slips off the screen. Someone asks for the roll to be moved up.

"Having found velocity we need distance." Lecturer asks for conjectures about integral of -32 t and v_o. The audience eventually arrives at the correct answers. Emphasises the constant of integration once more. Points out physical conditions not the integration determine the constants.

5.20 Lecturer recapitulates. "Finally we need to know what value of v_o will make Smax = 1000." Try $v_o = 250$, $v_o = 320$ etc. "Where is $\frac{ds}{dt} = 0$?" Finishes the problem. Students appear to have followed the argument reasonably well.

5.27 "Now we shall look at integration more systematically." Considers $y = 1, x, x^2, x^3$ and writes down $\frac{dy}{dx}$ in each case. The integral is established as the antiderivative, establishes results of $x + c$, $\frac{x^2}{2} + c$ etc. "What about x^n?" Defines the result as the "indefinite integral".

The pace is fairly brisk. Students are very busy taking notes. No-one has time to be distracted. Noticeable that some questions are meant to be rhetorical, a contrast to the tutorials.

Poses another problem: Car travelling at constant 30 mph, considers the graph and develops idea of area under the graph. Easy in this case because area is rectangular. "What about area of a circle?" "Let's look carefully at this problem of area."

5.35 What is area under $y = c$ and $y = mx$. "Any offers for $y = mx + c$?" Gets reply divide into rectangle and triangle. "How do I find area of triangle?" Student gets this wrong initially.

5.43 Students still seem fairly attentive. Not much noise or indication of restlessness. Integral defined as the area under the curve. Finds the expression $\frac{1}{2} mx^2 + cx$. Links antiderivative and area under the curve. Lecturer now says they will look at area under the curve $y = x^2$.

Students are answering questions which reveal they are following the development. Divides area into strips. If number of strips increases the approximation should get better. Derives $\frac{h}{2} (n^2h^2 + (n - 1)^2 h^2)$. Lecturer is asking questions but does not always get the answers he seeks e.g. "What is $(n - 1)^2$?" Students are very busy writing notes. This possibly is detracting from their ability to respond.

Long discussion of $\Sigma (n^2h^3 - n h^3 + \frac{h^3}{2})$, seen to involve the value of Σn. Lecturer cannot get anyone to give the correct value for this. Says they will finish this off next time, asks them to revise Σn and Σn^2 also. "The final step is then a limiting process."

5.55 The lecture ends. There are no students who stay behind to ask questions. Commenting afterwards to the observer the lecturer was a little unhappy about the final section of this lecture. He felt that some students had not been able to follow the argument very well and he made a mental note to look into this for the next time he gave the lecture.

Notes on Observation of a Tutorial in November 1979

The topic was Flowcharts. One or two students were without the booklet that accompanied the lectures. Tutor turns to the class discussion exercises. Draws the flow-chart in question (a) on the blackboard. (See Appendix 5) Question - "What is in the registers at the start?" Obtains answers of zero or rubbish.

Class go through the problem with tutor, the discussion flows smoothly. All students noted as paying attention and seem to be involved. Tutor is emphasising that contents of registers stay fixed until overwritten. The contents of all the registers are considered at every step.
Question - "Why is the answer correct?" Prompts review of method. All students appear happy with this problem.

Move on to question (b). Tutor writes shape of flow-chart on the blackboard without filling in any boxes. Some students not clear what is meant by n! Tutor explains why the calculation is necessary; gives example "How many ways of seating 15 students in 15 chairs?"

Goes through step-by-step, prompting students to suggest the operations. Answers seem to be occurring to students fairly readily.

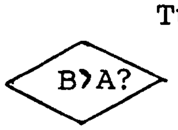
Tutor recapitulates by asking again why the right answer is obtained. Eventually is satisfied by the reply "'A' contains the numbers we need to multiply by to form the product."

Nearly all the students have taken some notes during this exercise. There was some hesitation about the technical detail, for example when the tutor asked what is this "bit" of the chart called, it took some time for someone to identify it as a "loop".

Discussion turns to question (d). Question - "Any suggestions." One student starts quite well but comes to a halt. The tutor is able to focus on the problem as being the form of the input. "Imagine what you would do" he prompts. Eventually, the students identify the correct procedure. Now, the effort concentrates on writing down the precise steps.

Tutor gets students to recognize the role of two registers - 'A' for largest number so far, 'B' for the next number tested.

Tutor draws the middle decision box on the board i.e.



There is a long discussion which illustrates the confusion that occurs between register name and register content. The steps are gradually built up, working from the middle.

Someone says they do not understand at this juncture. The tutor goes through the steps of the flow-chart once more and illustrates how it works with the list 4 3 9 7 11 3. At the end, all the students say they are clear about how the algorithm works.

Now turns to starting procedure. Question:- "Take the first two numbers, which goes in 'A', which in 'B'?" "It does not matter" says one student. This problem obviously is a source of some difficulty, the tutor resolves it by suggesting that since the first number is the largest so far, must it not go in 'A'? The conclusion of the chart presents little difficulty, students seem to understand where the solution will be found.

There is a discussion of the order in which instructions must appear in the chart. Question:- "Should the method deal with only one number?" The end of the tutorial has arrived, the three problems have taken up the available time. The tutor ends by commenting that problems may often be approached by looking at the loop in the middle first.

The tutor felt that the class had gone more or less as he would expect. The students had responded well and had been mainly attentive and shown a lively interest.

Appendix 7

An example of the self-study notes written for the pilot study.

(i) Objectives of the study of Rates of Change

(ii) Pages 1 to 11 of the study notes on differential calculus

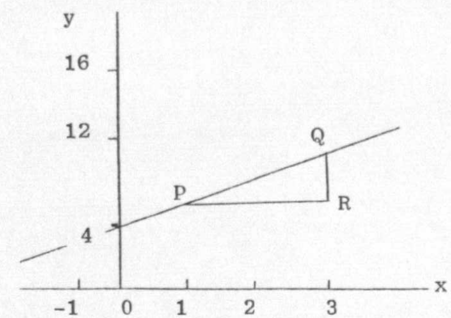
Rates of Change

Objectives of Study

To recognize rate of change, gradient and slope as synonymous.
To be able to define the rate of change of a linear function.
To relate rate of change to the "steepness" of the graph.
To sketch graphs to illustrate rates of change which are large, small or negative.
To find the rate of change of a linear function from the equation, if necessary by rearrangement.
To calculate numerically the rate of change of a simple non-linear function at given points.
To recognize the similarity between the algebraic method for finding the gradient and the numerical method and to be able to find the algebraic result for simple expressions like $y = x^2 + 2$.
To relate the rate of change at a point to the slope of the tangent at that point.
To define the law which relates the eq$^{\underline{n}}$ of the function to the derivative.
To apply the law of differentiation for powers of x to cases including negative and fractional powers.
To be able to apply the rules of differentiation of products, quotients and functions of a function to problems involving powers of x. To state the property of turning points in relation to the derivative.
To locate the turning points of functions involving powers of x.
To be able to state how a turning point may be classified by looking at the gradient in the neighbourhood.
To analyse a given function so that the turning points are located, classified and evaluated.

Rate of Growth

Many times in business calculations it is necessary to discuss rates of change, which for the moment we shall call rates of growth. To give a simple example, for a firm to say that its turnover has increased by £100,000 is not very significant, since the effect of this increase will be different if the firm is a small company in Bilston employing 50 people or ICI. However, if the managing director can report a 15% increase in turnover then he has cause for smug satisfaction. This may, you see, mean an increase of say, £25,000 or £5000,000 depending on the size of the organisation. So we need to define what is called the rate of growth. To do this, let us consider something familiar like a straight line graph e.g. $y = 3x + 4$.



Imagine this represents the growth of a company's turnover (highly unlikely!) Clearly we would be happier the steeper the line is. Indeed, if the line lay flat we should be mildly concerned, if it sloped downwards we might be deeply worried! So we already have a concept of a rate of growth - its the steepness of a line in the graphical picture. But this is a bit too inprecise for mathematical purposes. We often wish to compare different rates of growth and to do this we must quantify our idea, that is be able to express it numerically.

The way we do this is to measure the change that occurs in y when we make a change in x. Going back to the graph what happens when we change x from 1 to 3? Either by calculation or reading from the graph we find that y increases from 7 to 13. So a change of 2 in x produces a change of 6 in y (agreed?). Now we define the rate of growth as the ratio of these two changes:-

$$\text{rate of growth} = \frac{\text{change in y}}{\text{change in x}} = \frac{6}{2} = 3$$

2

Now choose two other points, say $x = 0$ and $x = 2$

when $x = 0$ $y = 4$ rate of growth $= \frac{10 - 4}{2 - 0} = \frac{6}{2} = 3$ again.

$x = 2$ $y = 10$

Try to find for yourself the rate of growth between $x = 2$ and $x = 3$ and between $x = 10$ and $x = 20$ [ANS 3 both times]

So, not only have we found out how to write down the rate of growth, we have discovered another most important fact - For a straight-line function the rate of growth is everywhere the same. Now this fits in with our concept doesn't it - in fact we might consider this a way of defining a straight line.

Good:- I want you now to work out the rates of growth of the following functions. Remember, the rate of growth for each function is a constant, so you may use whatever values of x you like to do your calculations.

(i) $y = x + 6$ (iii) $y = 4x + 2$ [ANS (i) 1 (ii) 2

(ii) $y = 2x + 1$ (iv) $y = 5x + 1$ [(iii) 4 (iv) 5

Make rough sketches of the graphs of these functions. Which has the steeper slope? Which has the least slope? Which has the greatest rate of growth? Which has the smallest rate of growth?

Try completing the following sentence "As the graph of the function gets steeper, so the rate of growth gets ----".

ANS [You should have entered "larger", "bigger" or "greater"]

If you look hard at the equations representing the functions and the figures for the rate of growth I think you will see a connection between the coefficients in the equation and the figure for the rate of growth.

Good - if you have spotted it you can write down, without calculation, the rate of growth of a linear function. Try these for example:

(i) $y = 3x - 5$ (iii) $y = 10x + 35$ [ANS (i) 3 (ii) 1

(ii) $y = x + 4$ (iv) $y = 5x - 92$ [(iii) 10 (iv) 5

What about $2y - 3x + 5 = 0$? You may be tempted to say the rate of growth is -3, but this would be wrong! What has happened to our nice conclusion? Well, look closely at this last expression, it is in a different form from the previous examples.

3

Look at it more closely $2y - 3x + 5 = 0$

rearranging $2y = 3x - 5$

$$y = \frac{3}{2}x - \frac{5}{2}$$

Now the rate of growth is actually $\frac{3}{2}$, which corresponds to the coefficient of x in the rearranged form. You see, we can only read off the rate of growth when the equation is in the proper form, which is

$$y = mx + c \quad \text{where m and c are constants.}$$

For the equation in this form, the rate of growth is the coefficient of x (that is, m). Notice that the required form has y as the subject of the expression. Here are two further examples:

(i) $6y - 12x + 15 = 0$

so $6y = 12x - 15$

$y = 2x - \frac{5}{2}$ rate of growth is 2.

(ii) $3y + 9x - 6 = 0$

$3y = -9x + 6$

$y = -3x + 2$ rate of growth is -3.

Find the rates of growth in the following cases:

(i) $3y - 3x - 9 = 0$ (iii) $2x + 3y + 9 = 0$

(ii) $4y + 8x + 15 = 0$ (iv) $9 - 2y + 4x = 0$

ANS (i) 1 (ii) -2 (iii) $-\frac{2}{3}$ (iv) 2

Well done. Notice an interesting fact, some of the rates of growth have turned out to be negative! Is this realistic? I hope you could describe what a negative rate of growth should mean. Try completing these sentences

(i) The graph of a function with a negative rate of growth will slope---------

(ii) The more negative the rate of growth the------the downward slope.

ANS (i) "downwards" (ii) "steeper" or "greater"

Now a change of name; we can't really go on calling a negative rate of growth a growth at all, it's a decline. So as a general term to cover both positive and negative values we shall use the name rate of change. Positive rates of change will be growth, negative rates of change will represent decline. Here is a summary of what we have covered so far:

1. For a linear function the rate of change is constant.

2. The rate of change = $\frac{\text{change in y-values}}{\text{change in x-values}}$ between two chosen points

 or $= \frac{y_Q - y_P}{x_Q - x_P}$ between two poinst P and Q

 or $= \frac{\Delta y}{\Delta x}$

All these last three expressions are different ways of saying the same thing.

3. For a linear function, if we express the equation in the form

 $y = mx + c$ the rate of change = m (the coefficient of x)

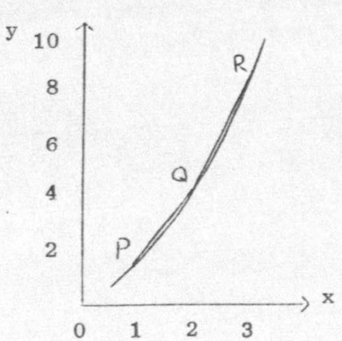
4. The magnitude of the rate of change is directly related to the steepness of the graph

 large m → steep line
 small m → nearly flat line
 negative m → line sloping downwards

Now do the first progress test. You must do well in this test before you go further.

Rate of Change of a Non-Linear Function

Let's start with a simple function which is not a straight line, say $y = x^2$



We shall try the method we used before i.e. rate of change = $\frac{\Delta y}{\Delta x}$

at Q x = 2 so y = 4
 P x = 1 y = 1
rate of change = $\frac{4 - 1}{2 - 1}$ = 3 between P and Q (1)

also R x = 3 so y = 9
rate of change = $\frac{9 - 4}{3 - 2}$ = 5 between Q and R (2)

We can see at once that the rate of change is not constant. This is quite reasonable if we look at the curve, it is clear that the steepness of the curve is continuously increasing. But it's not as simple as that! Look at the values we had in (1). This really represents the rate of change of the straight line joining P and Q, just as the result (2) in the rate of change of the straight line joining Q and R.

Look for a moment at the line PQ. Imagine it as a plank set against a hillside. At P the inclination of the plank is greater than that of the hill whereas at Q the reverse is true. So the answer we get for the line PQ is too big at P and too small at Q. It must be just right somewhere in between! In fact, it might give a reasonable answer half-way between i.e. the rate of change at x = 1·5 is about 3.

We can be more confident about this result the closer P and Q are together (because the closer P and Q the closer the line PQ approximates to the curve). It is interesting to note that this is how a computer is programmed to find a rate of change at a given point. It chooses two points either side of the given point but very close together and does exactly the same calculation as we did above.

Lets illustrate how it's done. Assume we want the rate of change of $y = x^2$ at the point x = 2·0.

Choose P at x = 1.9 then y = 3.61
 Q at x = 2.1 then y = 4.41 rate of change = $\frac{4.41-3.61}{2.1-1.9} = \frac{0.80}{0.20} = 4.00$

We might choose P and Q even closer to x = 2.

eg. at x = 1.99 y = 3.9601
 x = 2.01 y = 4.0401 rate of change = $\frac{4.0401-3.9601}{2.01-1.99} = \frac{0.08}{0.02} = 4.00$
 as before

But this of course, only gives us the rate of change at one point, so if say, we want the value at x = 3

6

choose x = 2.99 then y = 8.9401

x = 3.01 then y = 9.0601 rate of change = $\frac{9.0601-8.9401}{3.01 - 2.99} = \frac{0.12}{0.02} = 6.00$

and for the values at x = 1

choose x = 0.9 y = 0.81

x = 1.1 y = 1.21 rate of change = $\frac{1.21-0.81}{1.1-0.9}$ 1 2.00

Try this for yourself: What are the rates of change of $y = 2x^2$ at x = 2 and x = 3?

ANS (i) 8 (ii) 12

If you got that one right, try finding the rate of change of $y = x^2 + 2x$ at x = 1 and x = 2

ANS (i) 4 (ii) 6

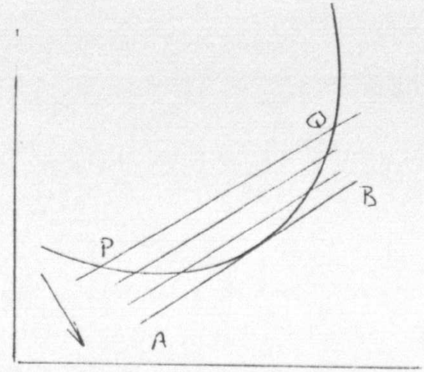
So we have a numerical method of finding the rate of change of any mathematical function that can be expressed as a formula. But there are snags with this method. Can you think of two?

Here are some possibilities: (i) the calculation has to be done for each individual value of x (ii) the calculation has to be done for each individual function (iii) there is an element of approximation, we strictly find the mean rate of change between P and Q.

We would like to find a method which is more general which does not involve the tedious calculations. However, before we do this we shall look at some new names:

When we draw a graph of a function, the rate of change is seen to be related to the steepness of the line, so the rate of change is also referred to as the gradient or slope. The three names are equivalent and can usually be interchanged. Now look at this diagram, there are a series of parallel lines drawn to interset the curve at two points P and Q which progressively move closer together as the lines move to the right.

7



What happens when P and Q are so close that you can't stick a pin between them? What is the name given to the line AB then? That's right, it is the tangent to the curve. But this is what we have been doing numerically, making P and Q as close as possible. So, we find the rate of change along the tangent at that point also. Hence, we can say that the gradient at a particular point on a curve is the same as the gradient of the tangent at the same point.

To see if you have grasped these points, fill in the missing words here:

(i) the gradient of the function is the other name of the--------- or-------------- .

(ii) along a curve, the gradient------- from one point to the next.

(iii) the slope of a curve at a point is the--------------as the slope of the--------- at the same point.

ANS (i) slope or rate of change (ii) variable (iii) same, tangent

Let us return to the problem which was mentioned earlier. We wish to avoid the numerical tedium. We can do this by doing the problem algebraically Don't panic! Take a simple case first, say $y = x^2$. Assume we want to find the rate of change at a point where x = k(say) As we did numerically, we choose two points P and Q either side of k, say at x = k-h and x = k+h.

Substitute for x in $y = x^2$

at Q $y_Q = (k+h)^2$

at P $y_P = (k-h)^2$

$\therefore$ rate of change = $\frac{(k+h)^2 - (k-h)^2}{(k+h) - (k-h)}$

$= \frac{(k^2+2kh+h^2) - (k^2-2kh+h^2)}{k+h - k+h}$

$= \frac{k^2+2kh+h^2-k^2 2kh-h^2}{2h} = \frac{4kh}{2h} = \frac{2k}{}$

This is a remarkable result! What it says is that if we want to know the rate of change at <u>any</u> point along $y = x^2$, the answer is <u>2 times the x-value at that point</u>.

Check the results we had earlier: when $x = 1$ gradient $= 2\times1 = 2$

when $x = 2$ gradient $= 2\times2 = 4$

when $x = 3$ gradient $= 2\times3 = 6$ and so on.

So, the problem of finding the rate of change anywhere on $y = x^2$ is solved once and for all. We can say that

for $y = x^2$, the rate of change is $2x$

What about $y = x^3$? Well we can do as we did for $y = x^2$

Take Q as $x = k+h$ then $y_Q = (k+h)^3 = k^3 + 3k^2h + 3kh^2 + h^3$

P as $x = k-h$ then $y_P = (k-h)^3 = k^3 - 3k^2h + 3kh^2 - h^3$

$$\text{rate of change} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{(k^3+3k^2h+3kh^2+h^3)-(k^3-3k^2h+3kh^2-h^3)}{(k+h)-(k-h)}$$

$$= \frac{6k^2h + 2h^3}{2h} = 3k^2 + h^2$$

As P and Q are brought closer together what happens to h? It gets very small doesn't it. In fact it can be made so small that h^2 can be ignored (You don't like that? Try putting $h = 0.001$ and see what happens to h^2)

So as $h \to 0$ (as we say, h tends to zero)

rate of change $\to 3k^2$

or if $y = x^3$, the rate of change is $3x^2$; when $x = 1$ slope $= 3$

$x = 2$ slope $= 12$

$x = -2$ slope $= 12$ etc.

Shall we see if you have got the hang of this algebric process? Try finding the expressions for the rates of change of the functions:

(i) $y = x^2 + 1$ (ii) $y = 2x$ (iii) $y = x^2 + 2x$

ANS(i) $\dfrac{(k+h)^2+1-[(k-h)^2+1]}{(k+h)-(k-h)} = \dfrac{4kh}{2h} = 2k$ or <u>$2x$</u>

(ii) $\dfrac{2(k+h)-2(k-h)}{(k+h)-(k-h)} = \dfrac{4h}{2h} = \underline{2}$ (a constant - remember?)

(iii) $\dfrac{(k+h)^2+2(k+h)-[(k-h)^2+2(k-h)]}{(k+h)-(k-h)} = \dfrac{4kh+4h}{2h} = 2k+2$ or <u>$2x+2$</u>

If you are still worried about this process you must ask advice, either from your colleagues or from a tutor.

You may feel that this remedy is worse than the complaint. However, do not give up yet, let us look a little more closely. First, a review of the situation. We no longer have to rework the gradient at each new point since we can now find a <u>general expression</u> for each given function. But clearly, working out the general expression in this way is hard going. It would be nice if we could <u>write them down</u>. Sounds unlikely, well let us assemble the evidence:

On the left are three functions of x, on the right expressions for their rates of change.

Function	Rate of Change
x	1
x^2	$2x$
x^3	$3x^2$

Look carefully at the expressions in the two columns, try and discover some relationship between them. If I now give you the function $y = x^4$, what do you think is the expression for its rate of change? And what about $y = x^5$?

ANS $4x^3$ and $5x^4$

Did you get them right? If you see how these results are arrived at, you will no longer have to do tedious numerical or algebraic manipulations.

In general, we say that if $y = x^n$, the rate of change of y with respect to x is given by nx^{n-1}. (It can be shown that n can be positive, negative or fractional)

More notation. It is too tedious to keep writing rate of change so we need a symbol to represent it now we have a simple way of producing the answer. Since from first principles, we found the $\frac{\text{change in } y}{\text{change in } x}$ and saw what happened when $h \to 0$, we shall use the symbol $\frac{dy}{dx}$ (pronounced d-y by d-x) for the rate of change.

Remember the connection between rate of change, gradient, slope and $\frac{dy}{dx}$.
The <u>process</u> of obtaining the general expression for the rate of change is called <u>differentiation</u> i.e. we differentiate a function to obtain its rate of change.

We can write down results like $y = x^2$ $\quad \frac{dy}{dx} = 2x$

$y = x^6 \quad \frac{dy}{dx} = 6x^5$

$y = x^{-1} \quad \frac{dy}{dx} = -x^{-2}$

Remember, we evaluate the gradient at any point by substituting the given value of x into the expression for $\frac{dy}{dx}$.

So if $y = x^6$ $\frac{dy}{dx} = 6x^5$ when $x = 2$ gradient is $6(2)^5 = 192$.

Only one problem remains, $y = x^5$ is fine, but what about $y = 3x^5+4x^3-6x$?
Two results are quoted here which I think you will find reasonable.

If not, discuss them with a tutor.

(i) If a function is multiplied by a constant, then so is the rate of change,
e.g. $y = x^3, \frac{dy}{dx} = 3x^2$, if $y = 4x^3$, $\frac{dy}{dx} = 4(3x^2) = 12x^2$.

(ii) If a function is the sum (or difference) of several terms, the rate of change is the sum (or difference) of the terms obtained by applying the rule to each term in turn.
e.g. if $y = x^3+x^2$ then $\frac{dy}{dx} = 3x^2+2x$

$y = 2x^5-\frac{1}{2}x^3$ then $\frac{dy}{dx} = 10x^4-\frac{3}{2}x^2$

By combining these results we can handle a large number of functions that occur in practice. Do not forget that the rule only applies to terms expressed in the form x^n. (So what about things like $y = \sqrt{x}$? Have you heard of $y = x^{1/2}$?)

Find the expressions that give the rate of change for each of the following functions of x:

(i) $y = x^2+2x$

(ii) $y = 2x^3-3x^2+x$

(iii) $y = 2x+2$ (what rate of change does a constant have?)

(iv) $y = \sqrt{x}$

(v) $y = \frac{1}{x}$ (express in the form $y=x^n$)

(vi) $y = \frac{3}{x^2}$

(vii) $y = x+\frac{1}{x}$

(viii) $y = 3x^3-2x+\frac{4}{x^2}$

(ix) $y = \frac{1}{\sqrt{x}}$

(x) $y = x^{5/2}$

ANS (i) $2x+2$ (ii) $6x^2-6x+1$ (iii) 2 (iv) $\frac{1}{2}x^{-1/2}$ or $\frac{1}{2}\frac{1}{x^{1/2}}$
(v) $-x^{-2}$ or $-\frac{1}{x^2}$ (vi) $-6x^{-3}$ (vii) $1-\frac{1}{x^2}$ (viii) $9x^2-2-\frac{8}{x^3}$
(ix) $-\frac{1}{2}x^{-3/2}$ or $-\frac{1}{2}\frac{1}{x^{3/2}}$ (x) $\frac{5}{2}x^{3/2}$

Now ask for the next progress test before advancing any further.

Appendix 8

8.1 The Mathematics Syllabus for B.A. Economics Undergraduates

The concept of a function and its graphical representation, exponential and logarithmic functions.

Differential calculus, rules for differentiation, higher order derivatives, implicit differentiation, relative maxima and minima. Applications.

Matrices, operations on matrices, the determinant and inverse matrix. Applications involving linear equations.

Functions of more than one variable, partial derivatives, maxima and minima.

Max/min with constraints, the Lagrange multiplier. Applications.

8.2 Categorization of Objectives in Differential Calculus Information

Define the rate of change of a linear function. Establish its constancy.

Introduce the notation involving $\frac{\Delta y}{\Delta x} = \frac{\text{"change in y"}}{\text{"change in x"}}$

Graphical notation, equivalence of terms "rate of change", "slope" and "gradient".

The rate of change and the form y = mx + c, establish m = gradient.

Numerical technique for expressing the gradient of a non-linear polynomial.

Description of the algebraic process as development of numerical approach. The limiting process.

Relationship between the gradient of the curve and the gradient of the tangent.

Define and illustrate product and quotient rules.

Develop the function of a function rule, the "chain" rule.

The terminology and methods for higher derivatives.

Define the derivatives of ln x and e^x.

Demonstrate the method of implicit differentiation.

Define terms that occur in economic applications e.g. marginal concept, average concept, elasticity of demand.

Use of Rules

Apply numerical techniques to simple polynomial functions.

Find gradient of linear function from the coefficients, if necessary, by rearranging.

Use algebraic technique to find gradient of simple powers of x and deduce the rule for the derivative of ax^n. Extension to any value of n.

Apply rules for the differentiation of products, quotients and functions of a function.

Find the turning points of a polynomial function.

Classify turning points either by considering gradients in the neighbourhood or by evaluating second derivatives.

Use rules to find rates of change of implicit functions.

Differentiate expressions involving logarithmic and exponential functions.

Application to Economic Problems

To relate the "marginal" concept to the rate of change, so that marginal cost, marginal profit, marginal propensity to save etc can be found by differentiating the appropriate functions.

To appreciate how price, cost, quantity of goods, revenue and profit are related so that, for example, revenue and profit may be expressed in terms of given information.

To be able to construct the average cost function from the cost function and appreciate that the turning points of the two are not related.

To apply the rules for finding maxima and minima to simple economic examples, e.g. minimum average cost, maximum profit.

To apply differentiation to the problem of calculating the elasticity of demand.

8.3 Examples from the Study Scheme

(i) Diagnostic Test.

(ii) Self-paced study notes and progress tests in differential calculus.

(iii) Problem sheets.

Diagnostic Test

Attempt ALL questions. Please show all working.

1. What is the value of $x^3 - 3x^2 - 2x + 6$

   (i) when $x = 2$ (ii) when $x = -1$?

2. Factorize the expression $x^2 - 5x + 6$

3. Solve the following equations

   (i) $3x + 18 = 0$
   (ii) $3(x + 6) = 12$
   (iii) $\frac{3}{x} = \frac{8}{3}$

4. Solve the simultaneous equations

   $3x + 2y = 7$
   $2x - y = 7$

5. Solve the quadratic equation

   $2x^2 + 2x - 3 = 0$

6. Draw a neat sketch of the graph of $y = 2x - 3$ between $x = -1$ and $x = 3$

7. (a) Write down the values of (i) $\log_{10} 100$ (ii) $\log_{10} \frac{1}{100}$.

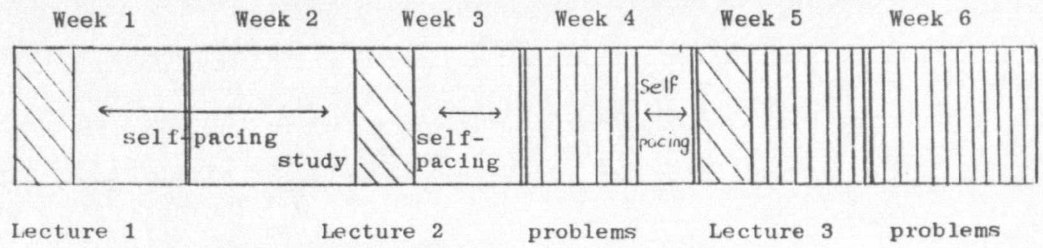
   (b) If $\log_{10} 3 = 0.4771$, what is the value of $\log_{10} 0.03$?

8. (a) Simplify (i) $a^3 \times a^5$ (ii) $x^3 \div x^5$ (iii) $\frac{a^2x^4}{a^4x^2}$

   (b) Express, using positive indices only,

   (i) a^{-2} (ii) $a^{-4/3}$ (iii) $\sqrt{a^5}$ (iv) $\frac{1}{a^{-1/2}}$

Rates of Change, Gradients and Differentiation



Lecture 1. Rates of change of linear and non-linear functions. Numerical approach. Drawback of the numerical method. Algebraic method, limiting process.

Lecture 2. Product, quotient and function of a function rules. Derivatives of e^x and lnx. Higher order derivatives

Lecture 3. Implicit differentiation.

Problems 1 Maxima and minima, applications to economic problems.

Problems 2 Cost, average cost, marginal cost, revenue, marginal revenue, elasticity of demand, profit under monopoly.

Self-pacing study 22 pages of notes to be worked through in this section. There are five progress tests.

It is important to successfully complete each progress test before starting the next section of work. Progress tests will only give a true indication of your mastery of the objectives if undertaken without any assistance.

Rates of Change, Gradients and Differentation

We have already encountered the ideas that rage of change =

$$\frac{\text{change in } y}{\text{change in } x}$$

which for a linear function turns out to be a constant. The "steeper" the line of the graph the greater is the "rate of change", "gradient" or "slope".

† Work out the rates of change of the following functions. Remember, for each function, the rate of change is constant, so you may choose whatever values of x you like for your calculations.

(i) $y = x + 6$ (iii) $y = 4x + 2$

(ii) $y = 2x + 1$ (iv) $y = 5x + 1$

(i) 1	(ii) 2	(iii) 4	(iv) 5

Make rough sketches of the graphs. Which has the steeper slope? Which has the least slope? Which has the greatest rate of change? Which has the smallest rate of change?

Look closely at the coefficients in the four equations above and then look at the figures you obtained for the gradients. Do you see any connection?

I hope so, because you will then be able to write down the rate of change of a linear function just by looking at the equation.

† Try these for example; look at the equations and write down the answer:

(i) $y = 3x - 5$ (iii) $y = 10x + 35$

(ii) $y = x + 4$ (iv) $y = 5x - 92$

The gradients are (i) 3 (ii) 1 (iii) 10 (iv) 5 respectively

What about $2y - 3x + 5 = 0$? You may be tempted to say that the gradient is -3, but this would be wrong. What has happened to our tidy rule? Notice that this last expression is in a different form from all our other equations. Let's try and put its form right by rearranging:

$$2y - 3x + 5 = 0$$
$$2y = 3x - 5$$
$$y = \frac{3}{2}x - \frac{5}{2}$$

And the gradient is in fact $\frac{3}{2}$. We can only read off the gradient when the equation is in the proper form, that is with y as the subject, (or expressed explicitly - remember ?)

thus $y = Mx + C$ then M (the coefficient of x) is the gradient of the line.

Here are two further examples:

(i) $6y - 12x + 15 = 0$
$6y = 12x - 15$
$y = 2x - \frac{5}{2}$ rate of change is 2

(ii) $3y + 9x - 6 = 0$
$3y = -9x + 6$
$y = -3x + 2$ rate of change is -3

† Find the rates of change in the following cases:

(i) $3y - 3x - 9 = 0$ (iii) $2x + 3y + 9 = 0$

(ii) $4y + 8x + 15 = 0$ (iv) $9 - 2y + 4x = 0$

(i) 1	(ii) -2	(iii) $-\frac{2}{3}$	(iv) 2

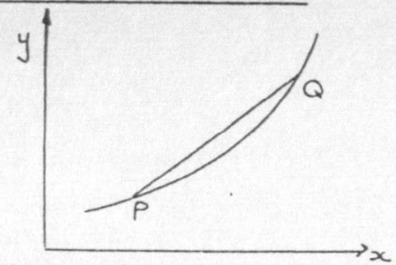
Well done so far! Here is a summary of what has been covered.

1. For a linear function, the rate of change is constant.
2. The rate of change = $\frac{\text{change in } y \text{ - values}}{\text{change in } x \text{ - values}} = \frac{Y_Q - Y_P}{X_Q - X_P} = \frac{\Delta y}{\Delta x}$ between two chosen points P and Q.
3. For a linear function, if the equation is expressed in the form $y = Mx + C$, the rate of change is M (the coefficient of x).
4. The magnitude of the slope is directly related to the steepness of the graph, large $M \rightarrow$ steep line, small $M \rightarrow$ nearly flat line, negative $M \rightarrow$ line slopes downwards.

Now do the first progress test. You must do well before proceeding further

3

Rate of change of a non-linear function



We are already familiar with a numerical technique of finding the average gradient between two points P and Q on a curve. This if at

Thus, if at P, x = 1.8 and at Q, x = 2.2 on the curve $y = x^2$

$y_P = 3.24$ $\qquad y_Q = 4.84$

rate of change $= \dfrac{4.84 - 3.24}{2.2 - 1.8} = \dfrac{1.60}{0.4} = 4.0$

But the rate of change is not constant and this method only gives the gradient along the line PQ and not along the curve. This is why P and Q must be fairly close to get a reasonable answer. Try these for yourself. Use the numerical approximation at each point. What are the rates of change of $y = 2x^2$ at x = 2 and x = 3 ?

What about the rates of change of $y = x^2 + 2x$ at x = 1 and x = 2?

(i) 8 and 12	(ii) 4 and 6

[If you are wrong here, look at H3 on the Supplementary Sheet]

When we draw the graph of a function, we see that the rate of change is directly related to the steepness of the curve, so the rate of change is often referred to as the gradient or slope. We also showed in the last lecture that the rate of change at any point on a curve is identical to the gradient of the tangent at that point.

Try filling the missing words in the following sentences:
(Don't peep at the answers!)

(i) The gradient of the function is another name for the ....
or ....

(ii) Along a curve, the gradient ...... from one point to the next.

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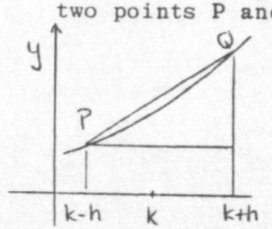
(iii) The slope of a curve at a particular point is the ....
as the slope of the .... to the curve at the same point.

(i) slope or rate of change	(ii) varies or changes
(iii) same, tangent	

Let us return to the problem of avoiding the numerical tedium of finding rates of change. We can do this by doing exactly the same thing as before, but this time algebraically. Let's take a simple example first, say $y = x^3$. Assume we want to find the rate of change at the point where (say) x = k. As we did numerically, choose two points P and Q either side, say at x = k - h and x = k + h



Substitute for x is $y = x^3$

at O $\qquad y_Q = (k+h)^3 = k^3 + 3k^2h + 3kh^2 + h^3$

at P $\qquad y_P = (k-h)^3 = k^3 - 3k^2h + 3kh^2 - h^3$

$\therefore$ rate of change $= \dfrac{y_Q - y_P}{x_Q - x_P}$

$= \dfrac{(k^3+3k^2h+3kh^2+h^3)-(k^3-3k^2h+3kh^2-h^3)}{(k+h)-(k-h)}$

removing brackets and collecting terms

$= \dfrac{6k^2h + 2h^3}{2h} = 3k^2 + h^2$

Now we can do something we couldn't do with the numerical result. we can let P and Q slide together - get closer and closer. What result will this have on the value of k? and the value of h?

Yes, that's right, k is fixed h gets nearer and nearer to zero. In fact, there is nothing to stop us making h smaller than any number we please! In mathematical notation we write $h \rightarrow 0$.

If we do this we get our rate of change equal to $3k^2$. This is a remarkable result! What it says is that to find the rate of change at any point on $y = x^3$, the answer is simply:

3 x (x-value at that point)^2 ie when x = 1 gradient = $3 \times 1^2 = 3$
when x = 2 gradient = $3 \times 2^2 = 12$
when x = 3 gradient = $3 \times 3^2 = 27$ etc

So the problem of finding the rate of change anywhere on $y = x^3$ is solved once and for all. We can say that

| for $y = x^3$, the rate of change is $3x^2$ |

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Let us illustrate a simpler case. Try the function $y = 3x + 2$
Take P as $(k - h)$ and Q as $(k + h)$ as before.
Then at Q $\quad y_Q = 3(k + h) + 2$

at P $\quad y_P = 3(k - h) + 2$

$$\text{rate of change} = \frac{3(k + h) + 2 - \{3(k - h) + 2\}}{(k + r) - (k - h)}$$
$$= \frac{3k + 3h + 2 - \{3k - 3h + 2\}}{k + h - k + h}$$
$$= \frac{6h}{2h} = 3 \quad \text{a constant!}$$

(Are you surprised by this? Think back to when we were talking about linear functions)

Usually our expressions don't work out as simply as this. Remember, any terms involving h can be made as small as we please by making $h \to 0$. Now you must have some practice at this for yourselves before we try and generalise our results.

† Find expressions for the rates of change of the functions:

(i) $y = x^2$ $\qquad$ (ii) $y = x^2 + 1$
(iii) $y = 2x$ $\qquad$ (iv) $y = x^2 + 2x$

(i) $\dfrac{(k+h)^2 - (k-h)^2}{(k+h) - (k-h)} = \dfrac{4kh}{2h} = 2k$ or $\underline{2x}$

(ii) $\dfrac{(k+h) + 1 - \{(k-h)^2+1\}}{(k+h)-(k-h)} = \dfrac{k^2+2kh+h^2+1 - \{k^2-2kh+h^2+1\}}{2h} = \dfrac{4kh}{2h} = 2k$ or $\underline{2x}$

(iii) $\dfrac{2(k+h)-2(k-h)}{(k+R)-(k-h)} = \dfrac{4h}{2h} = \underline{2}$

(iv) $\dfrac{(k+h)^2+2(k+h)-\{(k-h)^2+2(k-h)\}}{(k+h)-(k-h)} = \dfrac{4kh+4h}{2h} = 2k + 2$ or $\underline{2x + 2}$

If you are still worried about this process seek some advice, either from your colleagues or from a tutor.

You may feel that this remedy is worse than the complaint! However, let's look a little more closely at the results we have and try and find a general rule which would enable us to write down the expression for the gradient, just by looking at the original function . Look at this evidence below, we can assemble this from what we have done already.

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function of x	x	x^2	x^3
rate of change	1	$2x$	$3x^2$

† Try to discover some relationship between the top expression and the bottom.

What do you think is the rate of change of the function $y = x^4$? What about $y = x^5$? (The correct results are over the page - don't look until you have written down your conjecture!). What would you write down as the expression for the rate of change of x^n ?

Notation - "Rate of change" is too long, we need a symbol to represent "I have derived this algebraically by finding $\dfrac{\text{change in } y}{\text{change in } x}$ and letting $h \to 0$".
The symbol $\dfrac{dy}{dx}$ is used for the rate of change (pronounced "d-y by d-x").
This process of obtaining the general expression is called differentiation and the expression itself is called the derivative.

The results can be written down like this:

if $y = x^2$, $\dfrac{dy}{dx} = 2x$; if $y = x^6$, $\dfrac{dy}{dx} = 6x^5$; if $y = x^{-1}$, $\dfrac{dy}{dx} = -x^{-2}$
Remember, the value of the gradient at a point is found by substituting a value of x in $\dfrac{dy}{dx}$. Thus if $y = x^6$, when $x = 2$, $\dfrac{dy}{dx} = 6(2)^5 = 192$.

[If you cannot see how to write down the derivatives of x^2, x^6, x^{-2} etc. you must look at H1. on the supplementary sheet.]

Now $y = x^5$ is fine but what about something like $y = 3x^4 + 4x^3 - 5x$?

Here are two results quoted which you should find reasonable, If not discuss the point with the tutor:

(i) If a function is multiplied by a constant, so is the corresponding derivative.

e.g. $y=x^3$, $\dfrac{dy}{dx}=3x^2$ so if $y=4x^3$, $\dfrac{dy}{dx} = 4\times(3x^2)=12x^2$

(ii) If a function is the sum of several terms, the derivative is expressed as the sum of the individual derivative terms.

e.g. $y=x^3+x^2$ then $\dfrac{dy}{dx} = 3x^2+2x$

and $y=2x^5-\frac{1}{2}x^3$ $\quad \dfrac{dy}{dx} = 10x^4 - \dfrac{3}{2}x^2$

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$4x^3$ and $5x^4$ — what about $y=\sqrt{x}$ (Does $y=x^{\frac{1}{2}}$ help?)

† Find the expressions for the rates of change of the following functions of x:

(i) $y = x^2 + x$

(ii) $y = 2x^3 - 3x^2 + x$

(iii) $y = 2x + 2$ (Whate rate of change do you think a constant has?)

(iv) $y = \sqrt{x}$

(v) $y = \frac{1}{x}$ (Express first in the(x) form $y = x^n$)

(vi) $y = \frac{3}{x^2}$

(vii) $y = x + \frac{1}{x^2}$

(viii) $y = 3x^3 - 2x + \frac{4}{x^2}$

(ix) $y = \frac{1}{\sqrt{x}}$

$y = x^{\frac{5}{2}}$

(i)	$2x + 1$	(vi)	$\frac{-6}{x^3}$
(ii)	$6x^2 - 6x + 1$	(vii)	$1 \frac{-2}{x^3}$
(iii)	2	(viii)	$9x^2 - 2 \frac{-8}{x^3}$
(iv)	$\frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2x^{\frac{1}{2}}}$	(iv)	$-\frac{1}{2x^{\frac{3}{2}}}$
(v)	$-x^{-2}$ or $-\frac{1}{x^2}$	(x)	$\frac{5}{2}x^{\frac{3}{2}}$

■ Here is a summary of the work we have just covered. Make sure you can understand each section before going on to the progress test:

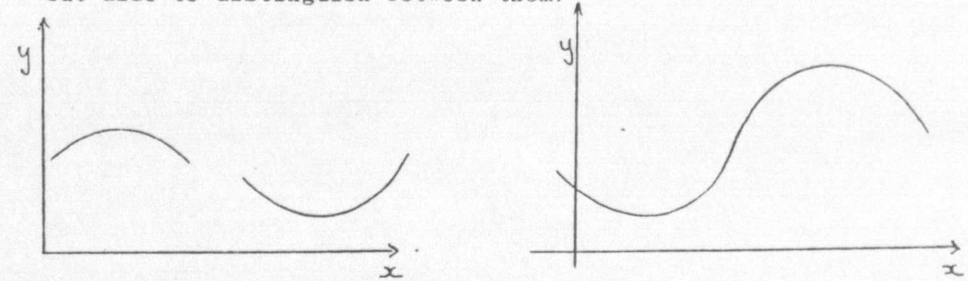
1. To find the rate of change of a function at a given point numerically.
2. To find a general expression for the rate of change algebraically from first principles (choose P and Q and let $h \to 0$)
3. To apply the general rule to expressions with terms that are (or can be written) in the form of $a.x^n$.

Now do the progress test

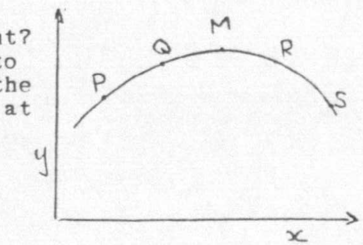
8

An Application of Differentiation

Very often in economic analysis we require to know when a particular function reaches a maximum or minimum. Graphically the two cases look like the figure below. Sometimes a function may contain both, so not only do we need to locate these points but also to distinguish between them.



First, how do we locate an extreme point? Look at the gradient of a curve close to a maximum point. What is the sign of the gradient at P? Is the gradient at Q greater or less than at P?
Now look at R and S. What do we know about the gradients at R and S?



† Try filling in the missing words:

The gradient at Q is .... because the function is increasing.

The gradient at Q is .... than the gradient at P.

The gradients at R and S are both .... because the function in this region is .... The gradient is less negative at ....

The missing words are "positive", "less", "negative, "decreasing", "R".

We can summarize the information:

Point	P	Q	R	S
Gradient	large +ve	small +ve	small -ve	large -ve

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What would happen if Q and R were closer to M? What do you think the rate of change is at M? What is the slope of the tangent at M? [If you are unsure of the answers, look at H2 on the supplementary sheet]

If we can find any points at which the rate of change is zero, we have the possibility of a maximim or minimum. Since the same condition rates at both points we cannot (as yet) distinguish them, so we shall use the general term of <u>extreme</u> or <u>turning</u> points. The first job is to locate them. <u>Let's</u> try it!

<u>Example 1</u>

$y = 6 + 4x - x^2$ so $\frac{dy}{dx} = 4 - 2x$

If we have a turning point, $\frac{dy}{dx} = 0$. Substitute this into the equation $0 = 4 - 2x$, solve this and <u>x = 2 marks a turning point</u>.

<u>Example 2</u>

$y = x^2 + 5x + 9$ so $\frac{dy}{dx} = 2x + 5$

At a turning point $0 = 2x + 5$

<u>$x = -2\frac{1}{2}$ marks a turning point</u>

<u>Example 3</u>

$y = x^3 + 12x^2 + 21x - 2$ so $\frac{dy}{dx} = 3x^2 + 24x + 21$

at a turning point $0 = 3x^2 + 24x + 21$

to aid the solution divide by 3 $0 = x^2 + 8x + 7$

and solving the quadratic <u>$x = -7$ or -1 marks two turning points</u>

† Try finding the turning points (if any) of the following functions:

(i) $y = x^2 - 2x$ (iii) $y = 4x - x^2 + 5$ (v) $y = \frac{1}{x} + x$

(ii) $y = x^2 - 6x + 11$ (iv) $y = x^3 - 12x$

(i) x = 1 (ii) x = 3 (iii) x = 2
(iv) x = 2 <u>and</u> -2 (most important) (v) x = 1 <u>and</u> -1

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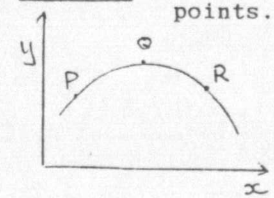
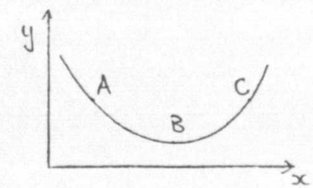
We can now locate extreme points but not distinguish between maxima and minima. Here are two methods of doing this.

<u>Method 1</u> - involves looking at the gradient at a few chosen points.

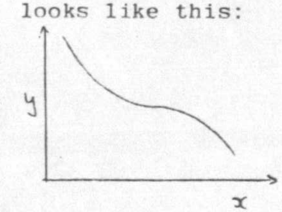




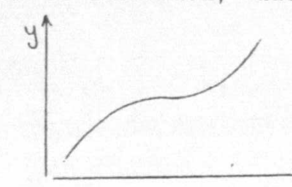
What sign will the gradient have (+ or -) at P, Q, R and A, B, C?

Point	P Q R	A B C
Sign of gradient	+ zero -	- zero +

Notice that the two patterns are different. There is a clear distinction between PQR (max) and ABC (min). It also enables us to identify what is called a point of inflexion, which looks like this:



OR



What is the pattern of signs in this case?

<u>Method 2</u> - This involves looking at the gradient of the gradient!

(We use the symbol $\frac{d^2y}{dx^2}$ for this and call it the second derivative because we must differentiate the function for $\frac{dy}{dx}$ to get it)

Notice: If $\frac{dy}{dx}$ goes from -ve to +ve it must be increasing and its gradient is +ve. If $\frac{dy}{dx}$ goes from +ve to -ve it must be decreasing and its gradient is -ve.

If we summarize

	MAX	MIN	
$\frac{dy}{dx}$	+ve zero -ve	-ve zero +ve	(1)
$\frac{d^2y}{dx^2}$	-ve	+ve	(2)

Hence, we can test using condition (1) or condition (2). Note that if $\frac{d^2y}{dx^2} = 0$, we can gain no useful information and condition (1) must be used. Let us now investigate the <u>nature</u> of the turning points we located in the previous examples.

1. $y = 6 + 4x - x^2$ $\quad \frac{dy}{dx} = 4 - 2x$ turning point at $x = 2$

Either: $\frac{d^2y}{dx^2} = -2$ which is -ve, hence <u>point is a maximum</u>

or : Choose points ither side of the turning point at $x = 2$, say $x = 1$ and $x = 3$

when $x = 1, \frac{dy}{dx} = 2$ (+ve)

$x = 3, \frac{dy}{dx} = -2$ (-ve) hence <u>point is a maximum</u>

2. $y = x^2 + 5x + 9$ $\quad \frac{dy}{dx} = 2x + 5$ $\quad \frac{d^2y}{dx^2} = 2$

Either: $\frac{d^2y}{dx^2}$ is +ve hence <u>point is a minimum</u>

Or: Choosing points either side of $x = -2\frac{1}{2}$, say $x = -3$ and $x = -2$

when $x = -3$ $\quad \frac{dy}{dx} = -1$ (-ve)

when $x = -2$ $\quad \frac{dy}{dx} = +1$ (+ve) hence <u>point is a minimum</u>

(Note! $x = -3$ is to the <u>left</u> of the point, $x = -2$ is to the <u>right</u>)

-3 -2 -1 0

3. $y = x^3 + 12x^2 + 21x - 2$ $\quad \frac{dy}{dx} = 3x^2 + 24x + 21$ $\quad \frac{d^2y}{dx^2} = 6x + 24$

Either: when $x = -1, \frac{d^2y}{dx^2} = 18$ (+ve) <u>point is a minimum</u>

when $x = -7, \frac{d^2y}{dx^2} = -18$ (-ve) <u>point is a maximum</u>

Or: choose points either side of $x = -1$ say $x = -2$ and $x = 0$

when $x = -2$ $\quad \frac{dy}{dx} = -15$

when $x = 0$ $\quad \frac{dy}{dx} = 21$ <u>point is a minimum</u>

choose points either side of $x = -7$, say $x = -8$ and $x = -6$

when $x = -8$ $\quad \frac{dy}{dx} = 21$

when $x = -6$ $\quad \frac{dy}{dx} = -15$ <u>point is a maximum</u>

To find the extreme values, substitute $x = -1$ and $x = -7$ in the original equation to find y.

so $y\ min = (-1)^3 + 12(-1)^2 + 21(-1) - 2 = -12$, at $x=-1$, $y=-12$)
and $y\ max = (-7)^3 + 12(-7)^2 + 21(-7) - 2 = 96$, at $x=-7$, $y=96$)

So this function has one maximum and one minimum. Go back to the exampled you did earlier and investigate the nature of the turning points. They are:

(i) $y = x^2 - 2x$ $\quad$ (iii) $y = 4x - x^2 + 5$ $\quad$ (v) $y = x + \frac{1}{x}$

(ii) $y = x^2 - 6x + 11$ $\quad$ (iv) $y = x^3 - 12x$

(i) y min = -1	(iv) y max = 16, y min = -16
(ii) y min = 2	(v) y max = -2, y min = 2 You may thing (v) a bit odd. Seek advice if you need it.
(iii) y max = 9	

■ From this section you should know:

1. How to locate the turing points in a function.
2. How to identify the nature of the turning points you have located.
3. How to evaluate the function at the turning points.

There is the third progress test waiting for you.

The Product, Quotient and Function of a Function Rules

So far, the method of finding the rate of change of any expression in the form ax^n has been developed. How could we deal with,

say, $\frac{x^3 + 3}{x^2 - 5x + 6}$ or $\sqrt{x^2 + x}$?

Some rules will now be demonstrated which enable us to deal with such examples. No proofs will be given. If you are interested, they will be found in any text-book on calculus e.g. Calculus by F Ayers (pub. McGraw - Hill).

The Product Rule

In the lecture, we discussed expressions such as

$y = uv$ where u and v are functions of x

then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Here are some examples:

1. $y = (x^2 + 2x + 6)(x^2 - 3x + 5)$*

here $u = x^2 + 2x + 6$ so $\frac{du}{dx} = 2x + 2$

$v = x^2 - 3x + 5$ so $\frac{dv}{dx} = 2x - 3$

hence $\frac{dy}{dx} = (x^2+2x+6)(2x-3) + (x^2-3x+5)(2x+2)$

simplifying $\frac{dy}{dx} = 2x^3+4x^2+12x-3x^2-6x-18+2x^3-6x^2+10x+2x^2-6x+10$

$\frac{dy}{dx} = 4x^3-3x^2+10x-8$

*You could multiply this out and then differentiate but the product rule is quicker and more general.

2. $y = (x^2-x+3)(\sqrt{x} + \frac{1}{\sqrt{x}})$

here $u = x^2 - x + 3$ so $\frac{du}{dx} = 2x - 1$

$v = x + \frac{1}{x}$ or $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ so $\frac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$\frac{dy}{dx} = (x^2-x+3)(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}) + (x^{\frac{1}{2}}+x^{-\frac{1}{2}})(2x-1)$

Simplify $\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} + x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

3. $y = (3-x^2)(9-2x+x^2)$

here $u = 3-x^2$ $\frac{du}{dx} = -2x$

$v = 9-2x+x^2$ $\frac{dv}{dx} = -2+2x$

$\frac{dy}{dx} = (3-x^2)(-2+2x)+(9-2x+x^2)(-2x)$

$= -6+6x+2x^2-2x^3-18x+4x^2-2x^3$

$= -6-12x+6x^2-4x^3$

In each example we have tidied up the answer by multiplying out the brackets and collecting terms.

† Try these examples for yourself.

(i) $y = (3x + 2)(x^2 + 1)$ (iii) $y = (x^3 + 5)(2x^2 - 4)$

(ii) $y = (1 - 3x)(3 + x^2)$

(i) $9x^2 + 4x + 3$	(ii) $-9x^2 + 2x - 9$	(iii) $10x^4 - 12x^2 + 20x$

Quotient Rule

If we have a function of the form

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$$y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This is more complicated than the product rule but we still only have to find $\frac{du}{dx}$ and $\frac{dv}{dx}$ separately. Let's look at some examples:

1. $y = \frac{x^2+3x-5}{x^2+x+1}$ $\quad u = x^2+3x-5 \quad \frac{du}{dx} = 2x+3$

$v = x^2+x+1 \quad \frac{dv}{dx} = 2x+1$

$$\frac{dy}{dx} = \frac{(x^2+x+1)(2x+3)-(x^2+3x-5)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{(2x^3+2x^2+2x+3x^2+3x+3)-(2x^3+6x^2-10x+x^2+3x-5)}{(x^2+x+1)^2}$$

$$= \frac{-2x^2+12x+8}{(x^2+x+1)^2}$$

Again we have simplified the result (watch those signs!)

2. $y = \frac{6-x^4}{3x+x^3}$ $\quad u = 6-x^4 \quad \frac{du}{dx} = -4x^3$

$v = 3x+x^3 \quad \frac{dv}{dx} = 3+3x^2$

$$\frac{dy}{dx} = \frac{(3x+x^3)(-4x^3)-(6-x^4)(3+3x^2)}{(3x+x^3)^2}$$

$$= \frac{(-12x^4-4x^6)-(18-3x^4+18x^2-3x^6)}{(3x+x^3)^2} = \frac{-x^6-9x^4-18x^2-18}{(3x+x^3)^2}$$

3. $y = \frac{\sqrt{x}}{6x+2}$ $\quad u = \sqrt{x}$ or $x^{\frac{1}{2}} \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$v = 6x+2 \quad \frac{dv}{dx} = 6$

$$\frac{dy}{dx} = \frac{(6x+2)(\frac{1}{2}x^{-\frac{1}{2}})-(x^{\frac{1}{2}})(6)}{(6x+2)^2}$$

$$= \frac{3x^{\frac{1}{2}}+x^{-\frac{1}{2}}-6x^{\frac{1}{2}}}{(6x+2)^2} \qquad = \frac{x^{-\frac{1}{2}}-3x^{\frac{1}{2}}}{(6x+2)^2}$$

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Try these examples

(i) $y = \frac{x^2+2}{4x-1}$ $\qquad$ (ii) $y = \frac{3x-4}{x^2+3}$

(iii) $y = \frac{6-x}{2-5x^2}$

(i) $\frac{4x^2-2x-8}{(4x-1)^2}$ $\qquad$ (ii) $\frac{-3^2+8x+9}{(x+3)^2}$

(iii) $\frac{-5x^2+60x-2}{(2-5x^2)^2}$

Function of a Function Rule

The name is given to expressions which are a function of another function!

e.g. $y = \sqrt{x^2+3x+1}$ y is a square-root function of a quadratic function

$y = \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3$ y is a cubic function of a function of square roots.

$y = (3x^2+2)^5$ y is a fifth-power function of a quadratic function.

How do we handle these horrors? Since we don't like the problem we're given, we invent some simpler ones.

1. consider $y = \sqrt{x^2+3x+1}$

Looking inside the square root we see a quadratic function. We take the quadratic and let:

$u = x^2+3x+1 \quad$ so $\frac{du}{dx} = 2x+3$

then $y = u^{\frac{1}{2}} \quad$ so $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

So, we can derive our result since:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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$= \frac{1}{2}u^{-\frac{1}{2}}(2x+3)$

Replace the function u by the corresponding function of x

then $\frac{dy}{dx} = \frac{1}{2}(x^2+3x+1)^{-\frac{1}{2}}(2x+3)$

Let's review the steps -

1. introduce a new variable equal to the innermost function
2. differentiate the two functions thus obtained
3. combine the two results
4. remove the substituted variable from the answer.

Here are the other two examples:

2. $y = (\sqrt{x}+\frac{1}{\sqrt{x}})^3$

Let $u = x^{\frac{1}{2}}+x^{-\frac{1}{2}}$ then $y = u^3$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$ $\frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}) = 3(x^{\frac{1}{2}}+x^{-\frac{1}{2}})^2(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}})$

3. $y = (3x^2+2)^5$

Let $u = 3x^2+2$ then $y = u^5$

$\frac{du}{dx} = 6x$ $\frac{dy}{du}$ $\frac{dy}{du} = 5u^4$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4(6x) = 30x(3x^2+2)^4$

Now try these examples.

(i) $y = (2x-7)^3$ (ii) $y = (5-x^2)^7$

(iii) $y = (x^2-3x)^5$ (iv) $y = \sqrt{6x+x^2}$

(i) $6(2x-7)^2$	(ii) $-14x(5-x^2)^6$
(iii) $5(x^2-3x)^4(2x-3)$	(iv) $(3+x)(6x+x^2)^{-\frac{1}{2}}$

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<u>Summary of Rules for Differentiation</u> [Learn these off by heart]

Function	Derivative
ax^n	anx^{n-1}
$u.v$	$u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$f(u)$ where $u=g(x)$	$\frac{df}{du} \cdot \frac{du}{dx}$

From this section you have now learnt:

1. To differentiate a product
2. To differentiate a quotient
3. To apply the function of a function rule.

Here are some revision questions: (Try them only if you feel you need to)

1. Differentiate: (i) $y = 5x^2-3x+4$

(ii) $z = \frac{3}{u^4}$

(iii) $s = \frac{1}{\sqrt{t}}$

2. Differentiate (i) $y = (3x-5)(2x^2+1)$

(ii) $w = \frac{3s-5}{2s^2+1}$

(iii) $s = (2t^2+1)^4$

3. Find the rate of change of y with respect to x when x = 2 given that:

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(i) $y = x^3+2$

(ii) $y = x^2-2x$

1.	(i) $10x-3$	2.	(i)	$18x^2-20x+3$
	(ii) $-12u^{-\frac{5}{3}}$		(ii)	$\frac{-6s^2+20s+3}{(2s^2+1)^2}$
	(iii) $-\frac{1}{2}t^{-\frac{3}{2}}$		(iii)	$16t(2t^2+1)^3$
3.	(i) 12			
	(ii) 2			

Now do progress test number four.

Higher Order Derivatives

You have been introduced to the notation for higher order derivatives.

Thus $\frac{dy}{dx}$ is the first derivative of y with respect to x.

$\frac{d^2y}{dx^2}$ is the second derivative of y with respect to x.

$\frac{d^3y}{dx^3}$ is the third derivative of y with respect to x.

Thus if $y = 3x^2-12x^2+2x-10$

$\frac{dy}{dx} = 9x^2-24x+2$

$\frac{d^2y}{dx^2} = 18x-24$

$\frac{d^3y}{dx^3} = 18$ etc.

and if $y = \ln x$ [This was referred to in the lecture, together with $y = e^x$]

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$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

$\frac{d^3y}{dx^3} = \frac{2}{x^3}$ etc

Try these examples for yourself:

1. Find the second derivatives of the following functions:

(i) $y = 4x^3-9x^2+10x-16$ (iv) $y = (2x+6)e^x$

(ii) $y = \sqrt{x}$ (v) $y = e^{x+4}$

(iii) $y = x + \frac{1}{x}$

(i)	$24x-18$	(ii)	$-\frac{1}{4}x^{-\frac{3}{2}}$
(iii)	$\frac{2}{x^3}$	(iv)	$(2x+10)e^x$
(v)	e^{x+4}		

Implicit Differentiation

So, far all we have said concerning differentiation assumes that the dependent variable can be expressed as an explicit function.

i.e. $y = f(x)$, $s = f(x)$, $y = f(s)$ etc.

What happens if we are given an implicit function

e.g. $x^2+2xy+y^2=0$?

It is not clear which variable is the independent one. Are we finding the rate of change of y with respect to x or the rate of change of x with respect to y?

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Remember to differentiate f(x) with respect to x (or f(θ) with respect to θ) should present no problem. The rules we have already learned can be applied.

so $\frac{d}{dx}(x^2+5x) = \frac{d}{dx}(x^2)+\frac{d}{dx}(5x) = 2x+5$

$\frac{d}{dx}(\frac{1}{x^2}) = -\frac{2}{x^3}$ etc

† Try answering these questions:

What is the rate of change of x^4 with respect to x?

What is the rate of change of $3\theta^3$ with respect to θ?

What is the rate of change of y^2 with respect ot x?

Why is the last question different from the other two? Why is the answer not 2y?

In the cases where the differentiation rules can be applied, the function is given in terms of the independent variable. In the last example we may assume y^2 is <u>some</u> function of x, but we do not know which.

What can be done about $\frac{d}{dx}(y^2)$? Well, we do know that just

as $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dt}(t^2) = 2t$, that $\frac{d}{dy}(y^2) = 2y$

Can we start from this? We use a method similar to the function of a function rule.

If we write $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2).\frac{dy}{dx} = 2y.\frac{dy}{dx}$ since we know $\frac{d}{dy}(y^2)$

We can apply this technique in conjunction with all the rules we have previously learned. Remember, the clue to the whole process is to write:

$$\left\| \frac{d}{dx}f(y) \text{ in the form } \frac{d}{dy}f(y).\frac{dy}{dx} \right\|$$

Example: to find $\frac{dy}{dx}$ given that $x^2+y^2= 12$

Differentiate with respect to x $\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = \frac{d}{dx}(12)$

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then $\underset{\text{(easy)}}{2x} + \frac{d}{dy}(y^2).\frac{dy}{dx} = 0$

so $2x + 2y\frac{dy}{dx} = 0$

Notice the equation still has to be rearranged to give $\frac{dy}{dx}$ explicitly.

hence $\underline{\frac{dy}{dx} = -\frac{x}{y}}$

Example: find $\frac{dy}{dx}$ given $x^2+6x+y^2-4y+10=0$

$\frac{d}{dx}(x^2)+\frac{d}{dx}(6x)+\frac{d}{dx}(y^2)-\frac{d}{dx}(4y)+\frac{d}{dx}(10)=0$

$2x+6+\frac{d}{dy}(y^2).\frac{dy}{dx}-\frac{d}{dy}(4y).\frac{dy}{dx}+0=0$

$2x+6+2y\frac{dy}{dx} - 4\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y-4) = -2x-6$

$\underline{\frac{dy}{dx} = \frac{-2x-6}{2y-4} \text{ or } \frac{-x-3}{y-2}}$

Example: find $\frac{dy}{dx}$ given $x^2+2xy+y^2=0$

$\frac{d}{dx}(x^2)+\frac{d}{dx}(2xy)+\frac{d}{dx}(y^2)=0$

Notice that the middle term is now a <u>product</u> and we must use the product rule.

$\frac{d}{dx}(x^2)+\overbrace{2x.\frac{d}{dx}(y)+y.\frac{d}{dx}(2x)}^{\text{product rule}}+\frac{d}{dx}(y^2) = 0$

$2x+2x.\frac{d}{dy}(y).\frac{dy}{dx}+y.2+\frac{d}{dy}(y^2).\frac{dy}{dx} = 0$

$2x+2x.\frac{dy}{dx}+2y+2y\frac{dy}{dx} = 0$

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$$(2x+2y)\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2(x+y)}{2(x+y)} \qquad [\text{signs!}$$

$$\therefore \frac{dy}{dx} = -1$$

Example: find $\frac{dy}{dx}$ given $\frac{x}{y} + \frac{y}{x} = x$

The quotient rule must be used for each of the two terms on the left.

i.e. $\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \right]$

$$\frac{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}{y^2} + \frac{x\frac{d}{dx}(y) - y\frac{d}{dx}(x)}{x^2} = \frac{d}{dx}(x)$$

$$\frac{y - x\frac{dy}{dx}}{y^2} + \frac{x\frac{dy}{dx} - y}{x^2} = 1$$

Multiply throughout by $x^2 y^2$ to remove the denominators:

$$x^2(y - x\frac{dy}{dx}) + y^2(x\frac{dy}{dx} - y) = x^2y^2$$

$$\frac{dy}{dx}(xy^2 - x^3) = x^2y^2 + y^3 - x^2y$$

$$\frac{dy}{dx} = \frac{(x^2y + y^2 - x^2)y}{(y^2 - x^2)x}$$

Here are some examples for you to try:

Find expressions for $\frac{dy}{dx}$ in the following cases:-

(i) $x^3+6x-y^2 = 4$ (vi) $\frac{y}{x} = x-6$

(ii) $x^2+y^2 = 2x$ (vii) $x^2+3xy^2+xy-y^3 = 0$

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(iii) $y(x+2) = 10$ (viii) $(2x+y)(x+2y) = 0$

(iv) $\frac{1}{x} + \frac{1}{y} = 2x$ (ix) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

(v) $y\,e^x = 3x$

(i)	$\frac{3x^2+6}{2y}$	(ii)	$\frac{1-x}{y}$	(iii)	$\frac{-y}{x+2}$	(iv)	$\frac{-y^2}{x^2}\ {}^-2y^2$
(v)	$3e^{-x} - y$	(vi)	$x + \frac{y}{x}$	(vii)	$\frac{2x - 3y^2 - y}{6xy + x - 3y^2}$		
(viii)	$\frac{-4x - 5y}{5x + 4y}$	(ix)	$\frac{-25x}{9y}$				

The objectives of studying this section are:

1. To know the derivatives of e^x and $\ln x$
2. To apply the function of a function technique to such problems as e^{3x}, $\ln(x^2 + 6)$ etc
3. To understand the notation for the higher derivatives and be able to find them if asked to do so.
4. To be able to apply the technique for implicit differentiation.

If you are completely happy about these objectives, try the fifth and final progress test for this section.

1

Rates of Change, Gradients and Differentiation

Supplementary Sheet

H1. To find an expression for the rate of change of x^3 (say), write down the index (3) as the coefficient and reduce the index by 1, making it 2. Thus we get $3x^2$.

The same rule applied to x^2 would give $2x^1$, or just $2x$, which is correct.

Hence, it is likely that x^4 has a rate of change expressed by $4x^3$ and x^5 by $5x^4$.

The general expression is to say that the rate of change of x^n is given by nx^{n-1}.

In words, to write down the rate of change of a power of x:

(i) Make the coefficient equal to the original index.

(ii) Reduce the index by 1.

This applies whatever the value of n.

Since, to apply this rule, the expression must be in the form x^n, some rearrangement may be necessary before differentiating.

e.g. for $\sqrt{x}$ and $\frac{1}{x^2}$

Remember: A reciprocal can be written as a negative index

e.g. $\frac{1}{x^2} = x^{-2}$, $\frac{1}{x} = x^{-1}$ and (N.B.) $\frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$, $\frac{1}{x^{-1}} = x$

Notice, only the sign changes when you take the reciprocal.

A root is written as a fractional index.

e.g. $\sqrt{x} = x^{\frac{1}{2}}$, $\sqrt[3]{x} = x^{\frac{1}{3}}$ etc

Sometimes these two rules are combined, so, for example

$$\frac{1}{\sqrt[3]{x}} = \frac{1}{x^{\frac{1}{3}}} = x^{-\frac{1}{3}}$$

If $y = \frac{1}{\sqrt[3]{x}}$, we can write $y = x^{-\frac{1}{3}}$ first

then $\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}}$

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H2. At a maximum or minimum, the tangent to the curve must be horizontal and the gradient or rate of change is zero. At all other points on the curve (apart from some points of inflexion) the gradient will be non-zero.

Hence, to locate the extreme points of a function, we must find where the gradient is zero.

i.e. $\frac{dy}{dx} = 0$ for $y = f(x)$

First, obtain the expression for $\frac{dy}{dx}$ by differentiation.

Put $\frac{dy}{dx} = 0$ and solve the resulting equation .

The roots of the equation (if any) are the locations of any turning points.

H3. For any given straight line, the gradient is always the same. So, between any two points P and Q, however close or distant, the gradient will be measured as the same value. However, with a curve the gradient at every point is different.

One way of assigning a magnitude to the gradient of a curve is to use a numerical approximation.

To do this choose two points P and Q on the curve, draw the straight line PQ and measure its gradient. (See figure below).

Notice that this represents the average gradient of the curve between P and Q and is a good estimate of its value at the mid-point between P and Q. In general, the closer P and Q are together, the better the approximation.

Hence to find the rate of change $y = 2x^2$ at $x = 2$, take P at $x = 1.9$ and Q at $x = 2.1$

then $y_Q = 2(2.1)^2 = 8.82$ rate of change $= \frac{8.82 - 7.22}{2.1 - 1.9} = \frac{1.60}{0.2} = 8.00$

$y_P = 2(1.9)^2 = 7.22$

Similarly for $x = 3$ take (say), P at $x = 2.9$ and Q at $x = 3.1$

then $y_Q = 2(3.1)^2 = 19.2$ rate of change $= \frac{19.22 - 16.82}{3.1 - 2.9} = \frac{2.4}{0.2} = 12.00$

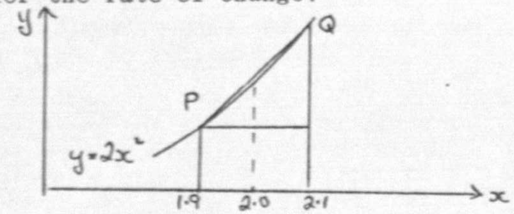
$y_P = 2.(2.9)^2 = 16.82$

Notice that it is necessary to carry out this calculation at every point where the gradient is required.

There are, therefore, two good reasons to find a general formula:

(i) the method involves some level of approximation

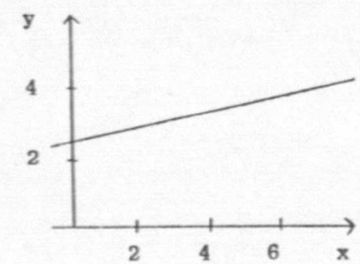
(ii) a lot of effort will be saved if we can write down <u>one</u> expression for the rate of change.



Rates of Change, Gradients and Differentiation

First Progress Test — **Test No. 1**

1. A function which is linear is such that when $x = 4$, $y = 9$ and when $x = 1$ $y = 2$. What is its rate of change?

2. 

In the figure the rate of change of the y with respect to x is:

(a) large positive (b) large negative
(c) small positive (d) small negative

which is the correct answer?

3. A linear function is such that when $x = 1$, $y = 8$ and when $x = 4$, $y = 1$. What is its rate of change?

4. If the gradient of a linear graph is 3 at the point where $x = 2$, what is the gradient (a) when $x = 5$ (b) when $x = -5$?

5. Find, by the shortest method you know, the gradient of the following functions, when x is the independent variable:

(i)	$y = x + 10$	(v)	$2y = 3 + x$
(ii)	$y = 5 - 3x$	(vi)	$x + y + 9 = 0$
(iii)	$y = x$	(vii)	$2x + 3y - 4 = 0$
(iv)	$y = 1\frac{1}{2} - \frac{1}{2}x$	(viii)	$4y - x = 3$

6. <u>Sketch</u> on the same axes, the graphs of (a) $y = 4 - x$ (b) $y = 2 - 3x$. Which has the steeper gradient?

Rates of change, Gradients and Differentiation

Second Progress Test — Test No. 1

1. Use a numerical technique to find the rates of change of the following functions at the points indicated

(i) $y = 2 + x - x^2$ at the point where $x = 2$
(ii) $y = 1 + x^3$ at the point where $x = 2$
(iii) $y = 2x^2 - 4$ at the point where $x = 0$ and at the point where $x = 2$

2. Find algebraically, from first principles, the expression for the rate of change of the function $y = 4x^2 + x$.

3. Find expressions for the rates of change of the following functions by applying the rule. (y is the dependent, x the independent variable)

(i) $y = 4 - 3x + 2x^2$
(ii) $y = x - \frac{1}{x}$
(iii) $y = 2x^3 - 3x^2 + 4x - 5$
(iv) $y = 2x^{-1} + \frac{1}{x^2}$
(v) $y = \sqrt{x}$
(vi) $y = \frac{1}{2\sqrt{x}}$
(vii) $y = \frac{1}{\sqrt{x}} + \frac{2}{x} - \frac{3}{x^2}$
(viii) $y = x^{1/4} + 2x^{3/4} - x^{-1/4}$

Rates of Change, Gradients and Differentiation

Third Progress Test — Test No. 1

1. Locate and determine the nature of any turning points in the following. Evaluate y at any points that you find:

(i) $y = x^2 + 2x - 3$
(ii) $y = x^3 + 2x^2 - 4x - 8$
(iii) $y = x^4 - 8x^2 + 16$

Rates of Change, Gradients and Differentiation

Fourth Progress Test — **Test No. 1**

1. Differentiate the following:

(i) $y = x + \frac{2}{x} - 3\sqrt{x}$

(ii) $s = t^3 - 3t^2 + 5t - 4$

(iii) $y = (x^2 + 3)(2x^2 - 5)$

(iv) $w = (z^2 + 2z + 1)(2z^2 - 3z - 5)$

(v) $y = \frac{2x + 3}{5x - 4}$

(vi) $s = \frac{t^2 + 2t}{2t^2 + 1}$

(vii) $y = \sqrt{x^2 + 3x + 5}$

(viii) $w = (z^2 + 5z + 1)^4$

(ix) $y = (2x + 8)\sqrt{x^2 + 3x}$

Rates of Change, Gradients and Differentiation

Fifth Progress Test — **Test No. 1**

1. Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the following cases

(a) $y = 4x^3 - 6x^2 + 12x$ (b) $y = \frac{3}{x} + \sqrt{x}$

2. Find expressions for the derivatives with respect to x of:

(a) e^x (d) $\ln x$
(b) e^{-x} (e) $\ln(2x + 1)$
(c) e^{2x}

3. Find $\frac{dy}{dx}$ for the following implicit relations

(a) $y^2 + x^2 = 6$

(b) $2x^2 + 3y^2 - 3x + 2y = 0$

(c) $\frac{1}{x} + \frac{1}{y} = 3x$

(d) $x^2 + 5xy + y^2 = 12$

4. Given that $y = xe^x$, prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - e^x = 0$

Problems Involving Calculus — B A Economics

1. The output Q varies as a function of input L given by $Q = 12 + 15L - L^2$. Find expression for both the average product and the marginal product. Find the value of L which maximises output and calculate the average and marginal products at this point.

2. The expression $Q = 150F + 9F^2 - F^3$ shows the variation of output Q with the level of input F. Find values of F & Q when the output is maximum. Calculate the values of the marginal product and the average product at this point.

3. The cost of operating a factory C is given in terms of the output (in Kgs of product) by $C = x^2 + 30x + 500$. Find the value of x for which the average cost is a minimum. When is the marginal cost equal to the average cost? Sketch the average and marginal cost function on the same diagram. [Take $0 \leqslant x \leqslant 80$].

4. (a) Write down the algebraic conditions for maximum profit (net revenue) given that the total revenue is $R = f_1(Q)$ and the cost function is $C = f_2(Q)$.

   (b) Find the best possible output of a firm whose total revenue and total cost are given by

$$R = 40Q - Q^2$$

$$C = 50 + 6Q$$

   Sketch these two functions.

5. A factory produces x tonnes of chemical fertilizer at a total cost given by $£C = x^3 - 30x^2 + 500x + 3000$

   The fertilizer is sold at £350 per tonne. What output will maximise the net revenue (profit).

   Explain what effect a change in fixed costs will have on the factory's optimum output.

6. A person can employ unskilled workers at \$60 per week and skilled workers at \$100 per week. If he can spend \$6000 per week on wages, express the wage constraint as an equation involving the number of unskilled workers (y) and the number of skilled workers (x) employed.

   The total output (Q) is a function of employees given by $Q = xy$. Three levels of output are proposed, Q = 500, 1000 and 1500. Find which value of Q makes the constraint line a tangent to the curve $y = \frac{Q}{x}$.

   Hence, or otherwise, find the maximum output which is possible with the given constraint.

   Sketch the three curves.

7. The total revenue (R) and the total costs (C) of a firm with output x (in 100's of units) are related by the function

$$R = 28x - x^2$$

$$C = x^3 - 4x^2 + 13x + 35$$

   (i) What output maximises total revenue?

   (ii) What output maximises profit?

   (iii) When is the marginal cost at a minimum?

   Sketch the total cost and total revenue functions for $0 \leqslant x \leqslant 15$.

Appendix 9

Questionnaires and Summary of Returns

Replies to Questionnaire I   B.A. Economics I   (November)

Figures for 1977 (N = 24), 1978 (N = 28) and 1979 (N = 18) are given in that order.

1. How important do you think a sound knowledge of mathematics is with respect to your degree? (Tick one box.)

very little	little	neutral	some importance	very important
0,0,1	0,0,0	0,0,0	12,8,4	12,20,13

2. Write below one reason why you think mathematics is important or unimportant on this course.

Typical response:

It was obviously important from the approach taken in their main subjects.

3. Do you feel concerned about your ability to make progress in mathematics?

concerned	no feelings	not concerned
20,24,15	4,2,1	0,2,1

4. Do you find mathematics a frustrating subject to study?

very frustrating	frustrating	no feelings	satisfying	very satisfying
2,2,2	7,6,4	1,3,3	11,13,7	3,2,2

5. Here are some reasons why people find difficulty at certain times. Tick not more than two reasons that you feel apply to you:

Lack of pre-requisite knowledge	9,12,10
Inability to cope with the logical structure	5, 0, 1
There is only one right answer and many ways of producing errors	4, 6, 6
Mathematics is too abstract and does not deal with real things	2, 2, 2
I seem to understand at the time but discover difficulties in tests	13,14, 6
It is not clear what has to be done or what has to be remembered	1, 4, 1
I get so worried about mathematics that I go to pieces	0, 3, 1

6. Is there enough revision before each progress test?

very little	little	about right	much	too much
0,0,2	6,5,4	18,19,10	0,2,1	0,0,0

7. Do the sections setting out the objectives (marked ■) give a good idea of what to expect in the tests?

Yes	No
18,24,14	2,3,3

8. Would it be better to place the answers to the exercises on the following pages rather than in boxes following each set of exercises?

Yes	No
13,13,10	14,15,8

9. Here are some specific topics from the first part of the course. Indicate how you found each one:

Recognition of linear, quadratic and cubic forms in given functions:

very hard	hard	average	easy	very easy
0,0,1	3,2,0	14,8,5	6,15,6	1,1,6

Graphical solution of cubic equations:

0,0,1	7,2,1	13,17,7	4, 5,5	0,2,4

Graphical solution of simultaneous equations:

0,0,0	3,3,1	14,15,5	7, 5,7	0,3,5

Naperian logarithms:

4,3,2	14,10,7	5,10,4	1, 3,5	0,1,0

Change of base of logarithms:

6,3,3	11,8,4	5,11,4	1, 4,6	0,0,1

Use of Naperian logarithms:

3,2,3	11,5,5	6,10,3	3, 7,5	0,1,2

Interest rates, geometric series etc:

0,1,1	6,3,2	12,13,5	6, 7,9	0,2,1

10. How do you find the discussion sessions before attempting the solution of problems?

very unhelpful	unhelpful	neutral	helpful	very helpful
1,0,2	1,1,1	3,3,4	15,13,6	4,8,5

11. If you cannot understand something in one of the problems, do you:

Ask a question of the lecturer	11, 6,2
Wait and have another think about it	6,11,6
Ask a colleague for advice	9,10,2
Look in a text-book	2, 0,8
None of these	0, 0,0

12. How do you feel about your progress on this course compared with that if it consisted of conventional lectures?

less progress	about the same	more progress
4,5,2	7,9,5	12,12,11

<u>Replies to Questionnaire II</u> <u>B.A. Economics I</u> (<u>February</u>)

Figures for 1979 (N = 26) and 1980 (N = 19) are given in that order.

1. How have you found the study of mathematics this last half-year?

very easy	easy	average	hard	very hard
1,1	1,3	13,10	10,4	1,1

2. What do you see as the main function of the lectures?

(Typical responses:)

To portray the basic ideas.

Give the central idea of each topic to help self-paced study.

To introduce a new or strange topic and capture the enthusiasm of students.

3. How do you rate the lectures in mathematics?

little use	some use	useful	very useful
2,1	6,1	12,11	6,5

4. Do the lectures fit in with the self-paced study?

very little	little	about right	well	very well
0,0	5,1	12,13	4,2	3,3

5. Is there any way the lectures could be improved?

(Typical responses:)

Take more time over economic applications.

Do all possible examples in lectures.

Make more interesting by advancing quicker through the principles.

By moving at a slower pace with more examples.

6. How do you rate self-paced study?

very poor	poor	average	good	very good
0,2	1,0	7,7	13,8	5,2

7. Is there any way it could be improved?

(Typical responses:)

The information should be initially covered in the lectures.

More exercises.

By adding the use of very simple economic applications.

8. Give one reason why you think performances in progress tests and assessment tests might be different.

(Typical responses:)

Progress tests are while the work is still fresh.

Time is more pressing.

A greater range of study is needed for assessment tests than for progress tests.

9. Do the time-tables for each block of work help you plan your studies?

little use	some use	very useful
7,6	6,10	2,3

10. Do you experience any difficulty in keeping up with the work in self-paced study?

very difficult	difficult	average	easy	very easy
1,1	8,1	12,9	4,5	1,3

11. How do you find applying the mathematics to economic examples?

very difficult	difficult	average	easy	very easy
6,2	11,10	8,5	1,1	0,0

12. Name what you think is the most difficult part of problem-solving.

(Typical responses:)

There is usually a twist before the problem can be solved.

Relating the words to the mathematical theory.

How to start off initially.

13. Does working in groups help in problem-solving?

very little	little	average	useful	very useful
3,2	3,7	5,2	12,6	3,0

14. How do you rate the assessment tests? (<u>Not</u> progress tests.)

very hard	hard	average	easy	very easy
1,1	8,10	14,6	2,2	1,0

15. What might (if necessary) improve your performance in assessment tests?

(Typical responses:)

I have to revise more often.

More exercises on problem-solving.

A clearer understanding of maths/economics relationship.

16. Is there the right amount of revision on the course?

very little	little	about right	much	too much
6,5	9,8	9,6	1,0	0,0

17. How do you find retaining your knowledge for use in tests?

very difficult	difficult	average	easy	very easy
4,3	8,4	12,8	1,4	1,0

18. In matrix algebra, how did you find having to discover your own rules?

very difficult	difficult	average	easy	very easy
1,0	7,8	10,8	7,3	1,0

19. Do you like the idea?

Yes	No
9,9	16,6

20. Did you find discovering your own rules rewarding experience?

little	some	very rewarding
6,1	20,10	1,6

21. Rate the following topics for the difficulty you encountered.

Implicit differentiation:

very hard	hard	average	easy	very easy
8,3	4,8	10,1	4,6	0,1

Problems involving max/min:

1,5	11,7	10,3	4,2	0,2

Matrix multiplication:

1,1	1,1	9,4	6,11	9,1

Cofactors of a matrix:

0,0	4,2	14,10	2,6	6,1

Determinants:

0,0	6,3	13,6	3,9	6,1

Solving equations by Cramer's Rule:

1,0	8,2	9,10	2,7	4,0

22. Are the number of lectures adequate?

very few	few	about right	too many	far too many
1,1	8,3	17,15	0,0	0,0

23. Is enough time devoted to self-paced study?

very little	little	about right	too much	far too much
1,0	5,3	18,13	1,1	1,2

24. Is enough time devoted to problem-solving?

very little	little	about right	too much	far too much
2,3	15,10	8,6	1,0	0,0

25. How do you rate your progress in mathematics?

very little	little	average	good	very good
1,1	5,2	14,11	5,4	1,1

Replies to Questionnaire III(a) B.A. Economics I (May)

This questionnaire was used in 1978 only (N = 11)

1. How have you found studying mathematics this last half-year?

very easy	easy	average	hard	very hard
0	1	4	5	1

2. Do you think the lectures are a useful part of the course?

little use	some use	very useful
0	1	10

3. How do you rate the time devoted to lectures?

need more	about right	need less
1	10	0

4. How do you rate the time devoted to self-paced study?

need more	about right	need less
2	9	0

5. How do you rate the time devoted to solving problems?

need more	about right	need less
4	7	0

6. Do you think more guidance and examples of problem-solving would help?

no advantage	some advantage	big advantage
0	2	9

7. When working from study notes, how do you rate your speed?

very slow	slow	about right	fast	very fast
0	3	8	0	0

8. How do you rate the assessment tests? (<u>Not</u> the progress tests.)

very hard	hard	average	easy	very easy
1	6	4	0	0

9. Give one suggestion that you think might help your performance in the assessment tests.

(Typical responses:)

Give more difficult questions before tests.

Less nerves.

More practice in problem-solving.

10. Do you discuss your mathematical problems with your colleagues?

never	rarely	sometimes	often
0	2	1	5

11. Do you still find gaps in your knowledge? Tick any of the following that present difficulty:

Algebraic manipulation	1
Solution of quadratic equations	1
Use of logarithms	8
Indices	1
Percentages	0

12. How difficult do you find applying mathematics to the solution of problems?

very difficult	difficult	average	easy	very easy
2	3	4	2	0

13. Here are some recently studied topics. Rate how difficult you found each one:

	very difficult	difficult	average	easy	very easy
Finding the inverse of a matrix:	0	3	6	2	0
Solving equations by matrix methods:	0	5	3	3	0
Partial differentiation:	1	2	4	4	0
Finding max/min in two variables:	0	3	6	1	0
Constrained optimisation (Lagrange-multiplier):	2	2	6	1	0

14. How do you rate your own progress on this course?

very little	little	average	good	very good
0	2	7	2	0

15. How do you feel about your ability in mathematics compared with the beginning of the year?

far less confident	less confident	about the same	more confident	much more confident
0	1	2	8	0

Replies to Questionnaire III(b) B.A. Economics I (May)

Figures for 1979 (N = 27) and 1980 (N = 17) given in that order.

1. How have you found the study of mathematics these last three months?

very easy	easy	average	hard	very hard
0,0	3,4	10,9	10,3	4,1

2. How do you feel about your progress in mathematics?

very little	little	average	good	very good
0,0	7,3	9,10	10,4	1,0

3. How do you rate the lectures in mathematics?

little use	some use	useful	very useful
2,1	6,2	13,8	6,6

4. How do you rate the self-paced study?

very poor	poor	average	good	very good
0,0	2,0	9,3	11,8	5,6

5. How do you find keeping up with the work in self-paced study?

very difficult	difficult	average	easy	very easy
1,0	5,0	17,10	4,5	1,1

6. How do the lectures fit with the self-paced study?

very little	little	about right	well	very well
0,0	4,0	15,10	8,7	0,0

7. Do the time-tables for each block of work help to plan your studies?

little use	some use	very useful
8,3	17,9	2,5

8. How difficult have you found problem-solving?

very difficult	difficult	average	easy	very easy
4,3	14,9	7,2	2,3	0,0

9. How helpful is working in groups with problem-solving?

very little	little	average	useful	very useful
0,0	10,8	3,4	12,5	2,0

10. Here are a list of difficulties encountered in problem-solving. Tick not more than two you feel apply to you:

Knowing how to begin	7,5
Translating economic terms into mathematical expressions	13,9
Remembering the exact mathematical method required	10,2
Deciding which method should be used	3,3
Problems seem to bear no relation to examples done in the notes	10,5
Difficulty in understanding the written question	5,4

11. Is the time allocated to problem-solving about right?

very little	little	about right	too much	far too much
0,1	10,8	17,7	0,0	0,1

12. Is the time allocated to self-paced study about right?

very little	little	about right	too much	far too much
0,0	2,0	22,13	2,3	0,1

13. Is the time allocated to lectures about right?

very little	little	about right	too much	far too much
1,1	5,4	20,11	0,0	1,0

14. Here are some recently studied topics. Rate how difficult you found each one:

Finding the inverse of a matrix:

very hard	hard	average	easy	very easy
1,1	6,1	11,6	7,6	1,1

Solving problems using matrix methods:

1,0	9,8	13,5	2,2	2,0

Partial differentiation:

0,0	2,4	11,6	10,5	4,0

Finding max/min in two variables:

0,1	10,5	12,7	4,1	0,1

Constrained Optimisation (Lagrange-multiplier):

4,1	11,5	9,6	2,0	1,1

15. Is there enough revision in the mathematics course?

very little	little	about right	too much	far too much
3,2	12,10	11,4	1,0	0,0

16. How do you rate your ability in mathematics now compared with early in the year?

much worse	worse	about the same	better	much better
0,0	2,0	9,4	11,10	5,2

Appendix 10

(i) The Prepared Questions for Student Interviews in November, February and May - Service Course in Mathematics for Economists

(The list of questions remained substantially the same throughout the year; students were interviewed once only. The major difference lay in those questions concerning the most recent topics to be studied.)

How important is mathematics on your present course? What reasons are there for studying it? Would you have studied mathematics as an option? Are there any parts of maths you would study out of interest?

What were your feelings about mathematics at school? Some students say maths is either rewarding or frustrating; what do you think? Does the examination at the end of the year pose any particular kind of threat? Is there any way in which mathematics appears different from other subjects?

(In November only) Can you point to any occasions when you experienced a lack of pre-requisite knowledge? (If necessary, suggest algebra, logarithms and solving equations.)

What do you see as the function of the lectures on this course? Should the number of lectures be changed? In what way? Can anything be done to make lectures and self-paced study fit together better? Would you like to see no lectures at all?

How do you rate self-paced study as a way of learning? Are the notes easy to follow? Can you point to any disadvantages of this way of studying? How much of a problem is falling behind in self-paced study? Why do you think the performances in progress tests and assessment tests are sometimes rather different? Does the time-table you are given help you to plan your work?

(The next section contained questions on specific parts of the syllabus e.g.)

Did you, in general, find calculus difficult or easy? Was enough time devoted to this subject?

How do you rate the following topics:

Use of product and quotient rules,

The function of a function rule,

Find maximum and minimum values,

Implicit differentiation?

How did you cope with the process of formulating your own rules? How well do you think this will help you to remember the rules?

How do you rate the difficulty of problem-solving? Can you point to particular advantages or disadvantages of working in groups? What do you find is the most difficult part of problem-solving? (If necessary, prompt by mentioning mathematical formulation.)

How do you rate your progress in mathematics? Can you suggest an alternative study method which might have been better? If you had a choice of study methods, which would you choose?

Appendix 10

(ii) Verbatim Transcript of an Interview with Four Economics Students in February 1979

Mathematical backgrounds:

(i) O-level, studied A-level but without success in examination.

(ii) O-level followed by Additional Mathematics in the fifth form.

(iii) O-level Cambridge papers, no mathematical studies since.

(iv) O-level Cambridge papers, no mathematical studies since.

Is mathematics important on your present course?

(i) Quite important. Have to know how to manipulate the figures properly. You must be adequate at mathematics. (Is the importance apparent in your other studies?) Yes.

(ii) I do not think it has been that apparent yet, but I see it becoming important.

(i) Something like financial analysis or accounting in the second year, then I think it will be very important.

(iii) I think it is important, particularly next year. At times they do not use mathematical language in other subjects but I think it will be used in the future.

(iv) Important in other subjects. For instance in the mid-sessional test in micro economics, we did quantity of supply which we had done in maths.

Would you have studied mathematics as an option?

(i) I do not know. I did not do maths at A-level because I wanted to, I did it because I had to and I do not know.

(<u>Have subsequent studies decided for you</u>?) Knowing what I know now, I probably would but a lot would depend on other choices.

(ii) If it is needed I would have done. (<u>What do you feel at the moment</u>?) We seem to spend a lot of time on it, it seems quite important.

(iii) Yes, I think so.

(iv) I do not know really, because I have not done maths for four or five years. Now I have taken it up again I think I would do it as an option.

<u>What do you think of mathematics as a subject, is it worth studying out of interest</u>?

(i) Sometimes, I do not know as a pure subject. I tend to be biased towards the practical, for example, I was interested in biology but I chose economics because I can do a job at the end of economics.

(ii) It can get tedious, revising when there is so much to do. It is quite interesting really.

(iii) It depends, what is the result? If you can cope it is interesting but if you cannot, then it is a very tedious one.

(iv) The subject is interesting, I do not find it distasteful.

<u>How did you feel about mathematics at school</u>?

(i) I did not particularly want to do maths. It was a choice between physics and maths, so I took maths. I liked doing maths at O-level. I did modern maths and had to change to pure maths. It was a big change, the teacher knew I did not want to do maths so he was not too bothered about me. I had a bit of a struggle for a while but now it is all there.

(ii) I was with a lot of people who really worked hard all the time and they inspired me. It really does help when you try and do all the exercises. Yes, I found it reasonably interesting.

(iii) I used to like maths best of all the subjects. I found it easy to study mathematics.

(iv) I liked maths but found it difficult at times. I coped with it, I quite liked it.

<u>Is mathematics rewarding or frustrating</u>?

(iv) It is rewarding when you can get the right answers. This happens reasonably often.

(iii) As (name) says, it is rewarding when you get the right answers, otherwise (pause). I think it is more rewarding than frustrating.

(ii) When I get something I cannot understand it is really annoying but when you understand it and have fathomed it out it is really good.

(i) I find that when you get to the end of a long problem and find you have got the right answer it is really rewarding, or when you walk out of an exam and know you could do the questions. (<u>Is it sometimes frustrating</u>?)

(i) Yes.

(ii) I find that. It should have been easy today, but was not. (They had just taken a test in statistics.)

<u>Does the final examination at the end of the year pose any kind of threat</u>?

(i) Yes, I do not like the idea of an examination at the end of the year. Right up to the moment I have sat it, it is there at the back of my mind, what if I fail?

(ii) The effect it has is to make me try and work harder.

(Does it worry you?) No, not at this stage it does not really worry me.

(iii) I think I am conscious of passing the exam but I do not see it as a threat, not yet.

(iv) It is always at the back of my mind that I must do well in maths. There is no particular stress produced by this.

What do you see as the function of the lectures?

(i) They seem to supplement the self-pacing notes. Very well I think. So far most of the stuff I have done before so it is fairly easy anyway.

(ii) Wrongly, I see them as the main part of the course. I ought to read the notes more and rely more on those. I do find it a help going through on the board.

(iii) I think you have got to explain the difficult parts of the notes.

(iv) I agree with (name), you can go through most things from the notes but the more difficult things I would like to see in a lecture. (Are the lectures doing their job?) Yes. (There were general signs of agreement with this.)

Should the number of lectures be changed?

(i) I do not know.

(ii) Spaced out more I think. On Tuesday morning it gets a bit monotonous at the end (of the class period). It is probably because it is the first thing coming back after the weekend.

(iii) I think they are sufficient.

(iv) I think it is about right but sometimes I get much further ahead in the notes. I go through the (progress) tests and then you come and go over what I have done before.

<u>Can anything be done to make the lectures more in step</u>?

(i) Possibly for the first three weeks, say, of each section of work, we had a one hour lecture and then self-pacing instead of missing say, the second week completely. (<u>Would the lectures always be ahead of the notes</u>?) I think they would tend to keep up. I find I have virtually finished and we only seem to be about half-way in the lectures.

(ii) I tend to stay with the lectures and I am a bit behind, if anything, with the progress tests. I think it is alright really. I am a bit hesitant with the progress tests. I keep thinking I will do badly in this when I take it and I put it off and leave it.

(iii) It is better if you do the lectures first, all the stuff we are going to cover and then start on the self-pacing notes.

(iv) I find sometimes the lectures are a bit behind. One thing is, after the lecture, you tell us up to what page you should go through the notes then we could have the next lecture and then we could go through the notes again. The notes and the lectures would be up to date.

<u>Let us talk about the self-pacing in calculus. Was it a reasonable way of studying</u>?

(i) Yes. With the calculus I had done it all before and could remember it so I virtually went through it, did the examples and sat the progress tests. I like it, a good idea. I find if I have half-an-hour to spare I sit and go through the self-pacing notes, otherwise I might watch the television.

(ii) They are well set out, they are very useful. Very reasonable way to study.

(iii) I think so, I like working from the notes.

(iv) Yes I like it (as a way of studying).

(All four interviewees had not met self-paced study before.)

Would you like to see more self-paced study?

(i) I think so for mathematics.

(ii) You could not have it with other subjects, really, arts and so forth.

Are there any snags with self-paced study?

(i) Only with the matrix algebra. I like a rule to be stated before me and not to have to work it out and not be sure of it until someone says it is right. I have a slight lack of confidence in myself, if I have worked it out.

(ii) Not really. Probably the lectures, if when we were doing products and output from a firm I did not think I would have understood that just from the notes.

(iii) No, I do not think so.

(iv) No, only in matrices, where I like to be given the formulae. I am not very confident myself sometimes.

Is there a problem with getting behind in self-paced study?

(i) I do not think there is provided you are interested and want to work. If you do not want to work you have a good excuse not to do any but if you want to work you have more opportunities to catch up.

(ii) Yes, I think there is extra chance to do more work. They are always there. (Can you plan your work?) Yes, I find so.

(iii) I think I heard someone complaining. They say they are a lot behind but I myself do not find any problem.

(iv) I think with most of them it is because they have not done enough work.

<u>In calculus, why do you think the performance in progress and assessment tests were different</u>?

(i) They were not using the notes properly, looking at the notes when doing the tests.

(ii) There is a great danger of this.

(i) If you use the tests properly they work well. They also give you some confidence because if you have passed the progress test you know you can do that section.

(iii) The difference might be the tension. You take the progress tests in class, but in the assessment tests you have the tension and the time difficulty, you might be careless.

(iv) Progress tests work reasonably well. I know some people will ask advice from someone who has done the tests.

<u>Was there sufficient time for calculus, were some people behind</u>?

(i) There was no problem because I had done it before.

(ii) It was not new to me although some parts were. I was short of time because I had not felt up to doing the progress tests. There was enough time really.

(iii) No, I found the time quite reasonable.

(iv) I was up to date by the time we finished. Just about the right time.

(ii) It might have helped if during a lecture you could have said you should now be up to such and such a progress test. (<u>Does the time-table help</u>?) Yes it does. (There was general agreement here.)

Was calculus a hard or easy subject?

(iv) Difficult for me particularly as I had not done maths for four years. Quite difficult.

(iii) I cannot say it was easy, but not very hard. About half-way.

How did you find the following topics?

Numerical technique for finding the rate of change?

(i) Easy.

(ii) Tedious.

(iii) Easy.

(iv) Tedious.

Algebraic technique for finding the rate of change?

(i) Alright.

(ii) The idea was alright, but I had to be careful with the signs in the algebra.

(iii) Alright.

(iv) Alright.

Product and quotient rules?

(i) Yes (these were no problem).

(ii) As long as you remembered the rules.

(iii) I was happy with these.

(iv) As long as you could remember the formulae.

Function of a function?

(i) This was no problem.

(ii) I had no problem.

(iii) I did reasonably.

(iv) Not very sure, but if I can do it carefully and slowly I might get it in the end.

Finding maxima and minima?

(i) This was no problem.

(ii) It was fairly straightforward.

(iii) It was not too bad.

(iv) No, it was not too bad.

Implicit Differentiation?

(i) It was no problem, I had done it before.

(ii) I was getting mixed up with it really, remembering where to put $^{dy}/dx$. I should not have found it difficult, if I had worked a bit harder maybe.

(iii) It was alright.

(iv) Yes, it was alright.

Problem solving, how did you cope with this?

(i) I found it not much of a problem. I can translate it straight into the mathematics.

(ii) I was a bit scared off at the start when I saw all the variations in the questions, but when you realise there are only a few laws really, it is alright.

(iii) I was alright, I found it quite straight-forward.

(iv) I found it quite difficult at first, which one to use. With a bit more practice it should be alright. (Are you improving with experience?) Yes.

How useful is working in groups?

(i) Useful, because sometimes you just cannot see one particular piece and someone else can see it and they can help you. (Do people communicate in groups?) It depends, in different groups. One group I sat there, having done the problem for about ten minutes before someone asked me a question. They did not communicate. Another group they communicated so much you could not do the problem. Depends who you are sitting with.

(ii) I think if that was the case (could not see the point) you would ask someone else anyway. I do not really see much point in working in groups. You either ask a friend or something. (Do not some people need encouragement?) I do yes, but I would ask somebody.

(iii) I do not really find it useful. I do ask people but working in groups in class I do not think is a good idea.

(iv) It is not very useful in class because if we do not know something we always ask someone. It would not be very useful in a lecture.

(ii) I think you hope when working in a group that someone will volunteer to help you and I do not think that is the case really unless they know you.

(The interviewees pointed out that group work was unfamiliar to them.)

(i) You seem to know the people in your particular tutorial group, you do not know the other people particularly well. (For principal subjects the class is divided into tutorial groups of 12.)

Do you find studying matrix algebra easy or hard?

(i) Easy, it is not a new subject, I did matrices at O-level and determinants at A-level. It seems in rather more detail than in A-level, we used a different method before.

(ii) It is a new subject to me, I have never done it before. I am finding it quite difficult, some of the rules. When you are multiplying out, where to put each element.

(iii) It is not really difficult but a bit complicated especially the product rule, you sometimes get things in the wrong place.

(iv) I find it easy except that it can be a bit confusing at times.

Forming the rules for yourself, how do you rate this approach?

(i) I find it disturbing. I do not particularly like it. I like someone to say that this is it or that is wrong. (Is it not satisfying?) Not really. It does not really help in remembering. If I found it for myself, then it says a bit further on "The rule is so and so", then I think I might remember it better because I know that I am right. There is always a nagging doubt.

(ii) I suppose I do find it satisfying. I can think of two instances and they seemed alright. (What about the multiplication rule?) Yes I saw that.

(iii) I do not find puzzling out the rules satisfying or rewarding. I do not like it this way.

(iv) I find it disturbing having to work out the rules. When I get the rules in the end I still like someone to tell me it is right. The trouble is I am not sure it is right, that is why (not particularly rewarding). It helps when one or two examples work out correctly but I still want the rules given to me.

(ii) There is enough scope for being frustrated when you know the rules without having to find them out.

How do you rate your progress in mathematics?

(i) I do not know how I should have done. I seem to have been getting good marks so I feel happy.

(ii) Not so well as I should have liked but I am working at it. I shall do the progress tests this time. It is a question of application.

(iii) Just about as I would have liked. I cannot really think of any way of improving it.

(iv) I do not know really, sometimes I find it easy, sometimes difficult. I think if you do lots of work and do the progress tests, then it will be alright.

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