



PAPER

Characterising small objects in the regime between the eddy current model and wave propagation

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Abstract

Being able to characterise objects at low frequencies, but in situations where the modelling error in the eddy current approximation of the Maxwell system becomes large, is important for improving current metal detection technologies. Importantly, the modelling error becomes large as the frequency increases, but the accuracy of the eddy current model also depends on the object topology and on its materials, with the error being much larger for certain geometries compared to others of the same size and materials. Additionally, the eddy current model breaks down at much smaller frequencies for highly magnetic conducting materials compared to non-permeable objects (with similar conductivities, sizes and shapes) and, hence, characterising small magnetic objects made of permeable materials using the eddy current at typical frequencies of operation for a metal detector is not always possible. To address this, we derive a new asymptotic expansion for permeable highly conducting objects that is valid for small objects and holds not only for frequencies where the eddy current model is valid but also for situations where the eddy current modelling error becomes large and applying the eddy approximation would be invalid. The leading-order term we derive leads to new forms of object characterisations in terms of polarizability tensor object descriptions where the coefficients can be obtained from solving vectorial transmission problems. We expect these new characterisations to be important when considering objects at greater stand-off distance from the coils, which is important for safety critical applications, such as the identification of landmines, unexploded ordnance and concealed weapons. We also expect our results to be important when characterising artefacts of archaeological and forensic significance at greater depths than the eddy current model allows and to have further applications parking sensors and improving the detection of hidden, out-of-sight, metallic objects.

1. Introduction

Improving the characterisation of metal objects from electromagnetic field perturbations, when their location is not in the immediate proximity of the exciting and sensing coils, is important for a range of applications. For example, while traditional walk-through metal detectors are used in transport hubs, court rooms, museums, galleries and at concerts and public events for identifying potential threat objects in close proximity (typically much less than a metre) of the coils, there is also considerable interest from security and police forces in being able to identify potential terrorist threat objects, including guns, knives and improvised weapons, at greater stand-off distances (in the order of several metres). The ability to characterise objects at greater stand-off distances could also bring benefits to other metal detection applications including improving safety for anti-personal landmine disposal teams, by identifying unexploded ordnance and landmines at greater distances, improving the detection of smaller, and deeper buried artefacts of archaeological significance and improving the detection of items in forensic



searches, which are beyond the reach of current metal detection technologies. There are further applications in parking sensors and in the detection of out-of-vision objects, such as cyclists, for large vehicles, such as lorries and buses.

The metal detection problem is usually described by the eddy current approximation of the Maxwell system, since the conductivities of the objects of interest are high and the frequencies of approximation are low. The eddy current approximation neglects the displacement currents in the Maxwell system, which is responsible for wave propagation and is often applied if the wavelength is large compared to the object size. Ammari, Buffa and Nédélec [1] show, depending on the topology of the object and its materials, that the eddy current approximation is either first- or second-order accurate when considering an asymptotic expansion of the electric and magnetic fields as the excitation frequency tends to zero. In practice, this means that the eddy current model will break down at a lower frequency for a horseshoe-shaped conductor compared to a similar sized solid sphere made of the same material. Since the wavelength needs to be large compared to the object size, this also means that the eddy current model breaks down at a lower frequency compared to non-magnetic objects of the same shape, size and conductivity. The modelling error in the eddy current approximation has been considered by Schmidt, Hiptmair and Sterz [12] who find that the approximation is accurate provided that

$$C_1 \epsilon_* \mu_* \omega^2 \alpha^2 \ll 1, \quad C_2 \frac{\omega \epsilon_*}{\sigma_*} \ll 1, \quad (1.1)$$

where ϵ_* , μ_* and σ_* denote the conducting object's permittivity, permeability and conductivity, respectively, ω is the angular frequency and α is a measure of the object size. The constants C_1 and C_2 depend on the topology of the object and, if they are of a moderate size, (1.1) reduces approximately down to the condition that the wavelength needs to be large compared to the object size (quasi-static limit) and $\epsilon_* \omega \ll \sigma_*$ (conductivities are large), respectively. They also provide a procedure for estimating C_1 and C_2 .

For the eddy current problem, Ammari, Chen, Chen, Volkov and Garnier [2] have derived the leading-order term in an asymptotic expansion for the perturbation of the magnetic field caused by the presence of a permeable conducting object as its size tends to zero for the eddy current problem. There are conditions imposed on the material parameters, which allows this result to be applied to the case of constant σ_* and μ_* as $\alpha \rightarrow 0$. We have shown, [5], that the leading-order term they have been derived can be simplified and provides an object characterisation, in terms of a rank 2 magnetic polarizability tensor (MPT) that depends on the object shape, object size, its material parameters and the frequency of excitation but is independent of its position. The object location is separated from the other object characterisation information with the background field being evaluated at the object position. In [8], we have extended the leading-order term obtained by Ammari et al. in [2] to a complete expansion of the perturbed magnetic field and introduced the concept of generalised MPTs, which provide improved object characterisation. We have derived alternative forms of the MPT and considered spectral behaviour in [9]. Together with Wilson, we have proposed an efficient computational procedure for the computation of the MPT spectral signatures [14] and shown how it can be used to characterise threat and non-threat objects of relevance to security screening [10] and proposed an approach to classification using machine learning [15].

For the full Maxwell system, Ammari, Vogelius and Volkov [3] have derived the leading-order term in an asymptotic expansion of the perturbed electric and magnetic fields caused by the presence of an object as its size tends to zero. In this case, there is no simplification in the Maxwell system and wave propagation effects are included. The resulting expansions shows that leading-order provides an object characterisation in terms of the simpler Pólya–Szegő tensor parameterised by contrasts in (complex) permittivity and permeability, which can be computed from a scalar transmission problem. Unlike the eddy current case, conditions are not imposed on the material parameters and, hence, ϵ_* , σ_* and μ_* are allowed to change as $\alpha \rightarrow 0$. In the limiting case of magnetostatics, it can be shown that the MPT simplifies to the simpler Pólya–Szegő tensor description [9], whose coefficients are obtained by solving scalar transmission problems, but, in the eddy current regime, the calculation of the MPT coefficients requires the solution of vectorial curl–curl transmission problems. However, it was not clear what form

the characterisation takes as the modelling error in the eddy current model becomes large and wave propagation effects become important.

Our present work contributes to understanding the regime where frequencies are small, but the displacement currents remain important through the following novelties:

1. We derive a new asymptotic formula for the magnetic field perturbation caused by the presence of a conducting permeable object as its size tends to zero. We establish the leading order for the full Maxwell system with specific constraints applied to the object’s materials and the frequency of excitation.
2. We obtain new forms of polarizability tensors for characterising objects, whose coefficients are obtained by post-processing solutions to vectorial transmission problems.
3. We show that our expansion reduces to the result obtained by Ammari et al. [2] if the eddy current approximation is made. Our new polarizability tensors reduce to an object characterisation using the MPT in this case.
4. Additionally, we obtain an expansion of the form obtained by Ammari et al. [3] if the conditions we impose are relaxed.

The material proceeds as follows: In Section 2, we introduce some notation that will aid the presentation of the material. Then, in Section 3, we introduce the mathematical model that will be of interest in this work. Section 4 presents our main result, the proof of which will be established subsequent sections. Section 5 establishes some energy estimates, Section 6 establishes an integral representation formula and Section 7 the remaining parts of the proof. Then, Section 8 presents alternative forms of our main result and presents simplifications and relaxation of assumptions that make comparisons with previous object characterisations possible.

2. Notation

For what follows it is beneficial to introduce the following notation: we will use boldface for vector quantities (e.g. \mathbf{u}) and denote by $\mathbf{e}_j, j = 1, 2, 3$ the units vectors associated with an orthonormal coordinate system. We denote the j -th component of a vector \mathbf{u} in this coordinate system by $(\mathbf{u})_j = \mathbf{u} \cdot \mathbf{e}_j = u_j$. We will use calligraphic symbols to denote rank 2 tensors, for example, $\mathcal{N} = \mathcal{N}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, where Einstein summation convention is implied, and denote their coefficients by \mathcal{N}_{ij} .

We recall that for $0 \leq \ell < \infty, 0 \leq p < \infty$,

$$\|\mathbf{u}\|_{W^{\ell,p}(B_\alpha)} := \left(\sum_{j=0}^{\ell} \int_{B_\alpha} |\mathbf{D}^j(\mathbf{u}(\mathbf{x}))|^p \, d\mathbf{x} \right)^{1/p},$$

where the derivatives are defined in a weak sense and

$$\|\mathbf{u}\|_{W^{\ell,\infty}(B_\alpha)} := \operatorname{ess\,sup}_{\mathbf{x} \in B_\alpha} \sum_{j=0}^{\ell} |\mathbf{D}^j(\mathbf{u}(\mathbf{x}))|.$$

3. Mathematical model

3.1 Governing equations

For linear materials, the time harmonic Maxwell equations are as follows:

$$\nabla \times \mathcal{E} = i\omega\mu_0 \mathcal{H}, \tag{3.1a}$$

$$\nabla \times \mathcal{H} = \sigma \mathcal{E} + \mathcal{J}_0 - i\omega\epsilon \mathcal{E}, \tag{3.1b}$$

$$\nabla \cdot (\epsilon \mathcal{E}) = \frac{1}{i\omega} \nabla \cdot (\sigma \mathcal{E}), \tag{3.1c}$$

$$\nabla \cdot (\mu \mathcal{H}) = 0, \tag{3.1d}$$

where \mathcal{E} and \mathcal{H} denote the complex amplitudes of the electric and magnetic field intensity vectors, respectively, for an assumed $e^{-i\omega t}$ time variation with angular frequency $\omega > 0$ and $i := \sqrt{-1}$. In addition, \mathcal{J}_0 denotes the complex amplitude of an external solenoidal current source, and the parameters ϵ , μ and σ denote the permittivity, permeability and conductivity, respectively, and satisfy

$$0 < \epsilon^{\min} \leq \epsilon \leq \epsilon^{\max} < \infty, \quad 0 < \mu^{\min} \leq \mu \leq \mu^{\max} < \infty, \quad 0 \leq \sigma \leq \sigma^{\max} < \infty. \quad (3.2)$$

As standard (e.g. Monk [11]), the scaled fields are introduced as $\mathbf{E}_\alpha = \epsilon_0^{1/2} \mathcal{E}$, $\mathbf{H}_\alpha = \mu_0^{1/2} \mathcal{H}$ and $\mathbf{J}_0 = \mu_0^{1/2} \mathcal{J}_0$, where $\epsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ H/m are the free space values of the permittivity and permeability, respectively, leading to

$$\nabla \times \mathbf{E}_\alpha = ik\tilde{\mu}_r \mathbf{H}_\alpha, \quad (3.3a)$$

$$\nabla \times \mathbf{H}_\alpha = \mathbf{J}_0 - ik\tilde{\epsilon}_r \mathbf{E}_\alpha, \quad (3.3b)$$

$$\nabla \cdot (\tilde{\epsilon}_r \mathbf{E}_\alpha) = 0, \quad (3.3c)$$

$$\nabla \cdot (\tilde{\mu}_r \mathbf{H}_\alpha) = 0, \quad (3.3d)$$

where

$$\tilde{\epsilon}_r := \frac{1}{\epsilon_0} \left(\epsilon + \frac{i\sigma}{\omega} \right), \quad \tilde{\mu}_r := \frac{\mu}{\mu_0}, \quad k := \omega(\epsilon_0 \mu_0)^{1/2},$$

and $\tilde{\epsilon}_r, \tilde{\mu}_r$ are, in general, functions of position.

3.2 Perturbed field formulation

We describe a single connected inclusion by $B_\alpha := \alpha B + \mathbf{z}$, which means that it could be thought of a unit-sized object B located at the origin, scaled by α and translated by \mathbf{z} . Its boundary, $\Gamma_\alpha := \partial B_\alpha$, is equipped with unit outward normal \mathbf{n} , and we assume that the object is homogeneous with material coefficients ϵ_* , μ_* and σ_* . The object is surrounded by an unbounded region of free space $B_\alpha^c := \mathbb{R}^3 \setminus \overline{B_\alpha}$ with material coefficients ϵ_0, μ_0 and $\sigma = 0$ so that, henceforth,

$$\tilde{\epsilon}_r(\mathbf{x}) := \begin{cases} \epsilon_r := \frac{1}{\epsilon_0} \left(\epsilon_* + \frac{i\sigma_*}{\omega} \right) & \mathbf{x} \in B_\alpha \\ 1 & \mathbf{x} \in B_\alpha^c \end{cases}, \quad \tilde{\mu}_r := \begin{cases} \mu_r := \frac{\mu_*}{\mu_0} & \mathbf{x} \in B_\alpha \\ 1 & \mathbf{x} \in B_\alpha^c \end{cases}.$$

Furthermore, we assume that \mathbf{J}_0 has support only in B_α^c and for it to be located away from B_α . The electric and magnetic fields obey the standard transmission conditions:

$$[\mathbf{n} \times \mathbf{E}_\alpha]_{\Gamma_\alpha} = \mathbf{0}, \quad (3.4a)$$

$$[\mathbf{n} \times \mathbf{H}_\alpha]_{\Gamma_\alpha} = \mathbf{0}, \quad (3.4b)$$

$$[\mathbf{n} \cdot \tilde{\epsilon}_r \mathbf{E}_\alpha]_{\Gamma_\alpha} = 0, \quad (3.4c)$$

$$[\mathbf{n} \cdot \tilde{\mu}_r \mathbf{H}_\alpha]_{\Gamma_\alpha} = 0, \quad (3.4d)$$

on Γ_α where $[\cdot]_{\Gamma_\alpha} = \cdot|_+ - \cdot|_-$ denotes the jump and subscript $+/-$ denotes the evaluation just outside/inside of B_α , respectively. By \mathbf{n}^+ and \mathbf{n}^- , we denote the unit outward normal pointing from the $+/-$ side of Γ_α into B_α and B_α^c , respectively, and with similar meanings on other interfaces. In absence of B_α , the background (incident) fields \mathbf{E}_0 and \mathbf{H}_0 satisfy (3.3) with $\tilde{\epsilon}_r = \tilde{\mu}_r = 1$ in \mathbb{R}^3 . The fields $\mathbf{E}_\Delta := \mathbf{E}_\alpha - \mathbf{E}_0$ and $\mathbf{H}_\Delta := \mathbf{H}_\alpha - \mathbf{H}_0$, which represent the perturbation in the electric and magnetic fields due to the presence of B_α , respectively, satisfy the radiation conditions:

$$\lim_{x \rightarrow \infty} x((\nabla \times \mathbf{E}_\Delta) \times \hat{\mathbf{x}} - ik\mathbf{E}_\Delta) = \mathbf{0},$$

$$\lim_{x \rightarrow \infty} x((\nabla \times \mathbf{H}_\Delta) \times \hat{\mathbf{x}} - ik\mathbf{H}_\Delta) = \mathbf{0},$$

where $x = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/x$. By eliminating \mathbf{H}_α , we arrive at the following transmission problem for \mathbf{E}_Δ :

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{E}_\Delta - k^2 \epsilon_r \mathbf{E}_\Delta = \nabla \times (1 - \mu_r^{-1}) \nabla \times \mathbf{E}_0 - k^2 (1 - \epsilon_r) \mathbf{E}_0 \quad \text{in } B_\alpha, \tag{3.5a}$$

$$\nabla \times \nabla \times \mathbf{E}_\Delta - k^2 \mathbf{E}_\Delta = \mathbf{0} \quad \text{in } B_\alpha^c, \tag{3.5b}$$

$$\nabla \cdot (\tilde{\epsilon}_r \mathbf{E}_\Delta) = 0 \quad \text{in } \mathbb{R}^3, \tag{3.5c}$$

$$[\mathbf{n} \times \mathbf{E}_\Delta]_{\Gamma_\alpha} = \mathbf{0} \quad \text{on } \Gamma_\alpha, \tag{3.5d}$$

$$[\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla \times \mathbf{E}_\Delta]_{\Gamma_\alpha} = -[\tilde{\mu}_r^{-1}]_{\Gamma_\alpha} \mathbf{n} \times \nabla \times \mathbf{E}_0 \quad \text{on } \Gamma_\alpha, \tag{3.5e}$$

$$\lim_{x \rightarrow \infty} x((\nabla \times \mathbf{E}_\Delta) \times \hat{\mathbf{x}} - ik \mathbf{E}_\Delta) = \mathbf{0}, \tag{3.5f}$$

for \mathbf{E}_Δ , while an analogous transmission problem could also be derived for \mathbf{H}_Δ . Given the solenoidal behaviour of \mathbf{E}_0 , the condition $\nabla \cdot (\tilde{\epsilon}_r \mathbf{E}_\Delta) = 0$ in \mathbb{R}^3 can be disregarded provided that $k^2 > 0$ is not an eigenvalue of (3.5), which we shall assume throughout.

The system (3.5) is set on an unbounded domain, and we follow the approach of Monk [11, Chapter 10, Chapter 11] to reduce this to a problem on a bounded domain. Compared to the static decay of the fields considered for the eddy current problem in [2], a different treatment is required before the weak variational statement can be introduced due to the presence of the radiation condition in (3.5f). We assume that there is a radius R_0 , centred on the origin, such that $B_\alpha \subset S_{R_0}$ with S_{R_0} being a solid sphere of radius R_0 . Next, we introduce a solid sphere S_R , with radius $R > R_0$, and let $\Omega := S_R$, which is a bounded domain. We denote the boundary of Ω by Σ and note that $\Gamma_\alpha \cap \Sigma = \emptyset$. This leads to the alternative transmission problem for \mathbf{E}_Δ :

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{E}_\Delta - k^2 \epsilon_r \mathbf{E}_\Delta = \nabla \times (1 - \mu_r^{-1}) \nabla \times \mathbf{E}_0 - k^2 (1 - \epsilon_r) \mathbf{E}_0 \quad \text{in } B_\alpha, \tag{3.6a}$$

$$\nabla \times \nabla \times \mathbf{E}_\Delta - k^2 \mathbf{E}_\Delta = \mathbf{0} \quad \text{in } \Omega \setminus \overline{B_\alpha}, \tag{3.6b}$$

$$\nabla \times \nabla \times \mathbf{E}_\Delta - k^2 \mathbf{E}_\Delta = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{3.6c}$$

$$[\mathbf{n} \times \mathbf{E}_\Delta]_{\Gamma_\alpha} = \mathbf{0} \quad \text{on } \Gamma_\alpha, \tag{3.6d}$$

$$[\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla \times \mathbf{E}_\Delta]_{\Gamma_\alpha} = -[\tilde{\mu}_r^{-1}]_{\Gamma_\alpha} \mathbf{n} \times \nabla \times \mathbf{E}_0 \quad \text{on } \Gamma_\alpha, \tag{3.6e}$$

$$[\mathbf{n} \times \mathbf{E}_\Delta]_\Sigma = \mathbf{0} \quad \text{on } \Sigma, \tag{3.6f}$$

$$[\mathbf{n} \times \nabla \times \mathbf{E}_\Delta]_\Sigma = \mathbf{0} \quad \text{on } \Sigma, \tag{3.6g}$$

$$\lim_{x \rightarrow \infty} x((\nabla \times \mathbf{E}_\Delta) \times \hat{\mathbf{x}} - ik \mathbf{E}_\Delta) = \mathbf{0}. \tag{3.6h}$$

By integrating over Ω , we have

$$\begin{aligned} & \int_{\Omega \setminus \overline{B_\alpha}} (\nabla \times \mathbf{E}_\Delta \cdot \nabla \times \bar{\mathbf{v}} - k^2 \mathbf{E}_\Delta \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x} + \int_{B_\alpha} (\mu_r^{-1} \nabla \times \mathbf{E}_\Delta \cdot \nabla \times \bar{\mathbf{v}} - k^2 \epsilon_r \mathbf{E}_\Delta \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x} \\ & + \int_\Sigma \mathbf{n}^- \times \nabla \times \mathbf{E}_\Delta \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \mathbf{d}\mathbf{x} \\ & = \int_{B_\alpha} ((1 - \mu_r^{-1}) \nabla \times \mathbf{E}_0 \cdot \nabla \times \bar{\mathbf{v}} - k^2 (1 - \epsilon_r) \mathbf{E}_0 \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x}, \end{aligned}$$

and, by applying the transmission condition in (3.6e), followed by the Dirichlet to Neumann map $\Lambda_e^{(k)}(\mathbf{n} \times \mathbf{E}_\Delta) := \mathbf{n} \times \frac{1}{ik} \nabla \times \mathbf{E}_\Delta$, in a similar manner to Monk [11, Equation (10.2), see also Chapter 11] (who instead of $\Lambda_e^{(k)}$ denotes the map by G_e), leads to the weak statement. Find $\mathbf{E}_\Delta \in H(\text{curl})$

$$\begin{aligned} & \int_{\Omega \setminus \overline{B_\alpha}} (\nabla \times \mathbf{E}_\Delta \cdot \nabla \times \bar{\mathbf{v}} - k^2 \mathbf{E}_\Delta \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x} + \int_{B_\alpha} (\mu_r^{-1} \nabla \times \mathbf{E}_\Delta \cdot \nabla \times \bar{\mathbf{v}} - k^2 \epsilon_r \mathbf{E}_\Delta \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x} \\ & + ik \int_\Sigma \Lambda_e^{(k)}(\mathbf{n}^- \times \mathbf{E}_\Delta) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \mathbf{d}\mathbf{x} \\ & = \int_{B_\alpha} ((1 - \mu_r^{-1}) \nabla \times \mathbf{E}_0 \cdot \nabla \times \bar{\mathbf{v}} - k^2 (1 - \epsilon_r) \mathbf{E}_0 \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x}, \tag{3.7} \end{aligned}$$

for all $\mathbf{v} \in H(\text{curl})$.

We also recall that

$$G_k(\mathbf{x}, \mathbf{y}) := \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},$$

is the Helmholtz’s free space Green’s function, which becomes the Laplace free space Green’s function if $k = 0$.

4. Main result

As mentioned in the introduction, Ammari, Buffa and Nédélec [1] have established that the eddy current model is a low-frequency approximation to the Maxwell system, which is first- or second-order accurate depending on the topology of the object and its materials. Furthermore, the conditions stated in (1.1) for the validity of the eddy current model depend on the B_α topology-dependent constants C_1 and C_2 , and Schmidt, Hiptmair and Sterz [12] have proposed a methodology for estimating their values. Nonetheless, the modelling error in the eddy current model increases as ω increases and will be large, and the eddy current model is invalid, as $C_1\epsilon_*\mu_*\omega^2\alpha^2 \rightarrow 1$. In this case, the formulae for computing the MPT coefficients [2, 5, 8], which is based on the eddy current model, will also need to be replaced.

Our goal is to derive an asymptotic formula for $\mathbf{H}_\Delta(\mathbf{x}) = (\mathbf{H}_\alpha - \mathbf{H}_0)(\mathbf{x})$ for \mathbf{x} away from B_α as $\alpha \rightarrow 0$ leading to a characterisation of B_α , which includes the situation where the modelling error in the eddy current model becomes large. To investigate this, we will require that $C(B)\epsilon_*\mu_*\omega_2\alpha^2 < 1$ and $C(\Omega)\epsilon_0\mu_0\omega^2R^2 < 1$, but, importantly, we will retain displacement current terms in order to investigate the form the characterisation takes when the eddy current model becomes invalid. Additionally, we introduce $\nu = O(1)$ as $\alpha \rightarrow 0$ where

$$\nu = \alpha^2k^2(\epsilon_r - 1) = \nu_r + i\nu_i, \quad \nu_r = (\epsilon_* - \epsilon_0)\mu_0\omega^2\alpha^2, \quad \nu_i = \sigma_*\mu_0\omega\alpha^2,$$

and $\mu_r = O(1)$. Our treatment will allow an extension of the results described in [2, 5, 8], which considered the eddy current model, where the quasi-static assumption (i.e. $\alpha \ll \lambda_0 = 2\pi/k$) and large conductivities ($\sigma_* \gg \epsilon_*\omega$) were assumed, and, instead, considered the regime where $\nu_i = O(1)$ and $\mu_r = O(1)$ as $\alpha \rightarrow 0$. Fixing $\nu = O(1)$, rather than $\nu_i = O(1)$, means that, in addition to including the case where σ_* and ω are constant, we include the situation where ϵ_* is constant. These quantities are also allowed to decrease with α , but not faster than $1/\alpha^2$.

We focus on $\mathbf{H}_\Delta(\mathbf{x}) = (\mathbf{H}_\alpha - \mathbf{H}_0)(\mathbf{x})$ to allow ease of comparison with these earlier results. Our main result is as follows:

Theorem 4.1. *For \mathbf{x} away from $B_\alpha = \alpha B + \mathbf{z}$, and the conditions stated in Section 4, the following expansion of $\mathbf{H}_\Delta(\mathbf{x}) = (\mathbf{H}_\alpha - \mathbf{H}_0)(\mathbf{x})$ holds*

$$\begin{aligned} (\mathbf{H}_\Delta(\mathbf{x}))_j &= -ik(\nabla_x G_k(\mathbf{x}, \mathbf{z}))_p \epsilon_{jpr} (\mathcal{A}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathcal{B}_{ri}(\mathbf{E}_0(\mathbf{z}))_i) \\ &\quad + (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{tm} \epsilon_{jts} C_{msi}(\mathbf{H}_0(\mathbf{z}))_i \\ &\quad + ((\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{jr} + k^2 \delta_{jr} G_k(\mathbf{x}, \mathbf{z})) \mathcal{N}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + (\mathbf{R}(\mathbf{x}))_j, \end{aligned} \tag{4.1}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{N} are polarizability tensors with coefficients:

$$\mathcal{A}_{ri} := \frac{ik\alpha^4(\epsilon_r - 1)}{2} \mathbf{e}_r \cdot \int_B (\mathbf{e}_i \times \boldsymbol{\xi} + \boldsymbol{\theta}_i) d\boldsymbol{\xi}, \tag{4.2a}$$

$$\mathcal{B}_{ri} := \alpha^3(\epsilon_r - 1) \mathbf{e}_r \cdot \int_B (\mathbf{e}_i + \boldsymbol{\phi}_i) d\boldsymbol{\xi}, \tag{4.2b}$$

$$\mathcal{C}_{msi} := -\frac{k^2\alpha^5(\epsilon_r - 1)}{2} \mathbf{e}_s \cdot \int_B \xi_m (\mathbf{e}_i \times \boldsymbol{\xi} + \boldsymbol{\theta}_i) d\boldsymbol{\xi}, \tag{4.2c}$$

$$\mathcal{N}_{ri} := \alpha^3(1 - \mu_r^{-1}) \mathbf{e}_r \cdot \int_B \left(\mathbf{e}_i + \frac{1}{2} \nabla \times \boldsymbol{\theta}_i \right) d\boldsymbol{\xi}, \tag{4.2d}$$

that depend on the solution to the transmission problems:

$$\nabla_{\xi} \times \mu_r^{-1} \nabla_{\xi} \times \theta_i - k^2 \alpha^2 \epsilon_r \theta_i = -k^2 \alpha^2 (1 - \epsilon_r) \mathbf{e}_i \times \xi \quad \text{in } B, \tag{4.3a}$$

$$\nabla_{\xi} \times \nabla_{\xi} \times \theta_i - k^2 \alpha^2 \theta_i = \mathbf{0} \quad \text{in } B^c := \mathbb{R}^3 \setminus \bar{B}, \tag{4.3b}$$

$$\nabla_{\xi} \cdot \theta_i = 0 \quad \text{in } \mathbb{R}^3, \tag{4.3c}$$

$$[\mathbf{n} \times \theta_i]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\xi} \times \theta_i]_{\Gamma} = -2(1 - \mu_r^{-1}) \mathbf{n} \times \mathbf{e}_i \quad \text{on } \Gamma := \partial B, \tag{4.3d}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_{\xi} \times \theta_i) \times \hat{\xi} - ik\alpha \theta_i \right) = \mathbf{0}, \tag{4.3e}$$

and

$$\nabla_{\xi} \times \mu_r^{-1} \nabla_{\xi} \times \phi_i - k^2 \alpha^2 \epsilon_r \phi_i = -k^2 \alpha^2 (1 - \epsilon_r) \mathbf{e}_i \quad \text{in } B, \tag{4.4a}$$

$$\nabla_{\xi} \times \nabla_{\xi} \times \phi_i - k^2 \alpha^2 \phi_i = \mathbf{0} \quad \text{in } B^c, \tag{4.4b}$$

$$\nabla_{\xi} \cdot \phi_i = 0 \quad \text{in } \mathbb{R}^3, \tag{4.4c}$$

$$[\mathbf{n} \times \phi_i]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\xi} \times \phi_i]_{\Gamma} = \mathbf{0} \quad \text{on } \Gamma, \tag{4.4d}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_{\xi} \times \phi_i) \times \hat{\xi} - ik\alpha \phi_i \right) = \mathbf{0}. \tag{4.4e}$$

The residual $\mathbf{R}(\mathbf{x})$ satisfies $|\mathbf{R}(\mathbf{x})| \leq C \left(\alpha^4 \|\mathbf{H}_0\|_{W^{2,\infty}(B_{\alpha})} + \alpha^4 k \left(|\epsilon_r - 1| + \alpha k^2 \left| 1 - \frac{1}{\epsilon_r} \right| \right) \|\mathbf{E}_0\|_{W^{1,\infty}(B_{\alpha})} \right)$.

The proof of this theorem follows in Section 7. Beforehand, we derive results that our proof will draw on.

5. Energy estimates

To arrive at an energy estimate for an eddy current problem, Ammari, Chen, Chen, Garnier and Volkov [2] have introduced the following equations:

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &:= \frac{1}{2} (\nabla_{\mathbf{z}} \times \mathbf{E}_0(\mathbf{z})) \times (\mathbf{x} - \mathbf{z}) + \frac{1}{3} \mathbf{D}_{\mathbf{z}} (\nabla_{\mathbf{z}} \times \mathbf{E}_0(\mathbf{z})) (\mathbf{x} - \mathbf{z}) \times (\mathbf{x} - \mathbf{z}) \\ &= \frac{ik}{2} \sum_{i=1}^3 (\mathbf{H}_0(\mathbf{z}))_i \mathbf{e}_i \times (\mathbf{x} - \mathbf{z}) + \frac{ik}{3} \sum_{ij=1}^3 \mathbf{D}_{\mathbf{z}} (\mathbf{H}_0(\mathbf{z}))_{ij} (\mathbf{x} - \mathbf{z})_j \mathbf{e}_i \times (\mathbf{x} - \mathbf{z}), \end{aligned} \tag{5.1}$$

which has the curl:

$$\begin{aligned} \nabla_{\mathbf{x}} \times \mathbf{F} &= \nabla_{\mathbf{z}} \times \mathbf{E}_0(\mathbf{z}) + \mathbf{D}_{\mathbf{z}} (\nabla_{\mathbf{z}} \times \mathbf{E}_0(\mathbf{z})) (\mathbf{x} - \mathbf{z}) \\ &= ik\mathbf{H}_0(\mathbf{z}) + ik \sum_{ij=1}^3 \mathbf{D}_{\mathbf{z}} (\mathbf{H}_0(\mathbf{z}))_{ij} (\mathbf{x} - \mathbf{z})_j \mathbf{e}_i. \end{aligned} \tag{5.2}$$

In the above equation, $\nabla_{\mathbf{x}} \times \mathbf{F}$ corresponds to the first two terms in a Taylor’s series expansion of $ik\mathbf{H}_0(\mathbf{x})$ about \mathbf{z} as $|\mathbf{x} - \mathbf{z}| \rightarrow 0$. We adopt a different form for \mathbf{F} as follows:

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &:= \frac{1}{2} (\nabla_{\mathbf{z}} \times \mathbf{E}_0(\mathbf{z})) \times (\mathbf{x} - \mathbf{z}) + \frac{1}{3} \mathbf{D}_{\mathbf{z}} (\nabla_{\mathbf{z}} \times \mathbf{E}_0(\mathbf{z})) (\mathbf{x} - \mathbf{z}) \times (\mathbf{x} - \mathbf{z}) + \mathbf{E}_0(\mathbf{z}) \\ &= \frac{ik}{2} \sum_{i=1}^3 (\mathbf{H}_0(\mathbf{z}))_i \mathbf{e}_i \times (\mathbf{x} - \mathbf{z}) + \frac{ik}{3} \sum_{ij=1}^3 \mathbf{D}_{\mathbf{z}} (\mathbf{H}_0(\mathbf{z}))_{ij} (\mathbf{x} - \mathbf{z})_j \mathbf{e}_i \times (\mathbf{x} - \mathbf{z}) + \mathbf{E}_0(\mathbf{z}), \end{aligned} \tag{5.3}$$

so that

$$\nabla_{\mathbf{x}} \times \mathbf{F} = ik\mathbf{H}_0(\mathbf{z}) + ik \sum_{ij=1}^3 \mathbf{D}_{\mathbf{z}} (\mathbf{H}_0(\mathbf{z}))_{ij} (\mathbf{x} - \mathbf{z})_j \mathbf{e}_i, \tag{5.4}$$

is still the first two terms in a Taylor’s series expansion of $ik\mathbf{H}_0(\mathbf{x})$ about \mathbf{z} as $|\mathbf{x} - \mathbf{z}| \rightarrow 0$. Note that $\mathbf{F}(\mathbf{z}) = \mathbf{E}_0(\mathbf{z})$ and $(\nabla_x \times \mathbf{F})(\mathbf{z}) = ik\mathbf{H}_0(\mathbf{z})$. We can also determine

$$\begin{aligned} \nabla_x \cdot \mathbf{F} &= -\frac{ik}{3} \sum_{i=1}^3 (\nabla_z \times \mathbf{H}_0(\mathbf{z}))_i \mathbf{e}_i \cdot (\mathbf{x} - \mathbf{z}) \\ &= \frac{k^2}{3} \sum_{i=1}^3 (\mathbf{E}_0(\mathbf{z}))_i \mathbf{e}_i \cdot (\mathbf{x} - \mathbf{z}) \\ &= \nabla^2 \phi_0, \end{aligned} \tag{5.5}$$

where, up to a constant,

$$\phi_0 = \frac{k^2}{18} ((\mathbf{E}_0(\mathbf{z}))_1(x_1 - z_1)^3 + (\mathbf{E}_0(\mathbf{z}))_2(x_2 - z_2)^3 + (\mathbf{E}_0(\mathbf{z}))_3(x_3 - z_3)^3). \tag{5.6}$$

Related to (3.7) in [2], and following a similar treatment to (3.7), we introduce \mathbf{w} as the unique solution to the weak problem. Find $\mathbf{w} \in H(\text{curl})$ s.t.

$$\begin{aligned} &\int_{\Omega \setminus \overline{B_\alpha}} (\nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{v}} - k^2 \mathbf{w} \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x} + \int_{B_\alpha} (\mu_r^{-1} \nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{v}} - k^2 \epsilon_r \mathbf{w} \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x} \\ &+ ik \int_{\Sigma} \Lambda_e^{(k)}(\mathbf{n}^- \times \mathbf{w}) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \mathbf{d}\mathbf{x} \\ &= \int_{B_\alpha} ((1 - \mu_r^{-1}) \nabla \times \mathbf{F} \cdot \nabla \times \bar{\mathbf{v}} - k^2(1 - \epsilon_r) \mathbf{F} \cdot \bar{\mathbf{v}}) \mathbf{d}\mathbf{x}, \end{aligned} \tag{5.7}$$

for all $\mathbf{v} \in H(\text{curl})$.

The following updated form of Lemma 3.2 from [2] relies on a Friedrichs’ type inequality that will allow us to estimate a divergence-free field $\mathbf{u} \in H(\text{curl}) \cap H(\text{div})$ in a bounded Lipschitz domain B_α in terms of its curl and $\mathbf{n} \cdot \mathbf{u}$ on Γ_α , for example, [11, Corollary 3.52, pg 72]:

$$\|\mathbf{u}\|_{L^2(B_\alpha)} \leq C(\|\nabla \times \mathbf{u}\|_{L^2(B_\alpha)} + \|\mathbf{n} \cdot \mathbf{u}\|_{L^2(\Gamma_\alpha)}), \tag{5.8}$$

which we assume holds with $\alpha = 1$ while, in general,

$$\|\mathbf{u}\|_{L^2(B_\alpha)} \leq C\alpha(\|\nabla \times \mathbf{u}\|_{L^2(B_\alpha)} + \|\mathbf{n} \cdot \mathbf{u}\|_{L^2(\Gamma_\alpha)}), \tag{5.9}$$

leading to:

Lemma 5.1. *Let \mathbf{w} be defined by (5.7). For $|\epsilon_r|$ such that $\frac{1}{|\epsilon_r|} \|\mathbf{n} \cdot (\mathbf{E}_\alpha|_+ - \mathbf{E}_0 - \mathbf{w})\|_{L^2(\Gamma_\alpha)} \leq \|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)}$, and the conditions stated in Section 4, there exists a constant C such that*

$$\begin{aligned} \|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)} &\leq C\alpha^{7/2} \mu_r (|1 - \mu_r^{-1}| + |\nu|) \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)}, \\ \left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)} &\leq C\alpha^{9/2} \mu_r (|1 - \mu_r^{-1}| + |\nu|) \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)}. \end{aligned}$$

Proof. For the eddy current model considered by [2], $\nabla \cdot \mathbf{E}_\alpha = \nabla \cdot \mathbf{E}_0 = 0$ in B_α and $\mathbf{n} \cdot \mathbf{E}_\alpha|_- = 0$ on Γ_α . Also, they have $\nabla \cdot \mathbf{w} = -\nabla \cdot \mathbf{F}$ in B_α and $\mathbf{n} \cdot \mathbf{w}|_- = -\mathbf{n} \cdot \mathbf{F}$ on Γ_α . Our situation is different, however, so that choosing $\mathbf{v} = \nabla \vartheta$ for some appropriate ϑ in (5.7) we find that

$$\nabla \cdot (\epsilon_r \mathbf{w}) = \nabla \cdot ((1 - \epsilon_r) \mathbf{F}) \quad \text{in } B_\alpha, \quad \mathbf{n} \cdot \mathbf{w}|_+ - \mathbf{n} \cdot \epsilon_r \mathbf{w}|_- = -(1 - \epsilon_r) \mathbf{n} \cdot \mathbf{F} \quad \text{on } \Gamma_\alpha.$$

We also know that for the problem we are considering

$$\nabla \cdot (\epsilon_r \mathbf{E}_\alpha) = 0 \quad \text{in } B_\alpha, \quad \mathbf{n} \cdot \mathbf{E}_\alpha|_+ - \mathbf{n} \cdot \epsilon_r \mathbf{E}_\alpha|_- = 0 \quad \text{on } \Gamma_\alpha.$$

We follow [2] and introduce ϕ_0 in a similar way such that $\nabla \cdot (\mathbf{E}_0 + \nabla \phi_0) = \nabla \cdot \mathbf{F}$ in B_α with $\mathbf{n} \cdot (\mathbf{E}_0 + \nabla \phi_0)|_- = \mathbf{n} \cdot \mathbf{F}$ on Γ_α , which is justified based on (5.6) and knowing that $\nabla \cdot \mathbf{E}_0 = 0$ in B_α . Since $\nabla \cdot (\mathbf{F} - \nabla \phi_0) = 0$ in B_α , then $\nabla \cdot ((1 - \epsilon_r)(\mathbf{F} - \nabla \phi_0)) = \nabla \cdot (\epsilon_r \mathbf{w} - (1 - \epsilon_r) \nabla \phi_0) = 0$ and $\nabla \cdot (\mathbf{w} - \frac{1 - \epsilon_r}{\epsilon_r} \nabla \phi_0) = 0$

in B_α . Then,

$$\nabla \cdot \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) = 0 \quad \text{in } B_\alpha, \tag{5.10}$$

and, by combining the above results, we also have

$$\mathbf{n} \cdot \left(\mathbf{E}_\alpha|_- - \mathbf{w}|_- - \mathbf{E}_0 - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) - \frac{1}{\epsilon_r} \mathbf{n} \cdot (\mathbf{E}_\alpha|_+ - \mathbf{w}|_+ - \mathbf{E}_0) = 0 \quad \text{on } \Gamma_\alpha. \tag{5.11}$$

For $|\epsilon_r| \rightarrow \infty$, then $\frac{1}{\epsilon_r} \mathbf{n} \cdot (\mathbf{E}_\alpha|_+ - \mathbf{w}|_+ - \mathbf{E}_0) = \frac{1}{\epsilon_r} \mathbf{n} \cdot (\mathbf{E}_\Delta|_+ - \mathbf{w}|_+) \rightarrow 0$ on Γ_α and, the presence of this term, means that rather than

$$\left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)} \leq C\alpha \|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)}, \tag{5.12}$$

we should use (5.8) leading to an estimate of the form:

$$\begin{aligned} \left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)} &\leq C\alpha \left(\|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)} \right. \\ &\quad \left. + \frac{1}{|\epsilon_r|} \|\mathbf{n} \cdot (\mathbf{E}_\alpha|_+ - \mathbf{E}_0 - \mathbf{w}|_+)\|_{L^2(\Gamma_\alpha)} \right). \end{aligned} \tag{5.13}$$

Next, we construct

$$\begin{aligned} &\int_{\Omega \setminus \bar{B}_\alpha} (\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}) \cdot \nabla \times \bar{\mathbf{v}} - k^2 (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}) \cdot \bar{\mathbf{v}}) \, \mathbf{d}\mathbf{x} \\ &+ \int_{B_\alpha} \left(\mu_r^{-1} \nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}) \cdot \nabla \times \bar{\mathbf{v}} - k^2 \epsilon_r \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) \cdot \bar{\mathbf{v}} \right) \, \mathbf{d}\mathbf{x} \\ &+ ik \int_\Sigma \Lambda_e^{(k)} (\mathbf{n}^- \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \, \mathbf{d}\mathbf{x} \\ &= \int_{B_\alpha} ((1 - \mu_r^{-1}) \nabla \times (\mathbf{E}_0 - \mathbf{F}) \cdot \nabla \times \bar{\mathbf{v}} - k^2 (1 - \epsilon_r) (\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}) \cdot \bar{\mathbf{v}}) \, \mathbf{d}\mathbf{x}. \end{aligned} \tag{5.14}$$

By noting the curl of the gradient of a scalar vanishes, we can write

$$\begin{aligned} &\int_{\Omega \setminus \bar{B}_\alpha} (\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}) \cdot \nabla \times \bar{\mathbf{v}} - k^2 (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}) \cdot \bar{\mathbf{v}}) \, \mathbf{d}\mathbf{x} \\ &+ \int_{B_\alpha} \left(\mu_r^{-1} \nabla \times \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) \cdot \nabla \times \bar{\mathbf{v}} - k^2 \epsilon_r \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) \cdot \bar{\mathbf{v}} \right) \, \mathbf{d}\mathbf{x} \\ &+ ik \int_\Sigma \Lambda_e^{(k)} (\mathbf{n}^- \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \, \mathbf{d}\mathbf{x} \\ &= \int_{B_\alpha} ((1 - \mu_r^{-1}) \nabla \times (\mathbf{E}_0 - \mathbf{F}) \cdot \nabla \times \bar{\mathbf{v}} - k^2 (1 - \epsilon_r) (\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}) \cdot \bar{\mathbf{v}}) \, \mathbf{d}\mathbf{x}. \end{aligned} \tag{5.15}$$

Choosing $\mathbf{v} = \begin{cases} \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 & \text{in } B_\alpha \\ \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} & \text{in } \Omega \setminus \overline{B_\alpha} \end{cases}$ and taking modulus of both sides of (5.15), we observe that the resulting left-hand side has the form $|Z| = (Z_r^2 + Z_i^2)^{1/2}$ where

$$\begin{aligned} Z_r &:= \|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(\Omega \setminus \overline{B_\alpha})}^2 - k^2 \|\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}\|_{L^2(\Omega \setminus \overline{B_\alpha})}^2 \\ &\quad + \mu_r^{-1} \left\| \nabla \times \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) \right\|_{L^2(B_\alpha)}^2 - k^2 \frac{\epsilon_*}{\epsilon_0} \left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)}^2 \\ &\quad + k \operatorname{Im} \left(\int_\Sigma \Lambda_e^{(k)} (\mathbf{n}^- \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \, d\mathbf{x} \right), \\ Z_i &:= -k^2 \frac{\sigma_*}{\omega \epsilon_0} \left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)}^2 \\ &\quad + k \operatorname{Re} \left(\int_\Sigma \Lambda_e^{(k)} (\mathbf{n}^- \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \, d\mathbf{x} \right), \end{aligned}$$

and clearly $|Z_r| < |Z|$. Considering the conditions needed for

$$\begin{aligned} &\|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(\Omega \setminus \overline{B_\alpha})}^2 - k^2 \|\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w}\|_{L^2(\Omega \setminus \overline{B_\alpha})}^2 > 0, \\ \mu_r^{-1} \left\| \nabla \times \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) \right\|_{L^2(B_\alpha)}^2 - k^2 \frac{\epsilon_*}{\epsilon_0} \left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)}^2 > 0, \end{aligned}$$

leads to $C(\Omega)\epsilon_0\mu_0\omega^2R^2 < 1$ and $C(B)\epsilon_*\mu_*\omega^2\alpha^2 < 1$ (assuming boundary terms can be absorbed) and, hence, the criteria in Section 4. In this case, we have

$$\begin{aligned} &\mu_r^{-1} \left\| \nabla \times \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w} - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) \right\|_{L^2(B_\alpha)} \\ &\leq |Z| = \left| \int_{B_\alpha} ((1 - \mu_r^{-1})\nabla \times (\mathbf{E}_0 - \mathbf{F}) \cdot \nabla \times \bar{\mathbf{v}} - k^2(1 - \epsilon_r)(\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}) \cdot \bar{\mathbf{v}}) \, d\mathbf{x} \right|. \end{aligned} \tag{5.16}$$

Also,

$$\left| \int_{B_\alpha} (1 - \mu_r^{-1})(\nabla \times (\mathbf{E}_0 - \mathbf{F}) \cdot \nabla \times \bar{\mathbf{v}}) \, d\mathbf{x} \right| \leq C|1 - \mu_r^{-1}|\alpha^{7/2} \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} \|\nabla \times \mathbf{v}\|_{L^2(B_\alpha)}, \tag{5.17}$$

and

$$\begin{aligned} \left| k^2 \int_{B_\alpha} (1 - \epsilon_r)(\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}) \cdot \bar{\mathbf{v}} \, d\mathbf{x} \right| &\leq k^2 |1 - \epsilon_r| \|\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}\|_{L^2(B_\alpha)} \|\mathbf{v}\|_{L^2(B_\alpha)} \\ &\leq C\alpha^2 k^2 |1 - \epsilon_r| \|\nabla \times (\mathbf{E}_0 - \mathbf{F})\|_{L^2(B_\alpha)} (\|\nabla \times \mathbf{v}\|_{L^2(B_\alpha)} + \|\mathbf{n} \cdot \mathbf{v}\|_{L^2(\Gamma_\alpha)}) \\ &\leq C\alpha^{7/2} |v| \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} (\|\nabla \times \mathbf{v}\|_{L^2(B_\alpha)} + \|\mathbf{n} \cdot \mathbf{v}\|_{L^2(\Gamma_\alpha)}), \end{aligned} \tag{5.18}$$

which follows since $\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}$ is divergence-free in B_α and has vanishing $\mathbf{n} \cdot (\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F})$ on Γ_α and so

$$\begin{aligned} \|\mathbf{E}_0 + \nabla \phi_0 - \mathbf{F}\|_{L^2(B_\alpha)} &\leq C\alpha \|\nabla \times (\mathbf{E}_0 - \mathbf{F})\|_{L^2(B_\alpha)} \leq C\alpha\alpha^{3/2}\alpha^2 \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} \\ &= C\alpha^{9/2} \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)}. \end{aligned} \tag{5.19}$$

Also, since \mathbf{v} is divergence-free, $\|\mathbf{v}\|_{L^2(B_\alpha)} \leq C\alpha(\|\nabla \times \mathbf{v}\|_{L^2(B_\alpha)} + \|\mathbf{n} \cdot \mathbf{v}\|_{L^2(\Gamma_\alpha)})$. Using (5.17) and (5.18) in (5.16), then

$$\begin{aligned} \mu_r^{-1} \left\| \nabla \times \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 - \mathbf{w} \right) \right\|_{L^2(B_\alpha)}^2 &\leq C|1 - \mu_r^{-1}| \alpha^{7/2} \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} \|\nabla \times \mathbf{v}\|_{L^2(B_\alpha)} \\ &\quad + C\alpha^{7/2} |\nu| \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} (\|\nabla \times \mathbf{v}\|_{L^2(B_\alpha)} + \|\mathbf{n} \cdot \mathbf{v}\|_{L^2(\Gamma_\alpha)}), \end{aligned} \tag{5.20}$$

and for $|\epsilon_r|$ such that $\|\mathbf{n} \cdot \mathbf{v}\|_{L^2(\Gamma_\alpha)} = \frac{1}{|\epsilon_r|} \|\mathbf{n} \cdot (\mathbf{E}_\Delta|_+ - \mathbf{w}|_+)\|_{L^2(\Gamma_\alpha)} \leq \|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)}$, then

$$\|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)} \leq C\mu_r \alpha^{7/2} (|1 - \mu_r^{-1}| + |\nu|) \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)}. \tag{5.21}$$

By additionally combining this with (5.13) completes the proof. □

Unlike the eddy current problem considered in [2], we are not guaranteed to have $\nabla_x \times \mathbf{H}_0(\mathbf{x}) = \mathbf{0}$ in B_α and so an integration by parts yields

$$\int_{B_\alpha} (1 - \mu_r^{-1}) \nabla \times \mathbf{F} \cdot \nabla \times \bar{\mathbf{v}} \, d\mathbf{x} = \int_{\Gamma_\alpha} [\tilde{\mu}_r^{-1} \nabla \times \mathbf{F} \times \mathbf{n}^-]_{\Gamma_\alpha} \cdot \bar{\mathbf{v}} \, d\mathbf{x} + \int_{B_\alpha} (1 - \mu_r^{-1}) \bar{\mathbf{v}} \cdot \nabla \times \nabla \times \mathbf{F} \, d\mathbf{x} \tag{5.22}$$

and, hence, the weak problem for \mathbf{w} becomes. Find $\mathbf{w} \in H(\text{curl})$ such that

$$\begin{aligned} &\int_{\Omega \setminus \bar{B}_\alpha} (\nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{v}} - k^2 \mathbf{w} \cdot \bar{\mathbf{v}}) \, d\mathbf{x} + \int_{B_\alpha} (\mu_r^{-1} \nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{v}} - k^2 \epsilon_r \mathbf{w} \cdot \bar{\mathbf{v}}) \, d\mathbf{x} \\ &\quad + ik \int_\Sigma \Lambda_e^{(k)} (\mathbf{n}^- \times \mathbf{w}) \cdot (\mathbf{n}^- \times \bar{\mathbf{v}}) \times \mathbf{n}^- \, d\mathbf{x} \\ &= \int_{\Gamma_\alpha} [\tilde{\mu}_r^{-1} \nabla \times \mathbf{F} \times \mathbf{n}^-]_{\Gamma_\alpha} \cdot \bar{\mathbf{v}} \, d\mathbf{x} - k^2 \int_{B_\alpha} ((1 - \epsilon_r) \mathbf{F} \cdot \bar{\mathbf{v}}) \, d\mathbf{x} \\ &\quad + \int_{B_\alpha} (1 - \mu_r^{-1}) \bar{\mathbf{v}} \cdot \nabla \times \nabla \times \mathbf{F} \, d\mathbf{x}, \end{aligned} \tag{5.23}$$

for all $\mathbf{v} \in H(\text{curl})$. This, in turn, motivates the strong form for \mathbf{w} as:

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{w} - k^2 \epsilon_r \mathbf{w} = (1 - \mu_r^{-1}) \nabla \times \nabla \times \mathbf{F} - k^2 (1 - \epsilon_r) \mathbf{F} \quad \text{in } B_\alpha, \tag{5.24a}$$

$$\nabla \times \nabla \times \mathbf{w} - k^2 \mathbf{w} = \mathbf{0} \quad \text{in } \Omega \setminus \bar{B}_\alpha, \tag{5.24b}$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega, \tag{5.24c}$$

$$\nabla \times \nabla \times \mathbf{w} - k^2 \mathbf{w} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \tag{5.24d}$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \tag{5.24e}$$

$$[\mathbf{n} \times \mathbf{w}]_{\Gamma_\alpha} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla \times \mathbf{w}]_{\Gamma_\alpha} = -(1 - \mu_r^{-1}) \mathbf{n} \times \nabla \times \mathbf{F} \quad \text{on } \Gamma_\alpha, \tag{5.24f}$$

$$[\mathbf{n} \times \mathbf{w}]_\Sigma = \mathbf{0}, \quad [\mathbf{n} \times \nabla \times \mathbf{w}]_\Sigma = \mathbf{0} \quad \text{on } \Sigma, \tag{5.24g}$$

$$\lim_{x \rightarrow \infty} x((\nabla \times \mathbf{w}) \times \hat{\mathbf{x}} - ik\mathbf{w}) = \mathbf{0}. \tag{5.24h}$$

which can also be expressed as:

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{w} - k^2 \epsilon_r \mathbf{w} = (1 - \mu_r^{-1}) \nabla \times \nabla \times \mathbf{F} - k^2 (1 - \epsilon_r) \mathbf{F} \quad \text{in } B_\alpha, \tag{5.25a}$$

$$\nabla \times \nabla \times \mathbf{w} - k^2 \mathbf{w} = \mathbf{0} \quad \text{in } B_\alpha^c, \tag{5.25b}$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } \mathbb{R}^3, \tag{5.25c}$$

$$[\mathbf{n} \times \mathbf{w}]_{\Gamma_\alpha} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla \times \mathbf{w}]_{\Gamma_\alpha} = -(1 - \mu_r^{-1}) \mathbf{n} \times \nabla \times \mathbf{F} \quad \text{on } \Gamma_\alpha, \tag{5.25d}$$

$$\lim_{x \rightarrow \infty} x((\nabla \times \mathbf{w}) \times \hat{\mathbf{x}} - ik\mathbf{w}) = \mathbf{0}. \tag{5.25e}$$

By introducing $\mathbf{w}(\mathbf{x}) = \alpha \mathbf{w}_0\left(\frac{\mathbf{x}-\mathbf{z}}{\alpha}\right) = \alpha \mathbf{w}_0(\boldsymbol{\xi})$, we see $\mathbf{w}_0(\boldsymbol{\xi})$ satisfies

$$\nabla_{\boldsymbol{\xi}} \times \mu_r^{-1} \nabla_{\boldsymbol{\xi}} \times \mathbf{w}_0 - k^2 \epsilon_r \alpha^2 \mathbf{w}_0 = (1 - \mu_r^{-1}) \nabla \times \nabla \times [\alpha \mathbf{F}] - k^2 \alpha^2 (1 - \epsilon_r) [\alpha^{-1} \mathbf{F}] \quad \text{in } B, \tag{5.26a}$$

$$\nabla_{\boldsymbol{\xi}} \times \nabla_{\boldsymbol{\xi}} \times \mathbf{w}_0 - k^2 \alpha^2 \mathbf{w}_0 = \mathbf{0} \quad \text{in } B^c, \tag{5.26b}$$

$$\nabla_{\boldsymbol{\xi}} \cdot \mathbf{w}_0 = 0 \quad \text{in } \mathbb{R}^3, \tag{5.26c}$$

$$[\mathbf{n} \times \mathbf{w}_0]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\boldsymbol{\xi}} \times \mathbf{w}_0]_{\Gamma} = -(1 - \mu_r^{-1}) \mathbf{n} \times \nabla \times [\alpha^{-1} \mathbf{F}] \quad \text{on } \Gamma, \tag{5.26d}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_{\boldsymbol{\xi}} \times \mathbf{w}_0) \times \hat{\boldsymbol{\xi}} - ik\alpha \mathbf{w}_0 \right) = \mathbf{0}, \tag{5.26e}$$

and, by introducing a bounded domain Ω_0 with boundary $\Sigma_0 := \partial\Omega_0$ the weak solution of (5.26) can be expressed in a similar way to (5.7).

Hence, Theorem 3.1 in [2], which follows directly from Lemma 5.1, is replaced by:

Theorem 5.2. For $|\epsilon_r|$ such that $\frac{1}{|\epsilon_r|} \|\mathbf{n} \cdot (\mathbf{E}_{\Delta}|_+ - \mathbf{w}|_+)\|_{L^2(\Gamma_\alpha)} \leq \|\nabla \times (\mathbf{E}_\alpha - \mathbf{E}_0 - \mathbf{w})\|_{L^2(B_\alpha)}$, and the conditions stated in Section 4, there exists a constant C such that

$$\left\| \nabla \times \left(\mathbf{E}_\alpha - \mathbf{E}_0 - \alpha \mathbf{w}_0 \left(\frac{\mathbf{x}-\mathbf{z}}{\alpha} \right) \right) \right\|_{L^2(B_\alpha)} \leq C \alpha^{7/2} \mu_r (|1 - \mu_r^{-1}| + |\nu|) \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)},$$

$$\left\| \mathbf{E}_\alpha - \mathbf{E}_0 - \alpha \mathbf{w}_0 \left(\frac{\mathbf{x}-\mathbf{z}}{\alpha} \right) - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right\|_{L^2(B_\alpha)} \leq C \alpha^{9/2} \mu_r (|1 - \mu_r^{-1}| + |\nu|) \|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)}.$$

From (5.3), we have

$$\alpha^{-1} \mathbf{F}(\alpha \boldsymbol{\xi} + \mathbf{z}) = \frac{ik}{2} \sum_{i=1}^3 (\mathbf{H}_0(\mathbf{z}))_i \mathbf{e}_i \times \boldsymbol{\xi} + \frac{ik\alpha}{3} \sum_{i,j=1}^3 (\mathbf{D}_z \mathbf{H}_0(\mathbf{z}))_{ij} \xi_j \mathbf{e}_i \times \boldsymbol{\xi} + \alpha^{-1} \sum_{i=1}^3 (\mathbf{E}_0(\mathbf{z}))_i \mathbf{e}_i,$$

and similarly

$$\mathbf{w}_0(\boldsymbol{\xi}) = \frac{ik}{2} \sum_{i=1}^3 (\mathbf{H}_0(\mathbf{z}))_i \boldsymbol{\theta}_i(\boldsymbol{\xi}) + \frac{ik\alpha}{3} \sum_{i,j=1}^3 (\mathbf{D}_z \mathbf{H}_0(\mathbf{z}))_{ij} \boldsymbol{\psi}_{ij}(\boldsymbol{\xi}) + \alpha^{-1} \sum_{i=1}^3 (\mathbf{E}_0(\mathbf{z}))_i \boldsymbol{\phi}_i(\boldsymbol{\xi}). \tag{5.27}$$

In the above equation, $\boldsymbol{\theta}_i(\boldsymbol{\xi})$ solves

$$\nabla_{\boldsymbol{\xi}} \times \mu_r^{-1} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i - k^2 \alpha^2 \epsilon_r \boldsymbol{\theta}_i = -k^2 \alpha^2 (1 - \epsilon_r) \mathbf{e}_i \times \boldsymbol{\xi} \quad \text{in } B, \tag{5.28a}$$

$$\nabla_{\boldsymbol{\xi}} \times \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i - k^2 \alpha^2 \boldsymbol{\theta}_i = \mathbf{0} \quad \text{in } B^c, \tag{5.28b}$$

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\theta}_i = 0 \quad \text{in } \mathbb{R}^3, \tag{5.28c}$$

$$[\mathbf{n} \times \boldsymbol{\theta}_i]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i]_{\Gamma} = -2(1 - \mu_r^{-1}) \mathbf{n} \times \mathbf{e}_i \quad \text{on } \Gamma, \tag{5.28d}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i) \times \hat{\boldsymbol{\xi}} - ik\alpha \boldsymbol{\theta}_i \right) = \mathbf{0}, \tag{5.28e}$$

$\boldsymbol{\psi}_{ij}(\boldsymbol{\xi})$ solves

$$\nabla_{\boldsymbol{\xi}} \times \mu_r^{-1} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\psi}_{ij} - k^2 \alpha^2 \epsilon_r \boldsymbol{\psi}_{ij} = (1 - \mu_r^{-1}) \mathbf{e}_j \times \mathbf{e}_i - k^2 \alpha^2 (1 - \epsilon_r) \xi_j \mathbf{e}_i \times \boldsymbol{\xi} \quad \text{in } B, \tag{5.29a}$$

$$\nabla_{\boldsymbol{\xi}} \times \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\psi}_{ij} - k^2 \alpha^2 \boldsymbol{\psi}_{ij} = \mathbf{0} \quad \text{in } B^c, \tag{5.29b}$$

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\psi}_{ij} = 0 \quad \text{in } \mathbb{R}^3, \tag{5.29c}$$

$$[\mathbf{n} \times \boldsymbol{\psi}_{ij}]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\psi}_{ij}]_{\Gamma} = -3(1 - \mu_r^{-1}) \mathbf{n} \times \xi_j \mathbf{e}_i \quad \text{on } \Gamma, \tag{5.29d}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_{\boldsymbol{\xi}} \times \boldsymbol{\psi}_{ij}) \times \hat{\boldsymbol{\xi}} - ik\alpha \boldsymbol{\psi}_{ij} \right) = \mathbf{0}, \tag{5.29e}$$

and $\phi_i(\xi)$ solves

$$\nabla_\xi \times \mu_r^{-1} \nabla_\xi \times \phi_i - k^2 \alpha^2 \epsilon_r \phi_i = -k^2 \alpha^2 (1 - \epsilon_r) \mathbf{e}_i \quad \text{in } B, \tag{5.30a}$$

$$\nabla_\xi \times \nabla_\xi \times \phi_i - k^2 \alpha^2 \phi_i = \mathbf{0} \quad \text{in } B^c, \tag{5.30b}$$

$$\nabla_\xi \cdot \phi_i = 0 \quad \text{in } \mathbb{R}^3, \tag{5.30c}$$

$$[\mathbf{n} \times \phi_i]_\Gamma = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_\xi \times \phi_i]_\Gamma = \mathbf{0} \quad \text{on } \Gamma, \tag{5.30d}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_\xi \times \phi_i) \times \hat{\xi} - ik\alpha \phi_i \right) = \mathbf{0}. \tag{5.30e}$$

Furthermore, given that the only source term is the gradient of a scalar, the solution to (5.30) can be expressed in terms of $\phi_i = (\epsilon_r - 1) \nabla \vartheta_i$ where ϑ_i solves

$$\nabla_\xi \cdot \epsilon_r \nabla_\xi \vartheta_i = 0 \quad \text{in } B, \tag{5.31a}$$

$$\nabla_\xi \cdot \nabla_\xi \vartheta_i = 0 \quad \text{in } B^c, \tag{5.31b}$$

$$[\vartheta_i]_\Gamma = 0 \quad \mathbf{n} \cdot \nabla_\xi \vartheta_i|_+ - \mathbf{n} \cdot \epsilon_r \nabla_\xi \vartheta_i|_- = \mathbf{n} \cdot \mathbf{e}_i \quad \text{on } \Gamma, \tag{5.31c}$$

$$\vartheta_i \rightarrow 0 \quad \text{as } |\xi| \rightarrow \infty. \tag{5.31d}$$

6. Integral representation formulae

A different integral representation formula for $\mathbf{H}_\Delta(\mathbf{x}) = (\mathbf{H}_\alpha - \mathbf{H}_0)(\mathbf{x})$ for \mathbf{x} away from B_α is required compared to that used in [2], since the eddy current approximation is no longer applied. The following is appropriate for describing the perturbed magnetic field outside of B_α and is the same as used in [6]:

$$\mathbf{H}_\Delta(\mathbf{x}) = \nabla_x \times \left(\int_{\Gamma_\alpha} G_k(\mathbf{x}, \mathbf{y}) \mathbf{n} \times \mathbf{H}_\Delta(\mathbf{y})|_+ d\mathbf{y} \right) - \frac{i}{k} \nabla_x \times \left(\nabla_x \times \int_{\Gamma_\alpha} G_k(\mathbf{x}, \mathbf{y}) \mathbf{n} \times \mathbf{E}_\Delta(\mathbf{y})|_+ d\mathbf{y} \right). \tag{6.1}$$

We transform this result to be expressed in terms of volume integrals over B_α and express the result in the lemma below.

Lemma 6.1. *An integral representation formula for $\mathbf{H}_\Delta(\mathbf{x}) = (\mathbf{H}_\alpha - \mathbf{H}_0)(\mathbf{x})$ for \mathbf{x} away from B_α expressed in terms of volume integrals over B_α is*

$$\begin{aligned} \mathbf{H}_\Delta(\mathbf{x}) = & -ik(\epsilon_r - 1) \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{E}_\alpha(\mathbf{y}) d\mathbf{y} + k^2(\mu_r - 1) \int_{B_\alpha} G_k(\mathbf{x}, \mathbf{y}) \mathbf{H}_\alpha(\mathbf{y}) d\mathbf{y} \\ & + (\mu_r - 1) \int_{B_\alpha} \mathbf{D}_y^2 G_k(\mathbf{x}, \mathbf{y}) \mathbf{H}_\alpha(\mathbf{y}) d\mathbf{y}. \end{aligned} \tag{6.2}$$

Proof. Using the transmission conditions (3.4), transforming the first term of (6.1) to a volume integral and then applying $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \nabla \cdot \mathbf{b} - \mathbf{b} \nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b}$ gives

$$\begin{aligned} \nabla_x \times \left(\int_{\Gamma_\alpha} G_k(\mathbf{x}, \mathbf{y}) \mathbf{n} \times \mathbf{H}_\Delta(\mathbf{y})|_+ d\mathbf{y} \right) &= \nabla_x \times \left(\int_{B_\alpha} \nabla_y \times (G_k(\mathbf{x}, \mathbf{y}) \mathbf{H}_\Delta(\mathbf{y})) d\mathbf{y} \right) \\ &= \nabla_x \times \left(\int_{B_\alpha} \nabla_y G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{H}_\Delta(\mathbf{y}) + G_k(\mathbf{x}, \mathbf{y}) \nabla_y \times \mathbf{H}_\Delta(\mathbf{y}) d\mathbf{y} \right) \\ &= \int_{B_\alpha} (\nabla_y G_k(\mathbf{x}, \mathbf{y}) (\nabla_x \cdot \mathbf{H}_\Delta(\mathbf{y})) - \mathbf{H}_\Delta(\mathbf{y}) \nabla_x \cdot \nabla_y G_k(\mathbf{x}, \mathbf{y}) \\ &\quad + (\mathbf{H}_\Delta(\mathbf{y}) \cdot \nabla_x) (\nabla_y G_k(\mathbf{x}, \mathbf{y})) - (\nabla_y G_k(\mathbf{x}, \mathbf{y}) \cdot \nabla_x) (\mathbf{H}_\Delta(\mathbf{y})) + \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \nabla_y \times \mathbf{H}_\Delta(\mathbf{y})) d\mathbf{y} \\ &= \int_{B_\alpha} (-k^2 \mathbf{H}_\Delta(\mathbf{y}) G_k(\mathbf{x}, \mathbf{y}) - \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) (\mathbf{H}_\Delta(\mathbf{y})) + \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \nabla_y \times \mathbf{H}_\Delta(\mathbf{y})) d\mathbf{y}, \end{aligned} \tag{6.3}$$

where $\nabla_x \cdot \nabla_y G_k(\mathbf{x}, \mathbf{y}) = -\nabla_x \cdot \nabla_x G_k(\mathbf{x}, \mathbf{y}) = k^2 G_k(\mathbf{x}, \mathbf{y})$ and $\nabla_x \nabla_y G_k(\mathbf{x}, \mathbf{y}) = -\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y})$ have been applied. Considering the second term in (6.1), and applying similar ideas, gives

$$\begin{aligned} \nabla_x \times \left(\nabla_x \times \int_{\Gamma_\alpha} G_k(\mathbf{x}, \mathbf{y}) \mathbf{n} \times \mathbf{E}_\Delta(\mathbf{y})_+ d\mathbf{y} \right) &= \nabla_x \times \left(\nabla_x \times \int_{B_\alpha} \nabla_y G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{E}_\Delta(\mathbf{y}) + G_k(\mathbf{x}, \mathbf{y}) \nabla_y \times \mathbf{E}_\Delta(\mathbf{y}) d\mathbf{y} \right) \\ &= \nabla_x \times \int_{B_\alpha} -k^2 \mathbf{E}_\Delta(\mathbf{y}) G_k(\mathbf{x}, \mathbf{y}) - \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) \mathbf{E}_\Delta(\mathbf{y}) + \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \nabla_y \times \mathbf{E}_\Delta(\mathbf{y}) d\mathbf{y} \\ &= \int_{B_\alpha} \left(-k^2 \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{E}_\Delta(\mathbf{y}) - \nabla_y \times \mathbf{E}_\Delta(\mathbf{y}) (\nabla_x \cdot \nabla_x G_k(\mathbf{x}, \mathbf{y})) + (\nabla_y \times \mathbf{E}_\Delta(\mathbf{y}) \cdot \nabla_x) \nabla_x G_k(\mathbf{x}, \mathbf{y}) \right) d\mathbf{y} \\ &= \int_{B_\alpha} \left(-k^2 \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{E}_\Delta(\mathbf{y}) + k^2 \nabla_y \times \mathbf{E}_\Delta(\mathbf{y}) G_k(\mathbf{x}, \mathbf{y}) + \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) (\nabla_y \times \mathbf{E}_\Delta(\mathbf{y})) \right) d\mathbf{y}. \end{aligned} \tag{6.4}$$

Using (6.3) and (6.4) in (6.1) gives

$$\begin{aligned} \mathbf{H}_\Delta(\mathbf{x}) &= -k^2 \int_{B_\alpha} \mathbf{H}_\Delta(\mathbf{y}) G_k(\mathbf{x}, \mathbf{y}) d\mathbf{y} - \int_{B_\alpha} \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) (\mathbf{H}_\Delta(\mathbf{y})) d\mathbf{y} + \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \nabla_y \times \mathbf{H}_\Delta(\mathbf{y}) d\mathbf{y} \\ &\quad - ik \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{E}_\Delta(\mathbf{y}) d\mathbf{y} - ik \int_{B_\alpha} \nabla_y \times \mathbf{E}_\Delta(\mathbf{y}) G_k(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\ &\quad - \frac{i}{k} \int_{B_\alpha} \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) (\nabla_y \times \mathbf{E}_\Delta(\mathbf{y})) d\mathbf{y}. \end{aligned} \tag{6.5}$$

Then, using (3.3), we have

$$\nabla \times \mathbf{E}_\Delta = ik\tilde{\mu}_r \mathbf{H}_\Delta + ik(\tilde{\mu}_r - 1)\mathbf{H}_0, \tag{6.6a}$$

$$\nabla \times \mathbf{H}_\Delta = -ik\tilde{\epsilon}_r \mathbf{E}_\Delta - ik(\tilde{\epsilon}_r - 1)\mathbf{E}_0, \tag{6.6b}$$

and inserting these into (6.5) completes the proof. □

7. Proof of Theorem 4.1

We write (6.2) as the sum:

$$\begin{aligned} \mathbf{H}_\Delta(\mathbf{x}) &= -ik(\epsilon_r - 1) \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \mathbf{E}_\alpha(\mathbf{y}) d\mathbf{y} + (\mu_r - 1) \int_{B_\alpha} (\mathbf{D}_y^2 G_k(\mathbf{x}, \mathbf{y}) + k^2 G_k(\mathbf{x}, \mathbf{y})) \mathbf{H}_\alpha(\mathbf{y}) d\mathbf{y} \\ &= \mathbf{T}_1 + \mathbf{T}_2. \end{aligned} \tag{7.1}$$

Approximation of \mathbf{T}_1

Let $\mathbf{T}_1 = \mathbf{T}_1^a + \mathbf{T}_1^b + \mathbf{T}_1^c + \mathbf{T}_1^d$ where

$$\mathbf{T}_1^a := -ik(\epsilon_r - 1) \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \left(\mathbf{E}_\alpha(\mathbf{y}) - \mathbf{E}_0(\mathbf{y}) - \alpha \mathbf{w}_0 \left(\frac{\mathbf{y} - \mathbf{z}}{\alpha} \right) - \frac{\epsilon_r - 1}{\epsilon_r} \nabla \phi_0 \right) d\mathbf{y},$$

$$\mathbf{T}_1^b := -ik(\epsilon_r - 1) \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times (\mathbf{E}_0(\mathbf{y}) + \nabla \phi_0 - \mathbf{F}(\mathbf{y})) d\mathbf{y} - ik \frac{(\epsilon_r - 1)}{\epsilon_r} \int_{B_\alpha} \nabla_x G_k(\mathbf{x}, \mathbf{y}) \times \nabla \phi_0 d\mathbf{y},$$

$$\mathbf{T}_1^c := -ik(\epsilon_r - 1) \int_{B_\alpha} (\nabla_x G_k(\mathbf{x}, \mathbf{y}) - \nabla_x G_k(\mathbf{x}, \mathbf{z}) + \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z})(\mathbf{y} - \mathbf{z})) \times \left(\mathbf{F}(\mathbf{y}) + \alpha \mathbf{w}_0 \left(\frac{\mathbf{y} - \mathbf{z}}{\alpha} \right) \right) d\mathbf{y},$$

$$\mathbf{T}_1^d := -ik(\epsilon_r - 1) \int_{B_\alpha} (\nabla_x G_k(\mathbf{x}, \mathbf{z}) - \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z})(\mathbf{y} - \mathbf{z})) \times \left(\mathbf{F}(\mathbf{y}) + \alpha \mathbf{w}_0 \left(\frac{\mathbf{y} - \mathbf{z}}{\alpha} \right) \right) d\mathbf{y}.$$

Using Theorem 5.2, we estimate

$$\begin{aligned} |\mathbf{T}_1^a| &\leq Ck|\epsilon_r - 1|\alpha^{3/2}\alpha^{9/2}\|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)}\mu_r(|1 - \mu_r^{-1}| + |\nu|) \\ &\leq C\alpha^4\|\mathbf{H}_0\|_{W^{2,\infty}(B_\alpha)}. \end{aligned}$$

Next, using (5.19) and (5.6) we estimate

$$\begin{aligned} |\mathbf{T}_1^b| &\leq Ck|\epsilon_r - 1|\alpha^{3/2}\alpha^{9/2}\|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} + Ck^3\left|1 - \frac{1}{\epsilon_r}\right|\alpha^{3/2}\alpha^{3/2}\alpha^2\|\mathbf{E}_0\|_{W^{1,\infty}(B_\alpha)} \\ &\leq C\left(\alpha^4\|\mathbf{H}_0\|_{W^{2,\infty}(B_\alpha)} + k^3\alpha^5\left|1 - \frac{1}{\epsilon_r}\right|\|\mathbf{E}_0\|_{W^{1,\infty}(B_\alpha)}\right). \end{aligned}$$

Then, using (5.3) and (5.27),

$$\begin{aligned} |\mathbf{T}_1^c| &\leq Ck|\epsilon_r - 1|\alpha^{3/2}\alpha^2\alpha^{3/2}\alpha\|\nabla \times \mathbf{E}_0\|_{W^{2,\infty}(B_\alpha)} \\ &\leq C\alpha^4\|\mathbf{H}_0\|_{W^{2,\infty}(B_\alpha)}. \end{aligned}$$

Similarly, $\mathbf{T}_1^d = \mathbf{T}_1^{d,1} + \mathbf{T}_1^{d,2} + \mathbf{T}_1^{d,3} + \mathbf{T}_1^{d,4} + \mathbf{T}_1^{d,5} + \mathbf{T}_1^{d,6}$ where

$$\begin{aligned} \mathbf{T}_1^{d,1} &:= -\frac{i k \alpha^4 (\epsilon_r - 1)}{2} \sum_{i=1}^3 \nabla_x G_k(\mathbf{x}, \mathbf{z}) \times \int_B (\mathbf{e}_i \times \boldsymbol{\xi} + \boldsymbol{\theta}_i) d\boldsymbol{\xi} (i k \mathbf{H}_0(\mathbf{z}))_i, \\ \mathbf{T}_1^{d,2} &:= -i k \alpha^3 (\epsilon_r - 1) \sum_{i=1}^3 \nabla_x G_k(\mathbf{x}, \mathbf{z}) \times \int_B (\mathbf{e}_i + \boldsymbol{\phi}_i) d\boldsymbol{\xi} (\mathbf{E}_0(\mathbf{z}))_i, \\ \mathbf{T}_1^{d,3} &:= -\frac{i k \alpha^5 (\epsilon_r - 1)}{3} \sum_{i,j=1}^3 \nabla_x G_k(\mathbf{x}, \mathbf{z}) \times \int_B (\xi_j \mathbf{e}_i \times \boldsymbol{\xi} + \boldsymbol{\psi}_{ij}) d\boldsymbol{\xi} (\mathbf{D}_z(i k \mathbf{H}_0(\mathbf{z})))_{ij}, \\ \mathbf{T}_1^{d,4} &:= \frac{i k \alpha^5 (\epsilon_r - 1)}{2} \sum_{i=1}^3 \int_B \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) \boldsymbol{\xi} \times (\mathbf{e}_i \times \boldsymbol{\xi} + \boldsymbol{\theta}_i) d\boldsymbol{\xi} (i k \mathbf{H}_0(\mathbf{z}))_i, \\ \mathbf{T}_1^{d,5} &:= -i k \alpha^4 (\epsilon_r - 1) \sum_{i=1}^3 \int_B \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) \boldsymbol{\xi} \times (\mathbf{e}_i + \boldsymbol{\phi}_i) d\boldsymbol{\xi} (\mathbf{E}_0(\mathbf{z}))_i, \\ \mathbf{T}_1^{d,6} &:= -\frac{i k \alpha^6 (\epsilon_r - 1)}{3} \sum_{i,j=1}^3 \int_B \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) \boldsymbol{\xi} \times (\xi_j \mathbf{e}_i \times \boldsymbol{\xi} + \boldsymbol{\psi}_{ij}) d\boldsymbol{\xi} (\mathbf{D}_z(i k \mathbf{H}_0(\mathbf{z})))_{ij}, \end{aligned}$$

Defining $\mathbf{R}(\mathbf{x})$ such that $|\mathbf{R}(\mathbf{x})| \leq C\left(\alpha^4\|\mathbf{H}_0\|_{W^{2,\infty}(B_\alpha)} + \alpha^4k\left(|\epsilon_r - 1| + k^2\alpha\left|1 - \frac{1}{\epsilon_r}\right|\right)\|\mathbf{E}_0\|_{W^{1,\infty}(B_\alpha)}\right)$, we see that $\mathbf{T}_1^a, \mathbf{T}_1^b, \mathbf{T}_1^c$ and $\mathbf{T}_1^{d,5}$ are elements of $\mathbf{R}(\mathbf{x})$ and, since $\mathbf{D}_z(i k \mathbf{H}_0(\mathbf{z}))$ is related to $\mathbf{E}_0, \mathbf{T}_1^{d,3}$ and $\mathbf{T}_1^{d,6}$ are also elements of $\mathbf{R}(\mathbf{x})$.

Remark 7.1. In the case of the eddy current regime, where (5.28) and (5.30) simplify to

$$\nabla_{\boldsymbol{\xi}} \times \mu_r^{-1} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i - \nu_i \boldsymbol{\theta}_i = \nu_i \mathbf{e}_i \times \boldsymbol{\xi} \quad \text{in } B, \tag{7.2a}$$

$$\nabla_{\boldsymbol{\xi}} \times \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i = \mathbf{0} \quad \text{in } B^c, \tag{7.2b}$$

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\theta}_i = 0 \quad \text{in } \mathbb{R}^3, \tag{7.2c}$$

$$[\mathbf{n} \times \boldsymbol{\theta}_i]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\theta}_i]_{\Gamma} = -2(1 - \mu_r^{-1}) \mathbf{n} \times \mathbf{e}_i \quad \text{on } \Gamma, \tag{7.2d}$$

$$\boldsymbol{\theta}_i = O(|\boldsymbol{\xi}|^{-1}) \quad \text{as } |\boldsymbol{\xi}| \rightarrow \infty, \tag{7.2e}$$

and

$$\nabla_{\xi} \times \mu_r^{-1} \nabla_{\xi} \times \phi_i - \nu_i \phi_i = \nu_i e_i \quad \text{in } B, \tag{7.3a}$$

$$\nabla_{\xi} \times \nabla_{\xi} \times \phi_i = \mathbf{0} \quad \text{in } B^c \tag{7.3b}$$

$$\nabla_{\xi} \cdot \phi_i = 0 \quad \text{in } \mathbb{R}^3, \tag{7.3c}$$

$$[\mathbf{n} \times \phi_i]_{\Gamma} = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_{\xi} \times \phi_i]_{\Gamma} = \mathbf{0} \quad \text{on } \Gamma, \tag{7.3d}$$

$$\phi_i = O(|\xi|^{-1}) \quad \text{as } |\xi| \rightarrow \infty, \tag{7.3e}$$

respectively, we find that $\mathbf{T}_1^{d,1} = \mathbf{T}_1^{d,2} = \mathbf{0}$ by using the above transmission problems and integration by parts. Similarly, it can be shown that $\mathbf{T}_1^{d,3} = \mathbf{0}$ by simplifying the transmission problem for ψ_{ij} in a similar way to the above.

In general, we write

$$\begin{aligned} \mathbf{T}_1 = & -ik e_j (\nabla_x G_k(\mathbf{x}, \mathbf{z}))_p \varepsilon_{jpr} (\mathcal{A}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathcal{B}_{ri}(\mathbf{E}_0(\mathbf{z}))_i) \\ & + e_j ((\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{\ell m} \mathcal{P}_{\ell mji}(\mathbf{H}_0(\mathbf{z}))_i + \mathbf{R}(\mathbf{x})), \end{aligned} \tag{7.4}$$

where

$$\mathcal{A}_{ri} := \frac{ik\alpha^4(\epsilon_r - 1)}{2} e_r \cdot \int_B (e_i \times \xi + \theta_i) d\xi, \tag{7.5a}$$

$$\mathcal{B}_{ri} := \alpha^3(\epsilon_r - 1) e_r \cdot \int_B (e_i + \phi_i) d\xi, \tag{7.5b}$$

$$\begin{aligned} \mathcal{P}_{\ell mji} &:= -\frac{k^2\alpha^5(\epsilon_r - 1)}{2} e_j \cdot \int_B e_{\ell} \times (\xi_m (e_i \times \xi + \theta_i)) d\xi \\ &= \varepsilon_{j\ell s} \mathcal{C}_{msi}, \end{aligned} \tag{7.5c}$$

$$\mathcal{C}_{msi} := \frac{1}{2} \varepsilon_{sj\ell} \mathcal{P}_{\ell mji} = -\frac{k^2\alpha^5(\epsilon_r - 1)}{2} e_s \cdot \int_B \xi_m (e_i \times \xi + \theta_i) d\xi, \tag{7.5d}$$

and the latter follows by considering the skew symmetry of \mathcal{P} w.r.t indices j and ℓ and applying similar arguments to [5]. Further simplifications are also possible if the eddy current approximation applies [5], and we shall consider these later in Section 8.

Approximation of \mathbf{T}_2

Let $\mathbf{T}_2 = (\mu_r - 1)(\mathbf{T}_2^a + \mathbf{T}_2^b + \mathbf{T}_2^c + \mathbf{T}_2^d)$ where

$$\mathbf{T}_2^a := \int_{B_{\alpha}} (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) + k^2 G_k(\mathbf{x}, \mathbf{y})) \mathbb{I} \left(\mathbf{H}_{\alpha}(\mathbf{y}) - \mu_r^{-1} \mathbf{H}_0(\mathbf{y}) - \mu_r^{-1} \mathbf{H}_0^* \left(\frac{\mathbf{y} - \mathbf{z}}{\alpha} \right) \right) d\mathbf{y},$$

$$\mathbf{T}_2^b := \mu_r^{-1} \int_{B_{\alpha}} (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{y}) - \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 (G_k(\mathbf{x}, \mathbf{y}) - G_k(\mathbf{x}, \mathbf{z}))) \mathbb{I} \left(\mathbf{H}_0(\mathbf{y}) + \mathbf{H}_0^* \left(\frac{\mathbf{y} - \mathbf{z}}{\alpha} \right) \right) d\mathbf{y},$$

$$\mathbf{T}_2^c := \mu_r^{-1} \int_{B_{\alpha}} (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 G_k(\mathbf{v}, \mathbf{z})) \mathbb{I} (\mathbf{H}_0(\mathbf{y}) - \mathbf{H}_0(\mathbf{z})) d\mathbf{y},$$

$$\mathbf{T}_2^d := \mu_r^{-1} \int_{B_{\alpha}} (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 G_k(\mathbf{x}, \mathbf{z})) \mathbb{I} \left(\mathbf{H}_0(\mathbf{z}) + \mathbf{H}_0^* \left(\frac{\mathbf{y} - \mathbf{z}}{\alpha} \right) \right) d\mathbf{y},$$

and $\mathbf{H}_0^*(\xi) = \frac{1}{ik} \nabla \times \mathbf{w}_0$. Following similar arguments to the above, and in [2], we have

$$|\mathbf{T}_2^a| \leq C\alpha^4 \|\mathbf{H}_0\|_{W^{2,\infty}(B_{\alpha})},$$

$$|\mathbf{T}_2^b| \leq C\alpha^4 \|\mathbf{H}_0\|_{W^{2,\infty}(B_{\alpha})},$$

$$|\mathbf{T}_2^c| \leq C\alpha^4 \|\mathbf{H}_0\|_{W^{2,\infty}(B_{\alpha})},$$

and we can express T_2^d as:

$$\begin{aligned} T_2^d &= \mu_r^{-1} \alpha^3 \left(\sum_{i=1}^3 (D_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 G_k(\mathbf{x}, \mathbf{z})) \mathbb{I} \int_B \left(\mathbf{e}_i + \frac{1}{2} \nabla \times \boldsymbol{\theta}_i \right) d\xi (\mathbf{H}_0(\mathbf{z}))_i \right) \\ &\quad + \mu_r^{-1} \alpha^4 \left(\sum_{i,j=1}^3 (D_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 G_k(\mathbf{x}, \mathbf{z})) \mathbb{I} \int_B \left(\xi_j \mathbf{e}_i + \frac{1}{3} \nabla \times \boldsymbol{\psi}_{ij} \right) d\xi (D_z(\mathbf{H}_0(\mathbf{z})))_{ij} \right) \\ &= \mu_r^{-1} \alpha^3 \sum_{i=1}^3 (D_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 G_k(\mathbf{x}, \mathbf{z})) \mathbb{I} \int_B \left(\mathbf{e}_i + \frac{1}{2} \nabla \times \boldsymbol{\theta}_i \right) d\xi (\mathbf{H}_0(\mathbf{z}))_i + \mathbf{R}(\mathbf{x}). \end{aligned}$$

So that

$$T_2 = (D_x^2 G_k(\mathbf{x}, \mathbf{z}) + k^2 G_k(\mathbf{x}, \mathbf{z})) \mathbb{I} e_j \mathcal{N}_{ji}(\mathbf{H}_0(\mathbf{z}))_i + \mathbf{R}(\mathbf{x}), \tag{7.6}$$

where

$$\mathcal{N}_{ji} := \alpha^3 (1 - \mu_r^{-1}) \mathbf{e}_j \cdot \int_B \left(\mathbf{e}_i + \frac{1}{2} \nabla \times \boldsymbol{\theta}_i \right) d\xi. \tag{7.7}$$

8. Simplifications and alternative forms

Lemma 8.1. Using $\boldsymbol{\phi}_i = (\epsilon_r - 1) \nabla \vartheta_i$ where ϑ_i solves (5.31) then \mathcal{B}_{ri} becomes

$$\mathcal{B}_{ri} = \alpha^3 \left((\epsilon_r - 1) |B| \delta_{ri} + (\epsilon_r - 1)^2 \int_B \mathbf{e}_r \cdot \nabla \vartheta_i d\xi \right), \tag{8.1}$$

which are also the coefficients of the symmetric Pólya–Szegő tensor $\mathcal{T}[\alpha B, \epsilon_r]$ of an object B_α for a contrast ϵ_r .

Proof. The proof directly follows from (7.5b) and (5.31). □

Lemma 8.2. Up to a residual term, \mathcal{C}_{msi} is skew-symmetric w.r.t. m and i and hence

$$\mathcal{C}_{msi} = \epsilon_{msr} \check{\mathcal{C}}_{ri} + R_{msi}, \quad \check{\mathcal{C}}_{ri} := -\frac{\nu \alpha^3}{4} \mathbf{e}_r \cdot \int_B \boldsymbol{\xi} \times (\boldsymbol{\theta}_i + \mathbf{e}_i \times \boldsymbol{\xi}) d\xi, \tag{8.2}$$

where $R_{msi} = O(\alpha^4)$ for fixed k as $\alpha \rightarrow 0$.

Proof. We first write the transmission problem for $\boldsymbol{\theta}_i$ in an alternative form (5.28):

$$\nabla_\xi \times \mu_r^{-1} \nabla_\xi \times \boldsymbol{\theta}_i - k^2 \alpha^2 \epsilon_r \boldsymbol{\theta}_i = -k^2 \alpha^2 (1 - \epsilon_r) \mathbf{e}_i \times \boldsymbol{\xi} \quad \text{in } B, \tag{8.3a}$$

$$\nabla_\xi \times \nabla_\xi \times \boldsymbol{\theta}_i - k^2 \alpha^2 \boldsymbol{\theta}_i = \mathbf{0} \quad \text{in } \Omega_0 \setminus \overline{B}, \tag{8.3b}$$

$$\nabla_\xi \cdot \boldsymbol{\theta}_i = 0 \quad \text{in } \Omega_0, \tag{8.3c}$$

$$\nabla_\xi \times \nabla_\xi \times \boldsymbol{\theta}_i - k^2 \alpha^2 \boldsymbol{\theta}_i = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_0}, \tag{8.3d}$$

$$\nabla_\xi \cdot \boldsymbol{\theta}_i = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_0}, \tag{8.3e}$$

$$[\mathbf{n} \times \boldsymbol{\theta}_i]_\Gamma = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_\xi \times \boldsymbol{\theta}_i]_\Gamma = -2(1 - \mu_r^{-1}) \mathbf{n} \times \mathbf{e}_i \quad \text{on } \Gamma, \tag{8.3f}$$

$$[\mathbf{n} \times \boldsymbol{\theta}_i]_{\Sigma_0} = \mathbf{0}, \quad [\mathbf{n} \times \nabla_\xi \times \boldsymbol{\theta}_i]_{\Sigma_0} = \mathbf{0} \quad \text{on } \Sigma_0, \tag{8.3g}$$

$$\lim_{\xi \rightarrow \infty} \xi \left((\nabla_\xi \times \boldsymbol{\theta}_i) \times \hat{\boldsymbol{\xi}} - i k \alpha \boldsymbol{\theta}_i \right) = \mathbf{0}, \tag{8.3h}$$

where Ω_0 and Σ_0 are introduced in a similar way to Ω and Σ in Section 3.2. Then, an integration by parts can be applied to (7.5) in a similar manner to the proof of Lemma 4.2 in [5] leading to

$$\begin{aligned}
 -\frac{2C_{msi}}{\alpha^3} &= \int_B \mathbf{e}_m \times \mathbf{e}_s \cdot \mu_r^{-1} \nabla \times \boldsymbol{\theta}_i d\xi - \alpha^2 k^2 \int_B \boldsymbol{\theta}_i \cdot \mathbf{e}_s \xi_m d\xi \\
 &\quad - 2[\tilde{\mu}_r^{-1}]_\Gamma \int_B \mathbf{e}_i \cdot \mathbf{e}_m \times \mathbf{e}_s d\xi - \int_\Gamma \mathbf{n}^+ \cdot \nabla \times \boldsymbol{\theta}_i \times (\xi_m \mathbf{e}_s)|_+ d\xi.
 \end{aligned} \tag{8.4}$$

Also,

$$\begin{aligned}
 \int_\Gamma \mathbf{n}^+ \cdot \nabla \times \boldsymbol{\theta}_i \times (\xi_m \mathbf{e}_s)|_+ d\xi &= - \int_{\Sigma_0} (\mathbf{n}^- \times (\xi_m \mathbf{e}_s)) \times \mathbf{n}^- \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_i|_- d\xi \\
 &\quad + \int_{\Omega_0 \setminus \bar{B}} (\alpha^2 k^2 \boldsymbol{\theta}_i \cdot \mathbf{e}_s \xi_m - \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times (\xi_m \mathbf{e}_s)) d\xi \\
 &= -ik\alpha \int_{\Sigma_0} (\mathbf{n}^- \times (\xi_m \mathbf{e}_s)) \times \mathbf{n}^- \cdot \Lambda_e^{(\alpha k)}(\mathbf{n}^- \times \boldsymbol{\theta}_i) d\xi \\
 &\quad + \int_{\Omega_0 \setminus \bar{B}} (\alpha^2 k^2 \boldsymbol{\theta}_i \cdot \mathbf{e}_s \xi_m d\xi - \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times (\xi_m \mathbf{e}_s)) d\xi.
 \end{aligned} \tag{8.5}$$

Thus, we find (8.2) holds with

$$R_{msi} = \frac{i\alpha^4 k}{2} \int_{\Sigma_0} (\mathbf{n}^- \times (\xi_m \mathbf{e}_s)) \times \mathbf{n}^- \cdot \Lambda_e^{(\alpha k)}(\mathbf{n}^- \times \boldsymbol{\theta}_i) d\xi - \frac{\alpha^5 k^2}{2} \mathbf{e}_s \cdot \int_{\Omega_0} \xi_m \boldsymbol{\theta}_i d\xi, \tag{8.6}$$

which is not skew-symmetric with respect to m and i , and consequently we see the estimate holds. \square

Corollary 8.3. *If v reduces to v_i , the transmission problem for $\boldsymbol{\theta}_i(\boldsymbol{\xi})$ provided in (5.28) reduces to (7.2) and $R_{msi} = 0$, since this is now identical to the case considered in Lemma 4.2 of [5].*

By differentiation, it is easily established that

$$\begin{aligned}
 \nabla_x G_k(\mathbf{x}, \mathbf{z}) &= \frac{(\mathbf{x} - \mathbf{z})}{4\pi |\mathbf{x} - \mathbf{z}|^2} \left(ik - \frac{1}{|\mathbf{x} - \mathbf{z}|} \right) e^{ik|\mathbf{x} - \mathbf{z}|}, \\
 \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) &= \frac{1}{4\pi} \left(\frac{1}{|\mathbf{x} - \mathbf{z}|^3} \left(\frac{3(\mathbf{x} - \mathbf{z}) \otimes (\mathbf{x} - \mathbf{z})}{|\mathbf{x} - \mathbf{z}|^2} - \mathbb{I} \right) - \frac{ik}{|\mathbf{x} - \mathbf{z}|^2} \left(\frac{3(\mathbf{x} - \mathbf{z}) \otimes (\mathbf{x} - \mathbf{z})}{|\mathbf{x} - \mathbf{z}|^2} - \mathbb{I} \right) \right. \\
 &\quad \left. - \frac{k^2}{|\mathbf{x} - \mathbf{z}|^3} (\mathbf{x} - \mathbf{z}) \otimes (\mathbf{x} - \mathbf{z}) \right) e^{ik|\mathbf{x} - \mathbf{z}|},
 \end{aligned}$$

and, by introducing $\mathbf{r} := \mathbf{x} - \mathbf{z}$, $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$, these derivatives can be expressed as:

$$\begin{aligned}
 \nabla_x G_k(\mathbf{x}, \mathbf{z}) &= \frac{\hat{\mathbf{r}}}{4\pi r} \left(ik - \frac{1}{r} \right) e^{ikr}, \\
 \mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}) &= \frac{1}{4\pi} \left(\frac{1}{r^3} (3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \mathbb{I}) - \frac{ik}{r^2} (3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \mathbb{I}) - \frac{k^2}{r} \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \right) e^{ikr}.
 \end{aligned}$$

Noting that $C_{msi} = \varepsilon_{msi} \check{C}_{ri} + R_{msi}$ by Lemma 8.2, we get

$$\begin{aligned}
 (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{\ell m} \varepsilon_{j\ell s} \varepsilon_{msr} \check{C}_{ri} &= -(\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{\ell m} \varepsilon_{sjt} \varepsilon_{smr} \check{C}_{ri} \\
 &= -(\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{\ell m} (\delta_{jm} \delta_{\ell r} - \delta_{jr} \delta_{\ell m}) \check{C}_{ri} \\
 &= (- (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{rj} + (\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{mm} \delta_{jr}) \check{C}_{ri} \\
 &= -(\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{jr} + k^2 \delta_{jr} G_k(\mathbf{x}, \mathbf{z}) \check{C}_{ri}.
 \end{aligned}$$

Furthermore, we introduce the rank 2 tensor \mathcal{M} with coefficients $\mathcal{M}_{ri} := \mathcal{N}_{ri} - \check{\mathcal{C}}_{ri}$, which is symmetric as the following lemma shows:

Lemma 8.4. *The tensor $\mathcal{M} = \mathcal{N} - \check{\mathcal{C}}$ is complex symmetric with coefficients satisfying $\mathcal{M}_{ri} = \mathcal{M}_{ir}$.*

Proof. By an application of an integration by parts to $\check{\mathcal{C}}_{ri}$, we find

$$\begin{aligned} \frac{-\check{\mathcal{C}}_{ri}}{\alpha^3} &= \frac{1}{4\nu} \int_B \nabla \times \mu_r^{-1} \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times \mu_r^{-1} \nabla \times \boldsymbol{\theta}_r \, d\xi \\ &\quad - \frac{1}{4} \left(\frac{\epsilon_r}{\epsilon_r - 1} - 1 \right) \left(\int_{\Omega_0} \tilde{\mu}_r^{-1} \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times \boldsymbol{\theta}_r \, d\xi - \int_{\Omega_0 \setminus \bar{B}} \alpha^2 k^2 \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_r \, d\xi - 2[\tilde{\mu}_r^{-1}] \int_B \mathbf{e}_r \cdot \nabla \times \boldsymbol{\theta}_i \, d\xi \right. \\ &\quad \left. - \int_{\Sigma_0} \boldsymbol{\theta}_r \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_i|_- \, d\xi \right) \\ &\quad - \frac{1}{4} \left(\frac{\epsilon_r}{\epsilon_r - 1} \right) \left(\int_{\Omega_0} \tilde{\mu}_r^{-1} \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times \boldsymbol{\theta}_r \, d\xi - \int_{\Omega_0 \setminus \bar{B}} \alpha^2 k^2 \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_r \, d\xi - 2[\tilde{\mu}_r^{-1}] \int_B \mathbf{e}_i \cdot \nabla \times \boldsymbol{\theta}_r \, d\xi \right. \\ &\quad \left. - \int_{\Sigma_0} \boldsymbol{\theta}_i \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_r|_- \, d\xi \right) \\ &\quad + \frac{1}{4} \left(\frac{\epsilon_r}{\epsilon_r - 1} - 1 \right) \left(\int_B \alpha^2 k^2 \epsilon_r \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_r \, d\xi \right), \end{aligned}$$

so that

$$\begin{aligned} \frac{\mathcal{N}_{ri} - \check{\mathcal{C}}_{ri}}{\alpha^3} &= \frac{1}{4\nu} \int_B \nabla \times \mu_r^{-1} \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times \mu_r^{-1} \nabla \times \boldsymbol{\theta}_r \, d\xi + \\ &\quad \frac{1}{4} \int_{\Omega_0} \tilde{\mu}_r^{-1} \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times \boldsymbol{\theta}_r \, d\xi + [\tilde{\mu}_r^{-1}]_\Gamma \int_B \delta_{ri} \, d\xi - \frac{\alpha^2 k^2}{4} \int_{\Omega_0} \tilde{\epsilon}_r \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_r \, d\xi - \frac{1}{4} \int_{\Sigma_0} \boldsymbol{\theta}_i \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_r|_- \, d\xi \\ &\quad - \frac{1}{2} \left(\frac{\epsilon_r}{\epsilon_r - 1} \right) \left(\int_{\Omega_0} \tilde{\mu}_r^{-1} \nabla \times \boldsymbol{\theta}_i \cdot \nabla \times \boldsymbol{\theta}_r \, d\xi - \int_{\Omega_0 \setminus \bar{B}} \alpha^2 k^2 \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_r \, d\xi - [\tilde{\mu}_r^{-1}]_\Gamma \int_B (\mathbf{e}_r \cdot \nabla \times \boldsymbol{\theta}_i + \mathbf{e}_i \cdot \nabla \times \boldsymbol{\theta}_r) \, d\xi \right. \\ &\quad \left. - \int_{\Sigma_0} \boldsymbol{\theta}_r \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_i|_- \, d\xi - \int_{\Sigma_0} \boldsymbol{\theta}_i \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_r|_- \, d\xi \right) \\ &\quad + \frac{1}{4} \left(\frac{\epsilon_r}{\epsilon_r - 1} \right) \left(\int_B \alpha^2 k^2 \epsilon_r \boldsymbol{\theta}_i \cdot \boldsymbol{\theta}_r \, d\xi \right), \tag{8.7} \end{aligned}$$

and, hence, $\mathcal{N}_{ri} - \check{\mathcal{C}}_{ri}$ is symmetric in r and i up to the term $\frac{\alpha^2}{4} \int_{\Sigma_0} \boldsymbol{\theta}_i \cdot \mathbf{n}^- \times \nabla \times \boldsymbol{\theta}_r|_- \, d\xi = \frac{i\alpha^4 k}{4} \int_{\Sigma_0} (\mathbf{n}^- \times \boldsymbol{\theta}_i) \times \mathbf{n}^- \cdot \Lambda_e^{(ak)}(\mathbf{n}^- \times \boldsymbol{\theta}_r) \, d\xi$; however, we note this remaining term can be absorbed in to R_{msi} . \square

It follows that an alternative form to (4.1) is

$$\begin{aligned} (\mathbf{H}_\Delta(\mathbf{x})) &= -i k e_j (\nabla_x G_k(\mathbf{x}, \mathbf{z}))_p \varepsilon_{jpr} (\mathcal{A}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathcal{B}_{ri}(\mathbf{E}_0(\mathbf{z}))_i) \\ &\quad + e_j ((\mathcal{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{jr} + k^2 \delta_{jr} G_k(\mathbf{x}, \mathbf{z})) \mathcal{M}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathbf{R}(\mathbf{x}) \\ &= \frac{e^{ikr}}{4\pi} \left\{ - \left(-\frac{k^2}{r} \hat{\mathbf{r}} \times (\mathcal{A}\mathbf{H}_0(\mathbf{z}) + \mathcal{B}\mathbf{E}_0(\mathbf{z})) - \frac{ik}{r^2} \hat{\mathbf{r}} \times (\mathcal{A}\mathbf{H}_0(\mathbf{z}) + \mathcal{B}\mathbf{E}_0(\mathbf{z})) \right) \right. \\ &\quad + \frac{1}{r^3} (3\hat{\mathbf{r}} \cdot (\mathcal{M}\mathbf{H}_0(\mathbf{z}))\hat{\mathbf{r}} - \mathcal{M}\mathbf{H}_0(\mathbf{z})) - \frac{ik}{r^2} (3\hat{\mathbf{r}} \cdot (\mathcal{M}\mathbf{H}_0(\mathbf{z}))\hat{\mathbf{r}} - \mathcal{M}\mathbf{H}_0(\mathbf{z})) \\ &\quad \left. - \frac{k^2}{r} (\hat{\mathbf{r}} \cdot \mathcal{M}\mathbf{H}_0(\mathbf{z}) - \mathcal{M}\mathbf{H}_0(\mathbf{z})) \right\} + \mathbf{R}(\mathbf{x}) \end{aligned}$$

$$\begin{aligned}
 &= \frac{e^{ikr}}{4\pi} \left\{ \frac{1}{r^3} (3\hat{\mathbf{r}} \cdot (\mathcal{M}\mathbf{H}_0(\mathbf{z}))\hat{\mathbf{r}} - \mathcal{M}\mathbf{H}_0(\mathbf{z})) \right. \\
 &\quad - \frac{ik}{r^2} (3\hat{\mathbf{r}} \cdot (\mathcal{M}\mathbf{H}_0(\mathbf{z}))\hat{\mathbf{r}} - \mathcal{M}\mathbf{H}_0(\mathbf{z}) - \hat{\mathbf{r}} \times (\mathcal{A}\mathbf{H}_0(\mathbf{z}) + \mathcal{B}\mathbf{E}_0(\mathbf{z}))) \\
 &\quad \left. - \frac{k^2}{r} (\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times (\mathcal{M}\mathbf{H}_0(\mathbf{z})) - \hat{\mathbf{r}} \times (\mathcal{A}\mathbf{H}_0(\mathbf{z}) + \mathcal{B}\mathbf{E}_0(\mathbf{z}))) \right\} + \mathbf{R}(\mathbf{x}), \tag{8.8}
 \end{aligned}$$

where we have assumed that $(\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{\ell m \epsilon_j l s} \mathbf{R}_{msi}(\mathbf{H}_0(\mathbf{z}))_i$ can be grouped with $\mathbf{R}(\mathbf{x})$ and have applied $\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{a}) = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{a}) - \mathbf{a}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{a}) - \mathbf{a}$.

Remark 8.5. Expression (8.8) bears close resemblance to the result in Theorem 4.1 of [6], which describes the field perturbation for wave-dominated problems, where, \mathcal{B} is the Pólya–Szegő tensor parameterised by ϵ_r , as expected, but, instead of a Pólya–Szegő tensor parameterised by μ_r we have the complex symmetric rank 2 tensor \mathcal{M} , which depends on μ_r and ν . Furthermore, we have the additional term $\mathcal{A}\mathbf{H}_0(\mathbf{z})$. We explore this connection further in later sections.

8.1 Quasi-static regime

Assuming that $\alpha \ll \lambda_0$ and noting that $(\mathbf{D}_x^2 G_k(\mathbf{x}, \mathbf{z}))_{jr} + k^2 \delta_{jr} G_k(\mathbf{x}, \mathbf{z}) = (\mathbf{D}_x^2 G_0(\mathbf{x}, \mathbf{z}))_{jr} + O(k)$ and $\nabla_x G_k(\mathbf{x}, \mathbf{z}) = \nabla_x G_0(\mathbf{x}, \mathbf{z}) + O(k)$ as $k \rightarrow 0$ then expression (8.8) reduces to

$$\begin{aligned}
 (\mathbf{H}_\Delta(\mathbf{x})) &= -ike_j (\nabla_x G_0(\mathbf{x}, \mathbf{z}))_p \epsilon_{jpr} (\mathcal{A}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathcal{B}_{ri}(\mathbf{E}_0(\mathbf{z}))_i) \\
 &\quad + e_j (\mathbf{D}_x^2 G_0(\mathbf{x}, \mathbf{z}))_{jr} \mathcal{M}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathbf{R}(\mathbf{x}),
 \end{aligned}$$

and, in the near field, the dominant response is

$$(\mathbf{H}_\Delta(\mathbf{x})) \approx e_j (\mathbf{D}_x^2 G_0(\mathbf{x}, \mathbf{z}))_{jr} \mathcal{M}_{ri}(\mathbf{H}_0(\mathbf{z}))_i, \tag{8.9}$$

where the coefficients of $\mathcal{M} = \mathcal{N} - \check{\mathcal{C}}$ depend on ϵ_* , μ_* , σ_* , α and ω and are obtained from Lemma 8.2 and (7.7) using (5.28). The expression (8.9) describes the magnetic field perturbation in terms of a complex symmetric rank 2 tensor \mathcal{M} , extending the MPT characterisation of objects considered in [5, 9] for the eddy current problem, to the regime where $\alpha \ll \lambda_0$, but without requiring $\sigma_* \gg \epsilon_* \omega$.

8.2 Eddy current regime

The eddy current regime is a low-frequency approximation of the Maxwell system where, in addition to the quasi-static approximation, we have $\sigma_* \gg \epsilon_* \omega$ and $\epsilon_* = \epsilon_0$ so that $\nu_i = O(1)$ and displacement currents are neglected. Hence, (5.28) reduces to (7.2) and (5.30) to (7.3). Then, by an application of an integration by parts on \mathcal{A}_{ri} and \mathcal{B}_{ri} in (7.5a) and (7.5b) gives $\mathcal{A}_{ri} = 0$ and $\mathcal{B}_{ri} = 0$. Furthermore, by Corollary 8.3, $\mathcal{C}_{msi} = \epsilon_{msr} \check{\mathcal{C}}_{ri}$ with $\mathbf{R}_{msi} = 0$ in this case and $\mathcal{M} = \mathcal{N} - \check{\mathcal{C}}$ becomes the MPT discussed in [5, 9]. Hence,

$$\begin{aligned}
 (\mathbf{H}_\Delta(\mathbf{x})) &= e_j (\mathbf{D}_x^2 G_0(\mathbf{x}, \mathbf{z}))_{jr} \mathcal{M}_{ri}(\mathbf{H}_0(\mathbf{z}))_i + \mathbf{R}(\mathbf{x}) \\
 &= \frac{1}{4\pi r^3} (3\hat{\mathbf{r}} \cdot (\mathcal{M}\mathbf{H}_0(\mathbf{z}))\hat{\mathbf{r}} - \mathcal{M}\mathbf{H}_0(\mathbf{z})) + \mathbf{R}(\mathbf{x}), \tag{8.10}
 \end{aligned}$$

where the coefficients of \mathcal{M} no longer depend on ϵ_* and are obtained using (7.2) instead of (5.28).

Remark 8.6. In practice, all metals have $\sigma_* \gg \epsilon_0 \omega$ over all the frequencies for which σ_* can be regarded as a constant so that $\epsilon_r \approx \epsilon_r - 1 \approx \sigma_*/(i\epsilon_0 \omega)$ holds over these frequencies and, for sufficiently small objects, these also extend to all the frequencies for which the quasi-static approximation holds. This

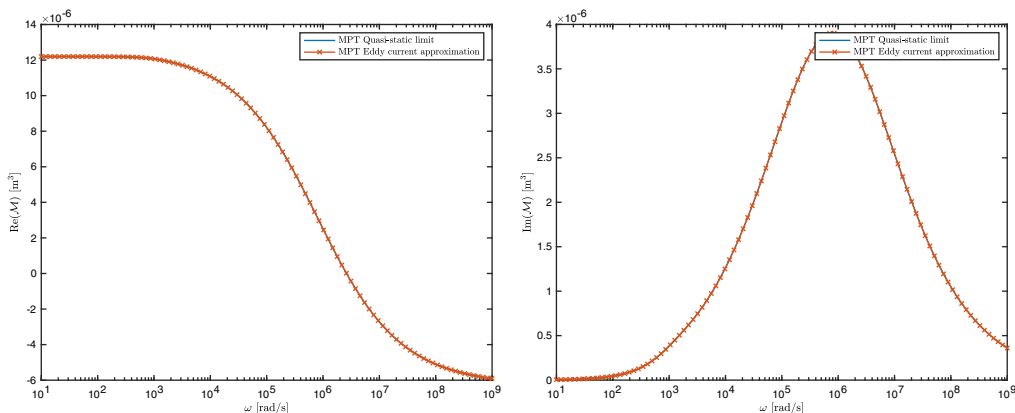


Figure 1. Comparison of the diagonal coefficients of \mathcal{M} for a conducting sphere with $\mu_r = 100$, $\sigma_* = 1 \times 10^6$ S/m and radius $\alpha = 0.01$ m for frequencies $1 \times 10^1 \leq \omega \leq 1 \times 10^9$ rad/s for results obtained using the quasi-static model and eddy current approximation.

means that the coefficients of \mathcal{M} obtained for the eddy current model using (7.2) are almost indistinguishable from those obtained using (5.28) provided that $\sigma_* \gg \epsilon_0 \omega$ and the object is sufficiently small. As an example, we compare the diagonal coefficients of \mathcal{M} obtained using the two models using the analytical solution of Wait [13] in Figure 1. Of course, by making the eddy current approximation, and using (7.2), then $\mathcal{A}_{ri} = 0$ and, using (7.3), $\mathcal{B}_{ri} = 0$ indicating that (8.9) is indeed the dominant response for the quasi-static regime for metallic objects, which holds not only in the near field.

8.3 Regime with non-constant parameters

In this regime, we no longer assume that the parameters are constant and relax the assumption of $\nu = O(1)$. First, we consider the case of $\sigma_* = 0$ and require the other parameters ($\epsilon_r = \epsilon_*/\epsilon_0$, μ_r) to change so that $k \rightarrow 0$ as $\alpha \rightarrow 0$. Second, we require the parameters ($\epsilon_r = 1/\epsilon_0(\epsilon_* + \sigma_*/(i\omega))$, μ_r and k) to change as $\alpha \rightarrow 0$.

8.3.1 Small k and $\sigma_* = 0$

We consider small k and $\sigma_* = 0$ so that $\epsilon_r = \epsilon_*/\epsilon_0$. In this case, we have $\theta_i = \tilde{\theta}_i^{(0)} + O(k)$ as $k \rightarrow 0$ where $\tilde{\theta}_i^{(0)}$ solves

$$\nabla_\xi \times \mu_r^{-1} \nabla_\xi \times \tilde{\theta}_i^{(0)} = \mathbf{0} \quad \text{in } B, \tag{8.11a}$$

$$\nabla_\xi \times \nabla_\xi \times \tilde{\theta}_i^{(0)} = \mathbf{0} \quad \text{in } B^c, \tag{8.11b}$$

$$\nabla_\xi \cdot \tilde{\theta}_i^{(0)} = 0 \quad \text{in } \mathbb{R}^3, \tag{8.11c}$$

$$[\mathbf{n} \times \tilde{\theta}_i^{(0)}]_\Gamma = \mathbf{0}, \quad [\mathbf{n} \times \tilde{\mu}_r^{-1} \nabla_\xi \times \tilde{\theta}_i^{(0)}]_\Gamma = -2(1 - \mu_r^{-1}) \mathbf{n} \times \mathbf{e}_i \quad \text{on } \Gamma, \tag{8.11d}$$

$$\tilde{\theta}_i^{(0)} = O(|\xi|^{-1}) \quad \text{as } |\xi| \rightarrow \infty, \tag{8.11e}$$

which allow us to deduce the following about the coefficients of \mathcal{M} :

Lemma 8.7. For small k and $\sigma_* = 0$, $\theta_i = \tilde{\theta}_i^{(0)} + O(k)$ as $k \rightarrow 0$ and $\mathcal{M}_{ri} = \mathcal{N}_{ri} - \check{\mathcal{C}}_{ri}$ reduces to $\mathcal{M}_{ri} = \mathcal{N}_{ri}^{(0)} + O(k)$ where

$$\mathcal{N}_{ri}^{(0)} = \frac{\alpha^3}{4} \int_{B \cup B^c} \tilde{\mu}_r^{-1} \nabla \times \tilde{\theta}_i^{(0)} \cdot \nabla \times \tilde{\theta}_r^{(0)} d\xi + [\tilde{\mu}_r^{-1}] \int_B \delta_{ri} d\xi, \tag{8.12}$$

which are also the coefficients of the Pólya–Szegő tensor $\mathcal{T}[\alpha B, \mu_r]$ of an object B_α for a contrast μ_r as defined, for example, in Lemma 3 of [7].

Proof. For the transmission problem (8.11), we can show that

$$\int_{B \cup B^c} \tilde{\mu}_r^{-1} \nabla \times \tilde{\theta}_i^{(0)} \cdot \nabla \times \tilde{\theta}_r^{(0)} \, d\xi = [\tilde{\mu}_r^{-1}] \int_B (\mathbf{e}_r \cdot \nabla \times \tilde{\theta}_i^{(0)} + \mathbf{e}_i \cdot \nabla \times \tilde{\theta}_r^{(0)}) \, d\xi, \tag{8.13}$$

which is obtained by integrating by parts $\int_{B \cup B^c} (\nabla \times \tilde{\mu}_r^{-1} \nabla \times \tilde{\theta}_i^{(0)} \cdot \tilde{\theta}_r^{(0)} + \nabla \times \tilde{\mu}_r^{-1} \nabla \times \tilde{\theta}_r^{(0)} \cdot \tilde{\theta}_i^{(0)}) \, d\xi = 0$.

Then, for small k , we have $\theta_i = \tilde{\theta}_i^{(0)} + O(k)$ and, by letting the solid sphere Ω_0 have radius that tends towards infinity, considering the far field decay of $\tilde{\theta}_i^{(0)}$ and using (8.13) in (8.7), we find that $\mathcal{N}_{ri} - \check{C}_{ri}$ reduces to $\mathcal{N}_{ri}^{(0)}$, and this is also equivalent to the coefficients of the Pólya–Szegő tensor $\mathcal{T}[\alpha B, \mu_r]$ of an object B for a contrast μ_r , as shown by Theorem 3.2 of [9] and Lemma 3 of [7]. Note that Lemma 3 of [7] considered the question of connectedness of B and showed that this reduction holds independent of the first Betti number of B . \square

By an integration by parts and then using $\theta_i = \tilde{\theta}_i^{(0)} + O(k)$, we get

$$\begin{aligned} \mathcal{A}_{ri} &:= \frac{i\alpha^3}{2\alpha k} \mathbf{e}_r \cdot \int_B (\nabla_\xi \times \mu_r^{-1} \nabla_\xi \times \theta_i - k^2 \alpha^2 \theta_i) \, d\xi \\ &= -\frac{i\alpha^4 k}{2} \mathbf{e}_r \cdot \int_{\Omega_0} \theta_i \, d\xi - \frac{\alpha^3}{2} \mathbf{e}_r \cdot \int_{\Sigma_0} \Lambda_e^{(\alpha k)} (\mathbf{n}^- \times \theta_i) \, d\xi = O(k), \end{aligned} \tag{8.14}$$

as $k \rightarrow 0$ by letting the solid sphere Ω_0 have radius that tends towards infinity and considering the far field decay of $\tilde{\theta}_i^{(0)}$.

Using the above results and Lemma 8.1 means that (8.8) now becomes

$$\begin{aligned} (\mathbf{H}_\Delta(\mathbf{x})) &= \frac{e^{ikr}}{4\pi} \left\{ \frac{1}{r^3} (3\hat{\mathbf{r}} \cdot (\mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) \hat{\mathbf{r}} - \mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) \right. \\ &\quad - \frac{ik}{r^2} (3\hat{\mathbf{r}} \cdot (\mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) \hat{\mathbf{r}} - \mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z}) - \hat{\mathbf{r}} \times (\mathcal{T}[\alpha B, \epsilon_r] \mathbf{E}_0(\mathbf{z}))) \\ &\quad \left. - \frac{k^2}{r} (\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times (\mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) - \hat{\mathbf{r}} \times (\mathcal{T}[\alpha B, \epsilon_r] \mathbf{E}_0(\mathbf{z}))) \right\} + \mathbf{R}(\mathbf{x}), \end{aligned} \tag{8.15}$$

which agrees with the known results for low-frequency scattering from dielectric and permeable bodies (e.g. [6]).

8.3.2 Small α and small ν

We now consider the case of small α and require k, ϵ_r, μ_r to change with α such that ν is also small. For small α, ν then $\theta_i = \tilde{\theta}_i^{(0)} + O(\alpha)$ as $\alpha \rightarrow 0$ where $\tilde{\theta}_i^{(0)}$ solves (8.11), which allow us to show the following about the coefficients of \mathcal{M} :

Lemma 8.8. For small α and ν , $\theta_i = \tilde{\theta}_i^{(0)} + O(\alpha)$ as $\alpha \rightarrow 0$ and $\mathcal{M}_{ri} = \mathcal{N}_{ri} - \check{C}_{ri}$ reduces to $\mathcal{M}_{ri} = \mathcal{N}_{ri}^{(0)} + O(\alpha^5)$ where

$$\mathcal{N}_{ri}^{(0)} = \frac{\alpha^3}{4} \int_{B \cup B^c} \tilde{\mu}_r^{-1} \nabla \times \tilde{\theta}_i^{(0)} \cdot \nabla \times \tilde{\theta}_r^{(0)} \, d\xi + [\tilde{\mu}_r^{-1}] \int_B \delta_{ri} \, d\xi, \tag{8.16}$$

which are also the coefficients of the Pólya–Szegő tensor $\mathcal{T}[\alpha B, \mu_r]$ of an object B_α for a contrast μ_r .

Proof. The proof is similar to the proof of Lemma (8.7) and again uses (8.13). For small α, ν , we have $\theta_i = \tilde{\theta}_i^{(0)} + O(\alpha)$ as $\alpha \rightarrow 0$ and find that $\mathcal{N}_{ri} - \tilde{\mathcal{C}}_{ri}$ reduces to $\mathcal{N}_{ri}^{(0)}$. □

Similarly, by an integration by parts and then using $\theta_i = \tilde{\theta}_i^{(0)} + O(\alpha)$, we get

$$\begin{aligned} \mathcal{A}_{ri} &:= \frac{i\alpha^3}{2\alpha k} \mathbf{e}_r \cdot \int_B (\nabla_{\xi} \times \mu_r^{-1} \nabla_{\xi} \times \theta_i - k^2 \alpha^2 \theta_i) d\xi \\ &= -\frac{i\alpha^4 k}{2} \mathbf{e}_r \cdot \int_{\Omega_0} \theta_i d\xi - \frac{\alpha^3}{2} \mathbf{e}_r \cdot \int_{\Sigma_0} \Lambda_e^{(\alpha k)} (\mathbf{n}^- \times \theta_i) d\xi = O(\alpha^4), \end{aligned} \tag{8.17}$$

as $\alpha \rightarrow 0$ by letting the solid sphere Ω_0 have radius that tends towards infinity and considering the far field decay of $\tilde{\theta}_i^{(0)}$.

Using the above results and Lemma 8.1 means that (8.8) now becomes

$$\begin{aligned} (\mathbf{H}_{\Delta}(\mathbf{x})) &= \frac{e^{ikr}}{4\pi} \left\{ \frac{1}{r^3} (3\hat{\mathbf{r}} \cdot (\mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) \hat{\mathbf{r}} - \mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) \right. \\ &\quad - \frac{ik}{r^2} (3\hat{\mathbf{r}} \cdot (\mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) \hat{\mathbf{r}} - \mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z}) - \hat{\mathbf{r}} \times (\mathcal{T}[\alpha B, \epsilon_r] \mathbf{E}_0(\mathbf{z}))) \\ &\quad \left. - \frac{k^2}{r} (\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times (\mathcal{T}[\alpha B, \mu_r] \mathbf{H}_0(\mathbf{z})) - \hat{\mathbf{r}} \times (\mathcal{T}[\alpha B, \epsilon_r] \mathbf{E}_0(\mathbf{z}))) \right\} + \mathbf{R}(\mathbf{x}), \end{aligned} \tag{8.18}$$

which agrees with the known results for scattering from small bodies [3, 4].

Remark 8.9. While the forms for $\mathbf{H}_{\Delta}(\mathbf{x})$ derived in (8.8) and (8.18) look very similar, the former holds for small k and requires $\sigma_* = 0$ and the latter is for the case of small α and small ν but does not require $\sigma_* = 0$.

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References

- [1] Ammari, H., Buffa, A. & Nédélec, J.-C. (2000) A justification of eddy currents model for the Maxwell equations. *SIAM J. Appl. Math.* **60**(5), 1805–1823.
- [2] Ammari, H., Chen, J., Chen, Z., Garnier, J. & Volkov, D. (2014) Target detection and characterization from electromagnetic induction data. *J. Math. Pures Appl.* **101**(1), 54–75.
- [3] Ammari, H., Vogelius, M. S. & Volkov, D. (2001) Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations. *J. Math. Pures Appl.* **80**, 769–814.
- [4] Ammari, H. & Volkov, D. (2005) The leading-order term in the asymptotic expansion of the scattering amplitude of a collection of finite number dielectric inhomogeneities of small diameter. *Int. J. Multiscale Comput. Eng.* **3**(3), 149–160.
- [5] Ledger, P. D. & Lionheart, W. R. B. (2015) Characterising the shape and material properties of hidden targets from magnetic induction data. *IMA J. Appl. Math.* **80**(6), 1776–1798.
- [6] Ledger, P. D. & Lionheart, W. R. B. (2015) The perturbation of electromagnetic fields at distances that are large compared with the object’s size. *IMA J. Appl. Math.* **80**(3), 865–892.
- [7] Ledger, P. D. & Lionheart, W. R. B. (2016) Understanding the magnetic polarizability tensor. *IEEE Trans. Magn.* **52**(5), 6201216.
- [8] Ledger, P. D. & Lionheart, W. R. B. (2018) Generalised magnetic polarizability tensors. *Math. Methods Appl. Sci.* **41**(8), 3175–3196.
- [9] Ledger, P. D. & Lionheart, W. R. B. (2020) The spectral properties of the magnetic polarizability tensor for metallic object characterisation. *Math. Methods Appl. Sci.* **43**(1), 78–113.

- [10] Ledger, P. D., Wilson, B. A., Amad, A. A. S. & Lionheart, W. R. B. (2021) Identification of metallic objects using spectral magnetic polarizability tensor signatures: Object characterisation and invariants. *Int. J. Numer. Methods Eng.* **122**(15), 3941–3984.
- [11] Monk, P. (2003) *Finite Element Methods for Maxwell's Equations*, Oxford Science Publications, Oxford, UK.
- [12] Schmidt, K., Sterz, O. & Hiptmair, R. (2008) Estimating the eddy-current modeling error. *IEEE Trans. Magn.* **44**(6), 686–689.
- [13] Wait, J. R. (1951) A conducting sphere in a time varying magnetic field. *Geophysics* **16**(4), 666–672.
- [14] Wilson, B. A. & Ledger, P. D. (2021) Efficient computation of the magnetic polarizability tensor spectral signature using proper orthogonal decomposition. *Int. J. Numer. Methods Eng.* **122**(8), 1940–1963.
- [15] Wilson, B. A., Ledger, P. D. & Lionheart, W. R. B. (2022) Identification of metallic objects using spectral magnetic polarizability tensor signatures: Object classification. *Int. J. Numer. Methods Eng.* **123**(9), 2076–2111.