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# Optimal control of the radius of a rigid circular inclusion in inhomogeneous two-dimensional bodies with cracks

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**Abstract.** A two-dimensional model describing the equilibrium state of a cracked inhomogeneous body with a rigid circular inclusion is investigated. The body is assumed to have a crack that reaches the boundary of the rigid inclusion. We assume that the Signorini condition, ensuring non-penetration of the crack faces, is satisfied. We analyze the dependence of solutions on the radius of rigid inclusion. The existence of a solution of the optimal control problem is proven. For this problem, a cost functional is defined by an arbitrary continuous functional, with the radius of inclusion chosen as the control parameter.

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## 1. Introduction

The problems related to deformation of composites containing both cracks and inclusions are subject of considerable scientific interest which is caused by growing trends in the applications of composites [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In particular, general representations of the solutions for a radial crack near a single and midway between two rigid inclusions are given in the paper [17]. Note that derivation of expressions was carried out under the assumption that the appropriate shear stresses on the crack faces are equal to zero. The plane problems for a cracked body with a rectilinear crack located midway between two circular elastic or rigid grains (inclusions) are investigated in [18]. The effect of a rigid elliptical inclusion on a straight crack was discussed in [19]. The interaction between an elliptical inclusion and a crack is analyzed in [20, 21]. For a more detailed review of studies related to the crack-inhomogeneity interaction, the interested reader can refer to the papers [22, 23].

It is well known that imposing of linear boundary conditions on the crack may lead to physical inconsistency of mathematical models since mutual penetration of the crack faces may happen [18, 24]. In recent years, a crack theory with non-penetration conditions has been under active study [25, 26, 27, 28, 29, 30]. This approach to solving crack problems is characterized by inequality type boundary conditions at the crack faces, is indeed what we employ in the present paper. Within this approach, various problems for bodies with rigid inclusions has been successfully formulated and investigated using variational methods, see for example [9, 25, 27, 31, 32, 33, 34]. In contrast to a previous study of an optimal control problem for a two-dimensional elastic body with a rigid delaminated inclusion, as considered in [31], we suppose that crack curve touches the inclusion's boundary only at the crack's tip. This means that displacements on the crack's faces are not required to have a prescribed structure of infinitesimal rigid displacements. Another difference between the problems that have been considered in [31] is that in the present work a family of rigid inclusions have not a fixed common boundary curve. The optimal control problem analyzed in this paper consists in the best choice of the radius $r^* \in [r_0, R]$ of the circular rigid inclusion. A cost functional is defined using an arbitrary continuous functional in the Sobolev space. The existence of the solution to the optimal control problem is proved. In addition, for a family of variational problems describing equilibrium of cracked bodies with inclusions of different radiuses $r \in [r_0, R]$, we prove the continuous dependence of the solutions with respect to the parameter $r$. The limit case of the control parameter $r_0 \to 0$ implies the change of topology, and it should be described by topological control, see e.g. [16] and the example of a circular hole at the tip of a crack in [35].

## 2. Family of equilibrium problems

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\Gamma \in C^{1,1}$. We consider the family of open balls $\{\omega_r\}$ of radius $r \in [r_0, R]$ such that

**a)** $\omega_{r'} \subset \omega_{r''}$ for all $r', r'' \in [r_0, R]$: $r' \leq r''$;

**b)** $\bar{\omega}_R \subset \Omega$;

**c)** the circles $\partial\omega_r$, $r \in [r_0, R]$, enclosing the balls $\omega_r$ intersect at a single point $P$ with coordinate $x_p = (x_{1p}, x_{2p})$ (see Fig. 1).

We consider a smooth curve $\gamma \subset \Omega$ that is without any self-intersections and has the following properties: $\bar{\gamma} \subset \Omega$, exactly one endpoint of $\gamma$ coincides with $P$ and has a non-zero angle with $\partial\omega_R$.

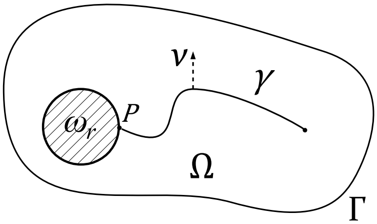


Fig. 1.

We assume that $\gamma$ can be extended in such a way that this extension crosses $\Gamma$ at two points, and $\Omega$ is divided into two subdomains $\Omega_1$ and $\Omega_2$ with Lipschitz boundaries $\partial\Omega_1$, $\partial\Omega_2$ and $meas(\Gamma \cap \partial\Omega_i) > 0$, $i = 1, 2$. This condition is sufficient for Korn's inequality to hold in the non-Lipschitz domain $\Omega_\gamma = \Omega \backslash \bar{\gamma}$. We denote by $W = (w_1, w_2)$ the displacement vector and also introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \quad \text{on} \quad \Gamma\}, \quad H(\Omega_\gamma) = H^{1,0}(\Omega_\gamma)^2.$$

The tensors describing the deformation of the elastic part of the inhomogeneous body may be introduced through

$$\varepsilon_{ij}(W) = \frac{1}{2}(w_{i,j} + w_{j,i}),\ i, j = 1, 2,$$

$$\sigma_{ij}(W) = c_{ijkl}\varepsilon_{ij}(W),\ i, j = 1, 2,$$

where $c_{ijkl}$ is the associated elasticity tensor, assumed as usual to be symmetric and positive definite, implying that

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i, j, k, l = 1, 2, \quad c_{ijkl} = const.,$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall \xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const., \quad c_0 > 0.$$

By virtue of the the assumption concerning the domain $\Omega_\gamma$, Korn's inequality may be assumed to hold in the form

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W)dx \geq c\|W\|^2_{H(\Omega_\gamma)} \ \ \forall W \in H(\Omega_\gamma), \tag{1}$$

with the constant $c > 0$ independent of $W$, see [24, 36].

**Remark 1.** *The inequality* (1) *yields the equivalence of the standard norm in* $H(\Omega_\gamma)$ *and the semi-norm determined by the left-hand side of* (1).

In order to formulate a mathematical model, we fix the parameter $r \in [r_0, R]$ and suppose that the ball $\omega_r$ models the rigid inclusion, with the domain $\Omega_\gamma \backslash \overline{\omega}_r$ corresponding to the elastic part of the body. To be precise, we have in mind that the rigid inclusion allows only a displacement $W|_{\omega_r} = \rho$ within the space, $R(\omega_r)$, of infinitesimal rigid displacements on $\omega_r$, where

$$R(\omega_r) = \{\rho = (\rho_1, \rho_2) \,|\, \rho(x) = b(x_2, -x_1) + (c_1, c_2);\, b, c_1, c_2 \in \mathbb{R},\ x \in \omega_r\},$$

see [15]. We further suppose that the curve $\gamma$ reaches the inclusion's boundary at the point $P$ and describes a crack in the undeformed state of the body.

The condition of mutual non-penetration of opposite faces of the crack is given in [29] and takes the form

$$[W]\nu \geq 0 \quad \text{on} \quad \gamma,$$

where $\nu = (\nu_1, \nu_2)$ is a unit normal to $\gamma$, $[v] = v|_{\gamma^+} - v|_{\gamma^-}$ is the jump of a function $v$ on $\gamma$. A zero Dirichlet boundary condition is imposed on the external boundary $\Gamma$. We now introduce the energy functional

$$\Pi(W, \Omega_\gamma) = \frac{1}{2} \int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W)dx - \int\limits_{\Omega_\gamma} FW dx, \tag{2}$$

where $F = (f_1, f_2) \in L^2(\Omega_\gamma)^2$ is the vector of prescribed exterior forces. The equilibrium problem of the cracked body may be formulated as the following minimization problem

$$\text{Find } U_r \in K_r, \quad \text{such that} \quad \Pi(U_r, \Omega_\gamma) = \inf_{W \in K_r} \Pi(W, \Omega_\gamma), \tag{3}$$

where

$$K_r = \{W \in H(\Omega_\gamma) \,|\, [W]\nu \geq 0 \quad \text{on} \quad \gamma,\ W|_{\omega_r} = \rho, \text{ where } \rho \in R(\omega_r)\}.$$

In [15] it has been established that the problem (3) is known to have a unique solution, $U_r \in K_r$; a solution which satisfies the variational inequality [15]

$$U_r \in K_r, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_r)\varepsilon_{ij}(W - U_r)dx \geq \int\limits_{\Omega_\gamma} F(W - U_r)dx \quad \forall W \in K_r. \tag{4}$$

## 3. Optimal control problem

We define the cost functional $J : [r_0, R] \to \mathbb{R}$ of an optimal control problem through use of the equality $J(r) = G(U_r)$, where $U_r$ is the solution of the problem (3) and $G(\chi) : H(\Omega_\gamma) \to \mathbf{R}$ is an arbitrary continuous functional, which is continuous in strong topology. As examples of such physically motivated functionals, we provide the following. The functional $G_1(W) = \|W - W_0\|_{H(\Omega_\gamma)}$

characterizes the deviation of the displacement vector from a given vector function $W_0$. A further functional

$$G_2(W) = \int\limits_{\Omega_\gamma} \Big\{ \frac{1}{2}\zeta_{,1}\, \sigma_{ij}(W)\varepsilon_{ij}(W) - \sigma_{ij}(W) w_{i,1}\zeta_{,j} \Big\} - \\ - \int\limits_{\Omega_\gamma} (\zeta f_i)_{,1}\, w_{i,1}, \quad \zeta \in C_0^\infty(\Omega), \tag{5}$$

is essentially a derivative of a potential energy functional with respect to the perturbation parameter of a rectilinear crack in the direction $x_1$, further details may be found in [29].

Consider the optimal control problem:

$$\text{Find} \quad r^* \in [r_0, R] \quad \text{such that} \quad J(r^*) = \sup_{r\in[r_0,R]} J(r). \tag{6}$$

**Theorem 1.** *There exists a solution of the optimal control problem* (6).

PROOF. Let $\{r_n\}$ be a maximizing sequence. By virtue of the boundedness of the segment $[r_0, R]$, we can extract a convergent subsequence $\{r_{n_k}\} \subset \{r_n\}$ such that

$$r_{n_k} \to r^* \quad \text{as} \quad k \to \infty, \quad r^* \in [r_0, R].$$

Without loss of generality, we assume that for sufficiently large $k$ it holds $r_{n_k} \neq r^*$. If this were not the case, there would exist a sequence $\{r_{n_l}\}$ such that $r_{n_l} \equiv r^*$, and therefore $J(r^*)$ is solution of (6). Consider the case of the subsequence $\{r_{n_k}\}$ satisfying $r_{n_k} \neq r^*$ for sufficiently large $k$. Now we take into account Lemma 2, proved below: the solutions $U_k$ of (3), corresponding to the parameters $r_{n_k}$, converge to the solution $U_{r^*}$ strongly in $H(\Omega_\gamma)$ as $k \to \infty$. This allows us to obtain convergence

$$J(r_{n_k}) \to J(r^*),$$

indicating that

$$J(r^*) = \sup_{r\in[r_0,R]} J(r).$$

The theorem is proved.

## 4. Auxiliary lemmas

Now we have to justify some auxiliary lemmas which had to used within the proof of the above theorem. In establishing the proof, we needed Lemma 2; however before proceeding further we prove the following lemma.

**Lemma 1.** *Let $r^* \in [r_0, R)$ be a fixed real number and let $\{r_n\} \subset [r^*, R]$ be a sequence of real numbers converging to $r^*$ as $n \to \infty$. Then for an arbitrary* *function* $W \in K_{r^*}$ *there exist a subsequence* $\{r_k\} = \{r_{n_k}\} \subset \{r_n\}$ *and a sequence of functions* $\{W_k\}$ *such that* $W_k \in K_{r_k}$, $k \in \mathbb{N}$ *and* $W_k \to W$ *weakly in* $H(\Omega_\gamma)$ *as* $k \to \infty$.

PROOF. First note that if there exists a subsequence $\{r_{n_k}\}$ such that $r_{n_k} = r^*$, then the assertion of the lemma holds for $W_k \equiv W$, $k \in \mathbb{N}$. Therefore, below we assume that $r_n > r^*$ for sufficiently large $n$. Denote by $\rho^* = W = (b^*x_2 + c_1^*, -b^*x_1 + c_2^*)$ the function describing the structure of $W$ in $\omega_{r^*}$. We extend the definition of $\rho^*$ to the whole domain $\Omega$ by the equality:

$$\rho^* = (b^*x_2 + c_1^*, -b^*x_1 + c_2^*), \quad x \in \Omega.$$

It is now necessary to fix an arbitrary value $r \in (r_0, R]$ and consider the following family of auxiliary problems:

$$\text{Find an element} \quad W_r \in K_r \quad \text{such that} \quad p(W_r) = \inf_{\chi \in K_r'} p(\chi), \tag{7}$$

where $p(\chi) = \int\limits_{\Omega_\gamma} \sigma_{ij}(\chi - W)\varepsilon_{ij}(\chi - W)dx$,

$$K_r' = \{\chi \in H(\Omega_\gamma) \mid \quad \chi = W \quad \text{on} \quad \gamma^\pm, \ \chi|_{\omega_r} = \rho^*\}.$$

It is easy to see that the functional $p(\chi)$ is coercive and weakly lower semi-continuous on the space $H(\Omega_\gamma)$. It can be verified that the set $K_t'$ is convex and closed in $H(\Omega_\gamma)$. These properties provide the existence of a unique solution $W_r$ of the problem (7), see [24]. The solution is characterized equivalently by the variational inequality

$$W_r \in K_r', \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(W_r - W)\varepsilon_{ij}(\chi - W_r)dx \ge 0 \quad \forall \chi \in K_r'. \tag{8}$$

Note that the solution $W_R$ of (8) for $r = R$ belongs to the set $K_r'$ with $r' \in (r_0, R]$. Substituting $W_R$ as test functions into (8), it is possible to establish $\forall r \in (r_0, R]$ that

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(W_r - W)\varepsilon_{ij}(W_R)dx + \int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W_r)dx \ge \int\limits_{\Omega_\gamma} \sigma_{ij}(W_r)\varepsilon_{ij}(W_r)dx.$$

Using Korn's inequality, we obtain from this relation the following uniform upper bound:

$$\|W_r\| \le c \quad \forall r \in (r_0, R].$$

It is therefore possible to extract from the sequence $\{W_{r_n}\}$ a subsequence $\{W_k\}$, defined by equalities $W_k = W_{r_{n_k}}$, $k \in \mathbf{N}$ (note that henceforth we define a sequence $\{r_k\}$ by the equality $r_k = r_{n_k}$), with $\{W_k\}$ weakly converging to some function $\widetilde{W}$ in $H(\Omega_\gamma)$.

It is now necessary to show that $\widetilde{W} = W$. By construction, $(W_k - W) \in H_0^1(\Omega_\gamma \backslash \overline{\omega}_{r^*})^2$ and consequently, bearing in mind the weak closeness of $H_0^1(\Omega_\gamma \backslash \overline{\omega}_{r^*})^2$, we have $(\widetilde{W} - W) \in H_0^1(\Omega_\gamma \backslash \overline{\omega}_{r^*})^2$. Now consider functions of the form $\chi_k^{\pm} = W_k \pm \alpha$, where $\alpha$ is the function defined by zero extension of some arbitrary function $\tilde{\alpha} \in C_0^\infty(\Omega_\gamma \backslash \overline{\omega}_{t^*})^2$ into $\Omega_\gamma$. One can observe that $\chi_k^{\pm} \in K'_{r_k}$ holds for sufficiently large $k$. It is now possible to substitute the elements of these sequences, $\chi_k^+$ and $\chi_k^-$, as test functions into the inequalities (8), revealing that

$$W_k \in K'_{r_k}, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(W_k - W)\varepsilon_{ij}(\alpha)dx = 0. \tag{9}$$

The function $\alpha$ is now fixed and by passing to the limit in (9) it is established that

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\widetilde{W} - W)\varepsilon_{ij}(\alpha)dx = \int\limits_{\Omega_\gamma \backslash \overline{\omega}_{r^*}} \sigma_{ij}(\widetilde{W} - W)\varepsilon_{ij}(\alpha)dx = 0 \quad \forall\, \alpha \in C_0^\infty(\Omega_\gamma \backslash \overline{\omega}_{r^*})^2.$$

Hence, by consideration of the density of $C_0^\infty(\Omega_\gamma \backslash \overline{\omega}_{r^*})$ in $H_0^1(\Omega_\gamma \backslash \overline{\omega}_{r^*})$, it is inferred that $\widetilde{W} - W = 0$ in $H_0^1(\Omega_\gamma \backslash \overline{\omega}_{r^*})^2$. Finally, by construction, the equality $\widetilde{W} = W$ is satisfied in $\omega_{r^*}$; in consequence $\widetilde{W} = W$ in $H(\Omega_\gamma)$ and we conclude that there is a sequence $\{W_k\}$ such that $W_k \in K_{r_k}$, $k \in \mathbb{N}$ and $W_k \to W$ weakly in $H(\Omega_\gamma)$ as $k \to \infty$. The Lemma is thus proved. We are now in a position to prove an auxiliary statement (Lemma 2) which was used in the proof of the Theorem 1.

**Lemma 2.** *Let $r^* \in [r_0, R]$ be a fixed real number. Then $U_r \to U_{r^*}$ strongly in $H(\Omega_\gamma)$ as $r \to r^*$, where $U_r$, $U_{r^*}$ are the solutions of (3), corresponding to parameters $r \in (r_0, R]$, $r^* \in [r_0, R]$.*

Proof. The proof of this lemma will be established by contradiction. To begin we assume that there exists a number $\epsilon_0 > 0$ and a sequence $\{r_n\} \subset (r_0, R]$ such that $r_n \to r^*$, $\|U_n - U_{r^*}\| \geq \epsilon_0$, where $U_n = U_{r_n}$, $n \in \mathbb{N}$ are the solutions of (3), corresponding to $r_n$.

In view of the fact that $W^0 \equiv 0 \in K_r$ for all $r \in [r_0, R]$, we can substitute $W = W^0$ in (4) for fixed $r \in (r_0, R]$, yielding

$$U_r \in K_r, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_r)\varepsilon_{ij}(U_r)dx \leq \int\limits_{\Omega_\gamma} FU_r dx \quad \forall\, r \in [r_0, R],$$

from which we conclude that for all $r \in [r_0, R]$ the following estimate holds

$$\|U_r\| \leq c,$$

for some constant $c > 0$ independent of $r$. Consequently, replacing $U_n$ by its subsequence if necessary, we can assume that

$$U_n \to \tilde{U} \quad \text{weakly in} \quad H(\Omega_\gamma) \text{ as } n \to \infty. \tag{10}$$

We will now show that $\tilde{U} \in K_{r^*}$. We first note that $U_n|_{\omega_{r_n}} = \rho_n \in R(\omega_{r_n})$ and that, in accordance with Sobolev's embedding theorem [24], we deduce that

$$U_n|_{\omega_{r^*}} \to \tilde{U}|_{\omega_{r^*}} \quad \text{strongly in } L_2(\omega_{r^*})^2 \text{ as } n \to \infty, \tag{11}$$

$$U_n|_\gamma \to \tilde{U}|_\gamma \quad \text{strongly in } L_2(\gamma)^2 \text{ as } n \to \infty. \tag{12}$$

Choosing a subsequence, if necessary, we assume that as $n \to \infty$ it holds $U_n \to \tilde{U}$ a.e. on $\omega_{r^*}$. This allows us to conclude that each of the numerical sequences $\{b^n\}$, $\{c_1^n\}$, $\{c_2^n\}$, defining the structure of $\rho_n$ on $\omega_{r_n}$, is bounded in its absolute value. Thus, we can extract subsequences (retain notation) such that

$$b^n \to b, \quad c_i^n \to c_i, \quad i = 1, 2, \quad \text{as} \quad n \to \infty.$$

We note that for the sequence $\{r_n\}$ there must exist either a subsequence $\{r_k\} \subset \{r_n\}$ converging to $r^*$ from the left or, if that is not the case, a subsequence $\{r_k\} \subset \{r_n\}$ such that $r_k \geq r^*$ for all $k \in \mathbf{N}$.

If a subsequence $\{r_k\} \subset \{r_n\}$, with $r_k \geq r^*$ for all $k \in \mathbf{N}$, exists then the following strong convergence

$$U_k|_{\omega_{r^*}} \to (bx_2 + c_1, -bx_1 + c_2), \tag{13}$$

in $L_2(\omega_{r^*})^2$ as $k \to \infty$, can readily be obtained. Thus, from a combination of (11) and (13) it follows that the inclusion $\tilde{U}|_{\omega_{r^*}} \in R(\omega_{r^*})$ must hold. Let us now consider the case of the subsequence $\{r_k\} \subset \{r_n\}$ converging to $r^*$ from the left, i.e. $r_k < r^*$ for all $k \in \mathbf{N}$ and $r_k \to r^*$ as $k \to \infty$. Under this assumption, for fixed $k' \in \mathbf{N}$ and the corresponding value $r' = r_{k'}$, we have

$$U_k|_{\omega_{r'}} \to (bx_2 + c_1, -bx_1 + c_2) \tag{14}$$

strongly in $L_2(\omega_{r'})^2$ as $k \to \infty$. It is possible to define a function $\mathcal{L} = bx_2 + c_1$ in $\omega_{r^*}$ and, because of (14), $u_{1k} \to \mathcal{L}$ strongly in $L^2(\omega_{r'})$ as $k \to \infty$. In view of the absolute continuity of the Lebesgue integral, for any $\epsilon > 0$ we can choose a number $k' \in \mathbf{N}$ large enough such that

$$\|\mathcal{L}\|_{L^2(\omega_{r^*} \setminus \omega_{r'})} < \sqrt{\epsilon}, \quad \|\tilde{u}_1\|_{L^2(\omega_{r^*} \setminus \omega_{r'})} < \sqrt{\epsilon},$$

where the value $r' = r_{k'}$ coincides with $k'$. Furthermore, using twice the triangle inequality, it follows that

$$\begin{aligned}\|u_{1k} - \mathcal{L}\|_{L^2(\omega_{r^*}\setminus\omega_{r'})} &\leq \|u_{1k}\|_{L^2(\omega_{r^*}\setminus\omega_{r'})} + \|\mathcal{L}\|_{L^2(\omega_{r^*}\setminus\omega_{r'})} \leq \\ &\leq \|\tilde{u}_1\|_{L^2(\omega_{r^*}\setminus\omega_{r'})} + \|u_{1k} - \tilde{u}_1\|_{L^2(\omega_{r^*}\setminus\omega_{r'})} + \|\mathcal{L}\|_{L^2(\omega_{r^*}\setminus\omega_{r'})} < \\ &< 2\sqrt{\epsilon} + \|u_{1k} - \tilde{u}_1\|_{L^2(\omega_{r^*})}.\end{aligned}$$

Consequently, it is established that

$$\begin{aligned}\|u_{1k} - \mathcal{L}\|^2_{L^2(\omega_{r^*})} &= \|u_{1k} - \mathcal{L}\|^2_{L^2(\omega_{r^*}\setminus\omega_{r'})} + \|u_{1k} - \mathcal{L}\|^2_{L^2(\omega_{r'})} \\ &< (2\sqrt{\epsilon} + \|u_{1k} - \tilde{u}_1\|_{L^2(\omega_{r^*})})^2 + \|u_{1k} - \mathcal{L}\|^2_{L^2(\omega_{r'})}. \quad (15)\end{aligned}$$

It is now noted that for sufficiently large $k$, the following estimates may be established

$$\|u_{1k} - \tilde{u}_1\|_{L^2(\omega_{r^*})} < \sqrt{\epsilon}, \quad \|u_{1k} - \mathcal{L}\|_{L^2(\omega_{r'})} < \sqrt{\epsilon}$$

allowing us to deduce that (15) is less than $10\epsilon$ and thus $u_{1k} \to \mathcal{L}$ strongly in $L^2(\omega_{r^*})$. Finally, based on the convergence (11), we deduce $\tilde{u}_1|_{\omega_{r^*}} = \mathcal{L}$ in $\omega_{r^*}$.

Analogously, we can derive

$$\tilde{u}_2|_{\omega_{r^*}} = -bx_1 + c_2 \quad \text{a.e. in} \quad \omega_{r^*},$$

whence the inclusion $\tilde{U}|_{\omega_{r^*}} \in R(\omega_{r^*})$ holds. As a result, in all possible cases we have $\tilde{U}|_{\omega_{r^*}} \in R(\omega_{r^*})$. It now remains to show that $\tilde{U}$ satisfies the inequality $[\tilde{U}]\nu \geq 0$ on $\gamma$. Bearing in mind the convergence (12), we can once again extract a subsequence satisfying $U_n|_\gamma \to \tilde{U}|_\gamma$ a.e. on $\gamma^\pm$. This allows us to pass to the limit through the following inequality

$$[U_n]\nu \geq 0 \quad \text{on} \quad \gamma.$$

This leads to $[\tilde{U}]\nu \geq 0$ on $\gamma$; therefore we have established that the inclusion $\tilde{U} \in K_{r^*}$.

Our next goals are to prove that $\tilde{U} = U_{r^*}$ and establish the existence of a sequence $U_n = U_{r_n}$, $n = 1, 2...$ of solutions strongly converging to $U_{r^*}$ in $H(\Omega_\gamma)$. Observe that, as $r_n \to r^*$, there must exist either a subsequence $\{r_{n_l}\}$ such that $r_{n_l} \leq r^*$ for all $l \in \mathbf{N}$ or, if that is not the case, a subsequence $\{r_{n_m}\}$, $r_{n_m} > r^*$ for all $m \in \mathbf{N}$. For this first case, we have the subsequence $\{r_{n_l}\} \subset (r_0, R]$ with the property $r_{n_l} \leq r^*$ for all $l \in \mathbf{N}$. For convenience, we denote this subsequence by $\{r_n\}$. Since $r_n \leq r^*$, it is noted that an arbitrary test function $W \in K_{r^*}$ also belongs to the set $K_{r_n}$. Consequently, it is possible to pass to the limit as $n \to \infty$ through the following inequalities with the test functions $W \in K_{r^*}$:

$$U_n \in K_{r_n}, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(W - U_n)dx \geq \int\limits_{\Omega_\gamma} F(W - U_n)dx, \quad r_n \in (r_0, r^*].$$

Now taking into account the weak convergence of $U_n$ to $\tilde{U}$ in $H(\Omega_\gamma)$, the limiting inequality takes the form

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(W - \tilde{U})dx \geq \int\limits_{\Omega_\gamma} F(W - \tilde{U})dx \quad \forall W \in K_{r^*}.$$

Due to the arbitrariness of $W \in K_{r^*}$ the last inequality is variational. Uniqueness of its solution yields the equality $\tilde{U} = U_{r^*}$. To complete the proof for the first case, we must establish the strong convergence $U_n \to U_{r^*}$. By substituting $W = 2U_r$ and $W = 0$ into the variational inequalities (4) for $r \in (r_0, R]$, we establish that

$$U_r \in K_r, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_r)\varepsilon_{ij}(U_r)dx = \int\limits_{\Omega_\gamma} FU_r dx \quad \forall\, r \in (r_0, R]. \tag{16}$$

In view of (4), the following relations may be established

$$U_r \in K_r, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_r)\varepsilon_{ij}(W)dx \geq \int\limits_{\Omega_\gamma} FW dx \quad \forall W \in K_r \tag{17}$$

which hold for all $r \in (r_0, R]$. The equalities (16), together with the weak convergence $U_n \to U_{r^*}$ in $H(\Omega_\gamma)$ as $n \to \infty$, imply that

$$\lim_{n\to\infty} \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)dx = \lim_{n\to\infty} \int\limits_{\Omega_\gamma} FU_n dx = \int\limits_{\Omega_\gamma} FU_{r^*} dx = \int\limits_{\Omega_\gamma} \sigma_{ij}(U_{r^*})\varepsilon_{ij}(U_{r^*})dx.$$

Since we have the equivalence of norms (see **Remark 1** in Section 2), one can see that $U_n \to U_{r^*}$ strongly in $H(\Omega_\gamma)$ as $n \to \infty$. Thus, in the first case we have obtained a contradiction to the assumption: $\|U_n - U_{r^*}\| \geq \epsilon$ for all $n \in \mathbf{N}$.

The second case is now considered. For convenience we keep the same notation for the subsequence. In doing so, we have $r_n \to r^*$ and $r_n > r^*$. Let us recall that by (10), we have $U_n \to \tilde{U}$ weakly in $H(\Omega_\gamma)$ as $n \to \infty$. We will in fact prove that $U_n \to \tilde{U}$ strongly in $H(\Omega_\gamma)$ as $n \to \infty$. In view of the weak convergence $U_n \to \tilde{U}$ in $H(\Omega_\gamma)$ as $n \to \infty$, the first relation in (16) may be used to show that

$$\lim_{n\to\infty} \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)dx = \int\limits_{\Omega_\gamma} F\tilde{U} dx. \tag{18}$$

Next, substituting $W = U_{r'} \in K_{r'} \subset K_r$, for arbitrary fixed numbers $r, r' \in (r_0, R]$ such that $r' \geq r$, in (17) as a test function, we arrive at the inequality

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(U_r)\varepsilon_{ij}(U_{r'})dx \geq \int\limits_{\Omega_\gamma} FU_{r'}dx.$$

We therefore conclude that for all $r_n$ and $r_m$ satisfying $r_n \leq r_m$ the following inequality is fulfilled

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(U_m)dx \geq \int\limits_{\Omega_\gamma} FU_m dx. \tag{19}$$

If we fix an arbitrary value $m$ in (19) and pass to the limit in the last relation as $n \to \infty$ it may be shown that

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(U_m)dx \geq \int\limits_{\Omega_\gamma} FU_m dx. \tag{20}$$

Passing to the limit in (20) as $m \to \infty$, confirms

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(\tilde{U})dx \geq \int\limits_{\Omega_\gamma} F\tilde{U}dx.$$

This inequality, the formula (18) and the weak lower semicontinuity of the bilinear form defined by the integral $\int\limits_{\Omega_\gamma} \sigma_{ij}(\cdot)\varepsilon_{ij}(\cdot)dx$ yield the following chain of relations

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(\tilde{U})dx \geq \int\limits_{\Omega_\gamma} F\tilde{U}dx = \lim_{n\to\infty} \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)dx \geq \int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(\tilde{U})dx,$$

indicating that

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(\tilde{U})dx = \lim_{n\to\infty} \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(U_n)dx.$$

Again, by the equivalence of norms (see **Remark 1** in Section 2) that $U_n \to \tilde{U}$ strongly in $H(\Omega_\gamma)$ as $n \to \infty$.

Now, let us prove that $\tilde{U} = U_{r^*}$. For this purpose we will analyze the variational inequality (4) and its limiting case. We can now apply the assertion of Lemma 1 to justify a passage to a limit in the variational inequalities. From Lemma 1, for any $W \in K_{r^*}$ there exist a subsequence $\{r_k\} = \{r_{n_k}\} \subset \{r_n\}$ and a sequence of functions $\{W_k\}$ such that $W_k \in K_{r_k}$ and $W_k \to W$ weakly in $H(\Omega_\gamma)$ as $k \to \infty$.

The properties established above for the convergent sequences $\{W_k\}$ and $\{U_n\}$ allow us to pass to the limit as $k \to \infty$ through following inequalities, derived from (4) for $r_k$ and with the test functions $W_k$:

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(U_k)\varepsilon_{ij}(W_k - U_k)dx \geq \int\limits_{\Omega_\gamma} F(W_k - U_k)dx.$$

As a result, we have

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\tilde{U})\varepsilon_{ij}(W - \tilde{U})dx \geq \int\limits_{\Omega_\gamma} F(W - \tilde{U})dx \quad \forall\, W \in K_{r^*}.$$

The unique solvability of this variational inequality implies that $\tilde{U} = U_{r^*}$.

Therefore, in either case, there exist a subsequence $\{r_{n_k}\} \subset \{r_n\}$ such that $r_k \to r^*$, $U_k \to U_{r^*}$ strongly in $H(\Omega_\gamma)$, which is a contradiction. The Lemma is thus proved.

## 5. Conclusion

In this paper, we have analyzed a family of variational problems describing equilibrium of cracked bodies with inclusions of different radii $r \in [r_0, R]$. The existence of the solution to the optimal control problem (6) is proved. For this problem, the cost functional $J(r)$ is defined by an arbitrary continuous functional, with $r$ the control parameter. Lemmas 1 and 2 establish a qualitative connection between the equilibrium problems for bodies with rigid circular inclusions of varying radii. This lemmas allow us to prove the strong convergence $U_r \to U_{r^*}$ in the Sobolev space $H(\Omega_\gamma)$, where $\{U_r\}$ are the solutions of (3) depending on the radius $r$.

The mathematical technique developed in the present work may be applied for another types of inhomogeneous bodies. For example, analogous problems for a three-dimensional elastic body may be investigated, as could the Reissner-Mindlin plate as well as Kirchhoff-Love plate reinforced by rigid inclusions. Within the theoretical framework of developed methodology, various cases of different rigid inclusion shapes could also be considered.

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